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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES *Volume XXXVII-2 (1996)*

CONSTRUCTING QUANTALES AND THEIR MODULES FROM MONOIDAL CATEGORIES

by Susan B. NIEFIELD

RESUME. Etant donné une catégorie monoïdale S, l'Auteur établit une adjonction entre la catégorie des catégories monoïdales sur S et la (duale de la) catégorie des monoïdes dans S. L'adjoint à droite associe à un monoïde *M* la catégorie des *M*-bimodules dans S. L'adjoint à gauche est donné par évaluation en l'objet unité *I*. Ce cadre permet une approche unifiée de nombreux exemples de quantales et de leurs modules, en montrant que toute catégorie monoïdale complète 'well-powered' avec factorisation est monoïdale sur la catégorie des sup-lattices, relativement au foncteur sous-objet *Sub*. En conclusion on donne des conditions pour que le quantale *Sub*(*I*) soit un locale.

1 Introduction

A **quantale** is a complete lattice Q together with an associative unital operation $\cdot$ such that $a \cdot (\bigvee_\alpha b_\alpha) = \bigvee_\alpha (a \cdot b_\alpha)$ and $(\bigvee_\alpha b_\alpha) \cdot a = \bigvee_\alpha (b_\alpha \cdot a)$. Equivalently, Q is a monoid in the category **Slat** of suplattices (i.e., complete lattices and sup preserving maps). The main examples of interest in this paper include the lattice $Id(R)$ of ideals of a ring R, the power set $\mathcal{P}(M)$ of a monoid M, and any locale L.

A **left Q-module** is a complete lattice X together with an associative unital action $\cdot$ of Q on X such that $a \cdot (\bigvee_\alpha x_\alpha) = \bigvee_\alpha (a \cdot x_\alpha)$ and $(\bigvee_\alpha a_\alpha) \cdot x = \bigvee_\alpha (a_\alpha \cdot x)$. Each of the main examples mentioned above has a related class of modules. The lattice $Sub_R(M)$ of submodules of a left R-module M, the power set $\mathcal{P}(X)$ of a left M-set X, and any locale over L are modules over $Id(R)$, $\mathcal{P}(M)$, and L, respectively. **Right Q-modules** and **Q-bimodules** are defined similarly.

The name "quantale" was proposed by Mulvey [12] to fill a noncommutative analogue in quantum logic of the role played by locales (i.e., complete Heyting algebras) in intuitionistic logic. Commutative quantales were considered earlier by Dilworth [3] under the name "multiplicative lattices" to provide an abstract setting in which to study the ideals of a commutative ring with unit. Continuing this approach in his duality theory for R-modules, Anderson [2] used modules over commutative quantales, calling them "fake modules" and "fake rings". Niefield and Rosenthal used the quantale $Id(R)$ in their work ([13]and [14]) on algebraic de Morgan's laws. Modules over a commutative quantale were also used by Joyal and Tierney [8] in their work on descent theory. The quantales $\mathcal{P}(M)$ arose in Lambek's [10] formal language theory and Girard's [6] linear logic. The left $\mathcal{P}(M)$-modules $\mathcal{P}(X)$ can be found in the labeled transition systems of Ambramsky and Vickers [1], but there the action was induced by a relation rather than a function.

One might ask how closely the three main examples of quantales and their modules are related to each other. For example, is it necessary to directly verify that the objects in question are in fact quantales or modules, or do they arise from a single construction? This question was partly answered (at least in the commutative case) by Niefield and Rosenthal [15] where a general construction of a commutative quantale from a (symmetric monoidal) closed category was presented. First, a lax adjunction was established between the lax comma category $\mathcal{C}/\!\!/\mathbf{Slat}$ of closed categories over **Slat** and (the dual of) the category of commutative quantales. The left adjoint was given by evaluating a closed functor $p: \mathbf{V} \longrightarrow \mathbf{Slat}$ at the unit object I. Then it was shown that the subobject functor $Sub_{\mathbf{V}}: \mathbf{V} \longrightarrow \mathbf{Slat}$ is closed, for certain suitable closed categories $\mathbf{V}$. In this case, the quantale produced by the adjunction was called the "ideals of $\mathbf{V}$". The quantales locales L and $Id(R)$ arose as the ideals of the categories $\mathbf{Sh}(L)$ and $\mathbf{Mod}(R)$ of sheaves on L and R-modules, respectively. Though it was not considered in [15], the commutative case of the third example (under consideration here) is also of this form, namely $\mathcal{P}(M)$ is the quantale of ideals of the category of M-sets.

The main goal of this paper is to generalize the results of [15] to include the quantales $Id(R)$ and $\mathcal{P}(M)$ in the noncommutative case. In doing so, it will be shown that the symmetric and closed assumptions are unnecessary, that the base category **Slat** can be replaced by any monoidal category with coequalizers which are preserved by $\otimes$, and that the adjunction need not be lax.

We begin (in §2) with a presentation of the adjunction, including a brief outline of the necessary background on monoidal categories, monoids, and bimodules. In Section 3, we show that the subobject functor $Sub_{\mathbf{V}}: \mathbf{V} \longrightarrow \mathbf{Slat}$ is monoidal for suitable monoidal categories $\mathbf{V}$, thus obtaining the desired examples of quantales. As a special case, we see that if M is a monoid in such a category $\mathbf{V}$, then the set $Id(M)$ of ideals of M is a quantale. We conclude (in §4) by showing that $Id(M)$ is a locale, whenever M is an idempotent monoid in $\mathbf{V}$.

2 The Adjunction

We begin this section with a review of monoids and modules in monoidal categories. Some of this background material is "folklore", but the basic definitions and a thorough treatment of coherence can be found in MacLane [11], Eilenberg and Kelly [4], and Kelly [9].

Recall that a **monoidal category** consists of a category $\mathbf{V}$, an object I (called the **unit**), a bifunctor $\otimes: \mathbf{V} \times \mathbf{V} \longrightarrow \mathbf{V}$ and three natural isomorphisms $\alpha: V \otimes (V' \otimes V'') \longrightarrow (V \otimes V') \otimes V''$, $\lambda: I \otimes V \longrightarrow V$, and $\rho: V \otimes I \longrightarrow V$ such that $\lambda_I = \rho_I$ and the pentagonal diagrams (in sense of MacLane [11]) as well as the triangular diagrams

$$\begin{array}{ccc} V \otimes (I \otimes V') & \xrightarrow{\alpha} & (V \otimes I) \otimes V' \\ & {\scriptstyle 1 \otimes \lambda} \searrow \quad \swarrow {\scriptstyle \rho \otimes 1} & \\ & V \otimes V' & \end{array}$$

commute.

Examples include cartesian categories with $\otimes = \times$ and $I = 1$, the terminal object 1, as well as **Ab** (i.e., abelian groups) and **Slat** with their usual tensor products and units $\mathbf{Z}$ and $\mathbf{2}$, respectively (see [8] for **Slat**). Similarly, the categories $\mathbf{Ab(E)}$ and $\mathbf{Slat(E)}$ of abelian groups and suplattices in a topos $\mathbf{E}$ are monoidal. Note that if L is a locale, then $\mathbf{Ab(Sh}(L))$ is the category of sheaves of abelian groups on L and $\mathbf{Slat(Sh}(L))$ is equivalent to the category of L-modules (see [8] for details). Finally, the category **Rel** of sets and relations is also monoidal via $\otimes = \times$, the cartesian product, but **Rel** is not cartesian closed since the cartesian product is not the categorical product in **Rel**.

A **monoid** in a monoidal category $\mathbf{V}$ is an object M together with morphisms $\cdot: M \otimes M \longrightarrow M$ and $e: I \longrightarrow M$ such that the usual associativity diagram (see MacLane [11]) and the unit diagram

$$\begin{array}{ccccc} I \otimes M & \xrightarrow{e \otimes 1} & M \otimes M & \xleftarrow{1 \otimes e} & M \otimes I \\ & {\scriptstyle \lambda} \searrow & \downarrow {\scriptstyle \cdot} & \swarrow {\scriptstyle \rho} & \\ & & M & & \end{array}$$

commute. Let $\mathbf{Mon(V)}$ denote the category of monoids and homomorphisms in $\mathbf{V}$, i.e., morphisms of $\mathbf{V}$ which preserve $\cdot$ and e.

Now, the unit I is clearly a monoid in $\mathbf{V}$ via $\lambda_I = \rho_I: I \otimes I \longrightarrow I$ and the identity $1: I \longrightarrow I$, by the coherence conditions in the definition of monoidal category. If fact, I is the initial object of $\mathbf{Mon(V)}$, since the unit $e: I \longrightarrow M$ is the unique homomorphism from I to M, for any monoid M in $\mathbf{V}$.

Note that **Mon(Sets)** is the usual category of monoids, **Mon(Ab)** is the category **Ring** of rings with unit, and **Mon(Slat)** is the category **Quant** of quantales. Also, **Mon(Ab(Sh(**L**)))** is the category of sheaves of rings on L.

Now, let M be a monoid in a monoidal category **V**. A **left** M**-module** in **V** is an object X together with a morphism $\cdot : M \otimes X \to X$ such that the following diagrams

$$\begin{array}{ccc} M \otimes (M \otimes X) & \xrightarrow{\alpha} & (M \otimes M) \otimes X \\ {\scriptstyle 1\otimes\cdot}\downarrow & & \downarrow{\scriptstyle \cdot\otimes 1} \\ M \otimes X & & M \otimes X \\ & {\scriptstyle\cdot}\searrow \quad \swarrow{\scriptstyle\cdot} & \\ & X & \end{array} \qquad \begin{array}{ccc} I \otimes X & \xrightarrow{e \otimes 1} & M \otimes X \\ & {\scriptstyle\lambda}\searrow \quad \swarrow{\scriptstyle\cdot} & \\ & X & \end{array}$$

commute. Right M-module and M-bimodules are defined similarly, with the additional commutative diagram

$$\begin{array}{ccc} M \otimes (X \otimes M) & \xrightarrow{\alpha} & (M \otimes X) \otimes M \\ {\scriptstyle 1\otimes\cdot}\downarrow & & \downarrow{\scriptstyle \cdot\otimes 1} \\ M \otimes X & & X \otimes M \\ & {\scriptstyle\cdot}\searrow \quad \swarrow{\scriptstyle\cdot} & \\ & X & \end{array}$$

in the latter case. Let $\mathbf{Mod}_M(\mathbf{V})$ denote the category of M-bimodules and homomorphisms in **V**, i.e., morphisms of **V** which are compatible with the action of M. Then it is not difficult to show that the functor $M \otimes - \otimes M: \mathbf{V} \to \mathbf{Mod}_M(\mathbf{V})$ is left adjoint to the forgetful functor $u: \mathbf{Mod}_M(\mathbf{V}) \to \mathbf{V}$.

For every object X of **V**, the morphism

$$\lambda: I \otimes X \to X \quad \text{and} \quad \rho: X \otimes I \to X$$

define left and right I-module structures on X, by the triangular diagram in the definition of monoidal category. Compatibility of the operations follows from the coherence theorem for monoidal categories, and so $\mathbf{Mod}_I(\mathbf{V}) \cong \mathbf{V}$.

Note that $\mathbf{Mod}_M(\mathbf{Sets})$ is the category of 2-sided M-sets, $\mathbf{Mod}_R(\mathbf{Ab})$ is the category of R-bimodules, $\mathbf{Mod}_Q(\mathbf{Slat})$ is the category of left Q-bimodules, and $\mathbf{Mod}_{\mathcal{R}}(\mathbf{Sh}(L))$ is the category of sheaves of $\mathcal{R}$-bimodules, with analogous results for right and left modules.

Proposition 2.1 *If* $\mathbf{V}$ *is a monoidal category with coequalizers which are preserved by* $\otimes$, *then* $\mathbf{Mod}_M(\mathbf{V})$ *is a monoidal category with unit object* M.

Proof. Given $X, X' \in \mathbf{Mod}_M(\mathbf{V})$, consider the coequalizer

$$X \otimes M \otimes X' \underset{\cdot \otimes 1}{\overset{1 \otimes \cdot}{\rightrightarrows}} X \otimes X' \xrightarrow{c} X \otimes_M X'$$

Since $M \otimes -$ preserves coequalizers, $X \otimes_M X'$ is a left M-module via the following diagram

$$\begin{array}{ccccc}
M \otimes X \otimes M \otimes X' & \underset{1 \otimes \cdot \otimes 1}{\overset{1 \otimes \cdot}{\rightrightarrows}} & M \otimes X \otimes X' & \xrightarrow{1 \otimes c} & M \otimes (X \otimes_M X') \\
{\scriptstyle \cdot \otimes 1} \downarrow & & \downarrow {\scriptstyle \cdot \otimes 1} & & \downarrow \\
X \otimes M \otimes X' & \underset{\cdot \otimes 1}{\overset{1 \otimes \cdot}{\rightrightarrows}} & X \otimes X' & \xrightarrow{c} & X \otimes_M X'
\end{array}$$

The right action is defined similarly and compatibility of the actions follows from that of X and X'. The associativity and unit isomorphisms are induced by those of $\mathbf{V}$ using the fact that $\otimes$ preserves coequalizers.

To obtain functorial monoid and module constructions, it is necessary to consider the appropriate functors between monoidal categories $\mathbf{V}$ and $\mathbf{V}'$. Recall that a **monoidal functor** is a functor $f: \mathbf{V} \to \mathbf{V}'$ together with a morphism $\phi^o: I' \to fI$ and a natural transformation $\phi: fV \otimes fV' \to f(V \otimes V')$ which are compatible with α, λ and ρ.

Given such a functor and a monoid M in $\mathbf{V}$, it is not difficult to show that fM is a monoid in $\mathbf{V}'$ via

$$fM \otimes fM \xrightarrow{\phi} f(M \otimes M) \xrightarrow{f(\cdot)} fM \qquad \text{and} \qquad I' \xrightarrow{\phi^o} fI \xrightarrow{f(e)} fM$$

In particular, for the associativity (respectively, unit) diagram one applies f to the associativity (respectively, unit) diagram for M and uses the naturality of ϕ, and compatibility of ϕ with α (respectively, λ and ρ). If $h: M \to M'$ is a homomorphism, then so is $fh: fM \to fM'$ (by naturality of ϕ), and so we obtain a functor $Mon(f): \mathbf{Mon}(\mathbf{V}) \to \mathbf{Mon}(\mathbf{V}')$.

Similarly, if X is an M-bimodule in $\mathbf{V}$, then it is not difficult to show that fX is an fM-bimodule in $\mathbf{V}'$ via

$$fM \otimes fX \xrightarrow{\phi} f(M \otimes X) \xrightarrow{f(\cdot)} fX$$

If $h: X \to X'$ is a homomorphism, then so is $fh: fX \to fX'$ (by naturality of ϕ), and so we obtain a functor $Mod(f): \mathbf{Mod}_M(\mathbf{V}) \to \mathbf{Mod}_{fM}(\mathbf{V}')$. If, in addition, $\mathbf{V}$ and $\mathbf{V}'$ have coequalizers which are preserved by $\otimes$, then $Mod(f)$ is monoidal via $1: fM \longrightarrow fM$ and the morphism $fX \otimes_{fM} fX' \longrightarrow f(X \otimes_M X')$ induced by the coequalizer defining $fX \otimes_{fM} fX'$ using the monoidality of f.

Let **Moncat** denote the 2-category of monoidal categories, monoidal functors, and monoidal natural transformations (i.e., 2-cells $\theta: f \Rightarrow f'$ which are compatible with ϕ^o and ϕ). If $\mathbf{S}$ is a fixed monoidal category, then $\mathbf{Moncat/S}$ denotes the usual 2-category of monoidal categories over $\mathbf{S}$, i.e., objects are monoidal functors $p: \mathbf{V} \longrightarrow \mathbf{S}$, morphisms are commutative triangles

$$\begin{array}{ccc} \mathbf{V} & \xrightarrow{\quad f \quad} & \mathbf{V}' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & \mathbf{S} & \end{array}$$

of monoidal functors, and 2-cells $\theta: f \Rightarrow f'$ are monoidal natural transformations such that $p'\theta = 1_p$.

Proposition 2.2 *If* $\mathbf{S}$ *is a monoidal category, then evaluation at the unit defines a 2-functor*

$$ev: \mathbf{Moncat/S} \longrightarrow \mathbf{Mon(S)}^{op}$$

where $\mathbf{Mon(S)}$ *is a discrete 2-category.*

Proof. First, $ev(p: \mathbf{V} \longrightarrow \mathbf{S}) = pI$ is a monoid in $\mathbf{S}$, since I is a monoid in $\mathbf{V}$. Given a monoidal functor $f: \mathbf{V} \longrightarrow \mathbf{V}'$ over $\mathbf{S}$, we know fI is a monoid in $\mathbf{V}'$. Since I' is the initial object of $\mathbf{V}'$, applying p' to the unique homomorphism $I' \longrightarrow fI$ yields the desired homomorphism $p'I' \longrightarrow p'fI = pI$. Finally, a monoidal natural transformation $\theta: f \Rightarrow f'$ over $\mathbf{S}$ induces the identity 2-cell since $p'\theta = 1_p$, and 2-functoriality easily follows.

Proposition 2.3 *If* $\mathbf{S}$ *is a monoidal category with coequalizers which are preserved by* $\otimes$*, then there is a functor*

$$mod: \mathbf{Mon(S)}^{op} \longrightarrow \mathbf{Moncat/S}$$

which takes a monoid M *to the forgetful functor* $u: \mathbf{Mod}_M(\mathbf{S}) \longrightarrow \mathbf{S}$.

Proof. Note that u is monoidal via $e: I \longrightarrow M$ and $c: X \otimes X' \longrightarrow X \otimes_M X'$. If $f: M \longrightarrow M'$ is a homomorphism, then the functor $f^*: \mathbf{Mod}_{M'}(\mathbf{S}) \longrightarrow \mathbf{Mod}_M(\mathbf{S})$ given by $f^*X = X$, with action by restriction of scalars, is a monoidal functor over $\mathbf{S}$ via $f: M \longrightarrow M'$ and $\phi: X \otimes_M X' \longrightarrow X \otimes_{M'} X'$ induced by the following coequalizer defining $X \otimes_M X'$

$$\begin{array}{ccccccc} X \otimes M \otimes X' & \xrightarrow{1 \otimes f \otimes 1} & X \otimes M' \otimes X' & \overset{1 \otimes \cdot}{\underset{\cdot \otimes 1}{\rightrightarrows}} & X \otimes X' & \xrightarrow{c} & X \otimes_M X' \\ & & & & & {\scriptstyle c'}\searrow & \downarrow{\scriptstyle \phi} \\ & & & & & & X \otimes_{M'} X' \end{array}$$

and compatibility easily follows.

Theorem 2.4 *If* **S** *is a monoidal category with coequalizers which are preserved by* $\otimes$, *then ev is left adjoint to mod.*

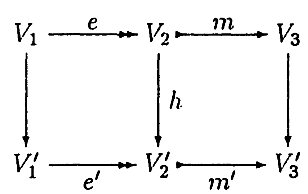
Proof. The identity $1: M \longrightarrow M$ is the counit $\epsilon_M: M \longrightarrow ev(mod(M))$. The unit $\eta_p: p \longrightarrow mod(ev(p))$ is the composite of the isomorphism $\mathbf{V} \cong \mathbf{Mod}_I(\mathbf{V})$ and $Mod(p): \mathrm{Mod}_I(\mathrm{V}) \longrightarrow \mathrm{Mod}_{pI}(\mathrm{S})$. The adjunction identities are easily established.

Applying the adjunction in the case where $\mathbf{S} = \mathbf{Ab}$, we see that if $p: \mathbf{V} \longrightarrow \mathbf{Ab}$ is monoidal, then there is a natural bijection between ring homomorphisms $R \longrightarrow pI$ and monoidal functors $\mathbf{V} \longrightarrow \mathbf{Mod}_R(\mathbf{Ab})$ over $\mathbf{Ab}$. In this (as well as the general) case, since $ev \circ mod = 1$, it follows that the functor *mod* is full and faithful.

3 The Subobject Functor

We will see that **Sets**, **Ab**, **Slat** and their related module categories can be viewed as monoidal categories over **Slat** by showing that the suboject functor $Sub_{\mathbf{V}}: \mathbf{V} \longrightarrow \mathbf{Slat}$ is monoidal for suitable categories **V**. Note that this is essentially the same construction used in [15] in the symmetric case, but the generalization is necessary so that we can consider categories of bimodules over (not necessarily commutative) monoids, rings, and quantales.

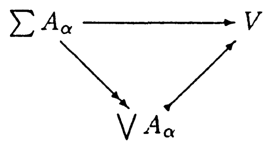
Suppose that **V** is any category, and let $\mathcal{E}$ and $\mathcal{M}$ be two classes of morphisms which each contain the isomorphisms and are closed under composition. Recall that $(\mathcal{E}, \mathcal{M})$ is called a **factorization system**, if every morphism f has a factorization $f = me$, where $e \in \mathcal{E}$ and $m \in \mathcal{M}$ such that every rectangular diagram of the following form admits a unique morphism h such that the resulting squares commute.



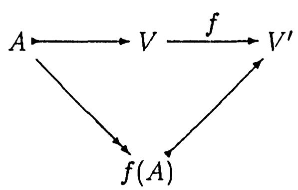
If every object V of **V** has only a set of $\mathcal{M}$-subobjects, then **V** is called **$\mathcal{M}$-well-powered** or simply **well-powered**. Note that a factorization system is called **proper** if every $e \in \mathcal{E}$ is an epimorphism and every $m \in \mathcal{M}$ is a monomorphism. For properties of proper factorization systems, we refer the reader to Freyd and Kelly [5]. Note that proper factorization systems were considered by Isbell [7] under the name **bicategorical structure**.

A monoidal category $\mathbf{V}$ which is well-powered with respect to a specified proper factorization system $(\mathcal{E}, \mathcal{M})$ such that $\mathcal{E}$ is closed under $\otimes$ will be called **a well-powered monoidal category with proper factorization.** If, in addition, $\mathbf{V}$ has colimits which are preserved by $\otimes$ in each variable, then $\mathbf{V}$ will be called a **suitable monoidal category.**

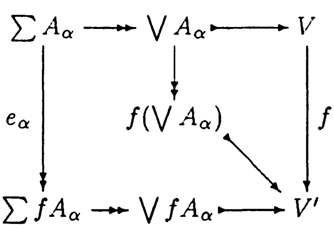
If V is an object of a suitable monoidal category $\mathbf{V}$, then the set $Sub_{\mathbf{V}}(V)$ of $\mathcal{M}$-subobjects of V is a complete lattice with $\bigvee A_\alpha$ given by



If $f: V \longrightarrow V'$, then the direct image $f: Sub_{\mathbf{V}}(V) \longrightarrow Sub_{\mathbf{V}}(V')$ given by



is sup-preserving, since $f(\bigvee A_\alpha) = \bigvee f(A_\alpha)$ by uniqueness of the factorization of the morphism $\sum A_\alpha \longrightarrow V'$ in the diagram



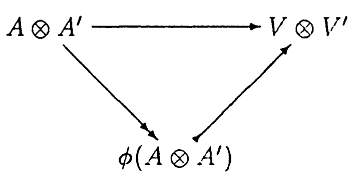
where $\sum e_\alpha \in \mathcal{E}$ since $\mathcal{E}$ is closed under coproducts. Thus, we get:

Proposition 3.1 *If* $\mathbf{V}$ *is a suitable monoidal category, then* $Sub_{\mathbf{V}}: \mathbf{V} \longrightarrow \mathbf{Slat}$ *is a monoidal functor.*

Proof. Functoriality follows from the universal property of the image factorization. The map $\phi^o: \mathbf{2} \longrightarrow Sub_{\mathbf{V}}(I)$ is the obvious one. To define

$$\phi_{VV'}: Sub_{\mathbf{V}}(V) \otimes Sub_{\mathbf{V}}(V') \longrightarrow Sub_{\mathbf{V}}(V \otimes V')$$

given $A \in Sub_{\mathbf{V}}(V)$ and $A' \in Sub_{\mathbf{V}}(V')$, consider



Bilinearity of ϕ follows from the universal property and the fact that $\otimes$ preserves coproducts and members of $\mathcal{E}$.

Since **Mon(Slat)** = **Quant**, we get the following corollary.

Corollary 3.2 *If* $\mathbf{V}$ *is a suitable monoidal category, then* $Sub_{\mathbf{V}}: \mathbf{V} \longrightarrow \mathbf{Slat}$ *restricts to functor* $Sub_{\mathbf{V}}: \mathbf{Mon}(\mathbf{V}) \longrightarrow \mathbf{Quant}$.

Thus, **Sets**, **Ab**, and **Slat** can be viewed as monoidal categories over **Slat** via the suboject functor, since each is easily seen to be a suitable monoidal category. When $\mathbf{V} = \mathbf{Sets}$, the subobject functor is the power set functor $\mathcal{P}: \mathbf{Sets} \longrightarrow \mathbf{Slat}$, and so by Corollary 3.2, we see that $\mathcal{P}(M)$ is a quantale for every monoid M and the direct image $f: \mathcal{P}(M) \longrightarrow \mathcal{P}(M')$ is a quantale homomorphism for every monoid homomorphism $f: M \longrightarrow M'$. When $\mathbf{V} = \mathbf{Ab}$, $Sub_{\mathbf{Ab}}: \mathbf{Ab} \longrightarrow \mathbf{Slat}$ associates to the group G the lattice $Sub_{\mathbf{Ab}}(G)$ of subgroups of G, and so by Corollary 3.2, we see that $Sub_{\mathbf{Ab}}(R)$ is a quantale for every ring R and the direct image $f: Sub_{\mathbf{Ab}}(R) \longrightarrow Sub_{\mathbf{Ab}}(R')$ is a quantale homomorphism for every ring homomorphism $f: R \longrightarrow R'$. However, in the case of rings, the quantale of interest (which we will consider later) is $Id(R)$ not $Sub_{\mathbf{Ab}}(R)$. When $\mathbf{V} = \mathbf{Slat}$, $Sub_{\mathbf{Slat}}: \mathbf{Slat} \longrightarrow \mathbf{Slat}$ associates to the suplattice X the lattice $Sub_{\mathbf{Slat}}(X)$ of subsuplattices of X, (i.e., $A \subseteq X$ which are closed under sups) and so by Corollary 3.2, we see that $Sub_{\mathbf{Slat}}(Q)$ is a quantale for every quantale Q and the direct image $f: Sub_{\mathbf{Slat}}(Q) \longrightarrow Sub_{\mathbf{Slat}}(Q')$ is a quantale homomorphism for every quantale homomorphism $f: Q \longrightarrow Q'$. As in the case of rings, the quantale of interest is $Id(Q)$ not $Sub_{\mathbf{Slat}}(Q)$.

Corollary 3.3 *If* $\mathbf{V}$ *is a suitable monoidal category, then*

$$Sub_{\mathbf{V}}: \mathbf{V} \longrightarrow \mathbf{Mod}_{Sub_{\mathbf{V}}(I)}(\mathbf{Slat})$$

is a monoidal functor.

Proof. This is the unit $\eta_{Sub_{\mathbf{V}}}$ of the adjunction in Theorem 2.3, in the case $\mathbf{S} = \mathbf{Slat}$.

Proposition 3.4 *If* $\mathbf{V}$ *is a suitable monoidal category, then so is* $\mathbf{Mod}_M(\mathbf{V})$.

Proof. First, we show that the restriction of $(\mathcal{E}, \mathcal{M})$ to $\mathbf{Mod}_M(\mathbf{V})$ is a factorization system which is clearly proper. Given a morphism $f: X \longrightarrow X'$ of M-bimodules, factor f as $X \xrightarrow{e} X'' \xrightarrow{m} X'$ in $\mathbf{V}$. Then X'' becomes an M-bimodule (such that e and m are M-homomorphisms) via the universal property of factorizations, since $\mathcal{E}$ is closed under $M \otimes -$ and $- \otimes M$. The universal property of factorizations in $\mathbf{Mod}_M(\mathbf{V})$ follows from that of $\mathbf{V}$, again since $\mathcal{E}$ is closed under $M \otimes -$ and $- \otimes M$. Next, it is not difficult to show that the restriction of $\mathcal{E}$ is closed under $\otimes_M$, since $\mathcal{E}$ is closed under $\otimes$. Also, $\mathbf{Mod}_M(\mathbf{V})$ is well-powered, since the forgetful functor $u: \mathbf{Mod}_M(\mathbf{V}) \longrightarrow \mathbf{V}$ is faithful. Finally, cocompleteness of $\mathbf{Mod}_M(\mathbf{V})$ follows from the fact that $\mathbf{V}$ is cocomplete and u has a left adjoint. Since $\otimes$ preserves colimits, it follows that colimits are formed in $\mathbf{V}$ and given the actions induced by those on the terms and $\otimes_M$ preserves colimits, to complete the proof.

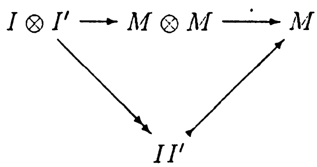
If M is a monoid in a suitable monoidal category $\mathbf{V}$ and X is an M-bimodule, let $Sub_M(X)$ denote the set of M-subbimodules of X, i.e., $Sub_{\mathbf{Mod}_M(\mathbf{V})}(X)$. In the case $X = M$, the M-submodules are called **ideals** of M and $Sub_M(M)$ is denoted by $Id(M)$. Note that this agrees with the usual notion of (2-sided) ideal when $\mathbf{V}$ is **Sets**, **Ab**, or **Slat**. Combining Corollary 3.3 and Propositions 3.4, we get:

Corollary 3.5 *If M is a monoid in a suitable monoidal category $\mathbf{V}$, then*

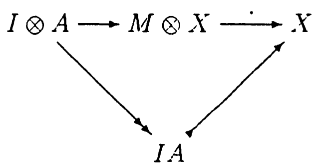
$$Sub_M: \mathbf{Mod}_M(\mathbf{V}) \longrightarrow \mathbf{Mod}_{Id(M)}(\mathbf{Slat})$$

is a monoidal functor.

It is not difficult to show that the product of I and I' in $Id(M)$ is given by



and the left action of $Id(M)$ on $Sub_M(X)$ is given by



with the right action is defined similarly.

Applying Corollary 3.5 to $\mathbf{V} = \mathbf{Ab}$, we see that if R is a ring and M is an R-bimodule, then the lattice $Sub_R(M)$ of M- submodules is an $Id(R)$-bimodule, and if $h: M \longrightarrow M'$ is a homomorphism of R-bimodules, then $h: Sub_R(M) \longrightarrow Sub_R(M')$ is a homomorphism of $Id(R)$-bimodules. Similar results hold for **Sets**, **Slat**, **Mon(Sh(L))**, **Ab(Sh(L))**, and **Slat(Sh(L))**. Note that the ideal construction is not functorial unless $\mathbf{V}$ is symmetric.

We conclude this section with a few remarks about applications to transition systems. Although this approach can also be used to obtain the right module structure of interest in the Abramsky and Vickers article [1], we must go beyond **Sets** to **Rel** to do so. As usual, morphisms of **Rel** will be denoted by $X \nrightarrow Y$ to distinguish them from functions. Since relations $X \nrightarrow Y$ correspond to functions $X \rightarrow \mathcal{P}(Y)$, it follows that the (monoidal) functor **Sets** $\rightarrow$ **Rel** is left adjoint to the power set (monoidal) functor $\mathcal{P}$: **Rel** $\rightarrow$ **Sets**. Moreover, one can show that **Rel** is a suitable monoidal category and that composing $\mathcal{P}$ with the subobject functor $\mathcal{P}$: **Sets** $\rightarrow$ **Slat** yields the subobject functor $Sub_{\mathbf{Rel}}$: **Rel** $\rightarrow$ **Slat**. However, this is not the monoidal functor of interest here. Instead, it is the (direct image) power set functor $\mathcal{P}$: **Rel** $\rightarrow$ **Slat**. Note that $\mathcal{P}$ is monoidal, since $\mathcal{P}(X) \otimes \mathcal{P}(Y) \cong \mathcal{P}(X \times Y)$ (see [8]).

Recall that a transition system labeled over a set Act is a set $Proc$ together with a relation $\rightarrow \subseteq Proc \times Act \times Proc$, whose elements are denoted $p \rightarrow^s q$. This relation clearly extends to $\rightarrow \subseteq Proc \times Act^* \times Proc$, where Act^* is the free monoid on Act. Thus, $\mathcal{P}(Proc)$ becomes a right $\mathcal{P}(Act^*)$-module via

$$X \cdot A = \{q \in Proc | \exists p \in X, s \in A \text{ such that } p \rightarrow^s q\}$$

Now, Act^* is a monoid in **Rel**, since the inclusion **Sets** $\rightarrow$ **Rel** is monoidal, and the relation $Proc \times Act^* \nrightarrow Proc$ given by $p \rightarrow^s q$ clearly gives $Proc$ the structure of a right M-module in **Rel**. Applying the right-sided version of the module construction to $\mathcal{P}$: **Sets** $\rightarrow$ **Rel**, we get a functor

$$Mod(\mathcal{P}): \mathbf{Mod}_{Act^*}(\mathbf{Rel}) \rightarrow \mathbf{Mod}_{\mathcal{P}(Act^*)}(\mathbf{Slat})$$

and hence, a right module $\mathcal{P}(Proc) \otimes \mathcal{P}(Act^*) \nrightarrow \mathcal{P}(Proc)$, which is easily seen to agree with the one described above.

4 Locales

It is well known that a quantale Q is a locale if and only if it is idempotent (i.e., $x^2 = x$ for all $x \in Q$) and the unit e is the top element $\top$. The commutative case is in [8], and the general case is similar. Since $e = \top$ implies that $x^2 \leq x$, it follows that Q is a locale if and only if $e = \top$ and $x \leq x^2$, for all $x \in Q$. A quantale such that $x \leq x^2$, for all $x \in Q$, will be called **subidempotent**.

In this section, we will consider conditions on a monoid M in a suitable monoidal category $\mathbf{V}$ under which the quantale $Sub_{\mathbf{V}}(M)$ is a locale. Special cases will include the locales $Sub(1)$ of subobjects of 1 in a Grothendieck topos, as well as $Id(M)$, $Id(R)$, and $Id(Q)$, for an idempotent monoid, ring, or quantale. In view of the above remarks, we would like to consider idempotent monoids in $\mathbf{V}$. This can be done with the aid of a diagonal morphism $M \rightarrow M \otimes M$. However, in most of the examples of interest, the diagonal is not actually a morphism in $\mathbf{V}$, but rather in **Sets**. Therefore, we must begin with a general setting in which idempotency makes sense.

Let $c: \mathbf{V} \longrightarrow \mathbf{C}$ be a monoidal functor, where $\mathbf{V}$ is a monoidal category and $\mathbf{C}$ is a cartesian closed category. If M is a monoid in $\mathbf{V}$, we say M is **idempotent relative to** c, if the composite

$$c(M) \xrightarrow{\Delta} c(M) \times c(M) \xrightarrow{\gamma} c(M \otimes M) \xrightarrow{c(\cdot)} c(M)$$

is the identity. Note that if $\mathbf{V}$ is **Sets**, **Ab**, **Slat**, or any of their related bimodule categories, the usual notion of an idempotent monoid agrees with the definition of idempotent relative to the forgetful functor $c: \mathbf{V} \longrightarrow$ **Sets**.

Note that for every object V of $\mathbf{V}$, c induces an order-preserving map $c_\#: Sub_{\mathbf{V}}(V) \longrightarrow Sub_{\mathbf{C}}(cV)$ given by

$$\begin{array}{ccc} cA & \longrightarrow & cV \\ & \searrow \quad \nearrow & \\ & c_\# A & \end{array}$$

In the case where $\mathbf{V}$ is **Sets**, **Ab**, **Slat**, or bimodules over a monoid in any of these categories, and c is the forgetful functor $u: \mathbf{V} \longrightarrow$ **Sets**, it is not difficult to show that $c_\#$ is also order-reflecting since it has a left inverse left adjoint.

Theorem 4.1 *If M is idempotent relative to a monoidal functor $c: \mathbf{V} \longrightarrow \mathbf{C}$ (where $\mathbf{V}$ and $\mathbf{C}$ are as above) and the induced maps $c_\#$ are order-reflecting, then $Sub_{\mathbf{V}}(M)$ is a subidempotent quantale and $Id(M)$ is a locale.*

Proof. Given a subobject A of M, consider the diagram

$$\begin{array}{ccccccc} cA & \longrightarrow & cA \times cA & \xrightarrow{\gamma} & c(A \otimes A) & \longrightarrow & c(A^2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ cM & \longrightarrow & cM \times cM & \xrightarrow[\gamma]{} & c(M \otimes M) & \longrightarrow & cM \end{array}$$

which is commutative by naturality of γ and definition of A^2. Since M is idempotent, we see that the morphism $cA \longrightarrow cM$ factors through $c(A^2) \longrightarrow cM$, and so by the universal property of factorization $c_\# A \leq c_\#(A^2)$. Thus, $A \leq A^2$, since $u_\#$ reflects order, and so $Sub_{\mathbf{V}}(M)$ is subidempotent. Since the unit M of $Id(M)$ is the top element and $Id(M) = Sub_{\mathbf{Mod}_M(\mathbf{V})}(M)$, to see that $Id(M)$ is a locale, it suffices to show that the hypotheses of the theorem hold for the composite $\mathbf{Mod}_M(\mathbf{V}) \xrightarrow{u} \mathbf{V} \xrightarrow{c} \mathbf{C}$, i.e., each $(cu)_\#$ is order-reflecting. To do so, it suffices to show that $u_\#: Sub_M(X) \longrightarrow Sub_{\mathbf{V}}(uX)$ has a left inverse. Given a subobject B of uX, let $u^\# B$ denote the image of the M-bimodule homomorphism $M \otimes B \otimes M \longrightarrow X$. Then $u^\#$ is clearly order-preserving. To see that $u^\# u_\# A = A$, consider the commutative diagram

$$\begin{array}{ccccc} M\otimes A\otimes M & \xrightarrow{e} & M\otimes u_{\#}A\otimes M & \longrightarrow & M\otimes X\otimes M \\ {\scriptstyle e'}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ & & u^{\#}u_{\#}A & \searrow & \\ A & & \longrightarrow & & X \end{array}$$

where $e \in \mathcal{E}$ since $M \otimes - \otimes M$ preserves members of $\mathcal{E}$ and $e' \in \mathcal{E}$ since $\mathcal{E}$ contains all retractions. Thus, $u^{\#}u_{\#}A = A$, since both are the image of the morphism $M \otimes A \otimes M \longrightarrow X$, to complete the proof.

Applying this theorem, we get the following results. If **C** is any suitable cartesian closed category (e.g., a Grothendieck topos), then $1_{\mathbf{C}}$ is suitable, and so $Sub(1)$ is a locale. If **V** is **Sets**, **Ab**, or **Slat**, then $Id(M)$, $Id(R)$, and $Id(Q)$ are locales whenever M, R, or Q is idempotent. Similar results hold for sheaves of rings.

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