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A THEORY OF ENRICHED SHEAVES

by Francis BORCEUX and Carmen QUINTEIRO*

Résumé

Nous travaillons sur une catégorie de base $\mathcal{V}$, régulière, localement de présentation finie, fermée monoidale symétrique, au sens de Kelly. Etant donné une petite $\mathcal{V}$-catégorie $\mathcal{C}$, nous définissons la notion de $\mathcal{V}$-topologie de Grothendieck sur $\mathcal{C}$ et prouvons l'existence et la $\mathcal{V}$-exactitude du foncteur faisceau associé correspondant. Il existe des bijections entre les $\mathcal{V}$-topologies de Grothendieck sur $\mathcal{C}$, les $\mathcal{V}$-localisations de la categorie $[\mathcal{C}^*, \mathcal{V}]$ des $\mathcal{V}$-prefaisceaux et les $\mathcal{V}$-opérations de fermeture universelle sur $[\mathcal{C}^*, \mathcal{V}]$.

Mots clés: Grothendieck topology, sheaf, locally presentable category, closed category, localization.

1 Introduction

The notions of locally finitely presentable category (see [5]) and symmetric monoidal closed category (see [3]) are now classical. Following a work of G.M. Kelly (see [9]) we fix a "locally finitely presentable symmetric monoidal closed category $\mathcal{V}$", meaning by this a category having all those properties and in which, moreover, it is assumed that a finite tensor product of finitely presentable objects is again finitely presentable (that is, the unit I of the tensor product is finitely presentable and when V, W are finitely presentable, so is $V \otimes W$).

An additional assumption we impose on $\mathcal{V}$ in this paper is its regularity in the sense of Barr (see [1]). This assumption implies at once that for every small $\mathcal{V}$-category $\mathcal{C}$, the $\mathcal{V}$-category $[\mathcal{C}^*, \mathcal{V}]$ of $\mathcal{V}$-presheaves on $\mathcal{C}$ has an underlying regular category.

This fixes the list of assumptions on our base category $\mathcal{V}$. We shall no longer recall them in the rest of the paper. Observe that the categories of sets, abelian groups, modules on a commutative ring, and all toposes of presheaves are instances of such categories $\mathcal{V}$. An arbitrary Grothendieck topos is cartesian closed regular but not locally *finitely* presentable in general. The category of small categories is locally finitely presentable cartesian closed, but not regular.

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In this paper, all categories, functors, limits, colimits and so on, will almost always be enriched over $\mathcal{V}$. For that reason and for the sake of brevity, we shall *always* omit the prefix $\mathcal{V}$-. We prefer using the term "ordinary" to emphasize, when some confusion could occur, the difference between a $\mathcal{V}$-structure and the ordinary underlying structure.

To continue the list of notation and terminology which apply to the whole paper, let us write $\mathcal{G}$ for the dense generating "set" of finitely presentable objects in $\mathcal{V}$ (the word "set" is used here in its loose meaning: the equivalence classes of objects in $\mathcal{G}$ constitute a set). Following an obvious intuition, we shall think an arrow $x: G \longrightarrow V$ in $\mathcal{V}$ as "an element of V at the level G"; the notation $x \in_G V$ will *always* indicate an arrow $x: G \longrightarrow V$ with $G \in \mathcal{G}$. Given a morphism $f: V \longrightarrow W$, the notation $f(x)$ will indicate the composite $f \circ x$. Moreover, when $\mathcal{X}$ is a tensored and cotensored category, we shall keep writing x for the various arrows

$$x: G \longrightarrow \mathcal{C}(X,Y), \quad x: G \otimes X \longrightarrow Y, \quad x: X \longrightarrow \{G,Y\}$$

corresponding to an element $x \in_G \mathcal{X}(X,Y)$.

The density of the generating family $\mathcal{G}$ means that defining an arrow $f: V \longrightarrow W$ in $\mathcal{V}$ is equivalent to defining a family of mappings

$$\forall G \in \mathcal{G} \quad (x \in_G V) \mapsto (f(x) \in_G W)$$

which is natural in G. Dense generating families are strong, implying that f is an isomorphism precisely when all those mappings are bijective. And finally the ordinary generating property means that $f, g: V \rightrightarrows W$ are equal when they give rise to the same such mappings, for all $G \in \mathcal{G}$.

By a finite category is meant a category $\mathcal{D}$ with finitely many objects and such that each $\mathcal{D}(X,Y) \in \mathcal{V}$ is finitely presentable. A finite indexing type is a functor $F: \mathcal{D} \longrightarrow \mathcal{V}$ defined on a finite category $\mathcal{D}$ and such that each $F(D) \in \mathcal{V}$ is finitely presentable. A finite limit is a limit weighted by a finite indexing type (see [9]). In particular, cotensoring with a finitely presentable object is a finite limit process.

Let us now introduce the various notions which will be studied in this paper. The first one is a straight generalization of the classical notion of localization (see [2], I-3.5.5).

Definition 1.1 *Let $\mathcal{C}$ be a small category. A localisation of the category $[\mathcal{C}^*, \mathcal{V}]$ of presheaves is a full reflective subcategory $\mathcal{L} \subseteq [\mathcal{C}^*, \mathcal{V}]$ whose reflection preserves finite limits.*

Our notion of Grothendieck topology is inspired from the classical one (see [2], III-3.2.4), but has reinforced axioms. The equivalence of our axioms with the classical ones, in the case $\mathcal{V} = \mathsf{Set}$ will be attested by lemma 1.6.

Definition 1.2 *Let $\mathcal{C}$ be a small category. A Grothendieck topology on $\mathcal{C}$ is the choice, for every object $C \in \mathcal{C}$, of a family $\mathcal{T}(C)$ of subobjects of the representable presheaf $\mathcal{C}(-,C)$. Those data must satisfy the following axioms:*

(T1) $\mathcal{C}(-,C) \in \mathcal{T}(C)$ *for every object* $C \in \mathcal{C}$;

(T2) given $R \in \mathcal{T}(C)$ *and* $f \in_G \mathcal{C}(D,C)$, *one has* $f^{-1}(R) \in \mathcal{T}(D)$, *where* $f^{-1}(R)$ *is defined by the following pullback:*

$$\begin{tikzcd} f^{-1}(R) \arrow[r] \arrow[d, tail] & \{G,R\} \arrow[d, tail] \\ \mathcal{C}(-,D) \arrow[r, "f"'] & \{G,\mathcal{C}(-,C)\}; \end{tikzcd}$$

(T3) given $S \in \mathcal{T}(C)$ *and a subobject* $R \rightarrowtail \mathcal{C}(-,C)$ *such that* $f^{-1}(R) \in \mathcal{T}(D)$ *for all* $f \in_G S(D)$, *one has* $R \in \mathcal{T}(C)$.

With a topology is associated the corresponding notion of sheaf; again, the coincidence with the classical notion in the case $\mathcal{V} = \mathsf{Set}$ is attested by lemma 1.6.

Definition 1.3 *Let* $\mathcal{C}$ *be a small category provided with a Grothendieck topology* $\mathcal{T}$. *A presheaf* $P \in [\mathcal{C}^*, \mathcal{V}]$ *is a* $\mathcal{T}$*-sheaf when, given* R *and* α *as in the following diagram*

$$\begin{tikzcd} R \arrow[r, tail, "r"] \arrow[d, "\alpha"'] & \mathcal{C}(-,C) \arrow[dl, dotted, "\beta"] \\ \{G,P\} & \end{tikzcd}$$

with $G \in \mathcal{G}$ *and* $(R,r) \in \mathcal{T}(C)$, *there exists a unique* β *such that* $\beta \circ r = \alpha$.

Our last notion could of course be defined in the more general context of a category with finite limits. The same comment as before applies to the case $\mathcal{V} = \mathsf{Set}$.

Definition 1.4 *Let* $\mathcal{C}$ *be a small category. A universal closure operation on* $[\mathcal{C}^*, \mathcal{V}]$ *is a process associating, with every subpresheaf* $R \rightarrowtail P$, *another subpresheaf* $\overline{R} \rightarrowtail P$. *Those data must satisfy the following axioms.*

(C1) $R \subseteq \overline{R}$,

(C2) $R \subseteq S \Rightarrow \overline{R} \subseteq \overline{S}$,

(C3) $\overline{\overline{R}} = \overline{R}$,

(C4) $f^{-1}(\{G, \overline{R}\}) = \overline{f^{-1}(\{G, R\})}$, *for all presheaves* P, Q, *subpresheaves* $R \rightarrowtail P$, $S \rightarrowtail P$ *and element* $f \in_G [Q, P]$, *where* $[Q, P] \in \mathcal{V}$ *indicates the object of natural transformations from* Q *to* P.

The purpose of the present paper is to prove the following theorem.

Theorem 1.5 *Let* $\mathcal{C}$ *be a small category. There is a bijection between*

(1) the localizations of $[\mathcal{C}^*, \mathcal{V}]$*;*

(2) the Grothendieck topologies on $\mathcal{C}$*;*

(3) the universal closure operations on $[\mathcal{C}^*, \mathcal{V}]$.

Through those bijections, the localization of $[\mathcal{C}^*, \mathcal{V}]$ *associated with a Grothendieck topology* $\mathcal{T}$ *on* $\mathcal{C}$ *is the category of sheaves on* $(\mathcal{C}, \mathcal{T})$.

To emphasize the fact that this theorem generalizes classical results in the cases of sets or abelian groups, it is useful to prove at once the following lemma.

Lemma 1.6 *Let* $\mathcal{C}$ *be a small category. The notions of Grothendieck topology* $\mathcal{T}$ *on* $\mathcal{C}$*, sheaf on* $(\mathcal{C}, \mathcal{T})$ *and universal closure operation on* $[\mathcal{C}^*, \mathcal{V}]$ *are equivalent to those obtained by restricting one's attention, in definitions 1.2, 1.3, 1.4, to those objects* G *which belong to an arbitrary strongly generating set of finitely presentable objects in* $\mathcal{V}$.

Proof Assume $\mathcal{G}' \subseteq \mathcal{G}$ is a strongly generating set and consider $G \in \mathcal{G}$. Since $\mathcal{V}$ is locally finitely presentable, G is a finite conical colimit of a diagram whose objects are in $\mathcal{G}'$ (see [5] or [2] II-5.2.6); let us write for short $G = \operatorname{colim}_{i \in I} G_i$. It follows at once

$$\{G, P\} \cong \{\operatorname{colim}_{i \in I} G_i, P\} \cong \lim_{i \in I} \{G_i, P\}$$

frow which already the conclusion in the case of sheaves.

Now consider presheaves two P, Q, an element $f \in_G [Q, P]$ and a subobject $r: R \rightarrowtail P$; write $s_i: G_i \longrightarrow G$ for the canonical morphisms of the colimit $G \cong \operatorname{colim}_{i \in I} G_i$. The consideration of the diagram

$$\begin{tikzcd}
f^{-1}(R) \arrow[r] \arrow[d, tail] & \{G, R\} \arrow[r, "\cong"] \arrow[d, tail] & \lim_{i \in I} \{G_i, R\} \arrow[r, "p_i = \{s_i, 1\}"] \arrow[d, tail] & \{G_i, R\} \arrow[d, tail, "\{G_i, r\}"'] \\
Q \arrow[r, "f"'] & \{G, P\} \arrow[r, "\cong"'] & \lim_{i \in I} \{G_i, P\} \arrow[r, "p_i = \{s_i, 1\}"'] & \{G_i, P\}
\end{tikzcd}$$

where p_i is a canonical projection of the limit, indicates that $f^{-1}(R) \cong \bigcap_{i\in I}(p_i \circ f)^{-1}(R)$.

The classical proof that a Grothendieck topology is stable under finite intersections (see [2] III-3.2.5) transposes here. In the previous argument and with the notation of definiton 1.2, choosing $P = \mathcal{C}(-, C)$ and $Q = \mathcal{C}(-, D)$ yields the conclusion in the case of Grothendieck topologies. In the same way the classical proof that univeral closure operations commute with finite intersections (see [2] I-5.7.2) applies here, forcing the conclusion in the case of definition 1.4. ∎

With lemma 1.6 in mind, let us re-read definitions 1.2, 1.3 and 1.4 in the case $\mathcal{V}$ = Set, choosing the singleton $G = \{*\}$ as single strong generator. We recapture the ordinary notions of Grothendieck topology, sheaf and universal closure operation. In the same way choosing $\mathcal{V}$ = Ab, the category of abelian groups, and $G = \mathbb{Z}$, the group of integers, as single generator, we recapture this time the notions of Gabriel localizing system, sheaf and universal closure operation in the additive context (see [4]).

Many proofs in the present paper mimic well-known proofs in the case $\mathcal{V}$ = Set. We shall focus our attention on those arguments where the translation from the Set-case to the $\mathcal{V}$-case requires some new techniques or ideas, leaving to the reader the adaptations of the classical proofs when those are straightforward. We use [2] as a reference for the Set-case and refer the reader to [8] for what concerns the enriched categories machinery. Detailed accounts of all proofs can be found in the thesis of the second author.

The particularities of the $\mathcal{V}$-case are multiple, even under the strong assumptions we have imposed on $\mathcal{V}$. First, one needs efficient techniques to handle the occurences of finitely presentable objects in the definitions of topology, sheaf and closure operation. Second, when expressing a presheaf as a colimit of representable ones, this colimit is no longer universal. Third, representable presheaves are no longer projective. It is on the manner to overcome these difficulties that we shall focus our attention.

2 From localizations to closure operations

We fix $\mathcal{V}$ and $\mathcal{G}$ as in the introduction. Moreover, we fix a small category $\mathcal{C}$ and a localization $\ell \dashv i: \mathcal{L} \rightleftarrows [\mathcal{C}^*, \mathcal{V}]$ of the category of presheaves on $\mathcal{C}$. Those ingredients will remain fixed through this section and will no always be recalled.

As in [2] I-5.7.11, given a subpresheaf $r: R \rightarrowtail P$, we define its closure via the following commutative diagram,

$$
\begin{tikzcd}
R \arrow[dr, tail] \arrow[drr, "\eta_R"] \arrow[ddr, tail, "r"'] & & \\
& \overline{R} \arrow[r] \arrow[d, tail] & i\ell R \arrow[d, tail, "i\ell r"] \\
& P \arrow[r, "\eta_P"'] & i\ell P
\end{tikzcd}
$$

where the square is a pullback and η is the unit of the adjunction $\ell \dashv i$.

Proposition 2.1 *The previous construction defines a universal closure operation on* $[\mathcal{C}^*, \mathcal{V}]$.

Proof All the arguments of [2] I-5.7.11 in the case of the universal closure operation associated with an ordinary localization apply here; this includes axioms (C1) to (C3) and the ordinary universality in axiom (C4).

Choose now a subobject $r: R \rightarrowtail P$ and an element $f \in_G [Q, P]$ and consider the commutative diagram

$$
\begin{tikzcd}
f^{-1}(\{G,R\}) \arrow[rrr] \arrow[dd, tail] \arrow[dr, tail] & & & \{G,R\} \arrow[dd, tail, "\{1,r\}"] \arrow[dl, tail] \\
& f^{-1}(\{G,\overline{R}\}) \arrow[r] \arrow[dl, tail] & \{G,\overline{R}\} \arrow[dr, tail] & \\
Q \arrow[rrr, "f"'] & & & \{G,P\}
\end{tikzcd}
$$

where both trapezes are pullbacks. Cotensoring with $G \in \mathcal{G}$ is a finite limit process, thus $i\ell\{G,P\} \cong \{G, i\ell P\}$ and $\eta_{\{G,P\}} = \{G, \eta_P\}$; analogously for R. On the other hand the endofunctor $\{G, -\}$ on $[\mathcal{C}^*, \mathcal{V}]$ preserves limits, since it has a left adjoint $G \otimes -$. With those observations in mind, applying the functor $\{G, -\}$ to the diagram defining $\overline{R}$ yields now $\{G, \overline{R}\} = \overline{\{G, R\}}$. The conclusion follows then from the ordinary universality of the closure operation in the previous diagram. ■

Associated with the universal closure operation, we have several classes of morphisms:

1. the dense monomorphisms $R \rightarrowtail P$, when $\overline{R} = P$,
2. the closed monomorphisms $R \rightarrowtail P$, when $\overline{R} = R$,
3. the bidense morphims $f: Q \longrightarrow P$, when both the image of f and the equalizer of its kernel pair are dense monomorphisms.

The universality of the closure operation and the regularity of $[\mathcal{C}^*, \mathcal{V}]$ imply at once that these classes are stable under pulling back and cotensoring with $G \in \mathcal{G}$. Moreover, the bidense monomorphisms are exactly the dense ones.

Our next proposition indicates in particular that a localization of $[\mathcal{C}^*, \mathcal{V}]$ is completely determined by the corresponding universal closure operation.

Proposition 2.2 *Let $\ell \dashv i: \mathcal{L} \underset{\longrightarrow}{\overset{\longleftarrow}{}} [\mathcal{C}^*, \mathcal{V}]$ be a localization of a presheaf category and consider the corresponding universal closure operation on $[\mathcal{C}^*, \mathcal{V}]$. The following conditions are equivalent for an object $P \in [\mathcal{C}^*, \mathcal{V}]$:*

(1) $P \in \mathcal{L}$ (up to an isomorphism);

(2) P is orthogonal to every bidense morphism;

(3) P is orthogonal to every dense monomorphism;

(4) $\{G, P\}$ is orthogonal to every dense subobject of a representable presheaf, for all $G \in \mathcal{G}$.

Moreover, a morphism f is bidense precisely when $\ell(f)$ is an isomorphism.

Proof We recall that P is orthogonal to $f: X \longrightarrow Y$ when each morphism $X \longrightarrow P$ factors uniquely through f.

Just condition (4) requires some attention; the rest is classical (see [2], I, 5.4 to 5.8). By axiom (C4), the density of the subobject $r: R \rightarrowtail \mathcal{C}(-, C)$ implies that of $\{1, r\}: \{G, R\} \rightarrowtail \{G, \mathcal{C}(-, C)\}$, for every $G \in \mathcal{G}$. The orthogonality of $\{G, P\}$ and r is equivalent to that of P and $\{G, r\}$, thus (3) implies (4).

Conversely assume condition (4) and consider $s: S \rightarrowtail Q$ in $[\mathcal{C}^*, \mathcal{V}]$, a dense subobject. Consider the canonical coend $Q = \int^C QC \otimes \mathcal{C}(-, C)$ (see [8] 3.17) and for each $C \in \mathcal{C}$, the canonical conical filtered colimit $QC = \operatorname{colim}_{i \in I} G_i$, with $G_i \in \mathcal{G}$ (see [5]). This yields the following diagram, where both squares are pullbacks.

$$\begin{tikzcd}
S_{C,i} \arrow[r, "\sigma'_{C,i}"] \arrow[d, tail, "s_{C,i}"'] & S_C \arrow[r, "\sigma'_C"] \arrow[d, tail, "s_C"] & S \arrow[d, tail, "s"] \\
G_i \otimes \mathcal{C}(-,C) \arrow[r, "g_i \otimes 1"'] & QC \otimes \mathcal{C}(-,C) \arrow[r, "\sigma_C"'] & Q
\end{tikzcd}$$

with thus each $s_{C,i}$ a dense monomorphism. By axiom (C4), the following pullback

$$\begin{tikzcd}
R_{C,i} \arrow[r, "h_{C,i}"] \arrow[d, tail, "r_{C,i}"'] & \{G_i, S_{C,i}\} \arrow[d, tail, "\{G, s_{C,i}\}"] \\
\mathcal{C}(-,C) \arrow[r, "\varepsilon_{C,i}"'] & \{G_i, G_i \otimes \mathcal{C}(-,C)\}
\end{tikzcd}$$

where $\varepsilon_{C,i}$ corresponds by adjunction to the identity on $G_i \otimes \mathcal{C}(-,C)$, yields a dense subobject $r_{C,i}: R_{C,i} \rightarrowtail \mathcal{C}(-,C)$. Given a morphism $\alpha: S \longrightarrow P$ yields now a composite

$$G_i \otimes R_{C,i} \xrightarrow{h_{C,i}} S_{C,i} \xrightarrow{\sigma'_{C,i}} S_C \xrightarrow{\sigma'_C} S \xrightarrow{\alpha} P.$$

This corresponds to a morphism $\alpha_{C,i}: R_{C,i} \longrightarrow \{G_i, P\}$ which, by assumption, extends as $\beta_{C,i}: \mathcal{C}(-,C) \longrightarrow \{G_i, P\}$ along $r_{C,i}$. We consider the corresponding morphisms $\beta_{C,i}: G_i \otimes \mathcal{C}(-,C) \longrightarrow P$. Since

$$QC \otimes \mathcal{C}(-,C) \cong (\operatorname{colim}_{i \in I} G_i) \otimes \mathcal{C}(-,C) \cong \operatorname{colim}_{i \in I}\big(G_i \otimes \mathcal{C}(-,C)\big)$$

and $Q \cong \int^C QC \otimes \mathcal{C}(-,C)$, checking the necessary naturalities yields the expected unique factorization $Q \longrightarrow P$. ∎

3 From closure operations to topologies

We fix again $\mathcal{V}$ and $\mathcal{G}$ as in the introduction. We fix also a small category $\mathcal{C}$ and a universal closure operation on the category $[\mathcal{C}^*, \mathcal{V}]$ of presheaves. Those data will remain fixed through this section and will not always be recalled.

The following result is an obvious consequence of the definitions.

Proposition 3.1 *The dense subobjects of the representable presheaves constitute a Grothendieck topology on $\mathcal{C}$.* ∎

We want now to prove that the original universal closure operation on $[\mathcal{C}^*, \mathcal{V}]$ is completely determined by the corresponding Grothendieck topology on $\mathcal{C}$. Since the closure of a subobject R is the biggest subobject in which R is dense, it follows at once that the closure operation is completely determined by the knowledge of the dense subobjects. So our goal is achieved by the following lemma.

Lemma 3.2 *For a subobject $r: R \rightarrowtail P$ in $[\mathcal{C}^*, \mathcal{V}]$, the following conditions are equivalent:*

(1) the subobject $R \rightarrowtail P$ is dense;

(2) for every object $C \in \mathcal{C}$ and every element $f \in_G \big[\mathcal{C}(-,C), P\big]$, the subobject $f^{-1}(R) \rightarrowtail \mathcal{C}(-,C)$ is dense.

Proof (1) $\Rightarrow$ (2) follows at once from axiom (C4). Assuming (2), observe first that $f^{-1}\big(\overline{R}\big)$ is both dense, since it contains $f^{-1}(R)$, and closed, since $\overline{R}$ is closed. Therefore $f^{-1}\big(\overline{R}\big) = \mathcal{C}(-,C)$. We write $r: R \rightarrowtail P$ and $\overline{r}: \overline{R} \rightarrowtail P$ for the subobjects involved.

Writing P as a canonical coend $P = \int^C PC \otimes \mathcal{C}(-,C)$ (see [8] 3.17), the conclusion follows in the case $\mathcal{V} = \mathsf{Set}$ from the universality of this coend. This universality does not hold here, but an alternative proof applies.

For each $C \in \mathcal{C}$ and each $x \in_G PC$, the composite

$$G \otimes \mathcal{C}(-,C) \xrightarrow{x \otimes 1} PC \otimes \mathcal{C}(-,C) \xrightarrow{s_C} P$$

is some f as in the statement, with s_C a canonical morphism of the coend. This composite factors through $\overline{R}$, since we know already that pulling back $\{G, \overline{R}\}$ along the corresponding morphism $\mathcal{C}(-,C) \longrightarrow \{G, P\}$ yields the identity. Via the Yoneda lemma, this means precisely that $x \in_G PC$ implies $x \in_G \overline{R}C$. Since $\overline{r}_C: \overline{R}C \rightarrowtail PC$ is a monomorphism, this means exactly that the mappings

$$\forall G \in \mathcal{G} \quad \big(x \in_G \overline{R}C\big) \mapsto \big(\overline{r}_C(x) \in_G PC\big)$$

are bijective; thus $\overline{r}_C$ is an isomorphism. ∎

4 From topologies to localizations

This is the core of the paper, the place where the role of finitely presentable objects in the definitions of section 1 takes its full strength. We fix again $\mathcal{V}$ and $\mathcal{G}$ as in the introduction. We fix also a small category $\mathcal{C}$ and a Grothendieck topology $\mathcal{T}$ on it. Those data will remain fixed through this section and will not always be recalled.

Following a classical approach for constructing the associated sheaf functor (see [10] 20.3 or [2] III-3.3), we prove first:

Theorem 4.1 *Given a presheaf $P: \mathcal{C}^* \longrightarrow \mathcal{V}$, the data*

$$\Sigma PC = \operatorname{colim}_{R \in \mathcal{T}(C)} [R, P]$$

for all $C \in \mathcal{C}$ extend as a presheaf $\Sigma P: \mathcal{C}^* \longrightarrow \mathcal{V}$ and, more generally, give rise to a functor

$$\Sigma: [\mathcal{C}^*, \mathcal{V}] \longrightarrow [\mathcal{C}^*, \mathcal{V}]$$

which preserves finite limits.

Proof By definition of a Grothendieck topology, the colimit in the statement is a conical filtered colimit. Therefore, once the other ingredients of the statement will be proved, the preservation of finite limits by Σ will follow at once from the the fact that in $\mathcal{V}$, conical filtered colimits commute with both conical finite limits and tensoring with a finitely presented object (see [9]).

Given $C, D \in \mathcal{C}$, we must construct first

$$\Sigma P_{C,D}: \mathcal{C}(C, D) \longrightarrow [\Sigma PD, \Sigma PC]$$

to make ΣP a functor. This reduces to constructing a natural family of mappings

$$\forall G \in \mathcal{G} \quad \big(x \in_G \mathcal{C}(C, D)\big) \mapsto \big(\Sigma P_{C,D}(x) \in_G [\Sigma PD, \Sigma PC]\big).$$

Consider the pullback

$$\begin{tikzcd} x^{-1}(R) \arrow[r, "x'"] \arrow[d, tail] & \{G, R\} \arrow[d, tail] \\ \mathcal{C}(-, C) \arrow[r, "x"'] & \{G, \mathcal{C}(-, D)\} \end{tikzcd}$$

for each $R \in \mathcal{T}(D)$. This yields composites

$$G \otimes [R, P] \xrightarrow{x' \otimes 1} [x^{-1}(R), R] \otimes [R, P] \longrightarrow [x^{-1}(R), P]$$

where the second arrow is composition. The corresponding morphisms

$$[R, P] \longrightarrow [G, [x^{-1}(R), P]] \xrightarrow{[1, \sigma_{P, x^{-1}(R)}]} [G, \Sigma PC],$$

where $\sigma_{P, x^{-1}(R)}$ is the canonical morphism of the colimit defining ΣPC, yield a factorization

$$\Sigma PD \cong \operatorname{colim}_{R \in \mathcal{T}(D)} [R, P] \longrightarrow [G, \Sigma PC].$$

This corresponds exactly to an element $\Sigma_{C,D}(x) \in_G [\Sigma PD, \Sigma PC]$. It is routine to check the axioms for ΣP being a functor.

To make Σ itself a functor, given presheaves P, Q on $\mathcal{C}$, we must still construct

$$\Sigma_{P,Q}\colon [P,Q] \longrightarrow [\Sigma P, \Sigma Q].$$

Using the same technique, it suffices to define a natural family of mappings

$$\forall G \in \mathcal{G} \quad \bigl(\alpha \in_G [P,Q]\bigr) \mapsto \bigl(\Sigma_{P,Q}(\alpha) \in_G [\Sigma P, \Sigma Q]\bigr).$$

Since $[\Sigma P, \Sigma Q]$ is defined as an end (see [8] 2.2), constructing $\Sigma_{P,Q}(\alpha)$ reduces to constructing a cone $G \longrightarrow [\Sigma PC, \Sigma QC]$, that is, a family $G \otimes \Sigma PC \longrightarrow \Sigma QC$, natural in $C \in \mathcal{C}$. Since tensoring with G preserves colimits, we must finally produce a family

$$G \otimes [R,P] \longrightarrow \Sigma QC$$

which is both natural in $C \in \mathcal{C}$ and $R \in \mathcal{T}(C)$. It suffices to consider the composites

$$G \otimes [R,P] \xrightarrow{\alpha \otimes 1} [P,Q] \otimes [R,P] \longrightarrow [R,Q] \xrightarrow{\sigma_{Q,R}} \Sigma QC$$

where the second arrow is composition and the third one is a canonical morphism of the colimit defining ΣQC. The rest is routine. ■

Proposition 4.2 *Given a presheaf P on $\mathcal{C}$ and via the Yoneda lemma, the canonical morphisms of the colimits in 4.1*

$$\sigma_{P,\mathcal{C}(-,C)}\colon PC \cong \bigl[\mathcal{C}(-,C), P\bigr] \longrightarrow \Sigma PC$$

define a natural transformation $\sigma_P\colon P \Rightarrow \Sigma P$ and further, a natural transformation

$$\sigma\colon \mathrm{id}_{[\mathcal{C}^*,\mathcal{V}]} \Rightarrow \Sigma.$$

These natural transformations satisfy the relations

$$\{G, \sigma_P\} = \sigma_{\{G,P\}}, \quad \Sigma * \sigma = \sigma * \Sigma$$

for all $P \in [\mathcal{C}^*, \mathcal{V}]$ and $G \in \mathcal{G}$.

Proof Proving that σ_P and σ are natural transformations is routine left to the reader: this reduces to the commutativity of some diagrams and it suffices to check this for all elements at all levels $G \in \mathcal{G}$.

The relation $\{G, \sigma_P\} = \sigma_{\{G,P\}}$ is an immediate consequence of the preservation of finite limits by Σ (see 4.1), since $\{G, -\}$ is a special instance of a finite limit.

The relation $\Sigma * \sigma = \sigma * \Sigma$ is more subtle. It reduces, for all $P \in [\mathcal{C}^*, \mathcal{V}]$, $C \in \mathcal{C}$, $G \in \mathcal{G}$ and $x \in_G \Sigma PC$, to proving that x has the same image by the two morphisms

$$\Sigma(\sigma_{P,C}),\ \sigma_{\Sigma P,C}\colon \Sigma PC \rightrightarrows \Sigma\Sigma PC.$$

But the morphism

$$x: G \longrightarrow \Sigma PC \cong \operatorname{colim}_{R \in \mathcal{T}(C)} [R, P]$$

factors through some term $[R, P]$ of the filtered colimit, because G is finitely presentable. With lemma 4.3 in mind, the rest is now straightforward. ■

At this stage, a comment is useful. Not assuming that $\mathcal{V}$ is locally finitely presentable, one could have given the definition of a Grothendieck topology (see 1.2) using for $\mathcal{G}$ an arbitrary dense generating class (for example, all objects of $\mathcal{V}$). It would have been possible to define accordingly Σ and σ as in propositions 4.1 and 4.2. But let us make a strong point that, in the previous proposition and in the next lemma, the finite presentation of G is crucial for proving the relation $\{G, \Sigma P\} \cong \Sigma\{G, P\}$ and further, the equality $\Sigma * \sigma = \sigma * \Sigma$. This equality is itself essential for getting, by iteration of Σ, the associated sheaf functor (see [7]). In fact, the full strength of our assumptions on $\mathcal{V}$ is constanly used in this section.

Lemma 4.3 *Consider a presheaf $P \in [\mathcal{C}^*, \mathcal{V}]$, an object $C \in \mathcal{C}$ and an element $R \in \mathcal{T}(C)$ of the topology. Given $x \in_G \Sigma PC$ and $\alpha \in_G [R, P]$, the commutativities of the two following diagrams are equivalent*

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[dr, "x"'] & {[R,P]} \arrow[d, "\sigma_{P,R}"] \\ & \Sigma PC \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\alpha"] \arrow[d, tail, "r"'] & \{G,P\} \arrow[d, "{\{1,\sigma_P\}}"] \\ \mathcal{C}(-,C) \arrow[r, "x"'] & \{G, \Sigma P\} \end{tikzcd}$$

where $\sigma_{P,R}$ indicates the canonical morphism of the colimit in 4.1.

Proof The commutativity of the square is equivalent to its commutativity at each object $D \in \mathcal{C}$ and thus further, to the coincidence of both composites on each element $y \in_F RD$, for every $F \in \mathcal{G}$. But since F is finitely presentable, saying that the two elements

$$\begin{aligned} &\big(x \circ r\big)_D(y),\ \big(\sigma_{\{G,P\}} \circ \alpha\big)_D(y) \\ &\qquad \in_F \{G, \Sigma P\}(D) \cong \Sigma\{G,P\}(D) \cong \operatorname{colim}_{S \in \mathcal{T}(D)} \big[S, \{G,P\}\big] \end{aligned}$$

coincide, implies that they coincide already on some term $\big[S, \{G, P\}\big]$ of the filtered colimit, with thus $S \in \mathcal{T}(D)$. Assuming the commutativity of the triangle, one gets at once the commutativity of the square by a straightforward expansion of the definitions, choosing S to be $\mathcal{C}(-, D)$ itself.

Conversely, let us assume the commutativity of the square and choose $R' \in \mathcal{T}(C)$ and $\beta \in_G [R', P]$ such that $x = \sigma_{P,R'}(\beta)$. If we prove that $x = \sigma_{P,R\cap R'}(\alpha|_{R\cap R'})$, it will follow at once that $x = \sigma_{P,R}(\alpha)$ by commutativity of the right hand side triangle in the following diagram

$$\begin{tikzcd}
G \arrow[r, "\alpha"] \arrow[dr, "x"'] & {[R,P]} \arrow[r] \arrow[d, "\sigma_{P,R}"] & {[R\cap R',P]} \arrow[dl, "\sigma_{P,R\cap R'}"] \\
& \Sigma PC. &
\end{tikzcd}$$

Composing the square with the inclusion $R\cap R' \subseteq R$, we are reduced to an analogous problem at the level $R \cap R'$, with this time $\alpha|_{R\cap R'}$ making the square commutative and $\beta|_{R\cap R'}$ making the triangle commutative. In other words and for the simplicity of notation, there is no restriction in assuming $R = R'$ in the previous discussion.

The thesis reduces now to $\sigma_{P,R}(\alpha) = \sigma_{P,R}(\beta)$, that is, by definition of the filtered colimit defining ΣPC, to the existence of $R'' \subseteq R$, $R'' \in \mathcal{T}(C)$, such that $\alpha|_{R''} = \beta|_{R''}$. In other words, we must prove that the equalizer k

$$\begin{tikzcd}
K \arrow[r, tail, "k"] & R \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & \{G,P\}
\end{tikzcd}$$

of α, β is still in $\mathcal{T}(C)$. Via axiom (T3), it suffices to prove that for every $D \in \mathcal{C}$ and $y \in_F RD$,

$$\begin{tikzcd}
y^{-1}(K) \arrow[r] \arrow[d, tail] & \{F,K\} \arrow[d, tail] \\
\mathcal{C}(-,D) \arrow[r, "y"'] & \{F,\mathcal{C}(-,C)\}
\end{tikzcd}$$

$y^{-1}(K) \in \mathcal{C}(-,D)$. This is achieved by observing that $S \subseteq y^{-1}(K)$, for every $S \in \mathcal{T}(D)$ given, as at the beginning of the proof, by the commutativity of the square in the statement. ■

We arrive now at the main theorem of this section: the so-called associated sheaf theorem. With lemma 4.3 at hand, translating the corresponding classical proof (see [2] III-3.3) is now straightforward routine left to the reader. By a separated presheaf, we mean as usual a presheaf which, in the conditions of definition 1.3, satisfies the uniqueness condition on β, but not necessarily the existence condition.

Theorem 4.4 *Let $\mathcal{C}$ be a small category, $\mathcal{T}$ a Grothendieck topology on $\mathcal{C}$ and $P \in [\mathcal{C}^*, \mathcal{V}]$ a presheaf on $\mathcal{C}$.*

(1) P is separated iff σ_P is a monomorphism;

(2) P is a sheaf iff σ_P is an isomorphism;

(3) ΣP is always separated;

(4) if P is separated, ΣP is a sheaf;

(5) the category of sheaves on $(\mathcal{C}, \mathcal{T})$ is a localization of $[\mathcal{C}^*, \mathcal{V}]$, with reflection $\Sigma\Sigma$. ■

Our next proposition implies in particular that the localization of $[\mathcal{C}^*, \mathcal{V}]$ described in the previous theorem characterizes completely the original topology $\mathcal{T}$ on $\mathcal{C}$.

Proposition 4.5 *For a subobject $r: R \rightarrowtail \mathcal{C}(-, C)$, the following conditions are equivalent:*

(1) $R \in \mathcal{T}(C)$

(2) $\ell(r)$ is an isomorphism,

where ℓ is the associated sheaf functor of theorem 4.4.

Proof Given $r: R \rightarrowtail \mathcal{C}(-, C)$, every sheaf is orthogonal to r, thus $\ell(r)$ is an isomorphism by a classical argument on orthogonal subcategories (see [6] or [2], I-5.4.10).

The proof of (2) $\Rightarrow$ (1), although inspired by that of III-3.5.1 in [2], is not at all a straightforward translation of it. Indeed, classically, the projectivity of the representable presheaves is used to reach the conclusion.

The first step consists in observing that for every presheaf P and every element $x \in_G \left[\mathcal{C}(-, C), \Sigma P\right]$, the pullback

$$\begin{tikzcd}
x^{-1}(\sigma_P(P)) \arrow[r] \arrow[d, tail] & \{G, \sigma_P(P)\} \arrow[d, tail] \\
\mathcal{C}(-, C) \arrow[r, "x"'] & \{G, \Sigma P\}
\end{tikzcd}$$

where $\sigma_P(P)$ is the image of $\sigma_P\colon P \longrightarrow \Sigma P$, yields $x^{-1}(\sigma_P(P)) \in \mathcal{T}(C)$. Indeed, by the Yoneda lemma, x corresponds to some element $\overline{x} \in_G \Sigma PC$, thus to some $y \in_G [R, P]$ for some $R \in \mathcal{T}(C)$. The commutativity of the previous square, via lemma 4.3, implies $R \subseteq x^{-1}(\sigma_P(P))$, thus $x^{-1}(\sigma_P(P)) \in \mathcal{T}(C)$.

Next we consider the composite

$$\mathcal{C}(-,C) \xrightarrow{\sigma_{\mathcal{C}(-,C)}} \Sigma\mathcal{C}(-,C) \rightarrowtail^{\sigma_{\Sigma\mathcal{C}(-,C)}} \Sigma\Sigma\mathcal{C}(-,C) \xrightarrow{(\Sigma\Sigma r)^{-1}} \Sigma\Sigma R$$

which, for simplicity, we write $\alpha\colon \mathcal{C}(-,C) \longrightarrow \Sigma\Sigma R$. We consider also the composite

$$R \xrightarrow{p} \!\!\!\!\rightarrow I \rightarrowtail^{i} \Sigma R \rightarrowtail^{\sigma_{\Sigma R}} \Sigma\Sigma R$$

where $p \circ i$ is the image factorization of σ_R. For simplicity again, we write $\beta = \sigma_{\Sigma R} \circ i$. By theorem 4.4, $\sigma_{\Sigma R}$ and $\sigma_{\Sigma\mathcal{C}(-,C)}$ are monomorphisms.

Considering the diagram

$$\begin{tikzcd}
\alpha^{-1}(\beta) \arrow[r, tail, "i'"] \arrow[d, "\alpha''"] & \alpha^{-1}(\Sigma R) \arrow[r, tail, "\sigma'"] \arrow[d, "\alpha'"] & \mathcal{C}(-,C) \arrow[d, "\alpha"] \\
I \arrow[r, tail, "i"] \arrow[rr, bend right, "\beta"'] & \Sigma R \arrow[r, tail, "\sigma_{\Sigma R}"] & \Sigma\Sigma R
\end{tikzcd}$$

where the squares are pullbacks, we shall first prove $\alpha^{-1}(\beta) \in \mathcal{T}(C)$. Putting $x = \alpha$ in the first step of the proof, we know already that $\alpha^{-1}(\Sigma R) \in \mathcal{C}(-,C)$. So, by axiom (T3), it remains to show that for every $f \in_G \alpha^{-1}(\Sigma R)(D)$, one has $f^{-1}(\alpha^{-1}(\beta)) \in \mathcal{C}(-,D)$.

$$\begin{tikzcd}
f^{-1}(\alpha^{-1}(\beta)) \arrow[r, tail, "i''"] \arrow[d, "f''"] & f^{-1}(\sigma_{\Sigma R}) \arrow[r, tail, "\sigma''"] \arrow[d, "f'"] & \mathcal{C}(-,D) \arrow[d, "f"] \\
\{G, \alpha^{-1}(\beta)\} \arrow[r, tail, "\{1,i'\}"'] & \{G, \alpha^{-1}(\Sigma R)\} \arrow[r, tail, "\{1,\sigma'\}"'] & \{G, \mathcal{C}(-,C)\}
\end{tikzcd}$$

Since $f \in_G \alpha^{-1}(\Sigma R)(D)$, one has $f^{-1}(\sigma_{\Sigma R}) = \mathcal{C}(-,D)$. But then, putting $x = f'$ in the first step of the proof, the consideration of the left hand square yields $f^{-1}(\alpha^{-1}(\beta)) \in \mathcal{T}(D)$.

To prove $R \in \mathcal{T}(C)$, by axiom (T3), we can now choose $g \in_G \alpha^{-1}(\beta)(D)$ and prove $g^{-1}(R) \in \mathcal{T}(D)$.

$$\begin{tikzcd}
g^{-1}(R) \arrow[rr] \arrow[d, tail] & & \{G,R\} \arrow[d, tail, "{\{1,r\}}"] \\
\mathcal{C}(-,D) \arrow[r, "g"'] & \{G,\alpha^{-1}(\beta)\} \arrow[r, tail] & \{G,\mathcal{C}(-,C)\}
\end{tikzcd}$$

To achieve this, we consider the composite

$$\mathcal{C}(-,D) \xrightarrow{g} \{G,\alpha^{-1}(\beta)\} \xrightarrow{\{1,\alpha\}} \{G,I\} \xrightarrow{\{1,i\}} \{G,\Sigma R\}$$

which determines an element $z \in_G \Sigma RD$, that is an element $y \in_G [S,R]$ for some $S \in \mathcal{T}(D)$, z and y being related by the diagrams of lemma 4.3. To prove $g^{-1}(R) \in \mathcal{T}(D)$, it suffices by axiom (T3) to prove $h^{-1}(g^{-1}(R)) \in \mathcal{T}(E)$ for each $E \in \mathcal{C}$, $F \in \mathcal{G}$ and $h \in_F S(E)$.

$$\begin{tikzcd}
h^{-1}(g^{-1}(R)) \arrow[rr] \arrow[d, tail] & & \{F,g^{-1}(R)\} \arrow[d, tail] \\
\mathcal{C}(-,E) \arrow[r, "h"'] & \{F,S\} \arrow[r, tail] & \{F,\mathcal{C}(-,D)\}
\end{tikzcd}$$

To prove this last fact, let us consider the following pentagon.

$$\begin{tikzcd}
& \{F,S\} \arrow[r, "{\{1,y\}}"] & \{F \otimes G,R\} \arrow[dd, "{\{1,r\}}"] \\
\mathcal{C}(-,E) \arrow[ur, "h"] \arrow[dr, "h"'] & & \\
& \{F,\mathcal{C}(-,D)\} \arrow[r, "g"'] & \{F \otimes G,\mathcal{C}(-,C)\}.
\end{tikzcd}$$

Using the naturality of σ and lemma 4.3, one verifies first that both ways along this pentagon are coequalized by the composite

$$\{F \otimes G,\mathcal{C}(-,C)\} \xrightarrow{\{1,\sigma_{\mathcal{C}(-,C)}\}} \{F \otimes G,\Sigma\mathcal{C}(-,C)\} \xrightarrow{\{1,\sigma_{\Sigma\mathcal{C}(-,C)}\}} \{F \otimes G,\Sigma\Sigma\mathcal{C}(-,C)\}$$

where as usual, we keep the same notation for various morphisms corresponding to each other via natural bijections. Now $\sigma_{\Sigma\mathcal{C}(-,C)}$ is a monomorphism by theorem 4.4, thus both ways along the pentagon are coequalized by $\{1, \sigma_{\mathcal{C}(-,C)}\}$. But $F \otimes G \in \mathcal{G}$ by assumption on $\mathcal{V}$ and therefore, by 4.1 and 4.2, this morphism $\{1, \sigma_{\mathcal{C}(-,C)}\}$ can be rewritten

$$\sigma_{\{F\otimes G, \mathcal{C}(-,C)\}}: \{F \otimes G, \mathcal{C}(-,C)\} \longrightarrow \Sigma\{F \otimes G, \mathcal{C}(-,C)\}.$$

Now observe that the definitions of Σ and σ imply at once that two morphisms of the form $u, v: \mathcal{C}(-,E) \rightrightarrows P$ are identified by $\sigma_P: P \longrightarrow \Sigma P$ precisely when u, v coincide on some $L \in \mathcal{T}(E)$. Therefore, both ways along the pentagon coincide on some $L \in \mathcal{T}(E)$. It follows easily that $L \subseteq h^{-1}(g^{-1}(R))$, yielding the conclusion. ■

5 Proof of theorem 1.5

Propositions 2.1, 3.1 and theorem 4.4 describe various correspondences from localizations of $[\mathcal{C}^*, \mathcal{V}]$ to universal closure operations on $[\mathcal{C}^*, \mathcal{V}]$, from universal closure operations on $[\mathcal{C}^*, \mathcal{V}]$ to Grothendieck topologies on $\mathcal{C}$ and from Grothendieck topologies on $\mathcal{C}$ to localizations of $[\mathcal{C}^*, \mathcal{V}]$. Moreover, proposition 2.2, lemma 3.2 and proposition 4.5 imply that these three correspondences are injective. To conclude the proof that all three correspondences are bijective, it suffices to prove that one of the three "cycles" is the identity.

Let us start with a topology $\mathcal{T}$ on the small category $\mathcal{C}$. We consider first the localization of $[\mathcal{C}^*, \mathcal{V}]$ given by the category of sheaves on $(\mathcal{C}, \mathcal{T})$. With that localization is associated a universal closure operation on $[\mathcal{C}^*, \mathcal{V}]$ and we must prove that a subobject $r: R \rightarrowtail \mathcal{C}(-,C)$ is dense for this closure operation precisely when $R \in \mathcal{T}(C)$. Being a dense monomorphism for the closure operation is obviously equivalent to being a bidense monomorphism thus, by 2.2, to $\ell(r)$ being an isomorphism. The conclusion follows then from 4.5. ■

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