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BOOLEANIZATION

by B. BANASCHEWSKI and A. PULTR

Résumé : Cet article est une étude des aspects différents de la procédure qui associe à un cadre L l'algèbre de Boole compléte $\mathfrak{B}L$ des éléments réguliers $a = a^{**}$. En particulier, nous étudions des applications entre cadres qui induisent des homomorphismes entre ces Booleanizations, et des propriétés de quelques foncteurs associés comprenant des homomorphismes faiblement ouverts. De plus, nous considérons des propriétés de limites et colimites dans le contexte de ces homomorphismes.

Recall that the well-known result in topology that the regular open subsets of any space form a complete Boolean algebra naturally extends to arbitrary frames, as originally observed by Glivenko [11] and later put in proper perspective by Isbell [13]. This associates with each frame L the complete Boolean algebra $\mathfrak{B}L$, consisting of the elements $a = a^{**}$, together with the homomorphism $\beta_L : \mathfrak{B}L \to L$ taking each element a to its double pseudocomplement a^{**}. This paper studies various aspects of this Booleanization. In particular, we investigate maps (not necessarily homomorphisms) between frames which induce a homomorphism between their Booleanization so that the correspondence $L \mapsto \mathfrak{B}L$ become functorial, or even a reflection. The frame homomorphisms arising in this context, called *weakly open* here are familiar from topos theory (Johnstone [15]) and have a natural topological origin connected with the Gleason cover (Mioduszewski - Rudolf [18]). We show that the reflection given by Booleanization for both

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the category $\mathbf{Frm}_{wo}$ of all frames and weakly open homomorphisms and the corresponding category of completely regular frames, does not have a left adjoint. This is of interest because in the case of uniform frames there is indeed such an adjoint, provided by completion (Banaschewski - Pultr [8]). Finally, concerning general properties of the category $\mathbf{Frm}_{wo}$, we prove that it has products and coequalizers, the former inherited from the category given by all frame homomorphisms but the latter not. It fails to have equalizers (Niederle [19]) ; in addition, we establish that certain frame coproducts are coproducts in $\mathbf{Frm}_{wo}$ while others are not.

0. Preliminaries

0.1 A *frame* is a complete lattice L satisfying the distributivity law $a \wedge \bigvee S = \bigvee\{a \wedge t \mid t \in S\}$ for all $a \in L$ and $S \subseteq L$, and a *frame homomorphism* $h : L \to M$ is a map preserving all finitary meets including the top 1, and arbitrary joins including the bottom 0. The resulting category will be denoted by

$$\mathbf{Frm}.$$

Thus, for instance, the lattice ΩX of open sets of a topological space is a frame, and if $f : X \to Y$ is a continuous map, $\Omega f : \Omega Y \to \Omega X$ defined by $\Omega f(U) = f^{-1}(U)$ is a frame homomorphism. For general facts concerning frames, see [14], [23].

0.2 Another example of a frame is a complete Boolean algebra. Note that *frame homomorphisms between Boolean algebras coincide with complete Boolean homomorphisms*, that is, they also preserve complements and arbitrary meets. On the other hand, although each frame is a *Heyting algebra* since the distributivity ensures the existence of an operation $a \to b$ for which $a \wedge b \leq c$ iff $a \leq b \to c$, frame homomorphisms do not generally preserve the operation $\to$, let alone arbitrary infinite meets.

0.3. The *pseudocomplement* of an element x of a frame L is $x^* = \bigvee\{y \mid y \wedge x = 0\}$, which is the largest y such that $y \wedge x = 0$. One has $x \leq x^{**}$ and $x^{***} = x^*$. We write $x \prec y$ if $x^* \vee y = 1$, and $x \prec\!\!\prec y$ if there are x_r for each rational r between 0 and 1 such that

$x = x_0, y = x_1$ and $x_r \prec x_s$ whenever $r < s$. A frame is said to be *regular* (*completely regular*) if

$$\forall a \in L,\ a = \bigvee\{x \mid x \prec a \ \ (x \prec\!\prec a)\}.$$

A frame L is called *compact* if for each subset $C \subseteq L$ such that $\bigvee C = 1$ there is a finite $E \subseteq C$ such that already $\bigvee E = 1$.

0.4. A frame homomorphism $h : L \to M$ is called *dense* if $h(a) = 0$ implies $a = 0$. An element $x \in L$ is *dense* if $x^{**} = 1$. A *compactification* of a frame L is a dense surjection $K \to L$ with compact regular K.

0.5. The system of all congruences on a frame L is again a frame and will be denoted by $\mathfrak{C}L$. The mapping $\nabla : L \to \mathfrak{C}L$ associating with $a \in L$ the congruence $\nabla(a) = \{(x, y) \mid x \vee a = y \vee a\}$ is a one-one frame homomorphism and each $\nabla(a)$ is *complemented* in $\mathfrak{C}L$; in fact, if $\Delta(a) = \{(x, y) \mid x \wedge a = y \wedge a\}$ then $\nabla(a) \vee \Delta(a) = 1$ and $\nabla(a) \wedge \Delta(a) = 0$ (see [14]).

0.6. For any frame L, the subset

$$\mathfrak{B}L = \{x \in L \mid x = x^{**}\},$$

is a complete Boolean algebra, with meet as in L and join $(\bigvee a_i)^{**}$, called the *Booleanization* of L. We will denote join in $\mathfrak{B}L$ by

$$\bigsqcup_{i \in J} a_i, \quad a \sqcup b, \quad a_1 \sqcup \cdots \sqcup a_n, \quad etc.$$

The map

$$\beta_L : L \to \mathfrak{B}L$$

given by $\beta_L(x) = x^{**}$ is a frame homomorphism. Obviously it is a dense surjection. (See [11], [13].)

0.7. In the last section we will make a few points on coproducts of frames. The reader can learn more about them e.g. in [14].

From category theory, only basics (as, say, in the first half of [17]) are assumed.

1. Booleanization as a functor

1.1. Although the construction $\mathfrak{B}$ and the homomorphisms $\beta_L : L \to \mathfrak{B}L$ are canonical in some sense, and β_L even has a certain universality property as the least dense surjection ([13]), one cannot extend $\mathfrak{B}$ to a functor on **Frm** behaving naturally with respect to the β_L. The following extends the fact on spaces from [15] (Lemma 3.2) for general frames :

Proposition. *Let $\varphi : L \to M$ be a frame homomorphism. Then there is a homomorphism $\psi : \mathfrak{B}L \to \mathfrak{B}M$ such that*

$$\begin{array}{ccc} L & \xrightarrow{\varphi} & M \\ {\scriptstyle \beta_L}\big\downarrow & & {\scriptstyle \beta_M}\big\downarrow \\ \mathfrak{B}L & \xrightarrow{\psi} & \mathfrak{B}M \end{array}$$

commutes iff for each $a \in L$, $\varphi(a^{**}) \leq \varphi(a)^{**}$.

PROOF: If ψ exists then $\varphi(a)^{**} = \beta\varphi(a) = \psi\beta(a) = \psi(a^{**})$, hence also $\varphi(a^{**})^{**} = \psi(a^{****}) = \psi(a^{**})$, and finally $\varphi(a^{**}) \leq \varphi(a^{**})^{**} = \psi(a^{**}) = \varphi(a)^{**}$. On the other hand if the condition is satisfied, it is easy to check that the formula $\psi(a) = \varphi(a)^{**}$ determines a homomorphism $\mathfrak{B}L \to \mathfrak{B}M$ with the desired property. □

Homomorphisms φ such that $\varphi(a^{**}) \leq \varphi(a)^{**}$ will be called *weakly open*. As β_L is onto, ψ in the diagram above is uniquely determined. We will denote it by $\mathfrak{B}\varphi$. Thus, we have the formula

$$\mathfrak{B}\varphi(a) = \varphi(a)^{**}.$$

1.2. The condition in 1.1 makes good topological sense. Its spatial counterpart has appeared in the literature under various names : *skeletal* in [18], *demi-open* in [12]. We prefer the term *weakly open* as it appears to us to be a particularly natural weakening of the openness condition (compare (7) below with 1.3); it should be noted, however, that this term has been used in [15] for what is called nearly open in [21] - see (1.3.1) below.

The following statement which summarizes some of the discussion from [7] may be useful.

Theorem. *The following conditions on a frame homomorphism φ are equivalent :*

(1) φ is weakly open,

(2) $\varphi(a^{**})^{**} = \varphi(a)^{**}$,

(3) $\varphi(a^*)^* \le \varphi(a)^{**}$,

(4) $\varphi(a^*)^* = \varphi(a)^{**}$,

(5) for each dense a, $\varphi(a)$ is dense.

If $\varphi = \Omega f$ for a continuous $f : X \to Y$, this is, further, equivalent to

(6) for each non-void open $V \subseteq X$, $\operatorname{int} \overline{f[V]}$ is non-void,

(7) for each open $V \subseteq X$ there is an open $U \subseteq Y$ such that $\overline{f[V]} = \overline{U}$.

1.3. Of course, $\mathfrak{B}\varphi$ can be defined for any choice of morphisms satisfying a condition stronger than weak openness ; notably for

(1.3.1) *nearly open homomorphisms*, satisfying

$$\varphi(a^*) = \varphi(a)^*,$$

which corresponds for spaces to the condition that, for each open U, $f[U]$ is dense in some open set - see [21],

(1.3.2) *feebly open homomorphisms*, the $\varphi : L \to M$ such that there is a mapping $\psi : M \to L$ such that

(α) $\psi(b) \ne 0$ for $b \ne 0$, and

(β) $\varphi(a) \wedge b \le \varphi(c)$ implies $a \wedge \psi(b) \le c$

(for spaces this corresponds to the condition that, for each open U, there is an open set dense in $f[U]$ - see [9]),

(1.3.3) *open homomorphisms*, that is, complete Heyting homomorphisms (see [7], [16]) which corresponds for spaces to the condition

for each open U, $f[U]$ is open.

Although the feebly open homomorphisms fit well into the topological picture, their algebraic nature seems to differ from that of the other three. They probably merit a separate study ; in this article we will mention them only in passing.

The categories of frames with weakly open, nearly open, feebly open, and open homomorphisms will be denoted respectively by

$$\mathbf{Frm}_{wo},\ \mathbf{Frm}_{no},\ \mathbf{Frm}_{fo},\ \mathbf{Frm}_{o}.$$

Thus, if $\mathfrak{B}\mathbf{Frm}$ is the category of Boolean frames, that is, complete Boolean algebras, we have :

For any of the categories $\mathcal{C}$ above, $\mathfrak{B}$ can be extended by the formula from 1.1 to a functor

$$\mathfrak{B} : \mathcal{C} \to \mathfrak{B}\mathbf{Frm}\ .$$

Note also that if L, M are Boolean, any frame homomorphism $\varphi : L \to M$ is complete Heyting ; thus,

$\mathfrak{B}\mathbf{Frm}$ *is a full subcategory of any of the $\mathcal{C}$.*

1.4. The question naturally arises whether we could not make $\mathfrak{B}$ functorial by restricting the objects rather than the morphisms - to be immediately dismissed, since we have :

Proposition. *If a frame L has the property that all $\varphi : L \to B$ into Boolean B are weakly open then L is Boolean.*

PROOF: Consider the composition

$$\varphi : L \xrightarrow{\nabla} \mathfrak{C}L \xrightarrow{\beta} \mathfrak{B}\mathfrak{C}L.$$

As each $\nabla(a)$ is complemented, φ is one-one. Now if $\varphi(a^{**}) \leq \varphi(a)^{**}$, we have $\nabla(a^{**}) \leq \nabla(a)$, hence ($\nabla$ is one-one) $a^{**} \leq a$, that is, $a^{**} = a$. □

2. Booleanization as reflection

2.1. Lemma. (see [15], p.227) *Each dense surjective homomorphism $\varphi : L \to M$ is nearly open.*

PROOF: Take $a \in L$. Choose a $b \in L$ such that $\varphi(a)^* = \varphi(b)$. We have $0 = \varphi(b) \wedge \varphi(a) = \varphi(b \wedge a)$. By density, $b \wedge a = 0$, that is, $b \leq a^*$ and we conclude that $\varphi(a)^* = \varphi(b) \leq \varphi(a^*) \leq \varphi(a)^*$. □

2.2. In particular, the $\beta_L : L \to \mathfrak{B}L$ are nearly (and hence also weakly) open. Thus, we easily conclude

Theorem. *The category* $\mathfrak{B}$**Frm** *is reflective in both* $\mathbf{Frm}_{wo}$ *and* $\mathbf{Frm}_{no}$*, with reflection functor* $\mathfrak{B}$.

2.3. A β_L is, however, seldom open. Here we characterize the frames L for which it is.

First (see, for instance, [7]), φ being open means there is $c \in L$ such that

$$\varphi(x) = \varphi(y) \text{ iff } c \wedge x = c \wedge y.$$

Thus, β_L is open iff there is a $c \in L$ such that

$$(*) \qquad c \wedge x = c \wedge y \text{ iff } x^{**} = y^{**}.$$

Theorem. *The following statements on a frame L are equivalent :*

(1) *$\beta_L : L \to \mathfrak{B}L$ is open,*
(2) *L has a smallest dense element,*
(3) *there is a dense $c \in L$ such that $\downarrow c = \{x \mid x \leq c\}$ is a Boolean algebra.*

PROOF: (1)⇒(2) : Take the c from $(*)$. In particular, $c \wedge x = c$ iff x is dense.

(2)⇒(3) : Let c be a smallest dense element, let $x \leq c$, If $y \wedge ((c \wedge x^*) \vee x) = 0$ we have $y \leq x^*$ and $y \wedge c \wedge x^* = y \wedge c = 0$ so that $y = 0$. Thus, $(c \wedge x^*) \vee x$ is dense and we have

$$c \leq (c \wedge x^*) \vee c \leq c$$

and hence $c \wedge x^*$ is the complement of x in $\downarrow c$.

(3)⇒(1) : Denote the complement of x in $\downarrow c$ by $\neg x$. As obviously

$$\bigvee\{y \mid 0 \leq y \leq c,\ y \wedge x = 0\} = \bigvee\{z \wedge c \mid z \wedge x = 0\},$$

we have

$$\neg x = x^* \wedge c$$

for any $x \in \downarrow c$. Since $\downarrow c$ is Boolean, we obtain, for any $x \in L$,

$$x \wedge c = \neg\neg(x \wedge c) = (x^* \wedge c)^* \wedge c \geq x^{**} \wedge c \geq x \wedge c$$

and hence $x \wedge c = \beta_L(x) \wedge c$. As $\beta_L(x) = \beta_L\beta_L(x)$ and $\beta_L(c) = 1$, we immediately infer that

$$x \wedge c = y \wedge c \text{ iff } \beta_L(x) = \beta_L(y).$$

□

2.4. Note : To have a better idea how rare the openness of β_L is: It is easy to prove that, for a T_0−space X, $\beta : \Omega(X) \to \mathfrak{B}\Omega(X)$ is open iff there is a dense open discrete subset $C \subseteq X$.

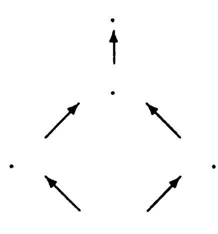
3. Non-existence of left adjoints

3.1. $\mathfrak{B}$, as a reflection functor, is a left adjoint in the weakly and nearly open cases. Now, in the metric and uniform setting, again with the choice of weakly open homomorphisms, $\mathfrak{B}$ is also a right adjoint ([6], [8]), and hence it is natural to wonder whether this might also hold for ordinary frames. In this section we show this is not the case. First, a simple reason for this at quite a general level.

Theorem. *For* $\mathcal{C}$=$\mathbf{Frm}_{wo}$, $\mathbf{Frm}_{no}$, $\mathbf{Frm}_{fo}$ *and* $\mathbf{Frm}_o$, $\mathfrak{B}$ *has no left adjoint.*

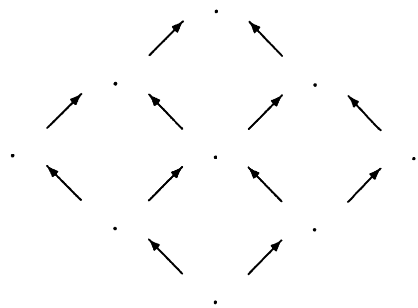
PROOF: Consider the two frames

L :

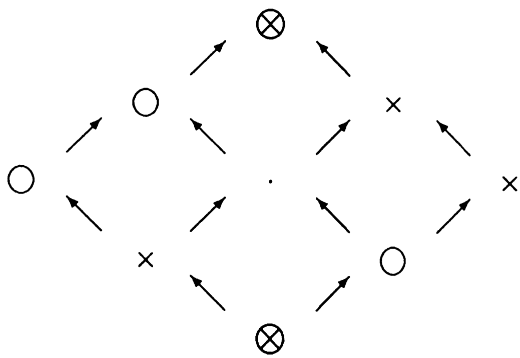


and

M :



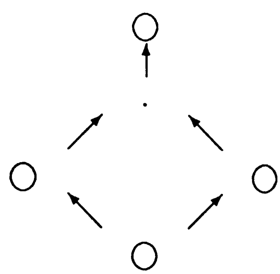
and the embeddings $\varphi, \psi : L \to M$ determined by the subframes of M indicated by ◯ and $\times$:



which are clearly weakly open.

Then, the equalizer of φ and ψ is the initial $2 \to L$. On the other hand, the Booleanizations of L and M are

$\mathfrak{B}L$:



and

$$\mathfrak{B}M \ : \quad \begin{tikzcd}[column sep=small, row sep=small]
 & & \bigcirc & & \\
 & \cdot \arrow[ur] & & \cdot \arrow[ul] & \\
\bigcirc \arrow[ur] & & \cdot \arrow[ul] \arrow[ur] & & \bigcirc \arrow[ul] \\
 & \cdot \arrow[ul] \arrow[ur] & & \cdot \arrow[ul] \arrow[ur] & \\
 & & \bigcirc \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

while $\mathfrak{B}\varphi = \mathfrak{B}\psi$ is the isomorphism $\mathfrak{B}L \to \mathfrak{B}M$; the equalizer of $\mathfrak{B}\varphi$ and $\mathfrak{B}\psi$ is therefore the identity $\mathfrak{B}L \to \mathfrak{B}L$ but $\mathfrak{B}(2 \to L) = 2 \to \mathfrak{B}L$.

(Note that φ and q do not preserve pseudocomplements; in particular, they are *not* open.) □

3.2. The theorem above leaves much to be desired. The initial question involved comparing the situation in frames with that in uniform frames. Now the underlying frames of uniform frames are of a very special nature : they are, as is well known, exactly the compactifiable frames, or, allowing the use of non-constructive principles, the completely regular frames ([4], [5], [22]). Thus, if the question is reinterpreted to ask whether the existence of a left adjoint to $\mathfrak{B}$ in the uniform case really depends on the uniform structure, as opposed to just the special nature of the frames in question, we should rather ask whether the restriction of $\mathfrak{B}$ to the completely regular frames,

$$\mathfrak{B} : \mathbf{CRegFrm}_{wo} \to \mathfrak{B}\mathbf{Frm} ,$$

has a left adjoint or not. This is the problem we shall deal with in the remainder of this section.

Note that a procedure similar to that in 3.1 would not apply here: in the regular case the β_L are monomorphisms by denseness, and hence if $\mathfrak{B}\varphi = \mathfrak{B}\psi$, one also has $\varphi = \psi$.

3.3. Now let

$$T : \mathfrak{B}\mathbf{Frm} \to \mathbf{CRegFrm}_{wo} \quad (\text{resp.}\mathbf{CRegFrm}_{no})$$

be a hypothetical left adjoint to $\mathfrak{B}$, with adjunction transformations

$$\varepsilon : T\mathfrak{B} \to \mathrm{Id} \quad \rho : \mathrm{Id} \to \mathfrak{B}T.$$

Let J : $\mathbf{\mathfrak{B}Frm} \to \mathbf{CRegFrm}_{wo}$ be the identical embedding, right adjoint to $\mathfrak{B}$. As now $\mathfrak{B} \circ T$ is a left adjoint to $\mathfrak{B} \circ J = \mathrm{Id}$, we conclude easily that

ρ is a natural equivalence.

In view of the adjunction identity

$$T \xrightarrow{T\rho} T\mathfrak{B}T \xrightarrow{\varepsilon T} T$$

we obtain further

Corollary. *Each $\varepsilon_{T(B)}$ is an isomorphism.*

3.4. Lemma. *If B is a Boolean algebra, ε_B is dense onto.*

PROOF: Since B is isomorphic to $\mathfrak{B}T(B)$ it suffices to prove that $\varepsilon_{\mathfrak{B}T(B)}$ is dense onto. Consider the commutative diagram

$$\begin{tikzcd} T\mathfrak{B}T(B) \arrow[r, "\varepsilon_{T(B)}"] \arrow[d, "\mathrm{id}_{T\mathfrak{B}T(B)}=T(\mathfrak{B}(\beta_{T(B)}))"'] & T(B) \arrow[d, "\beta_{T(B)}"] \\ T\mathfrak{B}(\mathfrak{B}T(B)) \arrow[r, "\varepsilon_{\mathfrak{B}T(B)}"] & \mathfrak{B}T(B) \end{tikzcd}$$

As $\varepsilon_{T(B)}$ is an isomorphism and $\beta_{T(B)}$ dense onto, so is $\varepsilon_{\mathfrak{B}T(B)}$. $\square$

3.5. Proposition. *A complete Boolean algebra is continuous iff it is atomic.*

PROOF: The "if" part being obvious, let B be a non-atomic complete Boolean algebra. Then $a = (\bigvee\{x \mid x \text{ atom in } B\})^*$ is not zero and hence we have a non-zero $b << a$. Further, as there are no atoms below a, we have a decreasing sequence

$$b = b_1 > b_2 > \cdots > b_n > \ldots .$$

Now the partition

$$a = (a \wedge b_1^*) \vee \bigvee_{n=1}^{\infty} (b_n \wedge b_{n+1})^* \vee \bigwedge_{n=1}^{\infty} b_n$$

obviously contains no finite cover of b, contradicting $b << a$, □

(This fact is implicit in the diagram of lattice properties on page 96 of [10]. There does not seem to be any reference to it in the text, though.)

3.6. Theorem. *Let $\mathcal{C}$ be* $\mathbf{CRegFrm}_{wo}$ *or* $\mathbf{CRegFrm}_{no}$. *Then* $\mathfrak{B} : \mathcal{C} \to \mathfrak{B}\mathbf{Frm}$ *has no left adjoint.*

PROOF: Suppose it has and use the notation of 3.2. For any $B \in \mathfrak{B}\mathbf{Frm}$ let $l : K \to T(B)$ be a compactification and put

$$k = (K \xrightarrow{l} T(B) \xrightarrow{\mathrm{id}=T(\beta_B)} T(\mathfrak{B}B) \xrightarrow{\varepsilon_B} B).$$

By 3.4 this is a compactification of B. We will show it is a smallest one, meaning that for any compactification $h : M \to B$ we have a $\varphi : K \to M$ such that

$$h \circ \varphi = k.$$

Indeed : As h is dense onto, $\mathfrak{B}h$ defined in 2.1 is also dense onto and hence an isomorphism by Booleanness. Then, for

$$\varphi = \varepsilon_M \circ T((\mathfrak{B}h)^{-1}) \circ T(\beta_B) \circ l,$$

we have

$$h \circ \varphi = h \circ \varepsilon_M \circ T((\mathfrak{B}h)^{-1}) \circ T(\beta_B) \circ l =$$
$$= \varepsilon_B \circ T(\mathfrak{B}h) \circ T((\mathfrak{B}h)^{-1}) \circ T(\beta_B) \circ l = k.$$

Now, a frame with a smallest compactification is continuous by [3], Proposition 4, but only atomic Boolean algebras possess this property by 3.5. □

4. Other maps

4.1. The argument from 3.2 - 3.6 cannot be used for the feebly open and open cases. Of compactifications we know only that they are nearly open (2.1), and indeed they often are not open, or feebly open. Furthermore, one cannot use the argument of 3.3 as β is not a reflection. On the other hand, if one asks just about the existence of a left adjoint, this is refuted for all cases by 3.1. Note that we have discussed the completely regular case primarily to contrast the frame situation with that of uniform frames, where the left adjoint exists in the weakly open case only anyway.

Still, a study of Booleanization as a functor from the category of Heyting algebras with special properties (and with complete Heyting homomorphisms $\equiv$ open frame homomorphisms) probably merits some interest, including, of course, the existence or non-existence of a *right* adjoint (recall 2.3).

4.2. Denote by $\mathfrak{J}L$ the ideal lattice of L, with intersection as meet and join given by

$$\bigvee_{i\in I} J_i = \{x_1 \vee \cdots \vee x_n \mid x_j \in J_{i_j}\},$$

and put, for $\varphi : L \to M$, $\mathfrak{J}\varphi(J) =\downarrow\varphi[J]$.

One easily sees that

$$J^* =\downarrow(\bigvee J)^*$$

for any $J \in \mathfrak{J}L$, hence $J^{**} =\downarrow(\bigvee J)^{**}$, and in particular $J^{**} =\downarrow\bigvee J$ if L is Boolean. Thus, for Boolean algebras we have a natural equivalence

$$\mu_B : B \cong \mathfrak{B}\mathfrak{J}B$$

defined by $\mu_B(b) =\downarrow b$. One also has somewhat canonical maps

$$\rho_L : \mathfrak{J}\mathfrak{B}L \to L$$

defined by $\rho_L(J) = \bigvee J$. These ρ_L are generally not homomorphisms. They do, however, satisfy "adjunction identities "

$$(\mathfrak{B}L \xrightarrow{\mu_{\mathfrak{B}(L)}} \mathfrak{B}\mathfrak{J}\mathfrak{B}(L) \xrightarrow{\mathfrak{B}(\rho_L)} \mathfrak{B}L) = \mathrm{id},$$

$$(\mathfrak{J}B \xrightarrow{\mathfrak{J}(\mu_B)} \mathfrak{J}\mathfrak{B}\mathfrak{J}(B) \xrightarrow{\rho_{\mathfrak{J}(B)}} \mathfrak{J}B) = \mathrm{id}$$

and therefore one is tempted to consider the category of frames with morphisms sufficiently relaxed to include these ρ_L. One can, for instance, take as morphisms those maps (not necessarily homomorphic) for which the square in 1.1 can be completed by a homomorphism ψ. (This results in the conditions: $\varphi(0) = 0$, $\varphi(1)^{**} = 1$, $\varphi(a \wedge b)^{**} = \varphi(a)^{**} \wedge \varphi(b)^{**}$ and $\varphi((\bigvee a_i)^{**})^{**} = (\bigvee \varphi(a_i)^{**})^{**}$; only the last relaxation is necessary for the ρ_L since they do preserve meets.) But this does not help either since the ρ_L do not form a transformation anyway. In the special case discussed in the next paragraph they do, but there they are actually homomorphisms.

4.3 A frame L is said to be *DeMorgan* (or *extremally disconnected*) if $a^* \vee a^{**} = 1$ for all $a \in L$, or, equivalently, if $(a^{**} \vee b^{**})^{**} = a^{**} \vee b^{**}$ for all $a.b \in L$. The category of all DeMorgan frames with weakly open homomorphisms will be denoted by $\mathbf{DMFrm}_{wo}$. Trivially, $\mathfrak{B}\mathbf{Frm}$ is a full subcategory of $\mathbf{DMFrm}_{wo}$ and $\mathfrak{B}$:$\mathbf{DMFrm}_{wo} \to \mathfrak{B}\mathbf{Frm}$ is a left adjoint to the embedding. We have

Theorem. *1. $\rho_L : \mathfrak{J}BL \to L$ is a frame homomorphism iff L is DeMorgan.*

2. The systems ρ and μ from 4.2 constitute an adjunction between $\mathfrak{B}$:$\mathbf{DMFrm}_{wo} \to \mathfrak{B}\mathbf{Frm}$ on the right and $\mathfrak{J}$:$\mathfrak{B}\mathbf{Frm} \to \mathbf{DMFrm}_{wo}$ on the left.

PROOF: 1. Since $\mathfrak{J}\mathfrak{B}L$ is the frame freely generated by the lattice $\mathfrak{B}L$, $\rho_L : \mathfrak{J}\mathfrak{B}L \to L$ is a frame homomorphism extending the identical embedding $\mathfrak{B}L \to L$ iff the latter is a lattice homomorphism, and this holds iff L is DeMorgan.

2. First note that, for any Boolean frame B, $\mathfrak{J}B$ is DeMorgan since $J^* \vee J^{**} = \downarrow(\bigvee J)^* \vee \downarrow\bigvee J = 1$. Further, for any $\varphi : B \to C$ in $\mathfrak{B}\mathbf{Frm}$, $\mathfrak{J}\varphi : \mathfrak{J}B \to \mathfrak{J}C$ is weakly open since an ideal J in B (or C) is dense iff $\bigvee J = 1$ while $\bigvee \mathfrak{J}\varphi(J) = \varphi(\bigvee J)$. Hence $\mathfrak{J}$ may be viewed as a functor $\mathfrak{B}\mathbf{Frm} \to \mathbf{DMFrm}_{wo}$.

On the other hand, $\mathfrak{B}$ is a functor on $\mathbf{DMFrm}_{wo}$ such that the identical embeddings $\mathfrak{B}L \to L$ are lattice homomorphisms natural in L : for any $\varphi : L \to M$ in $\mathbf{DMFrm}_{wo}$ and $a \in \mathfrak{B}L$, $\varphi(a) = \varphi(a)^{**}$

since $1 = a^* \vee a^{**} = a^* \vee a$ implies $\varphi(a) \vee \varphi(a^*) = 1$ and hence $\varphi(a) = \varphi(a^*)^*$. As a consequence, the $\rho_L : \mathfrak{J}\mathfrak{B}L \to L$ are also natural in L.

Finally, the adjunction identities

$$(\mathfrak{B}\rho_L)\mu_{\mathfrak{B}L} = \mathrm{id}_{\mathfrak{B}L}, \quad \rho_{\mathfrak{J}B}\mathfrak{J}\mu_B = \mathrm{id}_{\mathfrak{J}B}$$

are easily verified. □

For the somewhat related connection between $\mathbf{Frm}_{wo}$ and $\mathbf{DMFrm}_{wo}$ see [15].

5. Appendix : Some categorical properties

5.1. Since the categories $\mathbf{Frm}_{wo}$ and $\mathbf{Frm}_{no}$ contain $\mathfrak{B}\mathbf{Frm}$ as a reflective subcategory they cannot be cocomplete. $\mathbf{Frm}_{wo}$ is not complete either : By [19], some pairs of weakly open homomorphisms do not have equalizers in $\mathbf{Frm}_{wo}$.

5.2. $\mathbf{Frm}_{no}$ coincides with the category of complete distributive lattices with pseudocomplements, with homomorphisms preserving 0,1,$\wedge$, $\vee$ and *. $\mathbf{Frm}_o$ coincides with the category of complete Heyting algebras and complete Heyting homomorphisms. Thus, both of them are equationally presentable (in the terminology of [14] - see, e.g., [1]) and hence are complete with limits as in sets and therefore as in **Frm**.

5.3. Proposition. $\mathbf{Frm}_{wo}$ *has products and they are the usual frame products.*

PROOF: Let $p_i : L = \prod L_j \to L_i$ be a frame product. Then, for $x = (x_i)_i \in L$ obviously $x^* = (x_i^*)_i$, and hence the p_i are nearly open. Now consider weakly open $h_i : M \to L_i$. For the map $h : M \to \prod L_j$ satisfying $p_i h = h_i$, that is, $h(y) = (h_i(y))_i$, we have $h(y^{**}) = (h_i(y^{**}))_i \leq (h_i(y)^{**})_i = h(y)^{**}$. □

5.4. Proposition. *For any weakly open (nearly open, open) homomorphism $h : L \to M$, if $L \xrightarrow{p} K \xrightarrow{i} M$ is its usual image factorization, then p and i are also weakly open (nearly open, open).*

PROOF: As $\mathbf{Frm}_{no}$ and $\mathbf{Frm}_o$ are varieties of algebras, it suffices to prove the statement for weakly open h. For any dense $a \in L$, the fact that $p(a) = h(a)$ is dense in M trivially implies that it is dense in the subframe K of M. Further, if $h(a)$ is dense in K for some $a \in L$ then $h(a^*) = 0$ since $h(a) \wedge h(a^*) = 0$ in K. Now $h(a \vee a^*)$ is dense in M since h is weakly open, but this is just $h(a)$. Hence the identical embedding $i : K \to M$ is weakly open. □

5.5. Since any frame has, up to isomorphism, only a set of homomorphic images, and hence, a fortiori, only a set of homomorphic images in $\mathbf{Frm}_{wo}$, $\mathbf{Frm}_{no}$ or $\mathbf{Frm}_{o}$, a familiar argument constructs coequalizers in these categories from products and factorizations, and we have

Corollary. *Each of* $\mathbf{Frm}_{wo}$, $\mathbf{Frm}_{no}$ *and* $\mathbf{Frm}_{o}$ *has coequalizers.*

5.6. The coequalizers need not coincide with those in **Frm**. Here is an example covering all the three cases :

Let $L = \Omega(\mathbb{R})$ for the real line $\mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ sending x to $-x$, and $\varphi = \Omega(f)$, obviously an open homomorphism $L \to L$. Then, by regularity, the coequalizer of φ and id_L is a closed quotient of L and hence spatial. Thus, it corresponds to the equalizer $\{0\}$ of f and $\mathrm{id}_{\mathbb{R}}$ in spaces, making it the map $L \to \mathbf{2}$ given by $0 \in \mathbb{R}$, and this is obviously not weakly open since it takes the dense open $\mathbb{R} \setminus \{0\}$ to 0.

5.7. In the remaining three paragraphs we will add a few remarks on some coproducts in $\mathbf{Frm}_{wo}$. The following properties of coproducts will be used

(1) if $L \xrightarrow{i} L \oplus M \xleftarrow{j} M$ are the coproduct injections, $L \oplus M$ is $\bigvee-$ generated by the $x \oplus y = i(x) \wedge j(y)$,

(2) $x \oplus y = 0$ iff $x = 0$ or $y = 0$; consequently,

$$(x \oplus y)^* = (x^* \oplus 1) \vee (1 \oplus y^*)$$

and from this it is easy to infer that

$$(x \oplus y)^{**} = x^{**} \oplus y^{**}.$$

Lemma. *For any regular frame L, if $s = \bigvee\{x \oplus x^* \mid x \in L\}$ in $L \oplus L$ then $s^* = \bigvee\{a \oplus a \mid a$ atom in $L\}$.*

PROOF: To compute s^*, let $0 \neq a \oplus b$ be such that $(a \oplus b) \wedge s = 0$. As then also $(a \oplus b)^{**} \wedge s = (a^{**} \oplus b^{**}) \wedge s = 0$, we can assume that $a = a^{**}$ and $b = b^{**}$. Now $(a \oplus b) \wedge s = 0$ means that $(a \wedge x) \oplus (b \wedge x^*) = 0$ for all x, hence either $a \wedge x = 0$ or $b \wedge x^* = 0$, that is,

$$\text{either } a \leq x^* \text{ or } b \leq x^{**}, \text{ for any } x.$$

In particular for $x = a$ we obtain $b \leq a^{**} = a$ and similarly $a \leq b$ for $x = b^*$. Thus $a = b$, and it remains to prove that each such a is an atom. Let $0 < c \leq a$. By regularity, there is d such that $0 < d = d^{**} \prec c$ and by the property above,

$$a \leq d^* \text{ or } a \leq d.$$

Now, the first implies $d \leq a^*$, and as we already have $d \leq c \leq a$, this cannot be since $0 < d$. Thus, $a \leq d \leq c \leq a$, and hence $a = c$. □

5.8. As is well-known, the coproduct maps $L \xrightarrow{i} L \oplus M \xleftarrow{j} M$ are open for any frame coproduct $L \oplus M$ (see Pitts [20], also Joyal - Tierney [16]). This makes it natural to ask whether such a coproduct is also the coproduct in $\mathbf{Frm}_{wo}$, $\mathbf{Frm}_{no}$ or $\mathbf{Frm}_o$. The following proposition shows this is often not the case.

Proposition. *For* $\mathbf{K}$= $\mathbf{Frm}_{wo}$, $\mathbf{Frm}_{no}$, *or* $\mathbf{Frm}_o$ *the following are equiva lent for any Boolean L :*

(1) *L is atomic,*
(2) *$L \oplus M$ is the coproduct in $\mathbf{K}$ for any M,*
(3) *$L \oplus L$ the is coproduct in $\mathbf{K}$.*

PROOF: (1)⇒(2) : If X is the set of atoms of L then $L \cong \mathbf{2}^X$ and $L \oplus M \cong M^X$, with coproduct maps $2^X \to M^X$ determined by the initial $2 \to M$ and $M \to M^X$ the diagonal embedding, such that the homomorphism $h : M^X \to N$ for any given $f : L \to N$ and $g : M \to N$ is given by

$$h((a_s)_{s \in X}) = \bigvee\{g(a_s) \wedge f(s) \mid s \in X\}.$$

Further, $N \cong \prod\{\downarrow f(s) \mid s \in X\}$, taking $x \in N$ to $(x \wedge f(s))_{s\in X}$, and we may read h as

$$h((a_s)_{s\in X}) = (g(a_s) \wedge f(s))_{s\in X}.$$

Now, f is open because L is Boolean, and since relative pseudocomplements and meets in product frames are taken componentwise, this makes it immediately evident that h is weakly open, nearly open, or open whenever g is of the corresponding type.

(2)⇒(3) : trivial.

(3)⇒(1) : Since the codiagonal $\nabla : L \oplus L \to L$ is at least weakly open, $\nabla(s \vee s^*)$ is dense for s in 5.7, and thus $\nabla(s^*) = \nabla(s \vee s^*) = 1$ because $\nabla(s) = 0$ and L is Boolean. It follows by 5.7 that

$$1 = \bigvee\{\nabla(a \oplus a) \mid a \text{ atom in } L\} = \bigvee\{a \mid a \text{ atom in } L\},$$

showing L is atomic. □

Remark 1. Joyal and Tierney [16] characterize the complete atomic Boolean algebras as those frames for which (the initial map into L and) the codiagonal $L \oplus L \to L$ are open, giving part of the above (3)⇒(1).

Remark 2. If $L \oplus L$ is Boolean, for any frame L, then it is the coproduct in $\mathbf{Frm}_o$ while L itself is Boolean as homomorphic image of $L \oplus L$. Hence, by the proposition, L is then also atomic, and therefore

$L \oplus L$ is Boolean iff L is Boolean atomic

- a recent result of Dona Strauss.

5.9. Proposition. *For a finite frame L, all $L \oplus M$ are coproducts in* $\mathbf{Frm}_{wo}$.

PROOF: If $c_1, c_2, \ldots, c_n$ are the elements of L then any $u \in L \oplus M$ has the form $u = (c_1 \oplus z_1) \vee \cdots \vee (c_n \oplus z_n)$ with $z_k \in M$ so that (recall 5.7.(2))

$$u^* = (c_1 \oplus z_1)^* \wedge \cdots \wedge (c_n \oplus z_n)^* =$$

$$= ((c_1^* \oplus 1) \vee (1 \oplus z_1^*)) \wedge \cdots \wedge ((c_n^* \oplus 1) \vee (1 \oplus x_n^*)) =$$

$$= \bigvee\{c_S^* \oplus z_S^* \mid S \subseteq \{1,2,\ldots,n\}\}$$

where $c_S = \bigvee_{k\in S} c_k$, $z_S = \bigvee_{k\notin S} z_k$. Thus, u is dense iff all $c_S^* \oplus z_S^* = 0$, that is $c_S^* = 0$ or $z_S^* = 0$, and hence

for any S, either c_S or z_S is dense.

Now let $h : L \oplus M \to N$ result from weakly open $f : L \to N$ and $g : M \to N$. Then

$$h(u) = (f(c_1) \wedge g(z_1)) \vee \cdots \vee (f(c_n) \wedge g(z_n)) =$$

$$= \bigwedge\{f(c_S) \vee g(z_S) \mid S \subseteq \{1,2,\ldots,n\}\}$$

where each term in the latter meet is dense since either $f(c_S)$ or $g(z_S)$ has to be dense for each S. Thus, $h(u)$ is dense as a meet of finitely many dense elements. $\square$

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