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MEASURE CHARACTERISTICS OF COMPLEXES

by B. de SMIT

Résumé. Nous donnons une version du lemme du serpent en théorie de la mesure des groupes abéliens localement compacts. Cette version utilise la notion d'homomorphisme topologique strict et la notion d'exactitude de mesure. Grâce à ce lemme, les méthodes d'algèbre homologique peuvent être appliquées dans le contexte de l'analyse harmonique abstraite. On en déduit des résultats sur les caractéristiques des complexes de groupes abéliens localement compacts avec des mesures de Haar.

1. Introduction.

For a bounded complex of abelian groups with finite homology groups the Euler-Poincaré characteristic is the alternating product of the size of the homology groups. The object of this paper is to develop an analog of the Euler-Poincaré characteristic for complexes of locally compact abelian groups with Haar measures. This entails proving a measure-theoretic version of the snake lemma.

The main applications we have in mind are in algebraic number theory. Many important invariants one associates to number fields, such as the idele class group, are locally compact groups with a normalization of the Haar measure. One often manipulates with such groups by using exact sequences, and to keep track of what happens to the measures one needs a theory of measure characteristics. Recently there has been more interest in the Arakelov class group [8, III, (1.10)] of a number field. By using volume characteristics of complexes of such groups one can prove class number relations [4]. All results in this note will be formulated without any reference to number theory.

Under the assumption that all groups are countable at infinity, Oesterlé defines a volume characteristic of bounded complexes with finite homology groups [9]. By dropping these assumptions we can run into certain anomalies of a purely topological nature, namely non-strict homomorphisms. We will address this in the next section, and we will need some arguments concerning strictness that go slightly beyond Bourbaki [3, chap. III, §2.8].

In section 3 the measure-theoretic snake lemma is proved by a purely category-theoretic reduction to a much easier case of a 3×3 diagram (a short exact sequence of short exact sequences).

In section 4 we define measure characteristics of complexes and show that they are multiplicative over short measure exact sequences of complexes. As a special case we recover results used by Lang in his book on Arakelov geometry [7, chap. V, §2].

2. Strict morphisms of topological abelian groups.

All objects in this section are topological abelian groups and a morphism $f : A \to B$ is a continuous group homomorphism. We say f is *strict* if the map from A onto its image I in B (with relative topology from B) is an open map. In other words, f is strict if and only if the continuous bijection $A/\operatorname{Ker} f \xrightarrow{\sim} I$ is a homeomorphism. This definition of strictness can also be found in Bourbaki [3, chap. III, §2.8] and in [11, exp. 1, §3.1]. One word of caution: a composition of strict morphisms need not be strict, as one can see by considering the maps $\mathbb{Z}\sqrt{2} \subset \mathbb{R} \to \mathbb{R}/\mathbb{Z}$. A sequence of morphisms is said to be strict if all morphisms in the sequence are strict.

A continuous bijection from a compact to a Hausdorff topological space is a homeomorphism, so any morphism from a compact group to a Hausdorff group is strict. In the context of Oesterlé [9] one only considers Hausdorff locally compact abelian groups which are countable at infinity, i.e., a countable union of compact subsets. One can show that a morphism between such groups is strict if and only if it has a closed image [3, chap. IX, §5.3].

(2.1) Lemma. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be morphisms. If f is strict and surjective then*

$$g \text{ is strict} \iff gf \text{ is strict.}$$

If g is strict and injective then

$$f \text{ is strict} \iff gf \text{ is strict.}$$

Proof. For the first statement one notes that the open sets of B are exactly the images of open sets of A. The second statement follows from the fact that the map $f(A) \xrightarrow{g} g(f(A))$ is a homeomorphism. □

(2.2) Proposition. *Suppose we have a commutative diagram of morphisms*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi'} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi''} & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \end{array}$$

with strict exact rows. Let s be the group homomorphism $\operatorname{Ker}\varphi'' \to \operatorname{Coker}\varphi'$ *from the snake lemma; see* (3.5) *below. Then the following hold:*

(1) *if s is surjective and φ is strict, then φ'' is strict;*
(2) *if s is injective and φ is strict then φ' is strict;*
(3) *if s is the zero map and φ' and φ'' are strict, then φ is strict.*

Before giving the proof of (2.2), we point out an example of a diagram that shows how non-strict maps can occur in this context. All vertical maps are injective, and one map is non-strict. The map s is the zero map (which is injective).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\sqrt{2}} & & \downarrow{\scriptstyle\sqrt{2}} & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R}/\mathbb{Z} & \longrightarrow & 0 \end{array}$$

Proof of (2.2). The hypothesis that the rows are strict allows us to view the injections $A' \to A$ and $B' \to B$ as inclusions of topological groups.

We first make the following remark: if $X \subset Y \subset Z$ are inclusions of topological groups, then the canonical map $Y/X \rightarrowtail Z/X$ is strict. One can see this by writing the preimage U in Y of an open subset $\overline{U}$ of Y/X as $U = O \cap Y$ with O open in Z. We then have $U = (O+X) \cap Y$ so that $\overline{U}$ is the intersection of Y/X and the image of O in Z/X.

We first show (1). The surjectivity of the snake map means that $B' \subset \varphi(A)$. Since φ is assumed to be strict, the map $A \to \varphi(A)/B'$ is a strict surjection. By the remark above, the map $\varphi(A)/B' \to B/B'$ is strict, and with (2.1) we see that the map $A \to B/B'$ is strict. The map $B/B' \to B''$ is a topologically isomorphism and the map $A \to A''$ is strict and surjective. With (2.1) it follows that φ'' is strict.

The hypothesis of (2) implies that $\operatorname{Ker}\varphi \subset A'$. Again using the remark above, one sees that the map $A'/\operatorname{Ker}\varphi \to A/\operatorname{Ker}\varphi$ is strict. By applying (2.1) twice, and using that φ is strict, we see that the composition

$$A' \twoheadrightarrow A'/\operatorname{Ker}\varphi \longrightarrow A/\operatorname{Ker}\varphi \rightarrowtail B$$

is strict. Since $B' \to B$ is a strict injection, a third application of (2.1) shows that φ' is strict. This shows (2).

For (3) we need to show that φ is open on its image, so for an open neighborhood U of 0 in A we want that $\varphi(U)$ is a neighborhood of 0 in $\varphi(A)$. Let U_0 be an open neighborhood of 0 in A with $U_0 + U_0 \subset U$. Since φ' is strict, we have $\varphi(A' \cap U_0) = \varphi(A') \cap O$ for some open neighborhood O of 0 in B. Let O_0 be an open neighborhood of 0 in B with $O_0 - O_0 \subset O$. Let the subset X of $\varphi(A)$ be defined as $X = \varphi(A') + (O_0 \cap \varphi(U_0))$, then we have

$$\begin{aligned} X \cap O_0 &\subset (\varphi(A') \cap (O_0 - O_0)) + \varphi(U_0) \subset (\varphi(A') \cap O) + \varphi(U_0) \\ &= \varphi(A' \cap U_0) + \varphi(U_0) \subset \varphi(U). \end{aligned}$$

Thus, we are done if we can show that X is open in $\varphi(A)$. Denote the map $A \to A''$ by f. Since $U_0 \cap \varphi^{-1}(O_0)$ is open in A, and the map f is open, it follows that the set $V = f(U_0 \cap \varphi^{-1}(O_0))$ is open in A''.

Note that $X = \varphi(f^{-1}(V))$. Since φ'' is strict, the set $\varphi''(V)$ is open in $\varphi''(A'')$. Denoting the map $\varphi(A) \to \varphi''(A'')$ by g, we deduce that $g^{-1}(\varphi''(V)) = X + \operatorname{Ker} g$ is open in $\varphi(A)$. By some simple diagram chasing, one checks that our hypothesis that $s = 0$ is equivalent to $\operatorname{Ker} g = \varphi'(A')$. But we have $X = X + \varphi'(A')$, so this implies that X is open in $\varphi(A)$. □

(2.3) Proposition. *Suppose we have a commutative diagram of morphisms*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A' & \to & A & \to & A'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B' & \to & B & \to & B'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C' & \to & C & \to & C'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

in which all rows and columns are exact. If five of the six exact sequences are strict then so is the sixth.

Proof. The case that the sixth sequence is the middle row or middle column follows from (2.2) part (3).

So assume that the middle row and column are strict. It is clear from (2.1) that the maps out of A' and the maps into C'' are strict.

Consider the following diagram with strict exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B' & \longrightarrow & B' \oplus A & \longrightarrow & A & \longrightarrow & 0 \\ & & \Big\downarrow \text{id} & & \Big\downarrow f & & \Big\downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0, \end{array}$$

where $B' \oplus A$ has the product topology, and f is the sum map. The snake map for this diagram is the surjective zero map, so by (2.2) part (1) and (3) the strictness of f is equivalent to the strictness of the map $A \to B''$, and, by symmetry, it is also equivalent to the map $B' \to C$ being strict.

The fact that there is at most one non-strict row or column, and (2.1) now implies that $A \to B'$ or $B' \to C$ is strict. But then f is strict, so both $A \to B'$ and $B' \to C$ are strict. Using (2.1) again it follows that all incoming and outgoing maps of C' and A'' are strict homomorphisms. □

(2.4) Remark. Suppose all maps in the diagram in (2.3) are strict. We view A', A and B' as subgroups of B so that A'' and C' can be identified with subgroups of B/A'. The map f in the proof above is strict, and one can deduce that the canonical exact sequence

$$0 \to C' \oplus A'' \to B/A' \to C'' \to 0$$

is strict.

3. Haar measures and the snake lemma.

Let G be a Hausdorff locally compact topological abelian group. A Haar measure on G is a translation invariant non-zero measure on G, for which all open sets are measurable, and all compact sets have finite measure. If $\mathcal{F}(G)$ denotes the real vector space of real valued continuous functions on G with support inside a compact subset of G, then the Haar measure can be viewed as an $\mathbb{R}$-linear map $\mathcal{F}(G) \to \mathbb{R}$ sending f to $\int_G f(g)dg$. The Haar measure on G is unique up to multiplication by a positive real number. We refer to [5], section 11 and 15, and to Bourbaki [2, chap. VII] for the precise statements and proofs.

We say G is a *measured group* if G is a Hausdorff locally compact topological abelian group equipped with a choice of Haar measure. If G is a compact measured group then its *volume* is defined as the measure of the whole space, i.e., $\mathrm{vol}(G) = \int_G 1dg$. A strict morphism of measured groups has a closed kernel and image [3, chap. III, §3.3].

(3.1) Quotient measures. Suppose that H and G are measured groups, and that H is a closed subgroup of G. Then G/H is Hausdorff and locally compact, and we will show how to give it a natural Haar measure. A function $f \in \mathcal{F}(G)$ on G induces a function $\hat{f}$ on G/H defined by

$$\hat{f}(x) = \int_H f(\tilde{x} + h)dh,$$

where $\tilde{x}$ is a representative of x in G. Note that $\hat{f}(x)$ does not depend on the choice of $\tilde{x}$ as the measure on H is translation invariant. Furthermore, $\hat{f}$ has support inside a compact subset of G/H so $\hat{f} \in \mathcal{F}(G/H)$. The quotient measure on G/H is the unique Haar measure on G/H for which

$$\int_G f(g)dg = \int_{G/H} \hat{f}(x)dx.$$

(3.2) Measure characteristic of short exact sequences. Suppose we have a strict exact sequence of measured groups

$$(S) \qquad 0 \longrightarrow G' \longrightarrow G \xrightarrow{\varphi} G'' \longrightarrow 0.$$

The isomorphism $G' \xrightarrow{\sim} \operatorname{Ker}\varphi$ is a homeomorphism, so we can give $\operatorname{Ker}\varphi$ the Haar measure of G'. Uniqueness of the Haar measure implies that the topological isomorphism $G/\operatorname{Ker}\varphi \xrightarrow{\sim} G''$ identifies the quotient measure on $G/\operatorname{Ker}\varphi$ with c times the measure on G'' for a unique constant $c \in \mathbb{R}_{>0}$. This constant c is called the *measure characteristic* of (S), and we denote it by $\kappa(S)$. Oesterlé [10] calls c^{-1} the "Haar index." If $\kappa(S) = 1$ then we say (S) is a *measure exact* sequence. If G is compact, then so are G' and G'', and

$$\kappa(S) = \frac{\operatorname{vol}(G)}{\operatorname{vol}(G')\operatorname{vol}(G'')}.$$

The following proposition resembles Oesterlé [10, A.4.2]. Oesterlé imposes stronger topological conditions, but he also allows non-abelian groups.

(3.3) Proposition. *Suppose we have a commutative diagram*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_1^1 & \to & A_1^2 & \to & A_1^3 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_2^1 & \to & A_2^2 & \to & A_2^3 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_3^1 & \to & A_3^2 & \to & A_3^3 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

in which all rows and columns are strict short exact sequences of measured groups. Then we have

$$\kappa(A_\bullet^1)^{-1}\kappa(A_\bullet^2)\kappa(A_\bullet^3)^{-1} = \kappa(A_1^\bullet)^{-1}\kappa(A_2^\bullet)\kappa(A_3^\bullet)^{-1}.$$

Proof. We will identify A_1^1, A_1^2 and A_2^1 with their image in A_2^2. The idea is to compute the integral of a function $f \in \mathcal{F}(A_2^2)$ in two ways. For $a \in A_2^2$ and $(x, y) \in A_1^3 \times A_3^1$ we put

$$g_a(x, y) = \int_{A_1^1} f(a + \tilde{x} + \tilde{y} + z)dz,$$

where $\tilde{x}$ and $\tilde{y}$ are lifts in A_1^2 and A_2^1 of x and y. Note that $g_a(x, y)$ does not depend on the choice of the lifts. Now consider the strict exact sequence of (2.4):

$$0 \to A_1^3 \times A_3^1 \to A_2^2/A_1^1 \to A_3^3 \to 0$$

We have $g_a(x, y) = \hat{f}(\bar{a} + x + y)$, where $\hat{f} \in \mathcal{F}(A_2^2/A_1^1)$ is the function induced by f as in (3.1), and $\bar{a}$ is the image of a in A_2^2/A_1^1. It follows that $g_a \in \mathcal{F}(A_1^3 \times A_3^1)$. The statements of (3.1) now tell us that for $u \in A_3^3$ the integral

$$g(u) = \int_{A_1^3 \times A_3^1} g_{\tilde{u}}(x, y) d(x, y)$$

where $\tilde{u}$ is any lift of u in A_2^2, does not depend on the choice of $\tilde{u}$ and that we have $g \in \mathcal{F}(A_3^3)$.

We now rewrite the integral $\int_{A_2^2} f(a)da$ by first using the middle column, and then the outer rows:

$$\int_{A_2^2} f(a)da = \kappa(A_\bullet^2)\kappa(A_3^\bullet)\kappa(A_1^\bullet) \int_{A_3^3} \int_{A_3^1} \int_{A_1^3} g_{\tilde{u}}(x, y) dx dy du.$$

By Fubini's theorem [5, (13.8)] this is equal to

$$\kappa(A_\bullet^2)\kappa(A_3^\bullet)\kappa(A_1^\bullet) \int_{A_3^3} g(u) du.$$

The same is true with rows and columns switched. By choosing a function f whose integral over A_2^2 is not zero, we get the equality stated in (3.3). □

(3.4) Long measure exact sequences. Suppose we have a strict exact sequence

$$(A_\bullet) \qquad \cdots \to A_i \to A_{i+1} \to A_{i+2} \to \cdots$$

of measured groups, with almost all A_i equal to the zero group of volume 1. For each i, choose a Haar measure on the image B_i of A_{i-1} in A_i such that almost all B_i are the zero group of volume 1. We have strict short exact sequences of measured groups

$$(S_i) \qquad 0 \to B_i \to A_i \to B_{i+1} \to 0$$

and we define the measure characteristic $\kappa(A_\bullet)$ to be

$$\kappa(A_\bullet) = \prod_i \kappa(S_i)^{(-1)^i}.$$

This does not depend on the choice of measures on the B_i, because changing the measure of B_i by a factor $c \in \mathbb{R}_{>0}$ changes both $\kappa(S_i)$ and $\kappa(S_{i-1})$ by a factor c^{-1}. We say that $A_\bullet$ is *measure exact* if $\kappa(A_\bullet) = 1$. It is easy to see that that $\kappa(A_\bullet) = \prod_i \mathrm{vol}(A_i)^{(-1)^i}$ if all A_i are compact.

(3.5) The snake lemma. We briefly recall the snake lemma as given in Atiyah-Macdonald [1, prop. 2.10]; see also Lang [6, chap. II, §9]. Suppose we have a commutative diagram of abelian groups, with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & K_1 & & K_2 & & K_3 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_1 & \to & A_2 & \to & A_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & B_1 & \to & B_2 & \to & B_3 & \to & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & C_1 & & C_2 & & C_3 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The snake lemma asserts that we have a canonical exact sequence

$$(S) \qquad 0 \to K_1 \to K_2 \to K_3 \to C_1 \to C_2 \to C_3 \to 0,$$

where the "snake map" $K_3 \to C_1$ is defined as follows: take an element of K_3, find its image in A_3, lift it to A_2, map it to B_2, lift to B_1 and map to C_1.

(3.6) Theorem.

(1) *Suppose that the groups in the diagram are topological groups and that the maps are strict continuous homomorphisms. Then the maps in the snake sequence* (S) *are continuous and strict.*

(2) *Suppose that the diagram consists of measured groups and strict homomorphisms, and that the rows and columns are measure exact. Then the snake sequence* (S) *is also measure exact.*

Proof. Note that for continuity of the map $K_1 \to K_2$ we need strictness of the map $K_2 \to A_2$. For any commutative square of continuous homomorphisms of topological groups

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ \downarrow & & \downarrow \\ E' & \xrightarrow{\varphi'} & F' \end{array}$$

we have induced continuous maps $\operatorname{Ker}\varphi \to \operatorname{Ker}\varphi'$ and $\operatorname{Coker}\varphi \to \operatorname{Coker}\varphi'$. This shows continuity of all maps in (S) except for the snake map.

In a purely category-theoretic way one can build up the snake diagram from five diagrams of the type considered in (2.3) and (3.3). Thus the proof will be a reduction to (2.3) and (3.3).

First fix some notation: I_i is the image of A_i in B_i and for X equal to the letter K, I or C, we let KX be the kernel of $X_2 \to X_3$, and we let CX be the cokernel of $X_1 \to X_2$. We can choose topologies (measures) on these groups so that in the following two diagrams the short exact rows and columns are strict (measure exact):

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K_1 & \to & K_2 & \to & CK & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_1 & \to & A_2 & \to & A_3 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & I_1 & \to & I_2 & \to & CI & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \qquad \begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & KI & \to & I_2 & \to & I_3 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B_1 & \to & B_2 & \to & B_3 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & KC & \to & C_2 & \to & C_3 & \to & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Let H be the cokernel of the map $K_2 \to K_3$ and give it the topology (measure) that gives the following diagram strict (measure exact) rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & CK & \to & K_3 & \to & H & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & CK & \to & A_3 & \to & CI & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & I_3 & \to & I_3 & \to & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

We now deduce that the top row in the left diagram below is strict (measure exact), and that the bottom row in the right diagram is strict (measure exact)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_1 & \to & KI & \to & H & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_1 & \to & I_2 & \to & CI & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & I_3 & \to & I_3 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & I_1 & \to & I_1 & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & KI & \to & B_1 & \to & KC & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H & \to & C_1 & \to & KC & \to & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Our snake sequence now consists of four short strict (measure exact) sequences. This proves (3.6). □

(3.7) Remark. Suppose that the rows R_A and R_B and the columns P_1, P_2 and P_3 of the snake diagram are strict exact sequences of measured groups, without assuming that they are measure exact. In order to consider measure characteristics we need to fix the parity of indices in these sequences, so let the index be 1 at K_i for the columns, at A_1

and B_1 for the rows and at K_1 for the snake sequence (S). Then one has

$$\kappa(S) = \frac{\kappa(P_1)\kappa(P_3)}{\kappa(P_2)} \frac{\kappa(R_A)}{\kappa(R_B)}.$$

To see this one can either go through the proof of (3.6) again, or one can reduce to case we proved already by changing the measures on the groups in order to obtain measure exact rows and columns and keeping track of the effect of the measure changes on all measure characteristics involved.

(3.8) Remark. By modifying A_1 and B_3 one can easily generalize this snake lemma to the following diagram

$$\begin{array}{ccccccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & & & & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & & & & & \\
 & & & & & & K_1 & & K_2 & & K_3 & & & & & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & & & & & \\
0 \to A_n & \to & \cdots & \to & A_0 & \to & A_1 & \to & A_2 & \to & A_3 & \to & 0 & & & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & & & & & \\
 & & & & 0 & \to & B_1 & \to & B_2 & \to & B_3 & \to & B_4 & \to \cdots \to & B_m & \to & 0, \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & & & & & \\
 & & & & & & C_1 & & C_2 & & C_3 & & & & & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & & & & & \\
 & & & & & & 0 & & 0 & & 0 & & & & & &
\end{array}$$

where $n \le 1$ and $m \ge 3$. One obtains a snake sequence

$$0 \to A_n \to \cdots \to A_0 \to K_1 \to K_2 \to K_3$$
$$\to C_1 \to C_2 \to C_3 \to B_4 \to \cdots \to B_m \to 0.$$

Again, strictness of the diagram implies strictness of the snake sequence. In the measure-theoretic setting one can formulate the result as follows: if we fix indexing parity on the snake sequence by giving K_1 index 1, then the characteristic of the snake sequence is $\kappa(A_\bullet)\kappa(B_\bullet)^{-1}\kappa(P_1)\kappa(P_2)^{-1}\kappa(P_3)$ with P_i as in (3.7).

4. Measure characteristics on complexes.

A complex is a collection of abelian groups A_i with $i \in \mathbb{Z}$, with maps $d_i: A_i \to A_{i+1}$ such that $d_{i+1}d_i = 0$ for all i. By a measured complex we mean a complex $A_\bullet$ of measured groups for which all d_i are strict and almost all A_i are the zero group of volume 1. Suppose $A_\bullet$ is a measured complex. Let Z_i be the kernel of d_i, and let B_i be the image of d_{i-1}. The ith homology group of $A_\bullet$ is defined to be $H_i(A_\bullet) = Z_i/B_i$, and it is again locally compact. We have strict exact sequences

$$(S^i_\bullet) \qquad 0 \to Z_i \to A_i \to Z_{i+1} \to H_{i+1}(A_\bullet) \to 0.$$

If we fix indexing in $(S^i_\bullet)$ by $S^i_1 = Z_i$, and choose Haar measures on all Z_i and H_i (making almost all of them zero groups of volume 1) then we can use (3.4) to define the *measure characteristic* of $A_\bullet$ to be

$$\chi(A_\bullet) = \prod_i (\kappa(S^i_\bullet))^{(-1)^i}.$$

Now $\chi(A_\bullet)$ does not depend on the choice of measures on Z_i but it *does* depend on the measures on $H_i(A_\bullet)$.

(4.1) Exact sequences of complexes. Suppose we have a short measure exact sequence of measured complexes $0 \to A'_\bullet \to A_\bullet \to A''_\bullet \to 0$. By this we mean that for each i we have a sequence $0 \to A'_i \to A_i \to A''_i \to 0$ as in (3.2), such that the diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & A'_i & \longrightarrow & A_i & \longrightarrow & A''_i & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & A'_{i+1} & \longrightarrow & A_{i+1} & \longrightarrow & A''_{i+1} & \longrightarrow 0 \end{array}$$

commutes for each i. The most fundamental result of homological algebra [6, chap. XX, §2] says that a short exact sequence of complexes gives rise to a long exact sequence

$$(\mathcal{H}_\bullet) \qquad \cdots \to H_0(A'_\bullet) \to H_0(A_\bullet) \to H_0(A''_\bullet) \to H_1(A'_\bullet) \to \cdots .$$

(4.2) Theorem. *The sequence $(\mathcal{H}_\bullet)$ is strict. Moreover, if we fix indexing parity by setting $\mathcal{H}_0 = H_0(A'_\bullet)$, and we choose Haar measures on all homology groups (making almost all of them zero groups of volume 1), then*

$$\kappa(\mathcal{H}_\bullet) = \frac{\chi(A_\bullet)}{\chi(A'_\bullet)\chi(A''_\bullet)} .$$

Proof. The proof of (4.2) is by repeated application of the snake lemma. Let Z_i be the kernel of the map $A_i \to A_{i+1}$, and define Z'_i, and Z''_i similarly. Also choose Haar measures on all Z_i, Z'_i and Z''_i, giving almost all of them volume 1. We show by induction that for each i we have a strict long exact sequence

$$0 \to Z'_{i-1} \to Z_{i-1} \to Z''_{i-1} \to H_i(A'_\bullet) \to H_i(A_\bullet) \to H_i(A''_\bullet)$$
$$\to H_{i+1}(A'_\bullet) \to H_{i+1}(A_\bullet) \to H_{i+1}(A''_\bullet) \to \cdots$$

and we keep track of its measure characteristic. For sufficiently large i this sequence consists of only zero groups of volume 1 and for sufficiently small i it is identical to $\mathcal{H}_\bullet$. The reader may finish the proof by applying (3.8) to the diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & Z'_{i-1} & & Z_{i-1} & & Z''_{i-1} & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \to & A'_{i-1} & \to & A_{i-1} & \to & A''_{i-1} & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \to & Z'_i & \to & Z_i & \to & Z''_i & \to & H_{i+1}(A'_\bullet) & \to & H_{i+1}(A_\bullet) \to \cdots . \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & H_i(A'_\bullet) & & H_i(A_\bullet) & & H_i(A''_\bullet) & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array}$$

□

(4.3) Global measure characteristic. If the homology groups of a measured complex $A_\bullet$ are compact, then we can give them measure 1, and the corresponding value $\chi_{\text{gl}}(A_\bullet)$ of $\chi(A_\bullet)$ is called the *global measure characteristic.* This characteristic is the best measure of size, in the sense that we have

$$\chi_{\text{gl}}(A_\bullet) = \prod_i \text{vol}(A_i)^{(-1)^i}$$

if all A_i are compact. The theorem above implies that χ_{gl} is multiplicative over short measure exact sequences of measured complexes with compact homology. Moreover, strictness of the homology sequence implies that if two of the three complexes in the theorem have compact homology, then so does the third.

In Lang [7, chap. V, §2] this characteristic occurs in the following context. Let M be a finitely generated abelian group with a given Haar measure on $M \otimes_{\mathbb{Z}} \mathbb{R}$ (with the Euclidean topology). Giving M the discrete topology and counting measure, we can consider the complex $C_M : 0 \to M \to M \otimes_{\mathbb{Z}} \mathbb{R} \to 0$. The characteristic $\chi_{\text{gl}}(C_M)$ is the ratio of the covolume of the image of M in $M \otimes \mathbb{R}$ and the order of the torsion subgroup of M. Lang also shows additivity over short measure exact sequences.

(4.4) Local measure characteristic. If the homology groups of a measured complex $A_\bullet$ are discrete, then we can give them the counting measure, and the corresponding value $\chi_{\text{loc}}(A_\bullet)$ of $\chi(A_\bullet)$ is called the *local measure characteristic* of $A_\bullet$. This characteristic keeps track of local blow-up factors in the measure. For instance, if $A_1 = A_2 = \mathbb{R}/\mathbb{Z}$ with $d_1(x) = 2x$ and all other A_i are the zero group of volume 1, then $\chi_{\text{gl}}(A_\bullet) = 1$, but $\chi_{\text{loc}}(A_\bullet) = 2$. Note that χ_{loc} is multiplicative over short measure exact sequences of measured complexes with discrete homology. Again, strictness of the homology sequence implies that if two of the three complexes in the theorem have discrete homology, then so does the third.

If $A_\bullet$ has finite homology, then we have

$$\chi_{\text{gl}}(A_\bullet) = \chi_{\text{loc}}(A_\bullet) \cdot \chi_{\#}(A_\bullet),$$

where $\chi_{\#}$ is the usual Euler-Poincare characteristic given by

$$\chi_{\#}(A_\bullet) = \prod_i (\# H_i(A_\bullet))^{(-1)^i}.$$

If $A_\bullet$ is exact, then $\chi_{\text{gl}}(A_\bullet) = \chi_{\text{loc}}(A_\bullet) = \kappa(A_\bullet)$ (cf. (3.4)).

(4.5) Duality. Under Pontrjagin duality [5, chap. VI], compact groups are dual to discrete groups. The dual of a strict morphism is again strict [11, exp. 11, §6.1]. Moreover, the dual of a measured group has a dual measure, which is the measure for which the Fourier inversion formula holds. The dual of the volume-one-measure on a compact group is the counting measure on its dual. It is not hard to see that dualizing a measured short exact sequence inverts the measure characteristic. Taking the dual of a measured complex A with compact homology groups and replacing the indices i in the resulting sequence by $-i$, we obtain a measured complex $\hat{A}$ with discrete homology, and we have

$$\chi_{\text{loc}}(\hat{A}) = \chi_{\text{gl}}(A)^{-1}.$$

If the homology groups are finite we deduce that

$$\chi_{\text{gl}}(\hat{A})\chi_{\text{gl}}(A) = \chi_{\#}(A).$$

This statement can also be found in Oesterlé [9, §3].

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