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SOME REMARKS ON CONJUGACY CLASSES OF BUNDLE GAUGE GROUPS

by Matilde MARCOLLI

RESUME. Les G-fibrés principaux sur un CW-complexe B peuvent être divisés en classes d'équivalence, ou en classes de conjugaison de groupes de jauge (groupes d'auto-équivalence). Dans cet article, on étudie les connexions entre isomorphisme de groupes de jauge, relation de conjugaison et équivalence de fibrés.

On donne d'abord un exemple de deux fibrés qui ont des groupes de jauge isomorphes qui ne sont pas conjugués. On prouve alors que, sous certaines hypothèses sur le groupe G et sur la trivialisation locale des fibrés, deux fibrés sur un polyèdre fini B qui ont des groupes de jauge conjugués sont équivalents si et seulement si les fibrés produit fibré via l'inclusion du 3-squelette dans B sont équivalents.

1 Introduction

Let $\xi = (E, p, B)$ be a principal G-bundle, where G is a topological group. We assume that the base space B is a CW-complex. Let $\{U_\alpha, \phi_\alpha\}_{\alpha \in \Gamma}$ be the local trivialization with $\phi_\alpha : U_\alpha \times G \xrightarrow{\cong} p^{-1}(U_\alpha)$ and let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ the transition functions,

$$\phi_\beta(b, g) = \phi_\alpha(b, g_{\alpha\beta}(b)g).$$

The *gauge group* $\mathcal{G}(p)$ of the bundle is defined as the topological group of self equivalences $\Lambda : E \to E$ that commute with the map p, $p \circ \Lambda = p$ and have local expression

$$\Lambda\phi_\alpha(b, g) = \phi_\alpha(b, \lambda_\alpha(b)g)$$

with the transformation law

$$\lambda_\beta = g_{\beta\alpha}\lambda_\alpha g_{\alpha\beta}$$

where the product is pointwise in G. We give $\mathcal{G}(p)$ the topology induced from the compact–open topology of $\mathcal{M}(E, E)$.

A result of **[MP]** shows that the gauge groups $\mathcal{G}(p)$ of the principal G–bundles over B can be viewed as topological subgroups of the common *local gauge group*

$$\prod_\alpha \mathcal{M}(U_\alpha, G)$$

with the product topology and the compact–open topology on the factors, where $\mathcal{G}(p) \hookrightarrow \prod_\alpha \mathcal{M}(U_\alpha, G)$ is obtained by identifying a $\Lambda \in \mathcal{G}(p)$ with the collection $\{\lambda_\alpha \mid \lambda_\alpha \in \mathcal{M}(U_\alpha, G),\ \ \lambda_\beta = g_{\beta\alpha}\lambda_\alpha g_{\alpha\beta}\}$.

As subgroups of the same group we can divide the gauge groups into conjugacy classes: $\mathcal{G}(p) \sim \mathcal{G}(p')$ iff $\exists\{f_\alpha\} \in \prod_\alpha \mathcal{M}(U_\alpha, G)$ such that

$$\lambda'_\alpha = f_\alpha^{-1}\lambda_\alpha f_\alpha$$

for all $\{\lambda_\alpha\}$ that locally determine $\Lambda \in \mathcal{G}(p)$ and for all $\{\lambda'_\alpha\}$ corresponding to a Λ' of $\mathcal{G}(p')$.

The conjugacy relation is independent of the choice of trivialization. Suppose that a different set of transition functions for the two bundles, $\tilde{g}_{\alpha\beta}$, $\tilde{g}'_{\alpha\beta}$, is given. Then the relations

$$\tilde{g}_{\alpha\beta} = \eta_\alpha g_{\alpha\beta}\eta_\beta^{-1}$$

$$\tilde{g}'_{\alpha\beta} = \chi_\alpha g'_{\alpha\beta}\chi_\beta^{-1}$$

imply that the gauge groups $\mathcal{G}(p)$, $\mathcal{G}(p')$ are mapped into themselves by the inner automorphisms

$$\{\lambda_\alpha\} \mapsto \{\eta_\alpha\lambda_\alpha\eta_\alpha^{-1}\}$$

$$\{\lambda'_\alpha\} \mapsto \{\chi_\alpha\lambda'_\alpha\chi_\alpha^{-1}\}$$

and the conjugacy relation

$$\lambda'_\alpha = f_\alpha^{-1}\lambda_\alpha f_\alpha$$

is maintained by setting

$$\chi_\alpha \lambda'_\alpha \chi_\alpha^{-1} = g_\alpha^{-1} \eta_\alpha \lambda_\alpha \eta_\alpha^{-1} g_\alpha$$

where $g_\alpha = \eta_\alpha f_\alpha \chi_\alpha^{-1}$.

In the sequel we shall use a construction which associates to a principal G–bundle ξ a bundle with structural group the group of inner automorphisms of G; this new bundle is known as the *fundamental bundle* associated to ξ (see **[Hu]** for example).

Consider first the morphism which maps G into its group of automorphisms

$$\mu(g)(h) = g \cdot h \cdot g^{-1}$$

The exact sequence

$$0 \to Z(G) \hookrightarrow G \xrightarrow{\mu} A(G) \to 0$$

where $Z(G)$ is the centre of G defines $A(G)$, the group of *inner automorphisms* of G.

Let $\xi = (E, p, B)$ be given with transition functions $g_{\alpha\beta}$; call

$$g^A_{\alpha\beta}(b) = \mu(g_{\alpha\beta}(b))$$

the image in $A(G)$ of the transition functions.

The maps $g^A_{\alpha\beta} : U_\alpha \cap U_\beta \to A(G)$ behave like transition functions themselves, and thus uniquely determine a bundle over B which has fibre G and group $A(G)$; this is the *fundamental bundle* $A(\xi)$ associated to ξ.

An equivalence of two principal G–bundles over the same space B, $\xi = (E, p, B)$ and $\xi' = (E', p', B)$, is a map $\Lambda : E \to E'$, where $p' \circ \Lambda = p$, with the property that

$$\Lambda \phi_\alpha(b, g) = \phi'_\alpha(b, \lambda_\alpha(b) g)$$

$$\lambda_\beta = g'_{\beta\alpha} \lambda_\alpha g_{\alpha\beta}$$

Given ξ, ξ', two principal G–bundles over the same B, with transition functions $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ respectively, a construction analogous to that of the fundamental bundle leads to a criterion of equivalence, which we will refer to as *Hu's criterion* ([**Hu**]).

In fact considering the morphism of $G \times G$ into the group of homeomorphisms of G given by

$$\rho(g_1, g_2)(h) = g_1 \cdot h \cdot g_2^{-1}$$

we define the subgroup G^* of the homeomorphisms by means of the exact sequence

$$0 \to Z(G) \xrightarrow{\Delta} G \times G \xrightarrow{\rho} G^* \to 0$$

where $\Delta(g) = (g, g)$ is the diagonal map.

Thus a set of new transition functions is obtained as

$$g^*_{\alpha\beta}(b) = \rho(g_{\alpha\beta}(b), g'_{\alpha\beta}(b))$$

and the corresponding bundle with fibre G and group G^* is called the *Ehresmann bundle*, $\xi^*(\xi, \xi')$. We shall indicate the set of its sections by $\Gamma(B, \xi^*)$.

Hu's equivalence criterion states that there is a bijective correspondence between the equivalences of the two bundles and the sections of the Ehresmann bundle ([**Hu**] pg. 263). The Ehresmann bundle is in fact the bundle of morphisms of ξ into ξ', and the fundamental bundle $A(\xi)$ is the bundle of automorphisms of ξ.

That is, $\xi^*(\xi, \xi) = A(\xi)$; and therefore the gauge group $\mathcal{G}(p)$ is isomorphic to the group of sections $\Gamma(B, A(\xi))$.

As seen before, the equivalence condition for G–bundles is the relation between their transition functions

$$g'_{\beta\alpha} = \lambda_\beta g_{\beta\alpha} \lambda_\alpha^{-1}.$$

The conjucacy relation of gauge groups $\mathcal{G}(p) \sim \mathcal{G}(p')$ can also be described as a property of the transition functions of the two bundles, as shown in [**MP**] (pg. 237), with the following:

Theorem 1 *Let $\xi = (E, p, B)$ be a principal G–bundle such that*

(i) *the map $\mathcal{G}(p) \to G$ that sends $\Lambda \mapsto \Lambda\mid_{p^{-1}(b_0)}(e_G)$ is an epimorphism for all choices of $b_0 \in B$;*

(ii) *the covering $\{U_\alpha\}$ is such that every map $\bar{\lambda}_\alpha : U_\alpha \to A(G)$ has a lifting $\lambda_\alpha : U_\alpha \to G$, $\mu \circ \lambda = \bar{\lambda}$.*

Then the following statements are equivalent:

(a) $\mathcal{G}(p) \sim \mathcal{G}(p')$,

(b) $A(\xi)$ *is equivalent to* $A(\xi')$,

(c) $\forall \alpha, \beta \in \Gamma$

$$\exists \lambda_\alpha : U_\alpha \to G$$

$$\exists c_{\beta\alpha} : U_\alpha \cap U_\beta \to Z(G)$$

such that

$$g'_{\beta\alpha} = \lambda_\beta c_{\beta\alpha} g_{\beta\alpha} \lambda_\alpha^{-1}$$

2 Isomorphism of gauge groups and conjugacy relation

The purpose of this paragraph is to answer a question of **[MP]** (pg. 243): they ask for an example of two principal G–bundles whose gauge groups are isomorphic but not conjugate.

Lemma 1 *Let $\xi = (E, p, B)$ be a principal G–bundle. Then an automorphism $H : G \to G$ of the topological group G determines a bundle ξ^H that has gauge group $\mathcal{G}(p^H)$ isomorphic to $\mathcal{G}(p)$.*

Proof

Let $g_{\alpha\beta}$ be the transition functions of the bundle ξ. It is sufficient to note that

$$H \circ g_{\alpha\alpha} = e_G$$

$$H \circ g_{\alpha\beta} = (H \circ g_{\beta\alpha})^{-1}$$

$$H \circ g_{\alpha\beta} \cdot H \circ g_{\beta\gamma} = H \circ g_{\alpha\gamma}$$

if $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. Thus

$$g^H_{\alpha\beta} = H \circ g_{\alpha\beta}$$

are the transition functions of a G–bundle ξ^H.

Moreover H induces an isomorphism between the groups of sections

$$\Gamma(B, A(\xi)) \cong \Gamma(B, A(\xi^H))$$

$$\lambda_\beta = g_{\beta\alpha}\lambda_\alpha g_{\alpha\beta}$$
$$H \circ \lambda_\beta = g^H_{\beta\alpha}(H \circ \lambda_\alpha) g^H_{\alpha\beta}$$

Thus, as a consequence of Hu's criterion, the gauge groups $\mathcal{G}(p)$ and $\mathcal{G}(p^H)$ are isomorphic.

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The example we are looking for now is given as follows.

Let G be the dihedral group of order 4, given by the permutations of $\{1, 2, 3, 4\}$

$$\begin{array}{cccc} e & 1 & r & (1234) \\ r^2 & (13)(24) & r^3 & (4321) \\ h & (14)(23) & v & (12)(34) \\ m & (24) & n & (13) \end{array}$$

Consider the outer automorphism H given as a permutation by $(h, m)(v, n)$.

Now take $B = S^1$ with a covering $\{U_1, U_2\}$ given by

$$U_1 = \{\theta \mid \frac{\pi}{2} - \epsilon < \theta < \frac{3\pi}{2} + \epsilon\}$$

$$U_2 = \{\theta \mid \frac{3\pi}{2} - \epsilon < \theta \leq 2\pi\} \cup \{\theta \mid 0 \geq \theta < \frac{\pi}{2} + \epsilon\};$$

on the intersections U, V define the transition functions

$$g_U(b) \equiv h$$

$$g_V(b) \equiv e.$$

These uniquely determine a principal G–bundle ξ.

The bundle ξ^H has transition functions

$$g_U^H(b) \equiv m$$

$$g_V^H(b) \equiv e.$$

because of lemma 1 the gauge groups of ξ and ξ^H are isomorphic.

Suppose that they were conjugate. This means that, if $\mathcal{G}(p^H) = \{f_\alpha\}\mathcal{G}(p)\{f_\alpha\}^{-1}$, i.e.

$$\lambda_\alpha^H = f_\alpha \lambda_\alpha f_\alpha^{-1}$$

with $\{\lambda^H\}$ any element of $\mathcal{G}(p^H)$ and $\{\lambda_\alpha\}$ any element of $\mathcal{G}(p)$, then

$$f_\alpha g_{\alpha\beta} \lambda_\beta g_{\beta\alpha} f_\alpha^{-1} = g_{\alpha\beta}^H f_\beta \lambda_\beta f_\beta^{-1} g_{\beta\alpha}^H$$

i.e.

$$(f_\beta^{-1} g_{\beta\alpha}^H f_\alpha g_{\alpha\beta})\lambda_\beta = \lambda_\beta (f_\beta^{-1} g_{\beta\alpha}^H f_\alpha g_{\alpha\beta}).$$

Therefore $(f_\beta^{-1} g_{\beta\alpha}^H f_\alpha g_{\alpha\beta})$ commutes with all elements of G that are assumed as values of some local determination λ_α of a self equivalence of ξ.

Thus in our case we can write

$$g_{\beta\alpha}^H = f_\beta q_{\beta\alpha} g_{\beta\alpha} f_\alpha^{-1}$$

with $q_{\beta\alpha} : U_\alpha \cap U_\beta \to S \subset G$, where S is the subgroup $\{e, r^2, h, v\}$ of G : in fact the condition $\lambda_1 = g_U \lambda_2 g_U^{-1}$ forces $\lambda_i(b_0)$ to take values in S, for any choice of $b_0 \in U_i$, and any element which commutes with all elements of S has to be itself in S.

We have the following relations:

$$g_U^H = f_1 q_U g_U f_2^{-1}$$

$$g_V^H = f_2 q_V g_V f_1^{-1}$$

but a straightforward calculation shows that there can't exist any set of elements $f_1, f_2 \in G$ and $q_U, q_V \in S$ such that

$$m = f_2 \cdot q_V \cdot q_U \cdot h \cdot f_2^{-1}$$

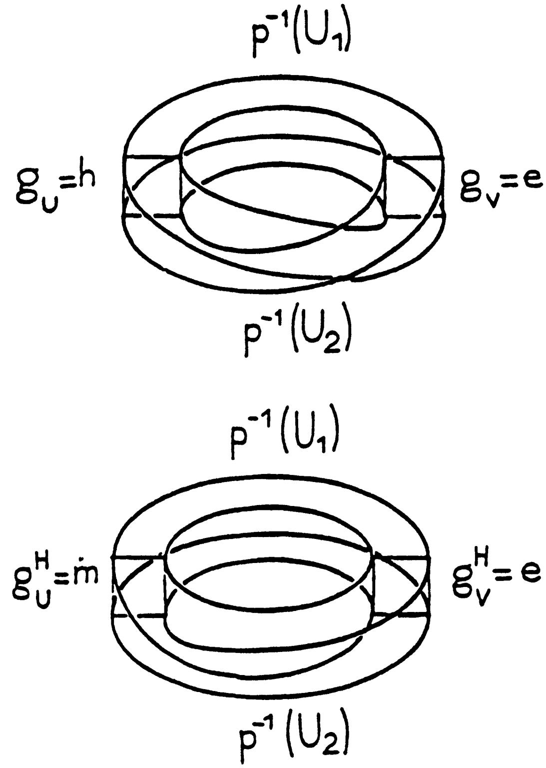

Figure 1: Two bundles with isomorphic gauge groups that are not conjugate

$$f_1 = f_2 \cdot q_V.$$

Under a more geometrical point of view, we can look, instead of principal bundles, at the associated bundles which have fibre the square and group G, where the action is permutation as in the definition.

The two bundles then look like figure 1.

Note: in this example we cannot use theorem 1, as the map $\mathcal{G}(p) \to G$

$$\Lambda \mapsto \Lambda\mid_{p^{-1}(b_0)}(e_G) = \lambda_i(b_0)$$

is not an epimorphism.

3 Conjugate gauge groups and equivalent bundles

It is trivially verified that equivalent bundles have conjugate gauge groups; however there are non equivalent bundles with conjugate gauge groups: the question that arises is how far from equivalence they can be.

Under some additional hypothesis it is possible to give an answer to such a question. We shall assume that the group G is a path connected topological group with a discrete centre, and that $H^1(U_\alpha, Z(G)) = 0$, an assumption that guarantees the existence of liftings $\lambda_\alpha : U_\alpha \to G$ for functions $\bar{\lambda}_\alpha : U_\alpha \to A(G)$: these assumptions imply conditions *(i)* and *(ii)* of theorem 1 Section 1, and therefore the result of **[MP]** holds true.

Hu's n–equivalence theorem

Before stating our result it is necessary to recall some tools used in **[Hu]**.

Suppose ξ and ξ' have transition functions $g_{\alpha\beta}$ and $g'_{\alpha\beta}$. Let $A(\xi)$ and $A(\xi')$ be the associated fundamental bundles, and $\xi^*(\xi, \xi')$ their Ehresmann bundle.

Suppose from now on that B is a polyhedron such that every simplex is contained in some U_α.

Definition 1 *Two such bundles are n–equivalent iff the pullback bundles via the inclusion map $i_n : B^n \to B$ of the nth skeleton in the base B are equivalent.*

By Hu's criterion n–equivalence of bundles corresponds to the existence of a section of the Ehresmann bundle over the nth skeleton.

Thus equivalence of two bundles can be proved by showing that a section over the $(n-1)$st skeleton can be extended to one over the nth $\forall n \geq 2$. 1–equivalence is obtained from path connectedness of G.

A condition for this to happen is found by **[Hu]** by means of *obstruction cocycles.*

Let's just recall the definition of such objects: an element $s = \{s_\alpha\}$ of $\Gamma(B^{(n-1)}, A(\xi))$, the group of sections of $A(\xi)$ over the $(n-1)$st skeleton $B^{(n-1)}$, defines a map

$$f_{s_\alpha} : \partial\sigma \to G$$
$$f_{s_\alpha}(b) = s_\alpha(b)$$

where σ is any n–simplex of B.

A homotopy class $[f_{s_\alpha}]$ is well defined, independent from the choice of U_α, because $s_\beta = g^A_{\beta\alpha} s_\alpha$ with $g^A_{\beta\alpha}$ defined on all σ; and it is an element of $\pi_{n-1}(G)$.

Thus a cocycle is defined with coefficients in $\pi_{n-1}(G)$, as

$$c^n_s(\sigma) = [f_{s_\alpha}]$$

and it determines a cohomology class $\{c^n_s\}$ in $H^n(B, \pi_{n-1}(G))$.

The *n-th obstruction set* of $A(\xi)$ is the set $\Omega^n(A(\xi))$ of all classes determined by sections $s \in \Gamma(B^{(n-1)}, A(\xi))$.

An analogous construction leads to the definition of $\Omega^n(A(\xi'))$ and $\Omega^n(\xi^*(\xi, \xi'))$.

The classes of obstruction cocycles of $A(\xi)$ and of $A(\xi')$ form two subgroups of $H^n(B, \pi_{n-1}(G))$, while $\Omega^n(\xi^*(\xi, \xi'))$ is a coset (**[Hu]** pg.268).

Theorem 2 *(**[Hu]**) Suppose that ξ^* has a section over the $(n-1)$–dimensional skeleton $B^{(n-1)}$ with $n \geq 2$, then it has a section on B^n iff*

$$\Omega^n(A(\xi)) = \Omega^n(\xi^*(\xi, \xi')) = \Omega^n(A(\xi'))$$

in $H^n(B, \pi_{n-1}(G))$.

Conjugacy relation and extension of sections

Theorem 3 *Let $\xi = (E, p, B)$, $\xi' = (E', p', B)$ be two principal G–bundles with conjugate gauge groups; we also assume that B and G satisfy the conditions stated at the beginning of the section; furthermore suppose that the open covering of the polyhedron B is such that $H^1(U_\alpha \cap U_\beta, Z(G)) = 0$.*

Suppose that the Ehresmann bundle $\xi^*(\xi,\xi')$ has a section over the 3–dimensional skeleton B^3, $s \in \Gamma(B^3,\xi^*)$, and that the following holds:
Base Point Condition*:*
$\forall U_\alpha\ \exists y_\alpha \in G$ *such that* $\forall \sigma \in B^3 \cap U_\alpha$ *there's a base point* $x_\sigma \in \sigma$ *such that* $s_\alpha(x_\sigma) = y_\alpha$.
Then ξ and ξ' are equivalent bundles.

Proof
By theorem 1 conjugacy of gauge groups has the same meaning as equivalence of the fundamental bundles. These are not principal bundles, but Hu's equivalence criterion extends to this case without any change.
This means that the Ehresmann bundle $\xi^*(A(\xi),A(\xi'))$ has a section. This has fibre $A(G)$, group $A^*(G)$ determined by

$$0 \to ZA(G) \xrightarrow{\Delta} A(G)\times A(G) \xrightarrow{\rho} A^*(G) \to 0,$$

and transition functions

$$g^{A^*}_{\alpha\beta} = \rho(g^A_{\alpha\beta}, g'^A_{\alpha\beta})$$

that act as $g^{A^*}_{\alpha\beta}(h) = g^A_{\alpha\beta}\cdot h\cdot g'^A_{\beta\alpha}$, $\forall h \in A(G)$.
By iteration we can construct the fundamental bundles $A^2(\xi)$ and $A^2(\xi')$ associated to $A(\xi)$ and $A(\xi')$. These have fibre $A(G)$, group $A^2(G)$ given by

$$0 \to ZA(G) \hookrightarrow A(G) \xrightarrow{\mu} A^2(G) \to 0$$

and transition functions

$$g^{A^2}_{\alpha\beta} = \mu(g^A_{\alpha\beta})$$

and

$$g'^{A^2}_{\alpha\beta} = \mu(g'^A_{\alpha\beta})$$

respectively, acting on $h \in A(G)$ as $g^A_{\alpha\beta}\cdot h\cdot g^A_{\beta\alpha}$.
To simplify notations we shall indicate in the following $\xi^*(\xi,\xi')$ as ξ^* and $\xi^*(A(\xi),A(\xi'))$ as ξ^*_A.
Because of the hypothesis that $Z(G)$ is discrete, $\forall q \geq 2$

$$\pi_q(G) \cong \pi_q(A(G))$$

Such isomorphism $\mu_n : \pi_{n-1}(G) \to \pi_{n-1}(A(G))$ induces an isomorphism of the cohomology groups

$$H^n(B, \pi_{n-1}(G)) \overset{\mu_n}{\cong} H^n(B, \pi_{n-1}(A(G)))$$

Now we prove the following lemma:

Lemma 2 *For every $n \geq 3$ we have*

$$\Omega^n(A(\xi)) \overset{\mu_n}{\cong} \Omega^n(A^2(\xi))$$

$$\Omega^n(A(\xi')) \overset{\mu_n}{\cong} \Omega^n(A^2(\xi')).$$

Proof

It is enough to show this in the case of $A(\xi)$.

The map $\mu \circ f_{s_\alpha} : \partial\sigma \to A(G)$ determines a cohomology class

$$\{[\mu \circ f_{s_\alpha}]\} = \{\mu_n[f_{s_\alpha}]\}$$

which is an element of $\Omega^n(A^2(\xi))$ because the transformation law

$$s_\beta = g^A_{\beta\alpha} s_\alpha$$

becomes

$$\mu(s_\beta) = g^A_{\beta\alpha}\mu(s_\alpha)g^A_{\alpha\beta}$$

We want to show that every map $f_{\bar{s}_\alpha} : \partial\sigma \to A(G)$, with $\bar{s} \in \Gamma(B^{(n-1)}, A^2(\xi))$, can be lifted to an $f_{s_\alpha} : \partial\sigma \to G$, such that $\mu \circ f_{s_\alpha} = f_{\bar{s}_\alpha}$, and $s \in \Gamma(B^{(n-1)}, A(\xi))$.

The hypothesis $H^1(U_\alpha \cap U_\beta, Z(G)) = 0$ implies that

$$g^A_{\alpha\beta} : U_\alpha \cap U_\beta \to A(G)$$

can be lifted to a function which takes values in G,

$$c_{\alpha\beta} g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$$

with $c_{\alpha\beta} : U_\alpha \cap U_\beta \to Z(G)$.

Moreover $\partial\sigma \cong S^{n-1}$ is a simply connected space ($n \geq 3$) and therefore every map $f_{\bar{s}_\alpha} : \partial\sigma \to A(G)$ can be lifted to the covering G. Let $f_\alpha : \partial\sigma \to G$ be the unique lifting of $f_{\bar{s}_\alpha}$ that preserves base points.

If $\partial\sigma \subset U_\alpha \cap U_\beta$ the transformation law

$$f_\beta = c_{\beta\alpha} g_{\beta\alpha} f_\alpha g_{\alpha\beta} c_{\alpha\beta} = g_{\beta\alpha} f_\alpha g_{\alpha\beta}$$

obtained as a base point preserving lifting of

$$\bar{s}_\beta = g^A_{\beta\alpha} \bar{s}_\alpha g^A_{\alpha\beta}$$

implies that the map f_α can be written as f_{s_α} with $s \in \Gamma(B^{(n-1)}, A(G))$.

As we have dealt with base point preserving maps, the homomorphism μ_n is injective.

From this we have

$$\Omega^n(A(\xi)) \stackrel{\mu_n}{\cong} \Omega^n(A^2(\xi))$$

$$\Omega^n(A(\xi')) \stackrel{\mu_n}{\cong} \Omega^n(A^2(\xi')).$$

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Lemma 3 *If there is a section of ξ^* over the third skeleton, then*

$$\Omega^n(\xi^*) \stackrel{\mu_n}{\cong} \Omega^n(\xi^*_A)$$

$\forall n \geq 3$.

Proof

As in the case of $\Omega^n(A(\xi))$ and $\Omega^n(A(\xi'))$, the correspondence

$$\mu_n : \Omega^n(\xi^*) \to \Omega^n(\xi^*_A)$$

associates to $s_\beta = g_{\beta\alpha} s_\alpha g'_{\alpha\beta}$ the section $\mu(s_\beta) = g^A_{\beta\alpha} \mu(s_\alpha) g'^A_{\alpha\beta}$.

The hypothesis that $\xi^*(\xi, \xi')$ has a section over the 3–skeleton implies

$$\Omega^3(A(\xi)) = \Omega^3(\xi^*(\xi, \xi')) = \Omega^3(A(\xi'))$$

But then

$$\Omega^3(\xi^*) \overset{\mu_3}{\cong} \Omega^3(\xi^*_A)$$

because of lemma 2.

Let's proceed by induction: suppose that

$$\Omega^n(\xi^*) \overset{\mu_n}{\cong} \Omega^n(\xi^*_A)$$

This means that every section on the $(n-1)$st skeleton

$$\bar{s}^{(n-1)}_\beta = g^A_{\beta\alpha}\bar{s}^{(n-1)}_\alpha g'^A_{\alpha\beta}$$

has a unique lifting

$$s^{(n-1)}_\beta = g_{\beta\alpha}s^{(n-1)}_\alpha g'_{\alpha\beta}$$

We want to show that every $\bar{s}^{(n)} \in \Gamma(B^n, \xi^*_A)$ has a lifting $s^{(n)} \in \Gamma(B^n, \xi^*)$, i.e. that

$$\Omega^{(n+1)}(\xi^*) \overset{\mu_{n+1}}{\cong} \Omega^{(n+1)}(\xi^*_A).$$

On every $\partial\sigma \subset U_\alpha \cap U_\beta$, σ $(n+1)$- simplex, a section

$$\bar{s}^{(n)}_\beta = g^A_{\beta\alpha}\bar{s}^{(n)}_\alpha g'^A_{\alpha\beta}$$

has a lifting

$$s^{(n)}_\beta = c_{\beta\alpha}c'_{\alpha\beta}g_{\beta\alpha}s^{(n)}_\alpha g'_{\alpha\beta}.$$

Because of unicity of liftings that preserve base points this has to coincide on $\partial\sigma \cap B^{(n-1)}$ with

$$s^{(n-1)}_\beta = g_{\beta\alpha}s^{(n-1)}_\alpha g'_{\alpha\beta}.$$

Therefore $c_{\beta\alpha}c'_{\alpha\beta} \mid_{\partial\sigma\cap B^{(n-1)}} = e_G$, but being $Z(G)$ discrete, on all $\partial\sigma$

$$c_{\beta\alpha}c'_{\alpha\beta} \equiv e_G.$$

But then every $\bar{s} \in \Gamma(B^{(n)}, \xi^*_A)$ determines an element of $\Omega^{(n+1)}(\xi^*)$ by the lifting

$$s^{(n)}_\beta = g_{\beta\alpha}s^{(n)}_\alpha g'_{\alpha\beta}.$$

Thus we have

$$\Omega^{(n+1)}(\xi^*) \overset{\mu_{n+1}}{\cong} \Omega^{(n+1)}(\xi^*_A).$$

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Now the theorem follows from the preceding lemmas, as by theorem 2 the existence of a section of $\xi^*(A(\xi), A(\xi'))$ implies that

$$\Omega^n(\xi^*_A) = \Omega^n(A^2(\xi)) = \Omega^n(A^2(\xi'))$$

$\forall n \geq 1$. We have

$$\Omega^n(\xi^*) = \Omega^n(A(\xi)) = \Omega^n(A(\xi')).$$

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4 Further remarks and examples

The condition on the base points, BPC of theorem 3, which is essential in the proof, in order to have uniqueness of the lifting $s^{(n)}_\alpha$, is rather general. It is trivially satisfied for instance, with all $y_\alpha = e_G$, whenever no $\partial\sigma \in B^3$ is entirely contained in some $U_\alpha \cap U_\beta$; and this is the case in many simple geometric examples.

Example 1 *Figure 2 gives an example of triangulation with a suitable choice of the base points in the case of a projective plane with respect to the trivializing open sets* $\{[x_0 : x_1 : x_2] \mid x_i \neq 0\}$*; an analogous polyhedral decomposition can be obtained for* RP^n*, with respect to the same kind of open cover, inductively from a triangulation of* S^n *which is invariant under the action of the antipodal map.*

Under some stronger assumption on the topology of the group G we have the following:

Corollary 1 *Let* G *be a Lie group;* $\xi = (E, p, B)$ *and* $\xi' = (E', p', B)$ *be two principal* G*–bundles on the polyhedron* B*, with conjugate gauge groups. Suppose they are 2–equivalent and the condition BPC is satisfied. Then they are equivalent.*

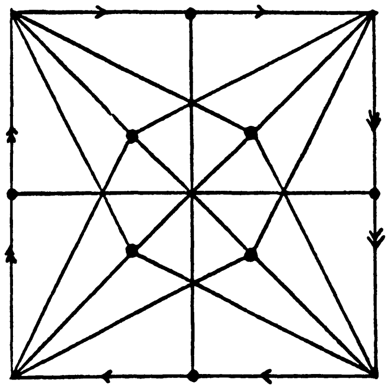

Figure 2: Base points in the triangulation of RP^2

Proof

The Lie group G has $\pi_2(G) = 0$; therefore

$$H^3(B, \pi_2(G)) = 0$$

and thus, if the Ehresmann bundle $\xi^*(\xi, \xi')$ has a section over the 2–dimensional skeleton, this can be extended to the three dimensional; by theorem 3 the bundles are equivalent.

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Corollary 2 *Let G be a Lie group with $\pi_1(G) = 0$ (e.g. $G = SU(N)$), if two principal G–bundles over a polyhedron B have conjugate gauge groups and the section over the 1–skeleton satisfies BPC, they are equivalent bundles.*

Proof

The existence of a section over the first skeleton B^1 is obtained from path connectedness of G. We have that $H^2(B, \pi_1(G)) = 0$: therefore there always exists a section over the 2–dimensional skeleton B^2. Apply corollary 1 to have n–equivalence for all $n \geq 3$.

***

Note moreover that theorem 3 cannot be restated with cellular instead of polyhedral structures: consider the following example.

Example 2 *Let ξ be the principal $SO(4)$-bundle over RP^4 associated to the vector bundle $H \oplus H \oplus H \oplus H$, with H the canonical line bundle. This is non-trivial since the Stiefel–Whitney class $w_4(\xi) \neq 0$. It can be shown that the associated fundamental bundle $A(\xi)$, which has group $PSO(4)$, is trivial, and that the pullback bundle via the inclusion of the cellular 2-skeleton $RP^2 \hookrightarrow RP^4$ is also trivial.*

That's not the case when restricted to the two dimensional polyhedral skeleton, according to theorem 3 and example 1.

We can provide many other bundles with isomorphic gauge groups that are not conjugate, using a method similar to the one used to construct the example of figure 1; also without assuming the group to be discrete. In this case however we have to consider a group G, which has more than one connected component, like in the following.

Consider the case $G = O(3,1)$, the Lorentz group in four dimensions, and $B = S^1$, with the same open covering given in the example described before. The group ha four path connected components and a discrete center, and the open sets of B are contractible.

Define the transition functions

$$g_U = \begin{pmatrix} I & \\ & -1 \end{pmatrix}$$

$$g'_U = \begin{pmatrix} -I & \\ & 1 \end{pmatrix}$$

and $g_V = g'_V = I$.

There's an isomorphism $L : H \to H'$, with H,H' subgroups of G containing the connected components of g_U and I, and of g_V and I respectively, such that $Lg_U = g'_U$, $Lg_V = g'_V$; lemma 1 extends to this case: the gauge groups are isomorphic.

If they were conjugate, by an argument similar to the example of section 2, we would have

$$g'_U = c_U \lambda_1 g_U \lambda_2^{-1}$$

$$g'_V = c_V \lambda_2 g_V \lambda_1^{-1}$$

with $c_U, c_V \in Z(G) = \{+I, -I\}$.

But the image of λ_i is contained in one of the four components of $O(3,1)$, and it is easy to check that this is not compatible with the relations above.

In such example the connected components play the same role as the discrete elements in the example of section 2 (picture 1).

But examples of bundles with isomorphic gauge groups that are non conjugate can be given also in the hypothesis of theorem 1. In order to see this we'll make use of the following lemma.

Lemma 4 *Given a path connected topological group, in general an outer automorphism need not be homotopically equivalent to the identity map.*

Assume that two bundles ξ, ξ' are given with an open cover of B made of contractible open sets U_α, and ξ' obtained from the transition functions $g'_{\alpha\beta} = H \circ g_{\alpha\beta}$, where $H : G \to G$ is an outer automorphism of G which is not homotopic to the identity map; thus for a suitable choice of the intersections $U_\alpha \cap U_\beta$, we have that the $g'_{\alpha\beta}$ are not homotopic to the $g_{\alpha\beta}$.

The gauge groups are isomophic because of lemma 1.

If the group is path connected with a discrete centre, the conditions (i) and (ii) of theorem 1 are satisfied; therefore the gauge groups are conjugate iff the following relation exists among the transition functions:

$$g'_{\alpha\beta} = \lambda_\alpha^{-1} c_{\alpha\beta} g_{\alpha\beta} \lambda_\beta$$

But since the centre $Z(G)$ is discrete, and G is path connected, the above relation implies that $g'_{\alpha\beta}$ are homotopic to $g_{\alpha\beta}$ as maps $U_\alpha \cap U_\beta \to G$, which is impossible according to the initial assumption.

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