CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

PAUL CHERENACK

Framed bicobordism

Cahiers de topologie et géométrie différentielle catégoriques, tome 36, nº 4 (1995), p. 291-321

<http://www.numdam.org/item?id=CTGDC_1995__36_4_291_0>

© Andrée C. Ehresmann et les auteurs, 1995, tous droits réservés.

L'accès aux archives de la revue « Cahiers de topologie et géométrie différentielle catégoriques » implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Volume XXXVI-4 (1995)

FRAMED BICOBORDISM

by Paul CHERENACK

Résumé. La Topologie algébrique a été à la source de la théorie des catégories et par le cobordisme Poincaré en un certain sens a fondé la topologie algébrique. Ici nous introduisons le cobordisme dans un cadre catégorique. En remplacant les sous-variétés encadrées par des applications encadrées, nous étendons la construction de Thom dans le cadre du cobordisme. Hardie a introduit la catégorie des paires d'homotopie pour étudier l'homotopie des groupes d'applications continues. Ici, nous considérons la catégorie des paires de cobordisme et étendons la construction de Thom dans ce cas. Boardman et Steer comparent les constructions de cobordisme encadré aux constructions en cohomotopie. Nous étendons ces correspondances au cas des paires pour les suspensions. Finalement, utilisant le travail de Hardie et Jansen, nous déterminons certains groupes de cobordisme encadré stables.

Introduction.

In §0 we make our basic definitions and prove some fundamental results. We first replace and extend the notion of framed manifold by introducing the notion of framed mapping, essentially using the Thom construction as motivation. In the same way the notion of framed cobordism of framed maps is introduced. Here, however, we are forced to use the reals **R** instead of the unit interval **I** and drop, for manifolds with boundary appearing in the definition of framed cobordism between framed maps, the requirement that submanifolds are neatly imbedded. We show, modelled on a proof in Hirsch [6] that framed cobordism of framed maps is essentially just homotopy. Some of the fundamental results shown in §0 seem to be known but for lack of an explicit reference we prove them here. We then introduce the homotopy pair category due to Hardie [6] and, correspondingly but with some necessary change, the cobordism pair category. We extend Thom's bijection relating framed cobordism classes in a smooth manifold X to the cohomotopy of X in the enlarged situation described in §0. Let P be a point of Σ^k, the k sphere. Then, Thom's construction tells us, without much difficulty, that there is a bijection

$$\alpha : \mathbf{PCOB}^k(X, P) \to \Pi^k(X, \infty),$$

where $\mathbf{PCOB}^k(X, P)$ is a set of proper framed maps replacing the set of framed submanifolds of X modulo framed cobordism in each case and $\Pi^k(X, \infty)$ is the cohomotopy of X. This result extends to the case where P is replaced by an arbitrary framed submanifold of X provided that the boundary of X is compact. The non-based version of this result follows more simply. Extending to the pair case, we show in Theorem 1.7: Let $g : (\Sigma^j, U) \to (\Sigma^k, P)$ be a framed map. Let $f : X \to Y$ be a closed embedding where Y does not have boundary or let $f : X \to Y$ be a submersion. Suppose that ∂X is compact or U is a one point set. There is a bijection

$$\alpha : \mathbf{CPC}^k(f, g) \to \mathbf{HPC}(f, g)$$

where $\mathbf{CPC}^k(f, g)$ is a subset of the hom-set $\mathbf{CPC}(f, g)$ of the cobordism pair category $\mathbf{CPC}$ and $\mathbf{HPC}(f, g)$ is the hom-set in the homotopy pair category $\mathbf{HPC}$. No counterexamples have been found to show that this result does not hold for arbitrary X, Y and f. In §2 we extend this last result to the case where $g : (W, D) \to (Z, E)$ is an arbitrary framed map where D and E are compact. We also obtain a bijection between the homotopy group $\Pi_n(Y, y_0)$ of a smooth manifold Y $(n > 0)$ and a certain class of framed maps from $\mathbf{R}^n$ to Y. Next, in §3, following the pattern developed by Boardman and Steer [1], in ordinary framed cobordism, we interpret suspension in pair homotopy in terms of pair cobordism. The methods here are not always straightforward but a more categorical approach sometimes makes them more natural. This list of interpretations can probably be extended to correspond to that treated in [1]. Finally, in our last section, using the work of Hardie and Jansen [5], we are able to state explicitly the nature of stable cobordism groups $F_k(g)$ where $g : \Sigma^3 \to \Sigma^2$ is the Hopf map. For function spaces the topology we refer to is the Whitney $\mathbf{C}^\infty$ or strong topology. The notation $f \bowtie_L A$ means that f is transversal to A at the points of L. If L is omitted, then $A = L$. We will refer the reader to specific parts of Hirsch [6] to justify the more difficult steps that we take; the reader should refer to [6] for basic definitions and results which may be used here without reference.

0. Basic definitions in framed bicobordism.

Let $f : X \to Y$ be a smooth map of manifolds. We assume at the outset that all of our manifolds have empty boundary. Suppose that U (resp., V) is a framed submanifold of X (resp., Y) such that $f^{-1}(V) = U$ and $f \bowtie V$. We do not assume that U or V are compact. Then, we write

$$f : (X, U) \to (Y, V)$$

and call f a framed map if via f the framing on V pulls back to the framing on U. The pair (X, U) is sometimes referred to as an inframed submanifold of X (meaning U is framed in X). We change the usual definition of framed cobordism which uses the unit interval in order to avoid talking later about boundaries of boundaries. Thus, two inframed manifolds (X, U) and (X, U') are framed cobordant if and only if there is an inframed submanifold $(X \times \mathbf{R}, U^*)$ such that $(X \times 0, (X \times 0) \cap U^*)$ (resp., $(X \times 1, (X \times 1) \cap U^*)$) is an inframed submanifold with framing induced from U^* isomorphic to (X, U) (resp., (X, U')).

Let$f : (X, U) \to (Y, V)$ and $g : (Y, V) \to (Z, W)$ be framed maps. Then, $f^{-1}(g^{-1}(W)) = U$ and $g \circ f \bowtie W$. Furthermore, via $g \circ f$ the framing on W pulls back to the framing on U. Thus, ordinary composition makes framed maps into a category **FR**, the category of framed maps. The definition here corresponds roughly with a definition in Stong [10,p.17]. Let$f : (X, U) \to (Y, V)$ and g: $(X, U') \to (Y, V')$ be framed maps. We say that f and g are framed cobordant or just cobordant and write $f\mathbf{cob}\ g$ if there is a framed cobordism between (Y, V) and (Y, V') and, for some framed cobordism V^* from V to V', there is a framed cobordism U^* from U to U' and a framed map

$$F \times \mathbf{R} : (X \times \mathbf{R}, U^*) \to (Y \times \mathbf{R}, V^*)$$

such that $F \times \mathbf{R}|_{X\times 0} = f$ and $F \times \mathbf{R}|_{X\times 1} = g$. Note that instead of writing the identity on $\mathbf{R}$ we simply write $\mathbf{R}$. Equivalently, one could use here the closed unit interval instead of the reals. For manifolds with boundary, $\mathbf{R}$ must be used (not $\mathbf{I}$), we require that, for maps such as f, the submanifolds U and V are neatly imbedded in X and Y (see [8]),respectively, but unfortunately must drop the requirement that U^* and V^* be neatly imbedded in $X \times \mathbf{R}$ and $Y \times \mathbf{R}$, respectively. In the next result we relate cobordism to homotopy.

Proposition 0.1. *Suppose that X and Y are inframed manifolds without boundary.*

a) Framed maps f and g from X to Y are framed cobordant (using $\mathbf{I}$ instead of $\mathbf{R}$) with U^* neatly imbedded in $X \times \mathbf{I}$ if and only if there is a neat cobordism between the codomains of f and g and a homotopy $G : X \times \mathbf{I} \to Y \times \mathbf{I}$ such that $G(x,0) = (f(x),0)$, $G(x,1) = (g(x),1)$ and $G(x,t) = (G'(x,t),t)$ for a continuous function $G' : X \times \mathbf{I} \to Y$. The cobordism between f and g can be chosen arbitrarily close to G in the strong topology .

b) If F and G define cobordisms (using $\mathbf{R}$) with neatly cobordant codomains between the framed maps f and g, and F and G are sufficiently close in the strong topology, then F and G are smooth homotopic to one another.

Proof. a) The one direction is clear. From a result in Thom [11]one can assume that G' is $\mathbf{C}^\infty$ and arbitrarily near the original G'. For a subset W of a set $Z \times \mathbf{I}$ we define $W_t = W \cap Z \times \{t\}$. One needs to show that one can find a smooth G of the form $G(x,t) = (G'(x,t),t)$ such that $G(x,0) = (f(x),0)$ and $G(x,1) = (g(x),1)$ where $G \bowtie V^*$. The form of G implies that $G^{-1}(V^*)$ is a neat inframed submanifold of $X \times I$ with framing induced from the framing of V^*.

Next, following Hirsch (see [8,p.75]), for smooth manifolds X and Y, a $\mathbf{C}^\infty$ mapping class on (X,Y) by definition is a function Υ on the set of tuples (L,U,V) where U is open in $X \times \mathbf{I}$, V is open in $Y \times \mathbf{I}$ and L is a closed subset of U. Let $\mathbf{C}^\infty(U,V)^{**}$ be the set

$$\{F \in \mathbf{C}^\infty(U,V)^* | F(x,0) = (f(x),0), F(x,1) = (g(x),1) \text{ if defined } \}$$

where $\mathbf{C}^\infty(U,V)^*$ is the set

$$\{F \in \mathbf{C}^\infty(U,V) | F(U_t) \subset V_t\}.$$

We will first assume that $\Upsilon(L, U, V) = \Upsilon_L(U, V) \subset \mathbf{C}^\infty(U, V)^{**}$ for all (L, U, V).

Next, we require a localization axiom: if $f_i \in \Upsilon(L_i, U_i, V_i) = \Upsilon_{L_i}(U_i, V_i)$, $f \in \mathbf{C}^\infty(U, V)^{**}$, $L \subset \cup L_i$ and $f = f_i$ on a neighborhood of L_i, then $f \in \Upsilon_L(U, V)$.

We finally require that Υ be rich: there are open covers $\mathcal{V}X$ and $\mathcal{V}Y$ of $X \times \mathbf{I}$ and $Y \times \mathbf{I}$ such that, for $K \subset U$ compact, one has $\Upsilon_K(U, V)$ is dense in $\mathbf{C}^\infty(U, V)^{**}$ for the weak topology. Let now $\Upsilon_L(U, V) = \{F \in \mathbf{C}^\infty(U, V)^{**} | F \bowtie_{L \cap U} V^* \cap V\}$.

One can show readily that Υ satisfies the localization axiom. Let $\mathcal{V}X$ consist of open sets of the form $U' \times \mathbf{J}$ where U' is an open coordinate neighborhood in X and $\mathbf{J}$ is an open interval or a half open sub-interval of $\mathbf{I}$ with the included endpoint 0 or 1. Let $\mathcal{V}Y$ be an atlas of open subsets for $Y \times \mathbf{I}$ which define submanifold charts on V^*. We prove the following lemma which shows that Υ is rich.

Lemma 0.2. *Let K be a compact set in a submanifold U of $X \times I$, V^* a submanifold of $\mathbf{R}^n \times \mathbf{I}$ with boundary, and $W \subset \mathbf{R}^n \times \mathbf{I}$ an open subset. Then, $\Upsilon_K(U, W)$ is dense in $\mathbf{C}^\infty(U, W)^{**}$ for the weak topology on both sets.*

Proof. Since $\mathbf{C}^\infty(U, W)^{**}$ is open in $\mathbf{C}^\infty(U, \mathbf{R}^n \times \mathbf{I})^{**}$, one can assume that $W = \mathbf{R}^n \times \mathbf{I}$. Thus, one needs to show that if $g \in \mathbf{C}^\infty(U, \mathbf{R}^n \times \mathbf{I})^{**}$, then g is in the closure of $\Upsilon_K(U, \mathbf{R}^n \times \mathbf{I})$. Multiplying by a suitable "bump" function one can replace g by an arbitrarily close function k which can be extended to an open coordinate neighborhood U' of $X \times \mathbf{R}$ and thus, after change of coordinate on U' and W, view k as a map $k = (k', 1) : \mathbf{R}^{m+1} \to \mathbf{R}^{n+1}$ with $V^* \subset \mathbf{R}^{n+1}$. Let L^1 denote the collection of $(m+1) \times 1$ matrices viewed as $\mathbf{R}^{m+1}$. Consider a map $M : \mathbf{R}^{m+1} \times L^1 \to \mathbf{R}^{n+1}$ defined by setting $M(x, t, A) = (k'(x, t) + Ax, t)$. It is not difficult to see that M is a submersion and hence transversal to V^* on $K \times L^1$. By Thom's Transversality Lemma there is a $C \in L^1$, arbitrarily small, such that the map M_C, which is M restricted to $A = C$, is transversal to V^* on $K \times C$. Thus, there is a map $h = (h', 1) : \mathbf{R}^{m+1} \to \mathbf{R}^{n+1}$, where $h'(x, t) = k'(x, t) + Cx$, transversal to V^* on K with C arbitrarily small and hence h arbitrarily close to k. Since, restricted to $t = 0$ and $t = 1$, k is transversal to V and V', respectively, one readily sees that k is transversal to V^* on a neighborhood N of $t = 0$ and $t = 1$ on K. If h is chosen sufficiently close to k, then one can find a "bump" function b which is 1 outside N, 0 on $t = 0$ and 1 and such that $p = bh + (1-b)k$ is transversal to V^* on K and at the same time agrees with k on $t = 0$ and $t = 1$. Since then $p \in \Upsilon_K(U, W)$ and p can be chosen arbitrarily close to k, we are done.

We globalize the last lemma:

Lemma 0.3. *With the strong topology, L closed and Υ rich, $\Upsilon_L(X \times \mathbf{I}, Y \times \mathbf{I})$ is open and dense in $\mathbf{C}^\infty(X \times \mathbf{I}, Y \times \mathbf{I})^{**}$.*

Proof. Since the result (see [8]) holds for a fixed parameter $t \in \mathbf{I}$ and $\mathbf{I}$ is compact, openess is clear. Let $f \in \mathbf{C}^\infty(X \times \mathbf{I}, Y \times \mathbf{I})^{**}$. Suppose that i and j range over Λ. Following [8], let $N = N(f; \Phi, \Psi, K, \epsilon)$ be a strong basic neighborhood of f with $\Phi = \{\phi_i, U_i\}$ a locally finite atlas on $X \times \mathbf{I}$, $K_i \subset U_i$ compact sets whose union contains L, $\Psi = \{\psi_i, V_i\}$ a family of charts on $Y \times \mathbf{I}$ such that $f(K_i) \subset V_i$ and $\epsilon = \{\epsilon_i\}$. Fix $j \in \Lambda$. Let $E = U_j \cap f^{-1}(V_j)$ and note that $K_j \subset E$. As Υ is rich, $\Upsilon_{K_j}(E, V_j)$ is dense in $\mathbf{C}^\infty(E, V_j)^{**}$. Choosing a $\mathbf{C}^\infty$ function λ with values in $\mathbf{I}$, compact support and 1 near K_j, for $g \in \Upsilon_{K_j}(E, V_j)$ sufficiently close to $f|_E$ and appropriate identification of V_j, one can define $h(x,t) = f(x,t) + \lambda(x,t)[g(x,t) - f(x,t)]$ if $x \in E$ and $f(x,t)$ otherwise. Then, as $g \to f|_E$, $h \to f$ in the strong topology. Thus, one can choose $h \in N$ and as $h = g$ near K_j, $h \in \Upsilon_{K_j}(X \times \mathbf{I}, Y \times \mathbf{I})$. It is not difficult to see that $\mathbf{C}^\infty(X \times \mathbf{I}, Y \times \mathbf{I})^{**}$ is a weakly closed subset of $\mathbf{C}^\infty(X \times \mathbf{I}, Y \times \mathbf{I})$ and hence Baire. But, then $\Upsilon_L(X \times \mathbf{I}, Y \times \mathbf{I}) = \cap_j \Upsilon_{K_j}(X \times \mathbf{I}, Y \times \mathbf{I})$ is dense in $\mathbf{C}^\infty(X \times \mathbf{I}, Y \times \mathbf{I})^{**}$. The proof of Lemma 0.3 is complete.

From Lemma 0.2 and 0.3 it follows that $\Upsilon_{X \times \mathbf{I}}(X \times \mathbf{I}, Y \times \mathbf{I})$ is dense in $\mathbf{C}^\infty(X \times \mathbf{I}, Y \times \mathbf{I})^{**}$ in the strong topology. Thus, arbitrarily near G in the strong topology, there is a $F \in \Upsilon_{X \times \mathbf{I}}(X \times \mathbf{I}, Y \times \mathbf{I})$ with the property that F is transversal to V^*. The proof of part a) of Proposition 0.1 is complete.

b) Let F and G define cobordisms between f and g, and F be sufficiently close to G in the strong topology. This implies, in particular, that G and F agree outside some compact set. From [8,p.38] one knows that, for ∂-manifolds (smooth manifolds with possible boundary) M and N, the set $\mathbf{Diff}(M, N)$ of diffeomorphisms from M to

N is open in $\mathbf{C}^\infty(M, \partial M, N, \partial N) = \{f \in \mathbf{C}^\infty(M, N) | f(\partial M) \subset \partial N\}$. Thus, restricting to boundary preserving smooth maps and proceeding as in [4,p.76], one sees that there is a tubular neighborhood of the graph of F (with F the zero section followed by projection to Y) such that G is a section of this tubular neighborhood followed by projection to Y.

Since F and G agree on $X \times 1 \cup X \times 0$ and preserve boundary ($t = 0$ and $t = 1$), there is a $\mathbf{C}^\infty$ homotopy between F and G obtained by deforming along the fibres of the tubular neighborhoods.

This completes the proof of Proposition 0.1.

Proposition 0.4. *Let X and Y be manifolds with boundary.*

a) Framed maps f and g from X to Y are framed cobordant using $\mathbf{R}$ *if and only if there is a cobordism (not necessarily neat) between the codomains of f and g and a homotopy $G : X \times \mathbf{I} \to Y \times \mathbf{I}$ such that $G(x,0) = (f(x),0)$, $G(x,1) = (g(x),1)$ and $G(x,t) = (G'(x,t),t)$ for some continuous function $G' : X \times \mathbf{I} \to Y$. The cobordism between f and g can be chosen arbitrarily close in the strong topology to G.*

b) If F and G define cobordisms with cobordant codomains between the framed maps f and g, and F and G are sufficiently close, then F and G are homotopic to one another and hence by the preceding statement cobordant.

Proof. For a) one proceeds as in the proof of Proposition 0.1 but, in Lemma 0.3, one must use the argument that Hirsch [8] gives for manifolds with boundary with small modification. The proof of b) follows from the following lemma. One can not apply the argument of Proposition 0.1, part b) since neither F nor G is known to preserve boundary.

We state two lemmas which will be useful later.

Lemma 0.5. *Let $g, f : X \to Y$ be continuous maps between* $\mathbf{C}^\infty$ *manifolds with boundary. If g is sufficiently near to f for the strong topology, then g can be chosen homotopic to f.*

Proof. Suppose first that Y has no boundary. Embed Y in $\mathbf{R}^p$ and give Y the structure of a Riemannian manifold. If g is sufficiently near f, then f and g differ on some compact set K. Cover $f(K)$ by finitely many geodesically convex open sets U_i [3,p.328] diffeomorphic to $\mathbf{R}^n$ where between any two points in a given U_i there is a unique geodesic. This is possible since $f(K)$ is compact. Notice that the intersection of finitely many geodesically convex open sets is again a geodesically convex open set. Suppose that g is sufficiently near f and thus for some x where $f(x)$ and $g(x)$ differ, $f(x)$ and $g(x)$ belong to U_i for some i. Let γ_x be the unique geodesic from $f(x)$ to $g(x)$. Let $d(u,v)$ be the topological metric corresponding to the Riemannian metric on Y. If $F(x,t)$ is the point on γ_x with $d(f(x),F(x,t)) = td(f(x),g(x))$, then $F(x,t)$ is continuous (see [3,p.328]) and defines a homotopy between f and g.

Suppose now that Y has boundary ∂Y. Put a collar C on ∂Y so that $Y \cup C$ is a manifold without boundary and there is a continuous map $\zeta : Y \cup C \to Y$ where $\zeta(y) = y$ if $y \in Y$ and $\zeta(y)$ is the projection of y onto ∂Y if $y \in C$. The maps f and g define maps $f', g' : X \to Y \cup C$. If f and g are sufficiently close, then $f(x)$ and $g(x)$ will lie in some geodesically convex open subset of $Y \cup C$, even if $f(x)$ and $g(x)$ belong to ∂Y, and one can then, as above, find a homotopy $F' : X \times \mathbf{I} \to Y \cup C$ between f' and g'. The homotopy $\zeta \circ F'$ is then the required homotopy between f and g.

To avoid encumbering detail, the preceding lemma will be used often without reference.

Lemma 0.6. *Let $f : X \to Y$ be a (resp., proper) continuous map between $\mathbf{C}^\infty$ manifolds where the boundary ∂X of X is compact. Suppose that U is a neat compact inframed submanifold of Y. Then, there is a (resp., proper) map $h : X \to Y$ arbitrarily close in the strong topology and hence homotopic to f such that $h \bowtie U$ and $h^{-1}(U)$ is a neat submanifold of X.*

Note that this result holds if U is a one point set without the requirement that ∂X be compact. See [1].

Proof. We assume that f is proper and find a h which is proper. The other case of the lemma follows without difficulty from this case. Since $\mathbf{C}^\infty(\partial X, Y)$ is dense in $\mathbf{C}^0(\partial X, Y)$ and the set of all maps in $\mathbf{C}^\infty(\partial X, Y)$ transversal to U forms an open dense subset of $\mathbf{C}^\infty(\partial X, Y)$, one can choose a $g \in \mathbf{C}^\infty(\partial X, Y)$ sufficiently close to $f|_{\partial X}$ and hence homotopic to $f|_{\partial X}$ such that g is proper, smooth and $g \bowtie U$. The manifold Y is given the structure of a Riemannian manifold. One chooses two sufficiently small closed collars C_1 and C_2 on ∂X with $C_1 \subset C_2$ and the fibres of C_1 extending to the fibres of C_2 such that if S and T are points of C_2 not in C_1 in the same fibre, then, with g close enough to $f|_{\partial X}$, $g \circ p'(S)$ (where p' is the projection of C_2 onto ∂X) and $f(T)$ can be joined by a unique geodesic. Suppose that $S \in (C_2 - C_1)^{cl} \cap C_1$, $T \in C_2 - C_2^{int}$, where int (resp., cl) denotes the topological interior (resp.,closure) operator, and Q is a point on a fibre of C_2 containing both S and T. Suppose that Q is a s-th of the distance from S to T for a suitable orthogonal structure on the collar C_2. We then let $k^\circ : X \to Y$ be the continuous map which is equal to $g \circ p'$ on C_1, which is equal f on the complement of C_2 and sends a point such as Q to the point a s-th of the distance from $g \circ p'(Q)$ to $f(Q)$ along a geodesic joining these points. By taking C_2 suitably small and g suitably close to $f|_{\partial X}$, one can make $k^\circ : X \to Y$ arbitrarily close to f and hence homotopic to it and proper(see [8,p.38]). Because ∂X is compact, k° is equal to f except on a compact set. By suitably smoothing k°, one finds a smooth map $k : X \to Y$ arbitrarily close to k° and hence homotopic to it and proper such that $k|_{\partial X} = g|_{\partial X} \bowtie U$. Since ∂X is compact, restriction to ∂X defines a continuous map $\Xi : \mathbf{C}^\infty(X,Y) \to \mathbf{C}^\infty(\partial X, Y)$ in the strong topology (see [8,p.64]). The set of maps B in $\mathbf{C}^\infty(\partial X, Y)$ transversal to U forms an open subset of $\mathbf{C}^\infty(\partial X, Y)$, in the strong topology, and then $(\Xi^{-1})(B)$ is open in $\mathbf{C}^\infty(X,Y)$ in the strong topology . Since the set $\mathcal{O}$ of proper maps in $\mathbf{C}^\infty(X,Y)$ forms an open subset, $k \in \mathcal{O} \cap (\Xi^{-1})(B)$ and the set of maps from X to Y transversal to U forms an open dense subset of $\mathbf{C}^\infty(X,Y)$, there is a proper h arbitrarily near k which is homotopic to k and thus to k° and f in turn such that $h \bowtie U$ and $h|_{\partial X} \bowtie U$.

Using the Thom construction, for some k, one can find a smooth proper map $f'' : Y \to \Sigma^k$ such that f'' is transversal to some point T and $(f'')^{-1}(T) = U$. This implies that $f'' \circ h$ and $f'' \circ h|_{\partial X}$ are transversal to T and thus, applying a result in [8,p.31], one sees that $h^{-1}(U) = (f'' \circ h)^{-1}(T)$ is a neat submanifold of X. This finishes the proof.

Clearly, the framed cobordism relation **cob** is then transitive. It is in some ways like the cobordism in Stong's paper [9] on cobordism of maps. The class of f under framed cobordism is denoted by $[f]$. If $[f]$ and $[g]$ are two framed cobordism classes, their composition $[g]\circ[f]$ is defined by setting $[g]\circ[f]=[g\circ f]$ if $g\circ f$ is defined (and a framed map). That the composition is well defined follows immediately from Proposition 0.4. Using this composition the framed cobordism classes of framed maps form a category **COB**, the framed cobordism category which is a quotient of **FR**. To see this one applies the object-free definition of category found in Herrlich [7,p.32].

First, notice that the identities and hence objects of **COB** are the framed cobordism classes of the objects of **FR**. Moreover, suppose that $I_U:(X,U)\to(X,U)$ and $I'_U:(X,U')\to(X,U')$ are the identity framed maps and (X,U) is framed cobordant to (X,U') via an inframed submanifold $U^*\subset X\times\mathbf{R}$. Then, I_U **cob** $I_{U'}$ via the framed map $I_{U^*}:(X\times\mathbf{R},U^*)\to(X\times\mathbf{R},U^*)$. Hence, the identities on **COB** contain all the identities between inframed cobordant submanifolds of a given manifold.

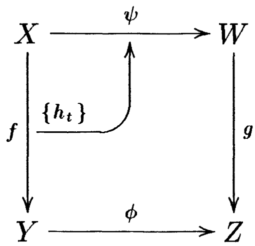
Second, the hom-sets are small since the framed maps between two smooth manifolds form a set.

Composition is clearly associative when defined. One needs only show that the matching condition holds. This follows in an evident fashion from the following lemma.

Lemma 0.7. *Suppose that $f:(X,U)\to(Y,V)$, $g:(Y,V')\to(Z,W)$ are framed maps and V is cobordant to V'. Then, there is a framed map $f':(X,U')\to(Y,V')$ cobordant to f.*

Proof. Using transversality results there is a map f' transverse to V' arbitrarily close to and hence homotopic to f. To complete the proof one applies Proposition 0.4.

We present briefly the definition of the homotopy pair category **HPC** following Hardie [6] which will with suitable adjustment motivate the definition of the framed cobordism pair category **CPC**. An object of the category **HPM** of homotopy pair mappings is a continuous map. Let $f:X\to Y$ and $g:W\to Z$ be objects in **HPM** . A morphism in **HPM** is a square

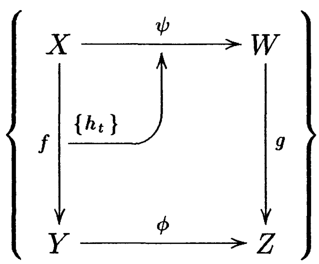


where a) ψ and ϕ are continuous; b) h_t is a homotopy between $\phi \circ f$ and $g \circ \psi$; and c) $\{h_t\}$ is the collection of homotopies from $\phi \circ f$ to $g \circ \psi$ homotopic to h_t. One sometimes expresses the morphism from f to g in **HPM** as a triple $(\phi, \psi, \{h_t\})$ and composition via

$$(\phi', \psi', \{h'_t\}) \circ (\phi, \psi, \{h_t\}) = (\phi' \circ \phi, \psi' \circ \psi, \{\phi' \circ h_t + h'_t \circ \psi\})$$

with $+$ denoting the joining of the homotopy relations.

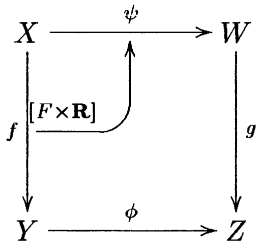
To obtain the homotopy pair category **HPC** we factor out the relation which identifies two morphisms in **HPM** from f to g of the form $(\phi_0, \psi_0, \{h_t\})$ and the form $(\phi_1, \psi_1, \{\phi_{1-t} \circ f + h_t + g \circ \psi_t\})$. A morphism in **HPC** is sometimes pictorially represented by the diagram



(1)

The category **HPC*** is formed like **HPC** but taking topological spaces with base points, base point preserving maps and homotopies where $\{h_t\}$ consists of all k_t such that there exists a $F : X \times \mathbf{I} \times \mathbf{I} \to Z$ with a) $F(x,t,o) = h(x,t)$, b)$F(x,t,1) = k(x,t)$ and c) $F(*,t,s) = *$ where $*$ represents the base point.

Let now an object of the framed cobordism pair mapping category **CPM** be a framed map. An arrow from the framed map $f : X \to Y$ to the framed map $g : W \to Z$ is a diagram



where a) ϕ and ψ are framed maps, and the composites $\phi \circ f$ and. $g \circ \psi$ aredefined in the category **FR** (but with the framings for $\phi \circ f$ and $g \circ \psi$ possibly different at X and Z) ; b) for the situation described in a), $g \circ \psi$ **cob** $\phi \circ f$ via a framed map $F \times \mathbf{R} : (X \times \mathbf{R}, U^*) \to (Y \times \mathbf{R}, V^*)$ where $F \times \mathbf{R}(x,t) = (F(x,t),t)$ and c) $[F \times \mathbf{R}]$ is the framed cobordism class of $F \times \mathbf{R}$. This arrow is denoted $(\phi, \psi, [F \times \mathbf{R}])$. Then, as in **HPM**,

$$(\phi', \psi', [F' \times \mathbf{R}]) \circ (\phi, \psi, [F \times \mathbf{R}]) = (\phi' \circ \phi, \psi' \circ \psi, [\sigma(\phi' \circ F + F' \circ \psi) \times \mathbf{R}])$$

where a) $\sigma(\phi' \circ F + F' \circ \psi)$ is a smooth map obtained by smoothing $\phi' \circ F + F' \circ \psi$; b)if the domain of $(\phi, \psi, [F \times \mathbf{R}])$ is $f : X \to Y$ and the range of $(\phi', \psi', [F' \times \mathbf{R}])$ is $h : W \to Z$, then $\sigma(\phi' \circ F + F' \circ \psi) \times \mathbf{R}$ defines a framed map and a framed cobordism from $h \circ \psi' \circ \psi$ to $\phi' \circ \phi \circ f$; and c) $\sigma(\phi' \circ F + F' \circ \psi)$ is homotopic to $\phi' \circ F + F' \circ \psi$. All of the above conditions can be achieved as a consequence of Proposition 0.4

The framed cobordism pair category **CPC** is formed from **CPM** as **HPC** was from **HPM**. We write more explicitly $(\phi, \psi, f, g, [F \times \mathbf{R}])$ for an arrow in **CPM**. In general we assume that the cobordism between two framed maps h_0 and h_1 has the form h_t. Using this notation, in order to form **CPC**, one factors out the relation γ on **CPM** identifying $(\phi_0, \psi_0, f_0, g_0, [F \times \mathbf{R}])$ and

$$(\phi_1, \psi_1, f_1, g_1, [\sigma(\phi_{1-t} \circ f_{1-t} + F \times \mathbf{R} + g_t \circ \psi_t)])$$

where $\sigma(g_{1-t} \circ \psi_{1-t} + F \times \mathbf{R} + \phi_t \circ f_t)$ is a framed map homotopic to $\phi_{1-t} \circ f_{1-t} + F \times \mathbf{R} + g_t \circ \psi_t$ defining a framed cobordism from $\phi_1 \circ f$ to $g \circ \psi_1$. The framed map $\sigma(\phi_{1-t} \circ f_{1-t} + F \times \mathbf{R} + g_t \circ \psi_t)$ exists and g is transitive because of Proposition 0.4.

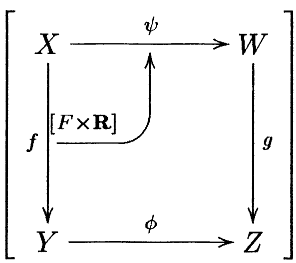
Let $[\phi, \psi, f, g, [F \times \mathbf{R}]]$ and $[\phi', \psi', f', g', [F' \times \mathbf{R}]]$ be the framed cobordism classes of $(\phi, \psi, f, g, [F \times \mathbf{R}])$ and $(\phi', \psi', f', g', [F' \times \mathbf{R}])$, respectively. Then, as usual, $[\phi, \psi, f, g.[F \times \mathbf{R}]] \circ [\phi', \psi', f', g', [F' \times \mathbf{R}]] = [\phi_0, \psi_0, f_0, g_0, [F_0 \times \mathbf{R}]] \circ [\phi'_0, \psi'_0, f'_0, g'_0, [F'_0 \times \mathbf{R}]] =$
$[(\phi_0, \psi_0, f_0, g_0, [F_0 \times \mathbf{R}]) \circ (\phi'_0, \psi'_0, f'_0, g'_0, [i' \times \mathbf{R}])]$ if

$$(\phi, \psi, f, g, [F \times \mathbf{R}]) \; \gamma \; (\phi_0, \psi_0, f_0, g_0, [F_0 \times \mathbf{R}])$$

and

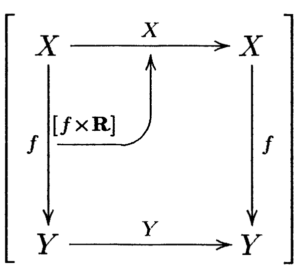
$$(\phi', \psi', f', g', [F' \times \mathbf{R}]) \; \gamma \; (\phi'_0, \psi'_0, f'_0, g'_0, [F_0 \times \mathbf{R}]),$$

g' is cobordant to f and the last composition is defined in **CPM**. As for **HPC**, we represent an element $[\phi, \psi, f, g, [F \times \mathbf{R}]]$ of **CPC** by the diagram



(2)

It is not difficult to see that the left identity for (2) (with a similar statement for the right identity) is



Using Lemma 0.7 it is easy to see that $[\phi, \psi, f, g.[F \times \mathbf{R}]]$ and $[\phi', \psi', f', g', [F' \times \mathbf{R}]]$ compose, as described above, if and only if $[g'] =$

$[f]$. Applying the object-free definition of category found in Herrlich [7,p.32], it is then clear that **CPM** is a category.

1. Extension of Thom's Theorem.

In this section we extend Thom's theorem relating framed cobordism classes to cohomotopy as found, for instance, in the work of Boardman and Steer [1]. In the next section we will state more general and more streamlined versions of these results.

For a category $\mathbf{A}$ we let $\mathbf{A}(X,Y)$ denote the collection of morphisms from X to Y with X and Y objects of $\mathbf{A}$. Let $\mathbf{TOPH}^*$ denote the based homotopy category and X^c denote the one-point compactification of a topological space X where $X^c = X \cup \{\infty\}$ has base point ∞. The set $\Pi^k(X,\infty) = \mathbf{TOPH}^*(X^c, \Sigma^k)$ is the k-compact cohomotopy of a smooth manifold X. Thom [11] has shown that there is a bijection between the set $F^k(X)$ of framed compact cobordism classes of a smooth manifold X and $\Pi^k(X,\infty)$.

Definition 1.1. *Let $\mathbf{FR}^k(X,U)$ (resp. $\mathbf{PFR}^k(X,U)$) be the collection of all (resp., proper) framed maps $f : (X,V) \to (\Sigma^k, U)$ where U together with a framing and k are fixed. Let $\mathbf{PCOB}^k(X,U)$ (resp. $\mathbf{COB}^k(X,U)$) be the image of $\mathbf{PFR}^k(X,U)$ (resp. $\mathbf{FR}^k(X,U)$) in $\mathbf{COB}(X,U)$.*

Note that since any point R of Σ^k can be smoothly rotated to any other point P, preserving orientation, $\mathbf{COB}^k(X,P)$ and $\mathbf{PCOB}^k(X,P)$ are independent of P. The set $\mathbf{FR}^k(X,P)$ for $P \in \Sigma^k$ consists of the codimension k framings on X. We will assume for general U that U does not contain the base point of Σ^k.

One can now show:

Proposition 1.2. *There is a bijection*

$$\alpha : \mathbf{PCOB}^k(X,U) \to \Pi^k(X,\infty),$$

if U is a one-point set or if ∂X is compact, induced by the Thom construction and essentially the Thom isomorphism (see [1] and [11]).

Proof. Let $U = \{P\}$. Let $f : (X, V) \to (\Sigma^k, U)$ be a proper framed map. The framing on U defines (see [1]) a base point preserving continuous map $f^c : X^c \to \Sigma^k$ and thus a homotopy class $\{f^c\}$ of basepoint preserving continuous maps from X^c to Σ^k. Suppose that g is cobordant to f and $g : (X, V) \to (\Sigma^k, P)$. Since U and V are cobordant by definition, f and g define, under the Thom bijection, the same homotopy class $\{f^c\} = \{g^c\}$ in $\Pi^k(X, \infty)$ and one has a map $\alpha : \mathbf{PCOB}^k(X, P) \to \Pi^k(X, \infty)$. For general U, let $P \in \Sigma^k$ not be the basepoint. Then, by Lemma 0.6, f is homotopic to a proper smooth map $f \bowtie P$ and thus there is a framed map
$\underline{f} : (X, \underline{f}^{-1}(P)) \to (\Sigma^k, P)$. One then defines, as above, $\alpha([f]) = \{\underline{f}^c\}$. To see that this definition is well defined, suppose that $[f] = [h]$. Then, f is homotopic to h and, thus, $\underline{f}$ is homotopic to $\underline{h}$. But, then, by Proposition 0.4, $\underline{f}$ is cobordant to $\underline{h}$ and, by the above, $\{\underline{f}^c\} = \{\underline{h}^c\}$.

Conversely, consider a basepoint preserving continuous map $f : X^c \to \Sigma^k$ sending the point at infinity to a point $Q \neq P$ in Σ^k and arising from the Thom construction. Let $U = \{P\}$. Then, because of the Thom isomorphism [1], one can assume that $f|X$ is proper, $f^{-1}(P)$ is an inframed submanifold of X and f is smooth on a tubular neighborhood of $f^{-1}(P)$. Furthermore, smoothing the map $f|_X$ and using the fact (see [8,p.41]) that proper smooth maps form an open subset of the collection of all smooth maps between 2 smooth manifolds, a smooth proper map $g : X \to \Sigma^k$ can be chosen arbitrarily near $f|_X$ and thus homotopic to it, by Lemma0.5, with $g \bowtie P$ and $V = g^{-1}(P) = f^{-1}(P)$. It follows that $g : (X, V) \to (\Sigma^k, P)$ is a framed map homotopic to $f|_X$. Thus, to f one can associate a cobordism class of maps $[g] \in \mathbf{PCOB}^k(X, P)$. Suppose that f is homotopic to $f' : X^c \to \Sigma^k$. If f' defines a proper framed map $g' : (X, V') \to (\Sigma^k, P)$ with g' homotopic to $f'|X$ and, because the homotopy from f to f' preserves basepoints, g homotopic to g', using Proposition 0.4, one can find a smooth homotopy F from g to g' such that $F \times \mathbf{R} : (X \times \mathbf{R}, V^*) \to (\Sigma^k \times \mathbf{R}, P \times \mathbf{R})$ is a framed map. But, then, g and g' are framed cobordant and a map $\beta : \Pi^k(X, \infty) \to \mathbf{PCOB}^k(X, P)$ is determined. The Thom construction shows that $\alpha \circ \beta$ is the identity map. The map β is surjective since every proper framed map $f : X \to \Sigma^k$ can be extended to X^c. For general U, one uses the fact that, from the Thom construction (see [1]), every homotopy class $\{f\}$ contains an element which restricts to a proper map h. One applies Lemma 0.6 to h' and obtains a proper framed map $h'' : (X, V) \to (\Sigma^k, U)$ homotopic to f with V neatly imbedded in X. It is not difficult to see then that $\alpha^{-1}(\{f\}) = [h'']$, which by Proposition 0.4 is independent of h''.

In a similar way, on approximating continuous maps by smooth maps, using standard transversality results, using Lemma 0.6 and then using Proposition 0.4, one can show:

Proposition 1.3. *There is a bijection*

$$\beta : \mathbf{COB}^k(X, U) \to \mathbf{TOPH}(X, \Sigma^k),$$

if ∂X is compact or U is a one point set, where the framed cobordism class of a framed map $f : (X, V) \to (\Sigma^k, U)$ is sent to the homotopy class $\{f\}$.

We write $\Pi(f, g) = \mathbf{HPC}(f, g)$, conforming to standard notation. Let $g : (\Sigma^j, U) \to (\Sigma^k, P)$ and f be fixed framed maps. Suppose that $\mathbf{CPC}^k(f, g)$(resp., $\mathbf{PCPC}^k(f, g)$ is the set of all elements $(\phi, \psi, f, g, [F \times \mathbf{R}])$ in $\mathbf{CPC}(f, g)$ such that $\phi \in \mathbf{FR}^k(X, P)$ and $\psi \in \mathbf{FR}^k(X, U)$ (resp., $\phi \in \mathbf{PFR}^k(Y, P)$ and $\psi \in \mathbf{PFR}^k(X, U)$) modulo γ. Let $1^k : \Sigma^k \to \Sigma^k$ be the identity map from the k sphere to itself.

We then show first a simple version of Theorem 1.7 below:

Theorem 1.4. *Let $f : X \to Y$ be a closed embedding where Y does not have boundary or let $f : X \to Y$ be a submersion. Let $j = k$ and $U = \{P\}$. There is a bijection*

$$\alpha : \mathbf{CPC}^k(f, 1^k) \to \Pi(f, 1^k)$$

extending Proposition 1.3.

Proof. Suppose that one is given a diagram in **CPC**

$$\left[\begin{array}{ccc} X & \xrightarrow{[\psi]} & \Sigma^k \\ {\scriptstyle f}\downarrow & [F\times\mathbf{R}] \nearrow & \downarrow{\scriptstyle 1^k} \\ Y & \xrightarrow{[\phi]} & \Sigma^k \end{array}\right] \tag{3}$$

Since in diagram (3) ψ is cobordant to $\phi \circ f$, there is a homotopy h_t from $\phi \circ f$ to ψ and, taking homotopy classes, a diagram

$$\left\{\begin{array}{ccc} X & \xrightarrow{\{\psi\}} & \Sigma^k \\ {\scriptstyle f}\downarrow & \{h_t\} \nearrow & \downarrow{\scriptstyle 1^k} \\ Y & \xrightarrow{\{\phi\}} & \Sigma^k \end{array}\right\} \tag{4}$$

in $\Pi(f, 1^k)$ where one readily sees that $\{h_t\}$ is depends only on the cobordism class $[F \times \mathbf{R}]$. The map α thus associates to diagram (3) diagram (4).

Conversely, suppose that one is given diagram (4) in **HPC**. Using Proposition 1.3, $\{\phi\}$ determines a framed cobordism class $\beta(\{\phi\}) = [\Phi]$ for some Φ and similarly $\{\psi\}$ determines $\beta(\{\psi\}) = [\Psi]$ for some Ψ. Let $V = f^{-1}(P)$ and $i : V \to Y$ be the inclusion. Suppose that f is a closed embedding. Then, $f \bowtie$V if $i \bowtie f(X)$. An open neighborhood S consisting only of closed embeddings (see [4,p.76]) of i in $\mathbf{C}^\infty(V, Y)$ can be identified with an open neighborhood Q of the graph of the 0 section in the set of sections $\mathbf{C}^\infty(U_i)$ of the closed tubular neighborhood U_i of the graph V_i of i (which is a closed smooth submanifold of $Y \times Y$). For this proceed as in Proposition 0.1,b) using the fact that, since Y has no boundary , the set $\mathbf{Diff}(Y, Y_i)$ of diffeomorphisms from Y to Y_i is open in $\mathbf{C}^\infty(V, V_i)$ (see [8,p.38]). Let $\rho : V \times Y \to Y$ be the projection.

The framing on V pulls back via ρ to a partial framing on V_i. By standard transversality results, there is a map h in S, arbitrarily close to i, with $h \bowtie V$ which after identification is a map h^* in $\mathbf{C}^\infty(U_i)$. The graph Γ_{th^*} of th^* acquires a partial framing by lifting the partial framing on V_i using th^* for $0 \leq t \leq 1$. The partial framing on Γ_{th^*} induces a framing $F_t(v) = (F_1(v,t), \cdots, F_k(v,t))$ on $\Lambda_t = \rho(\Gamma_{th^*})$, for h close enough to i, which depends continuously on t. Applying the Gram-Schmidt orthogonalisation procedure, one can assume that $F_t(v)$ is an orthonormal basis for each t and v. Using [8,p.116], it is clear that there is an isotopy $\{\underline{N}_t\}$ of closed tubular neighborhoods of radius $\delta_t > 0$ of the Λ_t. Let D^k be the closed k-dimensional unit disk. On each of the $\underline{N}_t$ one defines a map $\eta_t : \underline{N}_t \to D^k$ which sends a point (v_t, w_t) on the fiber of $\underline{N}_t$ above $v_t \in \Lambda_t$ to $(\frac{1}{\delta_t})u$ where u is the representation of W_t with respect to $F_t(v)$. Let $\omega_t = q \circ \eta_t$ where $q : D^k \to \Sigma^k$ contracts the boundary of D^k to a point $Q \neq P$ and we assume that $q(0) = P$. Now, Φ defines the framing $F(v,0)$ on V and so $\omega_0^{-1}(P) = V$. There is a homotopy L from ω_0 to Φ which is defined by mapping a point $X \in \underline{N}_0$ to a point a t-th of the distance from $\Phi(X)$ to $\omega_0(X)$ along the unique geodesic between $\Phi(X)$ and $\omega_0(X)$ and by mapping X in the complement of $\underline{N}_0$ to a point a t-th of the distance from $\Phi(X)$ to Q along the unique geodesic between $\Phi(X)$ and Q. Since ω_0 sends the complement of $\underline{N}_0$ to Q, it is clear that L is continuous. One can smooth ω_1 to obtain a smooth map ϕ' homotopic to ω_1 which is equal to ω_1 on a tubular neighborhood of ω_1. It then follows that $\phi' : (Y, h(V)) \to (\Sigma^k, P)$ is a framed map. But, since ϕ' is homotopic to ω_1, ω_1 is homotopic to ω_0 and ω_0 is homotopic to Φ, it follows that Φ is homotopic to ϕ' and hence by Proposition 0.1 cobordant to it. If f is a submersion, then $f : (X, f^{-1}(V)) \to (Y, V)$ is already a framed map and one lets $\Phi = \phi'$. Since $\phi' \circ f$ is homotopic to Ψ, by Proposition 1.3, $\phi' \circ f$ **cob** Ψ via a cobordancy $F \times \mathbf{R}$ and diagram (3) is obtained. One sees moreover , by Proposition 0.4, that $[F \times \mathbf{R}]$ is independent of the choice of h_t.

That α is a bijection follows then from Proposition 0.4 and Proposition 1.3.

Following immediately from this proof one obtains:

Corollary 1.5. *Suppose that Y does not have boundary. If, as con-*

jectured by Hirsch [8,p.84] $\{i \in \mathbf{C}^\infty(X,Y) | i \bowtie f\}$ *for proper* $f \in \mathbf{C}^\infty(X,Y)\}$ *is residual and open in* $\mathbf{C}^\infty(X,Y)$*, then if* f *is proper, Theorem 1.4 holds without the restriction that* f *be a closed embedding.*

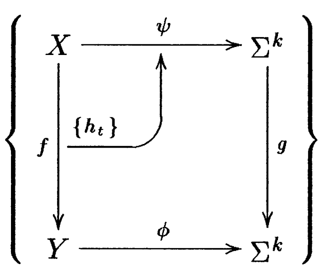
As shown by Hardie and Jansen in [5], $\mathbf{HPC}(f, 1^k)$ degenerates to $\mathbf{TOPH}(Y, \Sigma^k)$. In the same way here, since in diagram (3) $[\phi] \circ f$ determines $[\psi]$, one has:

Corollary 1.6. *With the conditions used in Theorem 1.4, the set* $\mathbf{CPC}^k(f, 1^k)$ *is bijectively equivalent to* $\mathbf{COB}^k(Y, P)$.

Let $g : (\Sigma^j, U) \to (\Sigma^k, P)$ be a framed map. Then, extending Theorem 1.4, one has:

Theorem 1.7. *Let* $f : X \to Y$ *be a closed embedding where* Y *does not have boundary or let* $f : X \to Y$ *be a submersion. Suppose that* ∂X *is compact or* U *is a one point set. There is a bijection* $\alpha : \mathbf{CPC}^k(f, g) \to \mathbf{HPC}(f, g)$ *defined in the same way as* α *in the proof of Theorem 1.4.*

Proof. Assuming that the definition of α is known, suppose that one has a diagram representing an element ζ of $\mathbf{HPC}(f, g) = \Pi(f, g)$:



Choose ϕ as in the proof of Theorem 1.4. Choose ψ so that $\psi \bowtie U$ and thus $\psi : (X, \psi^{-1}(U)) \to (\Sigma^k, U)$ is a framed map. Then, $g \circ \psi$ is homotopic to $\phi \circ f$ and hence cobordant (independent of h_t) to it. Thus, an element ζ^* in $\mathbf{CPC}(f, g)$ is determined and the association $\zeta \to \zeta^*$ inverts α.

One can prove a version of Theorem 1.7 extending Thom's theorem in the form below, using Proposition 1.2 instead of Proposition 1.3 and the evident modifications:

Theorem 1.8. *Let $f : X \to Y$ be a proper closed embedding where Y does not have boundary or let $f : X \to Y$ be a proper submersion. Suppose that U is a one-point set or ∂X is compact. With the evident choice of base points there is a bijection*

$$\alpha : \mathbf{PCPC}^k(f,g) \to \Pi^*(f,g) = \mathbf{HPC}(f^c,g),$$

where f^c aside from mapping the base point at infinity to the base point at infinity equals f.

Remark. Again Theorems 1.7 and 1.8 holds without the assumption that f is a closed embedding or a submersion if f is proper and Hirsch's conjecture, alluded to in Corollary 1.5, is true.

Examples

1) Let $X = Y = \Sigma^1$, $k = j = 1$, $f(z) = z$ and $g(z) = z^2$. Then, assuming transversality for the various mappings, set $g^{-1}(P) = \{P_1, P_2\}$, $\psi^{-1}(P_1) = \{Q_1, \cdots, Q_i\}$, $\psi^{-1}(P_2) = \{R_1, ..., R_j\}$ and $\phi^{-1}(P) = (\phi \circ f)^{-1}(P) = \{S_1, \cdots, S_m\}$. Note that $i = j$ [8,p.124]. Since $\phi \circ f$ is framed cobordant to $g \circ \psi$, $m = i + i$. It follows that $\Pi^*(f,g)$ is the subset of $\mathbf{Z} \times \mathbf{Z}$ consisting of all integer pairs (m, i) such that $m = 2i$ and thus isomorphic to $\mathbf{Z}$.
2) Let $X = Y$ be the Mobius strip without boundary. Then, X and Y can be viewed as the quotient of $\mathbf{I} \times (0,1)$ where, for $y \in (0,1)$, one identifies $(0, y)$ with $(1, y)$. Let $q : \mathbf{I} \times (0,1) \to X$ be the quotient map and X have the smooth structure induced from $\mathbf{I} \times (0,1)$. Suppose that $f(a) = a$ for $a \in X$ and $g(z) = z^2$. Consider the image L of $\{(o, y) | y \in (0,1)\}$ under q in X. By choosing arrows in one way or the other L can be made into a framed submanifold cobordant to L with the opposite orientation. It is also clear that L is cobordant to the image L' of $\{(1/2, y) | y \in (0,1)\}$ under q with the correct orientation. One can choose ψ so that $(g \circ \psi)^{-1}(P) = L \cup L'$ represents the trivial element of $\Pi^1(X, \infty)$ but with ψ not homotopically trivial. Suppose that $\phi^{-1}(P) = \emptyset$. If $(g \circ \psi')^{-1}(P) = \emptyset$ and ψ' is homotopically trivial, one has both $(\phi, \psi, [F \times \mathbf{R}])$ and $(\phi, \psi', [F' \times \mathbf{R}])$ in $\Pi^*(f,g)$ for suitable F and F'. Thus, $\mathbf{PCPC}^k(f,g)$ does not trivialize as in Corollary 1.6. It seems likely that $\Pi^*(f,g)$ can be identified with

the subset of $\mathbf{Z}_2 \times \mathbf{Z}_2$ consisting of all (m, i) such that $m = 2i = 0$ and thus with $\mathbf{Z}_2$.

2. General relations between CPC and HPC

We let $\mathbf{COB}(X, (Y, U))$ denote the union of the images of all $\mathbf{FR}((X, V), (Y, U))$ under the framed cobordism relation as V varies. To reduce the amount of language needed and relate some involved concepts we make the following definitions.

Definition 2.1. *Let D denote the quotient set of a set of structured maps where the underlying maps are $\mathbf{C}^r$ maps or framed maps ($0 \leq r \leq \infty$) but only one of these types from a smooth manifold X to a smooth compact manifold Y with basepoint y_0. Thus, D can be the quotient set of a set S whose elements are $\mathbf{C}^r$ maps alone for a fixed r or framed maps from X to Y alone. Let $D = S/R$. Let ES be the subset of S corresponding to $\mathbf{C}^r$ maps or framed maps from X to Y such that f extends to the one point compactification in a continuous way and maps the point at infinity to y_0. Next, let $ED(y_0)$ be the image of ES in D under the quotient map $S \to S/R$. Suppose now that $\mathbf{D}$ is a category. Then, we set $E\mathbf{D}(X, Y, y_0) = E\mathbf{D}(X, Y)(y_0)$. Let $P\mathbf{D}(X, Y, y_0)$, similarly, correspond to $\mathbf{C}^r$ maps or framed maps f from X to Y such that, for any compact subset K of Y, $f^{-1}(K)$ is compact and f maps the point at infinity to y_0.*

Remark. One can show that $f \in ES$ if and only if $f \in S$ and $f^{-1}(K)$ is compact for every compact set K with $y_0 \notin K$.

The following result can now be stated:

Theorem 2.2 *For smooth manifolds X and Y where X has compact boundary and U is compact:*

a) The set $\mathbf{COB}(X, (Y, U)) \cong \mathbf{TOPH}(X, Y)$ where $\cong$ denotes set equivalence.

b) Let $A = E$ or P. Let $y_0 \in Y$. Then, $A\mathbf{COB}(X, (Y, U), y_0) \cong A\mathbf{TOPH}(X, Y, y_0) \subset \mathbf{TOPH}(X^c, Y, y_0)$.

Proof We prove a). In the forward direction, $[f] \in \mathbf{COB}(X, (Y, U))$ implies $f : X \to Y$ is continuous and $f\mathbf{cob}\, g$ implies $f \simeq g$. In the reverse direction, for $\{f\} \in \mathbf{TOPH}(X,Y)$, f is homotopic to a smooth map g with $g \bowtie U$ and then $g : (X, g^{-1}(U)) \to (Y,U)$ is a framed map and $g^{-1}(U)$ is a neat submanifold of X, by Lemma 0.6. Using Proposition 0.4, it is clear that $[g]$ is independent of the choice of representative of $\{f\}$.

For b) and any A, in the usual way, the bijection in the forward direction is defined. In the reverse direction and any$A$ the fact that g (chosen sufficiently close to f) sends the point at infinity to y_0, provided that f sends the point at infinity to y_0, can be readily shown using the following result in [8, p.43]: For any sequence $\{g_n\}$ converging to f in $\mathbf{C}^r(X,Y)$ $(r \geq 0)$ in the strong topology, there exists a compact set $K \subset X$ such that, for almost all n, g_n is equal to f outside K. This allows one to show that g is extendable to the one point compactification and complete the proof if $A = E$. For $A = P$, one must show that g is proper if f is proper. Again, this follows from Lemma 0.6

The inclusion of b) follows immediately from definition.This completes the proof of the theorem.

We provide an example to show some uses for this theorem.

Example 2.3. Let Y be a compact connected smooth manifold and $n > 0$. Then, the n-th homotopy group $\Pi_n(Y, y_0)$ can be written

$$\Pi_n(Y,y_0) = \mathbf{TOPH}^*((\Sigma^n,*),(Y,y_0)) = E\mathbf{TOPH}(\mathbf{R^n},Y,y_0)$$

$$= E\mathbf{COB}(\mathbf{R^n},(Y,U),y_0).$$

Suppose that $[f]$,$[g] \in E\mathbf{COB}(\mathbf{R}^n,(Y,U),y_0)$. One identifies the domain of f (resp. g) with the set of points satisfying $x_n < 0$ (resp. $x_n > 0$). Let $f \coprod g$ be the extension of f and g to all points of $\mathbf{R}^n$ except those satisfying $x_n = 0$ and $f \coprod g$ maps those points satisfying $x_n = 0$ to y_0. Then, let h be $f \coprod g$ suitably smoothed and homotopic to $f \coprod g$. The addition in $\Pi_n(Y,y_0)$ transferred to $E\mathbf{COB}(\mathbf{R}^n,(Y,U),y_0)$ satisfies $[f] + [g] = [h]$. In fact, this addition can be shown to define a group structure in the usual way on $\mathbf{PCOB}(\mathbf{R}^n,(Y,U),y_0)$. One can ask:

$$\text{Is } \Pi_n(Y,y_0) = P\mathbf{COB}(\mathbf{R}^n,(Y,U),y_0)\ ?$$

If $Y = \Sigma^n$, the answer is no since, for any element $[f]$ of $P\mathbf{COB}(\mathbf{R}^n, (\Sigma^n, U), y_0)$, as $f^{-1}(\Sigma^n) = \mathbf{R}^n$, it is clear that $P\mathbf{COB}(\mathbf{R}^n, (\Sigma^n, U), y_0)$ is trivial. In fact, the author has yet to find a case where $P\mathbf{COB}(\mathbf{R}^n, (Y, U), y_0)$ is not trivial.

Let $g : (W, D) \to (Z, E)$ be a framed map. Let $\mathbf{CPC}(f, g)''$ consist of all squares (see diagram (2)) such that $[\phi] \in \mathbf{COB}(Y, (Z, E))$ and $[\psi] \in \mathbf{COB}(X, (W, D))$. Then, extending Proposition 1.3 to the pair category, one has:

Theorem 2.4 *Suppose that D and E are compact. Let $f : X \to Y$ be a closed embedding where Y does not have boundary or let $f : X \to Y$ be a submersion. Suppose that D is a one-point set or ∂X is compact. Suppose that E is a one-point set or ∂Y is compact. There is a bijection*

$$\alpha : \mathbf{CPC}(f, g)'' \to \Pi(f, g)$$

sending a framed cobordant commutative square (diagram(2)) to a corresponding homotopy commutative one (diagram(1)).

Proof. Again, the definition and existence of α is straightforward.

In the other direction, using Theorem 2.2, one can always find an element in the homotopy class of ψ in diagram (1) which is smooth and transversal to D; so on the side of diagram (1) with ψ there is little problem.

Let $\mu \in \{f\}$. One can suppose that μ is smooth and transversal to E. Let $C = \mu^{-1}(E)$. Then, $\mu : (Y, C) \to (Z, E)$ is a framed map. Since when f is a submersion, f is transversal to C, we need only consider the case where f is a closed embedding. There are up to diffeomorphism open tubular neighborhoods (defined by the framings) T_E and T_C of E and C, respectively, so that $\mu|_{T_C}$ can be viewed as a fibred map from T_C to T_E. Let $\rho : C \to T_C$ be a smooth section with ρ transverse to $f(X) \cap T_C$. The existence of ρ follows from an exercise in Hirsch [8,p.83(No. 13)]. One lets $B = f^{-1}(\rho(C))$ and then $f : (X, B) \to (Y, \rho(C))$ is a framed map. Let $T_{\rho(C)}$ be an open tubular neighborhoods of $\rho(C)$ strictly contained in an open tubular neighborhood $\underline{T}_C$ of C which is again strictly contained in T_C. There is a diffeomorphism κ of Y onto itself sending $T_{\rho(C)}$ to $\underline{T}_C$, $T_C - T_{\rho(C)}$ to $T_C - \underline{T}_C$ and which is the identity ouside T_C. Let $\mu' = \mu \circ \kappa$.

Then, clearly, $\mu' : (Y, \rho(C)) \to (Z, E)$ is a framed map. Since there is a smooth deformation of $\rho(C)$ onto C along the fibres of the tubular neighborhood T_C, the diffeomorphism κ can be chosen to depend smoothly on a parameter $t \in \mathbf{R}$ in such a way that, writing k_t to denote this dependence, then $\mu \circ k_1 = \mu'$ and $\mu \circ k_0 = \mu$. Using Proposition 0.4, μ is cobordant to μ' and hence $\mu' \in [\mu]$. Thus, on the bottom of diagram (2), one takes the framed map μ'.

Since $g \circ \psi$ and $\mu \circ f$ are homotopic framed maps, using Proposition 0.4 twice, one can associate to diagram (1) a unique diagram (2). In this way one defines a map β which associates to each diagram (1) a diagram (2). Clearly, the map β provides a right inverse to α. As β is onto, it and α are bijective.

Remark. Again, one can apply Hirsch's conjecture to prove the above result for proper f without the assumption that f is a closed imbedding.

3. Pair cohomotopy bicobordism conversion: suspension.

In analogy to the relation between such notions as suspension, track addition, etc. developed in cohomotopy and their corresponding representation in framed cobordism as put down in [1], we develop, as far as possible, the same relation (but extended) between framed bicobordism and pair cohomotopy in the case of suspension. Throughout this section $g : (\Sigma^j, U) \to (\Sigma^k, P)$ is a framed map.

The suspension of an inframed manifold $\alpha = (X, U)$ is the inframed manifold $E(\alpha) = (X \times \mathbf{R}, U)$ where U in $E(\alpha)$ has the framing of U in X plus the outward unit normal of X in $X \times \mathbf{R}$. Here, X is embedded in $X \times \mathbf{R}$, via the map sending x to $(x, 0)$.

Let $*$ denote the north pole of any sphere. Consider the map $q : \Sigma^k \times \Sigma^{k'} \to \mathbf{R}^{k+k'+1}$ sending $((x_1, \cdots, x_{k+1}), (y_1, \cdots, y_{k'+1}))$ to

$$(u/(1 - x_{k+1}))(x_1, \cdots, x_k, 0, \cdots) + (u/(1 - y_{k+1}))(0, \cdots, y_1, \cdots, y_{k'}, 0)$$

$$+(0, \cdots, 0, 1 - u)$$

where $u = (1 - x_{k+1})(1 - y_{k'+1})$. Notice that q is smooth. We show:

Lemma 3.1. *The smooth map* $q : \Sigma^k \times \Sigma^{k'} \to q(\Sigma^k \times \Sigma^{k'})$ *serves as the identification map for the smash product and is a diffeomorphism outside* $(\Sigma^k \times *) \cup (* \times \Sigma^{k'})$.

Proof. If x or y is $*$ (thus, defined by $x_{k+1} = 1$ or $y_{k'+1} = 0$), then $q(x, y) = (0, \cdots, 0, 1)$. The converse is also true. If neither x nor y is a north pole, then $u \neq 1$ and q is one-one. By definition of the smash product, there is a continuous bijective map

$$r : \Sigma^k \wedge \Sigma^k \to q(\Sigma^k \times \Sigma^{k'}).$$

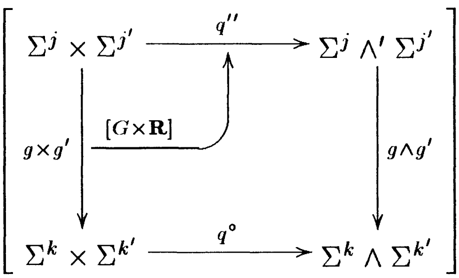
Since both $\Sigma^k \times \Sigma^{k'}$ and $q(\Sigma^k \times \Sigma^{k'})$ are compact Hausdorff, r is a homeomorphism and q is an identification as required. The inverse to q on $q(\Sigma^k \times \Sigma^{k'}) - \{(0, \cdots, 0, 1)\}$ sends $(a_1, \cdots, a_{k+k'+1})$ to $(P_k^{-1}(a_1/d, \cdots, a_k/d), P_{k'}^{-1}(a_{k+1}/d, \cdots, a_{k+k'}/d))$ where $d = 1 - a_{k+k'+1}$ and P_k is the inverse of the stereographic projection from the north pole. The proof of Lemma 3.1 is complete.

Let $g : (\Sigma^j, U) \to (\Sigma^k, P)$ and $g' : (\Sigma^{j'}, U') \to (\Sigma^{k'}, P')$ be framed maps. One would like to approximate the diagram

$$\begin{array}{ccc} \Sigma^j \times \Sigma^{j'} & \xrightarrow{q'} & \Sigma^j \wedge \Sigma^{j'} \\ {\scriptstyle g \times g'} \downarrow & & \downarrow {\scriptstyle g \wedge g'} \\ \Sigma^k \times \Sigma^{k'} & \xrightarrow{q} & \Sigma^k \wedge \Sigma^{k'} \end{array}$$

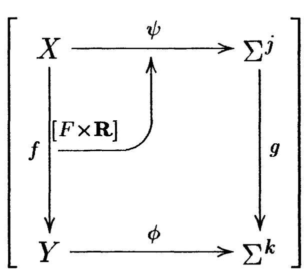
in **HPC**, where q' is the identification defining $\Sigma^j \wedge \Sigma^{j'}$, by a diagram in **CPC**. First, we replace $\Sigma^j \wedge \Sigma^{j'} = q'(\Sigma^j \times \Sigma^{j'})$ by $\Sigma^{j+j'}$ to which, in light of Lemma 3.1, it is homeomorphic and diffeomorphic outside the north pole and then view $g \wedge g' : \Sigma^{j+j'} \to \Sigma^{k+k'}$ as $q \circ (g \times g') \circ (q')^{-1}$ outside $(0, \cdots, 0, 1)$ and modulo a diffeomorphism. Next, we patch together, using partitions of unity, some constant map χ_1 from $\Sigma^{j+j'}$ to $\Sigma^{k+k'}$ defined on a sufficiently small euclidean neighborhood N of the north pole in $\Sigma^{j+j'}$, sending north pole to north pole, and the map χ_2 equal to $g \wedge g'$ on an open subset R of $\Sigma^{j+j'}$ not containing the north pole (where $\Sigma^{k+k'} = N \cup R$) by smoothing across the intersection $N \cap R$ where if N is small enough, then both χ_1 and χ_2 on $N \cap R$ map to the euclidean space which is $\Sigma^{k+k'}$ minus the south pole. Call the smooth patched map $g \wedge' g'$. If N and R are chosen properly, then, on open set H not containing the north pole, one has a) $q'(U \times U') \subset H$ b) $g \wedge' g'$ equals $g \wedge g'$; consequently $g \wedge' g'$ is transverse to $q(P \times P')$ and c) $g \wedge' g'$ is arbitrarily close to, thus homotopic to $g \wedge g'$. Thus, except for some smoothing around the north pole $g \wedge' g'$ equals $g \wedge g'$ and any two choices of $g \wedge' g'$ are homotopic.

In the same way one can approximate q' by a smooth map q'' equal to q' on an open set not meeting $\Sigma^k \times * \cup * \times \Sigma^{k'}$ and containing $U \times U'$, q'' is transversal to $q'(U \times U')$. As a consequence, one obtains framed maps $q'' : (\Sigma^j \times \Sigma^{j'}, U \times U') \to (\Sigma^{j+j'}, q''(U \times U'))$ and $g \wedge' g'$: $(\Sigma^{j+j'}, q''(U \times U')) \to (\Sigma^{k+k'}, q(P \times P))$. Different possibilities for q'' are cobordant. Similarly, q can be changed to q° along the bottom of the above diagram but $g \times g'$ need not be changed. One thus ends up in the evident way with a diagram in **CPC**:



where $F \times \mathbf{R}$ arises from the homotopy between $(g \wedge' g'') \circ q''$ and $q^\circ \circ (g \times g')$ which exists because of the above construction.

Consider the map ω :



in **CPC** where $\psi : (X, W) \to (\Sigma^j, U)$, $\phi : (Y, V) \to (\Sigma^k, P)$ and $f : (X, W') \to (Y, V)$ are framed maps. The suspension $E(\omega)$ of ω is the equivalence class of the outer rectangle in

$$\left[\begin{array}{ccccccc} X \times \mathbf{R} & \xrightarrow{\psi\times\mathbf{R}} & \Sigma^j \times \mathbf{R} & \xrightarrow{i_j} & \Sigma^j \times \Sigma^1 & \xrightarrow{q''} & \Sigma^{j+1} \\ \downarrow{\scriptstyle f\times\mathbf{R}} & [F\times\mathbf{R}\times\mathbf{R}] & \downarrow{\scriptstyle g\times\mathbf{R}} & & \downarrow{\scriptstyle g\times\Sigma^1} & [G\times\mathbf{R}] & \downarrow{\scriptstyle g\wedge'\mathbf{R}} \\ Y \times \mathbf{R} & \xrightarrow[\phi\times\mathbf{R}]{} & \Sigma^k \times \mathbf{R} & \xrightarrow[i_k]{} & \Sigma^k \times \Sigma^1 & \xrightarrow[q^\circ]{} & \Sigma^{k+1} \end{array}\right]$$

where i_j and i_k are open smooth immersions, the middle diagram being commutative defines an element of **CPM** and $G \times \mathbf{R}$ arises from the homotopy between $(g \wedge' \mathbf{R}) \circ q''$ and $q^\circ \circ (g \times \Sigma^j)$. Thus, suppose that $U^\circ = (q'' \circ i_j)(U \times 0)$. Then, $\mu = q'' \circ i_j \circ (\psi \times \mathbf{R})$ is transversal to U°, $\mu : (X \times \mathbf{R}, W \times 0) \to (\Sigma^{j+1}, U^\circ)$ is a framed map and μ is the top of the above diagram. As $q^\circ \circ i_k$ is an open embedding on a neighborhood of $P \times 0$, $P \wedge 0$ acquires a framing from the framing on $P \times 0$ as a subset of $\Sigma^k \times \mathbf{R}$. As $q'' \circ i_j$ is an open immersion on an open set containing $U \times 0$ and the framing on $P \times 0$ pulls back via $g \times \mathbf{R}$ to the framing on $U \times 0$, $U^\circ = (g \wedge' \mathbf{R})^{-1}(P \wedge 0)$ has the framing induced from $P \wedge 0$ via $g \wedge' \mathbf{R}$. Thus, $\lambda = (g \wedge' \mathbf{R}) \circ \mu : (X \times \mathbf{R}, W \times 0) \to (\Sigma^{k+1}, P \wedge 0)$ is a framed map where, as we noted above, $(X \times \mathbf{R}, W \times 0)$ is the suspension of (X, W). Similarly, along the bottom of the above diagram, one obtains a framed map

$$\nu = q^\circ \circ i_k \circ (\phi \times \mathbf{R}) : (Y \times \mathbf{R}, V \times 0) \wedge (\Sigma^{k+1}, P \wedge 0),$$

$$\beta = \nu \circ (f \times \mathbf{R}) : (X \times \mathbf{R}, \beta^{-1}(P \wedge 0)) \to (\Sigma^{k+1}, P \wedge 0)$$

is a framed map and

$$(X \times \mathbf{R}, \beta^{-1}(P \wedge 0)) = (X \times \mathbf{R}, (f \times \mathbf{R})((V \times 0)) = (X \times \mathbf{R}, W' \times 0)$$

is the suspension of (X, W'). Let $f : X \to Y$ be a proper closed embedding where Y does not have boundary or let $f : X \to Y$ be a proper submersion. Suppose that U is a one-point set or ∂X is empty. Let S denote the usual suspension functor on $\mathbf{TOP}^*$. The assignment

$\omega \to E(\omega)$ then, by Theorem 1.8, determines a map
$E : \Pi^*(f,g) \to \Pi^*(f \times \mathbf{R}, g \wedge' \mathbf{R}) = \mathbf{HPC}^*(Sf, g)$ of cohomotopy pair sets or, without the assumptions of Theorem 1.8, just a map
$E : \mathbf{CPC}^k(f,g) \to \mathbf{CPC}^{k+1}(f \times \mathbf{R}, g \wedge' \mathbf{R})$. On restricting to the case where f and g are identities and, applying Proposition 1.1, one obtains the usual suspension map:

$$\Pi^k(X,\infty) \cong F^k(X) \cong \mathbf{PCOB}^k(X,P) \to \Pi^{k+1}(SX,\infty).$$

Remark. Using the algebraic manipulation language MAPLE, one can show that if $k=1$, then $q(\Sigma^k \times \Sigma^1)$ has defining equation $2a^2b^2 + (2b^2+2a^2)(c-1)^2 + 8(c-1)^3 + 2(c-1)^4 = 0$. It follows then that $q(\Sigma^1 \times \Sigma^1)$ is an algebraic variety but not a manifold since $(0,0,1)$ is obviously a singular point of $q(\Sigma^k \times \Sigma^1)$.

4. Stable pair framed cobordism.

We let now $f = g : \Sigma^3 \to \Sigma^2$ be the Hopf map, which is a submersion, and use the paper [5] by Hardie and Jansen to provide an example of the cylinder web diagram with respect to **CPC** and then analagously calculate the stable group corresponding to
$P\mathbf{CPC}^2(g \times \mathbf{R}^k, g) \cong \Pi^*(g \times \mathbf{R}^k, g)$. We let $.h$ denote the function induced by precomposition with h and $h.$ the function defined by postcomposition with h. The cylinder web diagram in [5] modified to $\mathbf{CPC}^*$, where

i) as usual the cohomotopy set
$\Pi^k_\infty(X) = \Pi^k(X,\infty) = \mathbf{TOPH}^*(X^c, \Sigma^k) \cong \mathbf{PCOB}^k(X,P)$,
ii) the horizontal and vertical arrows are described in terms of mappings between hom-sets in $\mathbf{TOPH}^*$,
iii) $Pg : \mathbf{CP}^1 \to C_g = \mathbf{CP}^2$ can be viewed as the inclusion identifying $\mathbf{CP}^1$ with a projective line in $\mathbf{CP}^2$,
iv) Qg is the map shrinking $\mathbf{CP}^1$ as imbedded by Pg in $\mathbf{CP}^2$ to a point and
v) $\underline{X}$ is X minus its basepoint and we write simply $\mathbf{P}^j$ instead of $\mathbf{CP}^j$ for complex j-dimensional projective space,

is given below

$$
\begin{array}{ccccccc}
\cdots\Pi^2_\infty(\mathbf{R}^5) & \xrightarrow{.SQg} & \Pi^2_\infty(\underline{\mathbf{P}}^2\times\mathbf{R}) & \xrightarrow{.SPg} & \Pi^2_\infty(\underline{\mathbf{P}}^1\times\mathbf{R}) & \xrightarrow{.Sg} & \Pi^2_\infty(\mathbf{R}^4) \\
\downarrow{\scriptstyle Pg.} & & \downarrow{\scriptstyle Pg.} & & \downarrow{\scriptstyle Pg.} & & \downarrow{\scriptstyle Pg.} \\
\cdots\{S^2\Sigma^3,\mathbf{P}^2\} & \xrightarrow{.SQg} & \{S\mathbf{P}^2,\mathbf{P}^2\} & \xrightarrow{.SPg} & \{S\mathbf{P}^1,\mathbf{P}^2\} & \xrightarrow{.Sg} & \{S\Sigma^3,\mathbf{P}^2\} \\
\downarrow{\scriptstyle S^{-1}.Qg.} & & \downarrow{\scriptstyle S^{-1}.Qg.} & & \downarrow{\scriptstyle S^{-1}.Qg.} & & \downarrow{\scriptstyle S^{-1}.Qg.} \\
\cdots\Pi^3_\infty(\mathbf{R}^3\times\mathbf{R}) & \xrightarrow{.Qg} & \Pi^3_\infty(\underline{\mathbf{P}}^2) & \xrightarrow{.Pg} & \Pi^3_\infty(\underline{\mathbf{P}}^1) & \xrightarrow{\cdot g} & \Pi^3_\infty(\mathbf{R}^3) \\
\downarrow{\scriptstyle g\cdot} & & \downarrow{\scriptstyle g\cdot} & & \downarrow{\scriptstyle g\cdot} & & \downarrow{\scriptstyle g\cdot} \\
\cdots\Pi^2_\infty(\mathbf{R}^3\times\mathbf{R}) & \xrightarrow{.Qg} & \Pi^2_\infty(\underline{\mathbf{P}}^2) & \xrightarrow{.Pg} & \Pi^2_\infty(\underline{\mathbf{P}}^1) & \xrightarrow{\cdot g} & \Pi^2_\infty(\mathbf{R}^3) \\
\downarrow{\scriptstyle Pg.} & & \downarrow{\scriptstyle Pg.} & & \downarrow{\scriptstyle g\cdot} & & \downarrow{\scriptstyle Pg.} \\
\cdots\{S^1\Sigma^3,\mathbf{P}^2\} & \xrightarrow{.Qg} & \{\mathbf{P}^2,\mathbf{P}^2\} & \xrightarrow{.Pg} & \{\mathbf{P}^1,\mathbf{P}^2\} & \xrightarrow{\cdot g} & \{\Sigma^3,\mathbf{P}^2\}
\end{array}
$$

Viewed as a diagram in **TOPH**, one can apply repeatedly the suspension S to the above diagram. Taking direct limits one ends up with the stable cylinder web diagram where we let

$$F_n(X,\Sigma^m) = \lim_{j\to\infty} F_{m+j}(X\times\mathbf{R}^{n+j}) \cong \lim_{j\to\infty} \Pi^{m+j}(X\times\mathbf{R}^{n+j},\infty)$$

$$= \lim_{j\to\infty} \{S^{j+n}X^c,\Sigma^{j+m}) = G_n(X^c,\Sigma^m).$$

The isomorphism here is that of groups (see [1]). The groups $F_n(\mathbf{R}^k,\Sigma^3)$ (resp., $F_m(\mathbf{R}^l,\Sigma^2)$) are all isomorphic via the Thom construction to Ω^{fr}_{n+k-3}, the group of framed cobordism classes of framed $n+j-2$-dimensional manifolds, (resp.Ω^{fr}_{l+m-2}). See [10]. The stable cylinder web diagram is then

$$
\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots F_{n+1}(\mathbf{R}^3,\Sigma^3) & \xrightarrow{.Qg} & F_n(\underline{\mathbf{P}}^2,\Sigma^3) & \xrightarrow{.Pg} & F_n(\mathbf{R}^2,\Sigma^3) & \xrightarrow{\cdot g} & F_n(\mathbf{R}^3,\Sigma^3)\cdots \\
g\cdot\downarrow & & g\cdot\downarrow & & g\cdot\downarrow & & g\cdot\downarrow \\
\cdots F_{n+1}(\mathbf{R}^3,\Sigma^2) & \xrightarrow{.Qg} & F_n(\underline{\mathbf{P}}^2,\Sigma^2) & \xrightarrow{.Pg} & F_n(\mathbf{R}^2,\Sigma^2) & \xrightarrow{\cdot g} & F_n(\mathbf{R}^2,\Sigma^2)\cdots \\
Pg.\downarrow & & Pg.\downarrow & & Pg.\downarrow & & Pg.\downarrow \\
\cdots G_{n+1}(\Sigma^3) & \xrightarrow{.Qg} & G_n(\mathbf{P}^2) & \xrightarrow{.Pg} & G_n(\mathbf{P}^1) & \xrightarrow{\cdot g} & G_n(\Sigma^3)\cdots \\
Qg.\downarrow & & Qg.\downarrow & & Qg.\downarrow & & Qg.\downarrow \\
\cdots F_{n+1}(\mathbf{R}^3,\Sigma^3) & \xrightarrow{.Qg} & F_n(\underline{\mathbf{P}}^2,\Sigma^3) & \xrightarrow{.Pg} & F_n(\mathbf{R}^2,\Sigma^3) & \xrightarrow{\cdot g} & F_n(\mathbf{R}^3,\Sigma^3)\cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow
\end{array}
$$

where $G_i(X) = G_i(X,\mathbf{P}^2)$. Using the above cylinder web diagram, the stable groups $\pi_{k+2}(g) = G_k(\mathbf{CP}^2,\Sigma^2)$ have been calculated in [5] and this calculation implies immediately the following table of results:

Proposition 4.1

n	=	0	1	2	3	4	5	6
$F_{n-2}(\mathbf{CP}^2,\Sigma^2)$	=	$\mathbf{Z}$	0	$\mathbf{Z}$	$\mathbf{Z}_4+\mathbf{Z}_3$	0	$\mathbf{Z}_4+\mathbf{Z}_3$	$\mathbf{Z}_3$

n	=	7	8	9
$F_{n-2}(\mathbf{CP}^2,\Sigma^2)$	=	$\mathbf{Z}_1+\mathbf{Z}_3+\mathbf{Z}_5$	$\mathbf{Z}_4$	$\mathbf{Z}_{16}+\mathbf{Z}_3+\mathbf{Z}_5$

n	=	10	11	12
$F_{n-2}(\mathbf{CP}^2,\Sigma^2)$	=	$\mathbf{Z}_3$	$\mathbf{Z}_8+\mathbf{Z}_2+\mathbf{Z}_9+\mathbf{Z}_7$	$\mathbf{Z}_3$

Let $(S^jg)^\wedge$ be a framed map where the underlying map is homotopic to S^jg. One can also define

$$F_k(g) = \lim_{j\to\infty} \mathbf{PCPC}^2(g\times\mathbf{R}^{k+j},(S^jg)^\wedge) \cong \lim_{j\to\infty} \pi(g\times\mathbf{R}^{k+j},S^jg).$$

If the isomorphism induces a group structure on $F_k(g)$, using the results of [4], one then obtains:

Proposition 4.2

n	$= -3$ *or less*	-2	-1	0	1	2	3
$F_k(g) =$	0	$\mathbf{Z}$	0	$\mathbf{Z}+\mathbf{Z}$	$\mathbf{Z}_3+\mathbf{Z}_2+\mathbf{Z}_4$	$\mathbf{Z}_2$	$2\mathbf{Z}_8+2\mathbf{Z}_3$

n	$= 4$	5	6	7	8
$F_k(g) =$	$\mathbf{Z}_2$	$\mathbf{Z}_{16}+\mathbf{Z}_3+\mathbf{Z}_5$	$\mathbf{Z}_2+\mathbf{Z}_4$	$2\mathbf{Z}_{16}+2\mathbf{Z}_3+2\mathbf{Z}_5$	$2\mathbf{Z}_2+2\mathbf{Z}_3$

However, the group structure on $F^k(g)$ should be given in the first instance by extending the pair cohomotopy bicobordism conversion of §3.

5. Bibliography

[1] J. M. Boardman and B. Steer, On Hopf Invariants, Comment. Math. Helv. 42(1967), 180-221.

[2] J. Dieudonne, A History of Algebraic and Differential Topology. 1900-1960, Birkhuser(1989).

[3] Y. Chocquet-Bruhat and C. deWitt-Morette with M. Dillard-Bleick, Analysis, Manifolds and Physics, North-Holland Publishing Co. (1982).

[4] M. Golubitsky and VGuillemin, Stable Mappings and Their Singularities, Springer Verlag(1973).

[5] K. A. Hardie and A.V. Jansen, Exact Sequences in Stable Homotopy Pairs, Trans. Amer. Math. Soc. (2)285(1984),803-816.

[6] K. A. Hardie, On the Category of Homotopy Pairs, Topology Appl. 4 (1982), 59-69.

[7] H. Herrlich, Abstract and Concrete Categories, John Wiley and Sons,Inc. (1990).

[8] M. W. Hirsch, Differential Topology, Springer Verlag(1976).

[9] R. Stong, Cobordism of Maps, Topology 5 (1966),245-258.

[10] R. Stong, Notes on Cobordism Theory., Princeton University Press (1968a).

[11] R. Thom, Quelques proptits globales des varitis diffrentiables, Comment. Math. Helv. 28(1954),17-68.

Department of Mathematics, University of Cape Town, 7700 Rondebosch, South Africa

e-mail: chair@maths.uct.ac.za