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**A SUSPENSION THEOREM FOR THE
PROPER HOMOTOPY AND
STRONG SHAPE THEORIES**

by *C. ELVIRA-DONAZAR and L.J. HERNANDEZ-PARICIO*

Résumé. Nous présentons une extension du théorème de suspension de Freudenthal pour la catégorie de systèmes inverses d'espaces et comme conséquences nous avons des théorèmes de suspension pour l'homotopie propre et pour la théorie de la forme forte.

Abstract. We extend the Freudenthal suspension theorem to the category of towers of spaces and as consequences we obtain suspension theorems for proper homotopy and strong shape theories.

Key words: Tower of spaces, prospaces, model structure, suspension theorem, proper homotopy, strong shape, cofibration, fibration, weak equivalence.

AMS classification numbers: 55P40, 55P55, 54C56, 55Q07

Introduction.*

In 1936, Hans Freudenthal [12] proved that the transformation defined by the suspension functor becomes an isomorphism (or epimorphism) under adequate dimension and connectivity conditions. In this paper we extend the Freudenthal suspension theorem to the homotopy category of towers of pointed spaces $Ho(towTop_*)$.

An application of this extended version is obtained when we consider the Edwards-Hasting embedding theorem [10] of the proper homotopy category of locally compact Hausdorff σ -compact spaces into the category $Ho(towTop_*)$. This embedding preserves suspensions for nice spaces, see [7]. Consequently a suspension theorem in the proper setting is also proved. We have not developed a suspension theorem for the "global" category $Ho(towTop_*, Top_*)$, but it can be checked that the proof given in this paper also works in this category. Therefore there is also a corresponding suspension theorem for global proper homotopy.

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Another application of this extended version of the suspension theorem is given for strong shape theory. By considering the Vietoris functor, first introduced by Porter [20], we also have an embedding of the pointed strong shape category into $Ho(proTop_*)$. The Porter–Vietoris prospace VX of a metrisable compact space is isomorphic to a tower of pointed spaces in $Ho(proTop_*)$. Therefore if we restrict ourselves to metrisable compact spaces, we have an embedding of the category of pointed strong shape of metrisable compact spaces into $Ho(towTop_*)$. The Vietoris–Porter functor also preserves suspensions, see [10], and then we also have a suspension theorem for pointed strong shape.

The paper is divided in three sections. The first section is concerned with the categories used in this paper and some of their properties. For example, we consider $Ho(towTop_*)$ as the category of fractions obtained by the inversion of weak equivalences. Edwards and Hastings proved that if we have in a category \mathcal{C} a closed model structure (satisfying the condition N) then $tow\mathcal{C}$ and $pro\mathcal{C}$ inherit induced closed model structures. A similar result was proved by Porter [22–23] for the homotopy structure defined by Brown.

There are two well known structures of closed model categories in Top_* , the structure given by Quillen [29] and the structure given by Strøm [33]. The formal inversion of the respective families of weak equivalences produces the categories $Ho_{Str}(Top_*)$ and $Ho_{Quillen}(Top_*)$. As consequence of the Edwards–Hastings methods we also have the categories $Ho_{Str}(towTop_*)$ and $Ho_{Quillen}(towTop_*)$. On the other hand, there is another different notion of weak equivalence defined by Grossman that gives a new closed model structure on towers of simplicial sets and induces the category of fractions $Ho(towSS_*)$. In section 1, we analyse some relationships between these categories of fractions. The basic references for this section are the monograph of Edwards and Hastings [10] and the papers of Porter [20–28] and Grossman [14–16].

In section 2, we develop a proof of the suspension theorem. We establish the suspension theorem for towers of pointed CW –complexes. The main difference with the Freudenthal theorem for standard homotopy is that the dimension condition is stronger. However, when we consider towers of CW –complexes, whose bonding morphisms are cellular inclusions of CW –complexes and whose limit is trivial, we have conditions similar to those of the standard suspension Freudenthal theorem. This implies that we have similar conditions for the suspension theorem in the proper setting but stronger conditions must be considered for the suspension theorem in strong shape context.

The last section is devoted to obtaining the suspension theorem in the proper and strong shape settings from the suspension theorem for towers of pointed spaces. As a consequence of a Grossman result the connectivity conditions can be given

in terms of towers of homotopy groups or in terms of Grossman homotopy groups. These Grossman homotopy groups appear as Brown homotopy groups in the proper setting and as Quigley inward groups in the strong shape category.

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1. Preliminaries.

In this section we recall some of the notions and results that will be used in this paper. The structure of closed model category given by Quillen and the notion of procategory introduced by Grothendieck are basic tools of this work. One of the main results that we use is the Edwards-Hastings embedding of the proper homotopy category into the homotopy category of prospace.

a) Model categories.

This structure was introduced by Quillen [29]. It provides sufficient conditions on a category to develop a homotopy theory. Next we give some of the notions that we are going to use.

An ordered pair of morphisms (i, p) is said to have the lifting property if for any commutative diagram

$$\begin{array}{ccc}
 A & \xrightarrow{u} & Y \\
 \downarrow i & & \downarrow p \\
 X & \xrightarrow{v} & B
 \end{array}$$

there is a morphism $f: X \rightarrow Y$ such that $fi = u$ and $pf = v$. A map i has the left lifting property with respect to a class, P , of maps if (i, p) has the lifting property for all p in P , similarly a map p has the right lifting property with respect to a class, I , if for each i in I , (i, p) has the lifting property.

A closed model category consists of a category \mathcal{C} and three distinguished classes of morphisms, fibrations, cofibrations and weak equivalences, satisfying certain basic

properties (axioms) that guarantee the existence of the basic constructions of a homotopy theory, see [29].

A morphism which is both fibration (resp. cofibration) and weak equivalence is said to be a trivial fibration (resp. trivial cofibration). The initial object of \mathcal{C} is denoted by \emptyset and the final object by $*$. An object X of \mathcal{C} is said to be fibrant if the morphism $X \rightarrow *$ is a fibration and it is said to be cofibrant if $\emptyset \rightarrow X$ is a cofibration.

Definition 1. Let X be an object of \mathcal{C} , a cylinder for X is a commutative diagram

$$\begin{array}{ccc} X \vee X & \xrightarrow{\partial_0 + \partial_1} & X' \\ \nabla \downarrow & \nearrow p & \\ & & X \end{array}$$

where $\partial_0 + \partial_1$ is a cofibration, p is a weak equivalence and $p(\partial_0 + \partial_1) = id_X + id_X = \nabla$

Dually to this we have the cocylinder.

Definition 2. A cocylinder for an object X of \mathcal{C} is a commutative diagram

$$\begin{array}{ccc} & & X'' \\ & \nearrow s & \downarrow (d_0, d_1) \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

where (d_0, d_1) is a fibration, s is a weak equivalence and $(d_0, d_1)s = (id_X, id_X) = \Delta$.

Given a model category \mathcal{C} , the category obtained by formal inversion of the weak equivalences is denoted by $Ho(\mathcal{C})$.

The category \mathcal{C} is said to be pointed if the initial and final objects are isomorphic. This object is usually denoted by $*$ and it is called the zero object.

Given a morphism $f: X \rightarrow Y$ in a pointed category, the fibre is defined to be the fibre product $* \times_Y X$ and the cofibre is the coproduct $* \vee_X Y$.

Let X be a cofibrant object, a suspension SX of X is the cofibre of $\partial_0 + \partial_1: X \vee X \rightarrow X'$, where X' is a cocylinder for X . If Y is a fibrant object of \mathcal{C} , a loop object for Y is the fibre of $Y'' \xrightarrow{(d_0, d_1)} Y \times Y$ where Y'' is a cocylinder for Y .

The constructions S and Ω induce functors $S: Ho(\mathcal{C}) \rightarrow Ho(\mathcal{C})$ and $\Omega: Ho(\mathcal{C}) \rightarrow Ho(\mathcal{C})$ such that S is left adjoint to Ω .

The following closed model categories will be used in this paper.

1) The category of simplicial sets SS . Quillen [29] gave the following structure to the category of simplicial sets: A map $f: X \rightarrow Y$ is a fibration if f has the right lifting property with respect to $\Delta(n, k) \rightarrow \Delta[n]$ for $0 \leq k \leq n$ and $n > 0$, where $\Delta(n, k)$ is the simplicial subset generated by the faces $\{\partial_i \Delta[n] \mid 0 \leq i \leq n \ i \neq k\}$ of the standard n -simplex $\Delta[n]$. A map $f: X \rightarrow Y$ is said to be a trivial fibration if f has the right lifting property with respect to $\dot{\Delta}[n] \rightarrow \Delta[n]$ for $n \geq 0$, where $\dot{\Delta}[n]$ is the simplicial subset generated by the faces of $\Delta[n]$. A map $i: A \rightarrow B$ is said to be a cofibration if it has the left lifting property with respect to trivial fibrations and it is said to be a trivial cofibration if it has the left lifting property with respect to fibrations. A map $f: X \rightarrow Y$ is a weak equivalence if f can be factored as $f = pi$ where i is a trivial cofibration and p is a trivial fibration.

2) The category of topological spaces Top with the Quillen [29] structure. A map $p: E \rightarrow B$ is said to be a fibration if p has the right lifting property with respect to $D^n \rightarrow D^n \times I$, $x \rightarrow (x, 0)$, for $n \geq 0$, where D^n is the standard n -disk, and I denotes the unit interval. A map $f: X \rightarrow Y$ is a weak equivalence if for any $q \geq 0$ and $x \in X$ the induced map $\pi_q(f): \pi_q(X, x) \rightarrow \pi_q(Y, fx)$ is an isomorphism. A map $A \rightarrow X$ is a cofibration if it has the left lifting property with respect to any trivial fibration (fibration and weak equivalence).

3) The category of topological spaces Top with the Strøm [33] structure. A map $p: E \rightarrow B$ is said to be a fibration if it has the right lifting property with respect to the maps $\partial_0: X \rightarrow X \times I$, $x \rightarrow (x, 0)$. A map $i: A \rightarrow X$ is said to be a cofibration if it is a closed map and it has the left lifting property with respect to

$d_0: Y^I \longrightarrow Y, d_0(\alpha) = \alpha(0)$, where Y^I is the standard cocylinder of Y . Finally a map $f: X \longrightarrow Y$ is said to be a weak equivalence if it is a homotopy equivalence.

Let $Ho_{Quillen}(Top)$ and $Ho_{Str}(Top)$ denote the categories of fractions obtained by considering the Quillen structure and the Strøm structure, respectively. If $Ho_{Str}(Top)/CW$ denotes the full subcategory of $Ho_{Str}(Top)$ determined by the spaces that admit a CW decomposition, we have that $Ho_{Str}(Top)/CW$ and $Ho_{Quillen}(Top)$ are equivalent categories.

If $Sin: Top \longrightarrow SS$ is the singular functor and $R: SS \longrightarrow Top$ is the realisation functor, the equivalence above is given by the induced functors

$$inclusion: Ho_{Str}(Top)/CW \longrightarrow Ho_{Quillen}(Top),$$

$$R Sin: Ho_{Quillen}(Top) \longrightarrow Ho_{Str}(Top)/CW.$$

b) *Procategories.*

The category $pro\mathcal{C}$, where \mathcal{C} is a given category, was introduced by A. Grothendieck [17]. Some properties of this category can be seen in the appendix of [1] the monograph [10] or in the books [19] and [9].

The objects of $pro\mathcal{C}$ are functors $X: I \longrightarrow \mathcal{C}$, where I is a small left filtering category and the set of morphisms from $X: I \longrightarrow \mathcal{C}$ to $Y: J \longrightarrow \mathcal{C}$ is given by

$$pro\mathcal{C}(X, Y) = \lim_j \operatorname{colim}_i \mathcal{C}(X_i, Y_j)$$

A morphism from X to Y can be represented by $(\{f_j\}, \varphi)$, where $\varphi: J \longrightarrow I$ is a map and each $f_j: X_{\varphi(j)} \longrightarrow Y_j$ is a morphism of \mathcal{C} such that if $j \longrightarrow j'$ is a morphism of J , there are $i \in I$ and morphisms $i \longrightarrow \varphi(j)$, $i \longrightarrow \varphi(j')$ such that the composite $X_i \longrightarrow X_{\varphi(j)} \longrightarrow Y_j \longrightarrow Y_{j'}$ is equal to the composite $X_i \longrightarrow X_{\varphi(j')} \longrightarrow Y_{j'}$.

The results of this paper will be developed for the category $tow\mathcal{C}$ which is the full subcategory of $pro\mathcal{C}$ determined by the objects indexed by \mathbf{N} the “small category” of non-negative integers.

Edwards and Hastings [10], proved that if \mathcal{C} is a closed model category satisfying some additional condition (condition N), then $pro\mathcal{C}$ and $tow\mathcal{C}$ inherit closed model structures. As a consequence we can use the categories $Ho_{Quillen}(proTop)$, $Ho_{Str}(proTop)$, $Ho_{Quillen}(towTop)$, etc., and the corresponding pointed versions. We also have that $Ho_{Str}(towTop)/towCW$ the full subcategory determined by towers of CW -complexes is equivalent to $Ho_{Quillen}(towTop)$.

In this paper, we will use the comparison theorem of Edwards and Hastings: If \mathcal{C} is a pointed simplicial closed model category, the following sequence is exact

$$\begin{aligned} 0 &\longrightarrow \lim_k^1 \operatorname{colim}_j \{Ho(\mathcal{C})(SX_j, Y_k)\} \\ &\longrightarrow Ho(\operatorname{tow}\mathcal{C})(\{X_i\}, \{Y_k\}) \\ &\longrightarrow \operatorname{tow}Ho(\mathcal{C})(\{X_i\}, \{Y_k\}) \longrightarrow 0. \end{aligned}$$

We also need the closed model structure of $\operatorname{tow}SS$ given by Grossman and the corresponding pointed version in $\operatorname{tow}SS_*$. To see an exact description of these different closed model categories we refer the reader to [10] and [14]. Let $f = \{f_i: X_i \rightarrow Y_i\}_{i \in \mathbf{N}}$ a levelwise map in $\operatorname{tow}SS$ (or $\operatorname{tow}SS_*$). The map f is said to be a strong cofibration if for each $i \in \mathbf{N}$, $f_i: X_i \rightarrow Y_i$ is a cofibration. Similarly, it is defined a strong weak equivalence. The map f is said to be a strong fibration if for each $i \in \mathbf{N}$, $f_i: X_i \rightarrow Y_i$ and the induced map $X_{i+1} \rightarrow X_i \times_{Y_i} Y_{i+1}$ are fibrations. The notion for cofibration given by Edwards and Hastings agrees with the notion given by Grossman. A cofibration is a retract of a strong cofibration. For the closed model structure considered by Edwards and Hastings, the class of weak equivalences is the saturation of the class of strong weak equivalences. Grossman takes as weak equivalences those morphisms which induce isomorphisms in the homotopy progroups, for a more precise definition see [14]. We note that a weak equivalence in the sense of Edwards-Hastings is always a weak equivalence in the sense of Grossman. A Grossman level fibration is a strong fibration $f = \{f_i: X_i \rightarrow Y_i\}$ such that for each i there exists an $n(i)$ such that $\pi_q(X_i) \rightarrow \pi_q(Y_i)$ is an isomorphism for $q > n(i)$. A Grossman fibration is a retract of a level fibration. On the other hand, a fibration in the sense of Edwards and Hastings is a retract of a strong fibration. We note that a fibration in the sense of Grossman is a fibration in the sense of Edwards-Hastings. As a consequence of the relation between the two notions of weak equivalence, the inclusion (identity) functor induces a functor on the localized categories:

$$\operatorname{Inc}: Ho_{\operatorname{Quillen}}(\operatorname{tow}SS) \longrightarrow Ho_{\operatorname{Grossman}}(\operatorname{tow}SS)$$

where $Ho_{\operatorname{Quillen}}(\operatorname{tow}SS)$ denotes the category obtained by considering the Quillen structure on SS and the corresponding Edwards-Hastings extension for $\operatorname{tow}SS$. Given an object X of $\operatorname{tow}SS$, the map $X \rightarrow *$ can be factored as the composite $X \rightarrow R_G X \rightarrow *$, where $X \rightarrow R_G X$ is a Grossman trivial cofibration and $R_G X \rightarrow *$ is a Grossman fibration. The object $R_G X$ can be obtained from $X = \{X_i\}$ by killing higher homotopy groups, $\{cosk_i^R X_i\}$, and replacing bonding maps by bonding fibrations. Thus we have an induced functor

$$R_G: Ho_{\operatorname{Grossman}}(\operatorname{tow}SS) \longrightarrow Ho_{\operatorname{Quillen}}(\operatorname{tow}SS)$$

These functors satisfy that

$$Ho_{Quillen}(towSS)(X, R_G Y) \cong Ho_{Grossman}(towSS)(X, Y).$$

c) *Categories of spaces and proper maps and the Edwards–Hastings embedding.*

Definition 1. *A continuous map $f: X \rightarrow Y$ between topological spaces is said to be proper if for every closed compact subset K of Y , $f^{-1}K$ is a compact subset of X . Two proper maps $f, g: X \rightarrow Y$ are said to be properly homotopic if there is a homotopy $F: X \times I \rightarrow Y$ from f to g which is proper.*

Let P denote the category of Hausdorff, locally compact topological spaces with proper maps. Dividing by proper homotopy relations we have the proper homotopy category $\pi_0(P)$.

We also consider categories of spaces and germs of proper maps, see [10], that can be defined by considering the category of right fractions, see [13], defined by the class Σ of cofinal inclusions. An inclusion $j: A \rightarrow X$ is said to be cofinal if $cl(X - A)$ is a closed compact subset of X . The category $P\Sigma^{-1}$ is also denoted by P_∞ . There is also the corresponding notion of proper homotopy between germs of proper maps and we also have the category of proper homotopy at infinity $\pi_0(P_\infty)$.

A closed map $i: A \rightarrow X$ is said to be a cofibration if it has the proper homotopy extension property. A rayed space (X, α) is a space with a proper map $\alpha: J \rightarrow X$, where $J = [0, +\infty)$ is the half real line. A proper map preserving the ray is said to be a proper map between rayed spaces. It is said that (X, α) is well rayed if $\alpha: J \rightarrow X$ is a cofibration. We denote by P_J the category of well rayed spaces (X, α) where X is in P . In a similar way we can define the category $(P_J)_\infty$ and the corresponding proper homotopy categories $\pi_0(P_J)$, $\pi_0((P_J)_\infty)$.

We will also work with some full subcategories of P , in particular P_σ denotes the full subcategory of Hausdorff, locally compact, σ -compact spaces.

Given a well rayed space (X, α) , Edwards and Hastings defined the end prospace of (X, α) by

$$\varepsilon(X, \alpha) = \{(cl(X - K) \cup \alpha(J), \alpha(0))\}.$$

If X is an object in P_σ , then there is an increasing sequence of compact subsets

$$\emptyset = K_0 \subset \text{int } K_1 \subset K_1 \subset \text{int } K_2 \subset \dots$$

such that $X = \bigcup_0^\infty K_i$. If $X_i = cl(X - K_i)$, the prospace $\varepsilon(X, \alpha)$ is isomorphic to the end tower $\{(X_i \cup \alpha(J), \alpha(0)) \mid i = 0, 1, 2, \dots\}$. Therefore ε is a functor from $((P_\sigma)_J)_\infty \rightarrow \text{towTop}_*$. Edwards and Hastings proved that the induced functor:

$$\varepsilon: \pi_0((P_\sigma)_J)_\infty \rightarrow \text{HoStr}(\text{towTop}_*)$$

is a full embedding.

Notice that for a well rayed space (X, α) in $((P_\sigma)_J)_\infty$ we have the morphism $\alpha: (J, id_J) \rightarrow (X, \alpha)$ that induces a promap $\varepsilon\alpha: \varepsilon(J, id_J) \rightarrow \varepsilon(X, \alpha)$. It is clear that $\varepsilon(J, id_J) = \{\dots \rightarrow (J, 0) \rightarrow (J, 0) \rightarrow (J, 0)\}$. Since the constant map $J \rightarrow *$ is a homotopy equivalence in $\text{Ho}(\text{Top}_*)$, it follows that $\varepsilon(J, id_J) \rightarrow *$ is a weak equivalence in towTop_* . Consider the pushout

$$\begin{array}{ccc} \varepsilon(J, id_J) & \xrightarrow{\varepsilon\alpha} & \varepsilon(X, \alpha) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \varepsilon'(X, \alpha) \end{array}$$

in which $\varepsilon\alpha$ is a cofibration and $\varepsilon(J, id_J) \rightarrow *$ is a weak equivalence. Therefore we obtain that $\varepsilon(X, \alpha) \rightarrow \varepsilon'(X, \alpha)$ is a weak equivalence, where

$$\varepsilon'(X, \alpha) = \{\dots \rightarrow (X_2/\text{ray}, *) \rightarrow (X_1/\text{ray}, *) \rightarrow (X/\text{ray}, *)\}.$$

Consequently, because we work with well rayed spaces we can replace the functor ε by ε' and we also have that $\varepsilon': \pi_0((P_\sigma)_J)_\infty \rightarrow \text{HoStr}(\text{towTop}_*)$ is a full embedding.

It is interesting to consider well rayed spaces X such that εX is isomorphic in $\text{HoStr}(\text{towTop}_*)$ to a tower of CW-complexes X^\dagger . If X and Y are well rayed spaces of this type, then

$$\begin{aligned} \pi_0(((P_\sigma)_J)_\infty)(X, Y) &\cong \text{HoStr}(\text{towTop}_*)(\varepsilon X, \varepsilon Y) \cong \text{HoStr}(\text{towTop}_*)(X^\dagger, Y^\dagger) \\ &\cong \text{HoQuillen}(\text{towTop}_*)(X^\dagger, Y^\dagger) \cong \text{HoQuillen}(\text{towTop}_*)(\varepsilon X, \varepsilon Y). \end{aligned}$$

This means that if we confine ourselves to these spaces in the Edwards-Hastings embedding the Quillen model structure can be used instead of the Strøm structure.

Therefore it will be useful to consider the full subcategory $(CWP_\sigma)_J$ of $(P_\sigma)_J$ determined by well rayed spaces (X, α) such that X admits a CW-decomposition

with $\alpha(J)$ as a subcomplex and X has a sequence of subcomplexes $\{X_i \mid i \geq 0\}$ such that for each $i \geq 0$, $cl(X - X_i)$ is compact, $X_{i+1} \subset intX_i$ and $\bigcap_{i=0}^{\infty} X_i = \emptyset$.

2. A theorem of Freudenthal type for $towTop_*$

First we analyse some properties for cylinders, cocylinders, suspension and loop objects in the category $towTop_*$.

If K is a pointed CW -complex, a cylinder for K is given by the commutative diagram

$$\begin{array}{ccc}
 K \vee K & \xrightarrow{\partial_0 + \partial_1} & K \otimes [0, 1] \\
 \nabla \downarrow & \swarrow p & \\
 & & K
 \end{array}$$

where $K \otimes [0, 1] = K \times [0, 1] / * \times [0, 1]$, $\partial_0 + \partial_1$ is a cofibration and p is a weak equivalence. This diagram is a cylinder for the Strøm structure and for the Quillen structure.

For a tower X of pointed CW -complexes, if we apply levelwise the above decomposition, we obtain a commutative diagram

$$\begin{array}{ccc}
 X \vee X & \xrightarrow{\partial_0 + \partial_1} & X \otimes [0, 1] \\
 \nabla \downarrow & \swarrow p & \\
 & & X
 \end{array}$$

where $\partial_0 + \partial_1$ is a cofibration, p is a weak equivalence in $towTop_*$.

For the Strøm structure of Top_* the remarks above can be extended to towers of well pointed spaces.

Now for a well pointed space L , consider the commutative diagram

$$\begin{array}{ccc}
 L^{[0,1]} & & \\
 s \uparrow & \searrow & (d_0^L, d_1^L) \\
 L & \xrightarrow{\Delta} & L \times L
 \end{array}$$

where $L^{[0,1]}$ is the standard space of continuous maps from $[0, 1]$ to L provided with the compact-open topology. If $l \in L$, $s(l)$ is the constant path equal to l and for $\alpha \in L^{[0,1]}$, $d_0^L(\alpha) = \alpha(0)$, $d_1^L(\alpha) = \alpha(1)$. In this diagram, s is a pointed homotopy equivalence, and (d_0^L, d_1^L) is a pointed Hurewicz fibration (that is, a fibration in the sense of Strøm). Note that (d_0^L, d_1^L) is also a pointed Serre fibration (that is, a fibration in the sense of Quillen). Recall that if L is Hausdorff, then s is also a trivial cofibration in the sense of Strøm.

If X is a tower of well pointed spaces, applying levelwise the decomposition above, we have the diagram

$$\begin{array}{ccc}
 X^{[0,1]} & & \\
 s \uparrow & \searrow & (d_0, d_1) \\
 X & \xrightarrow{\Delta} & X \times X
 \end{array}$$

Each level of the promap (d_0, d_1) is a fibration, but (d_0, d_1) need not be a fibration in the sense of Edwards-Hastings. In order to obtain a cocylinder for X , the morphism (d_0, d_1) can be factored as composition of a trivial cofibration i and a fibration (d''_0, d''_1)

$$\begin{array}{ccc}
 X^{[0,1]} & \xrightarrow{i} & X'' \\
 s \uparrow & \searrow (d_0, d_1) & \downarrow (d''_0, d''_1) \\
 X & \xrightarrow{\Delta} & X \times X,
 \end{array}$$

we also can assume that i is a level morphism and each level is a weak equivalence. Because i is a weak equivalence and (d''_0, d''_1) is a fibration, the diagram above is a cocylinder for X . If X is a tower of well pointed Hausdorff spaces, then i is also a Strøm cofibration.

Now for each $j \geq 0$, consider the diagram

$$\begin{array}{ccccc}
 {}^l\Omega X_j & \longrightarrow & X_j^{[0,1]} & \xrightarrow{(d_0^{X_j}, d_1^{X_j})} & X_j \times X_j \\
 \downarrow & & \downarrow i_j & & \downarrow id \\
 (\Omega X)_j & \longrightarrow & (X'')_j & \xrightarrow{(d''_0, d''_1)_j} & (X \times X)_j
 \end{array}$$

where $(d_0^{X_j}, d_1^{X_j}), (d''_0, d''_1)_j$ are fibrations in Top_* in the sense of Strøm (or Hurewicz), ${}^l\Omega X_j$ is the fibre of $(d_0^{X_j}, d_1^{X_j})$ for each $j \geq 0$ and ΩX is the fibre of (d''_0, d''_1) . By Proposition 5 Ch.I §.3 of [29], we have that ${}^l\Omega X_j \rightarrow (\Omega X)_j$ is a weak equivalence. Therefore if we write ${}^l\Omega X = \{{}^l\Omega X_j\}$, we have that ${}^l\Omega X \rightarrow \Omega X$ is a weak equivalence. Then the loop functor of $Ho(towTop_*)$ can be defined by extending levelwise the loop functor of $Ho(Top_*)$.

Remark. Given $X, Y \in towTop_*$, we shall denote

$$Ho(towTop_*)(X, Y) = [X, Y].$$

The category obtained from $towTop$ dividing by homotopy relations will be denoted by $\pi_0(towTop_*)$. Notice that if X is a tower of well-pointed spaces and Y is fibrant, then

$$Ho_{Str}(towTop_*)(X, Y) \cong \pi_0(towTop_*)(X, Y).$$

If X is a tower of pointed CW-complexes and Y is fibrant in the sense of Quillen, then

$$Ho_{Quillen}(towTop_*)(X, Y) \cong \pi_0(towTop_*)(X, Y).$$

Next we analyse the Freudenthal theorem for the category $Ho_{Quillen}(towTop_*)$ giving enough conditions to obtain an isomorphism $S: [X, Y] \rightarrow [SX, SY]$.

For $q \geq 0$ the functor π_q induces a functor $tow\pi_q$. Then for a given object $X = \{\dots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0\}$ of $towTop_*$, $tow\pi_q X$ denotes the inverse system

$\{\dots \rightarrow \pi_q X_2 \rightarrow \pi_q X_1 \rightarrow \pi_q X_0\}$. Let $Top_*^{\mathbf{N}}$ be the category of towers of pointed spaces and level morphisms; that is, the objects are functors of the form $\mathbf{N} \rightarrow Top_*$ and the morphisms are natural transformations between these functors. It is clear that we have a natural functor $Top_*^{\mathbf{N}} \rightarrow towTop_*$.

Lemma 1. Consider in $Top_*^{\mathbf{N}}$ the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & E \\ \downarrow i & & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

satisfying the following properties:

1) For each $i \geq 0$, X_i is a CW-complex with $\dim X_i \leq n + 1$, X_{i+1} is a subcomplex of X_i , A_i is a subcomplex of X_i and $\bigcap_{i=0}^{\infty} X_i = *$.

2) For each $i \geq 0$, $p_i: E_i \rightarrow B_i$ is a (Serre) fibration and B_i is 0-connected. Suppose also that for $q \leq n$ $tow\pi_q F$ is trivial, where $F = \{F_i\}$ and F_i is the fibre of p_i .

Then there is a morphism $h: X \rightarrow E$ in $towTop_*$ such that $ph = f$ and $hi = g$ (in $towTop_*$).

Proof. Since $tow\pi_q F \cong 0$ for $q \leq n$, we can find a subtower $\{F_{\varphi_i}\}$ such that for every $i \geq 0$ and $q \leq n$, $\pi_q F_{\varphi(i+1)} \rightarrow \pi_q F_{\varphi(i)}$ is trivial. Therefore after reindexing we can assume that we have the additional hypothesis that for $i \geq 0$ and $q \leq n$, $\pi_q(F_{i+1}) \rightarrow \pi_q(F_i)$ is trivial.

The lifting is going to be constructed by induction on the dimension of the skeletons of X . Since each B_i is 0-connected and $p_i: E_i \rightarrow B_i$ is a fibration, it follows that each p_i is a surjective map. Therefore given a 0-cell D^0 in $X_i \setminus (X_{i+1} \cup A_i)$ we can find $h_i(D^0) \in E_i$ such that $p_i h_i(D^0) = f_i(D^0)$. Using the elements $h_j(D^0)$, $j \geq i$ and the bonding maps of E , we can define a coherent level lifting $h_i: X_i^0 \rightarrow E_i$.

In the next step we do not obtain a level lifting, but it is possible to construct a sequence of coherent maps $h_i: X_{i+1}^1 \rightarrow E_i$, $i \geq 0$. Given a 1-cell D^1 in

$X_{i+1} \setminus (X_{i+2} \cup A_{i+1})$, we can apply that $\pi_0(F_{i+1}) \rightarrow \pi_0(F_i)$ is trivial to obtain the desired lifting.

Repeating this argument with higher homotopy groups and taking into account that $\dim X \leq n+1$, we finally obtain a sequence of coherent maps $h_i: X_{i+n+1} \rightarrow E_i$ that defines a lifting $h: X \rightarrow E$ in the category towTop_* .

Lemma 2. *Let $f: Y \rightarrow Z$ be a morphism in towTop_* between towers of pointed 0-connected CW-complexes. If for every $q \geq 0$, f induces an isomorphism $\text{tow}\pi_q f$, then $Sf: SY \rightarrow SZ$ satisfies the same property; that is, $\text{tow}\pi_q Sf$ is isomorphism for $q \geq 0$.*

Proof. Consider the singular functor

$$\text{Sin}: \text{HoQuillen}(\text{towTop}_*) \rightarrow \text{HoQuillen}(\text{towSS}_*)$$

and the “inclusion functor”

$$\text{Inc}: \text{HoQuillen}(\text{towSS}_*) \rightarrow \text{HoGrossman}(\text{towSS}_*).$$

Because *Sin* and *Inc* preserve cofibrations and weak equivalences we have that *Sin* and *Inc* commute with suspensions functors. By the conditions of the hypothesis *Inc Sin(f)* is a Grossman weak equivalence. Therefore the suspension $S(\text{Inc Sin}(f))$ is also a Grossman weak equivalence. However $S(\text{Inc Sin}(f)) = \text{Inc Sin}(Sf)$. This implies that $Sf: SY \rightarrow SZ$ is such that $\text{tow}\pi_q Sf$ is an isomorphism for $q \geq 0$.

Given a tower of CW-complexes $X = \{\cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0\}$ we denote $\dim X = \sup\{\dim X_i \mid i \geq 0\}$.

Lemma 3. *Let $f: Y \rightarrow Z$ be a morphism in towTop_* between towers of 0-connected spaces such that $\text{tow}\pi_q f$ is an isomorphism for $0 \leq q < 2n - 1$ and epimorphism for $q = 2n - 1$. Assume that X is a tower of pointed CW-complexes such that for each $i \geq 0$, X_{i-1} is a subcomplex of X_i and $\bigcap_{i=0}^{\infty} X_i = *$. Then if $\dim X < 2n - 1$, $f_*: [X, Y] \rightarrow [X, Z]$ is an isomorphism and if $\dim X \leq 2n - 1$, f_* is an epimorphism.*

Proof. Because $\text{Top}_*^{\mathbb{N}}$ has a closed model structure, see [10], we can consider a

commutative diagram

$$\begin{array}{ccc}
 Y & \xrightarrow{u} & E \\
 f \downarrow & & \downarrow p \\
 Z & \xrightarrow{v} & B
 \end{array}$$

where E, B are fibrant, p is a level morphism and each $p_i: E_i \rightarrow B_i$ is a fibration and u and v are weak equivalences.

Let F_i denote the fibre of $p_i: E_i \rightarrow B_i$ and $F = \{F_i\}$. From the hypothesis conditions on $\text{tow}\pi_q f$ we have that $\text{tow}\pi_q F \cong 0$ for $q \leq 2n - 2$. Since u, v are weak equivalences the map $[X, Y] \xrightarrow{f_*} [X, Z]$ is isomorphic to $[X, E] \xrightarrow{p_*} [X, B]$. Since X is cofibrant and E, B are fibrant $[X, E]$ and $[X, B]$ can be realised as sets of homotopy classes. Now from Lemma 1, we have that if $\dim X < 2n - 1$, p_* is an isomorphism and if $\dim X \leq 2n - 1$, then p_* is an epimorphism.

Lemma 4. *Given a tower of pointed spaces Y such that $\text{tow}\pi_q Y \cong 0$ for $q \leq n - 1$ ($n \geq 1$), there is a tower of pointed 0-connected CW-complexes Y' and a level morphism $Y' \rightarrow Y$ such that $\text{tow}\pi_q Y' \rightarrow \text{tow}\pi_q Y$ is an isomorphism for $q \geq 0$ and for each $i \geq 0$ and $q \leq n - 1$, $\pi_q(Y'_i) \cong 0$.*

Proof. Given $Y = \{Y_i\}$, we can apply the singular functor $\text{Sin}: \text{towTop}_* \rightarrow \text{towSS}_*$, $\text{Sin}Y = \{\text{Sin}Y_i\}$ and by considering the coskeleton functor $\text{cosk}_n: \text{towSS}_* \rightarrow \text{towSS}_*$, we have the level morphism

$$\{\text{Sin}Y_i\} \rightarrow \{\text{cosk}_n \text{Sin}Y_i\}.$$

Let $F_i \rightarrow \text{Sin}Y_i \rightarrow \text{cosk}_n \text{Sin}Y_i$ denote the homotopy fibre of $\text{Sin}Y_i \rightarrow \text{cosk}_n \text{Sin}Y_i$. We have that $\pi_q F_i = 0$ for $q \leq n - 1$ and $\pi_q F_i \rightarrow \pi_q \text{Sin}Y_i$ is an isomorphism for $q \geq n$. This implies that $\text{tow}\pi_q \{F_i\} \rightarrow \text{tow}\pi_q \{\text{Sin}Y_i\}$ is an isomorphism for $q \geq 0$. Applying now the realization functor R we have that the morphism

$$\{RF_i\} \rightarrow \{R\text{Sin}Y_i\} \rightarrow \{Y_i\}$$

induces isomorphisms on $\text{tow}\pi_q$ for $q \geq 0$ and for each $i \geq 0$ and $q \leq n - 1$, $\pi_q(RF_i) \cong 0$. Then defining $Y'_i = RF_i$, $Y' = \{Y'_i\}$ is the desired tower.

Theorem 1. *Let X be a tower of pointed CW -complexes such that for each $i \geq 0$, X_{i+1} is a subcomplex of X_i and $\bigcap_{i=0}^{\infty} X_i = *$. Suppose also that Y is a tower of pointed CW -complexes such that $\text{tow}\pi_q Y \cong 0$ for $q \leq n - 1$. Then if $\dim X < 2n - 1$, the suspension map*

$$S: [X, Y] \longrightarrow [SX, SY]$$

is an isomorphism and if $\dim X \leq 2n - 1$, then S is an epimorphism.

Proof. Since $\text{tow}\pi_0 Y \cong 0$, Y can be considered up to isomorphism in towTop_* as a tower of pointed 0-connected CW -complexes. By Lemma 4, there is a tower Y' and a morphism $f: Y' \longrightarrow Y$ such that $\text{tow}\pi_q Y' \longrightarrow \text{tow}\pi_q Y$ is an isomorphism for $q \geq 0$ and such that $\pi_q Y'_i = 0$ for $i \geq 0$ and $q \leq n - 1$. By Lemma 2, $Sf: SY' \longrightarrow SY$ induces an isomorphism $\text{tow}\pi_q Sf$ for $q \geq 0$. Now consider the commutative diagram

$$\begin{array}{ccc} [X, Y'] & \xrightarrow{f_*} & [X, Y] \\ \downarrow & & \downarrow \\ [SX, SY'] & \xrightarrow{(Sf)_*} & [SX, SY]. \end{array}$$

By Lemma 3, f_* and $(Sf)_*$ are isomorphisms. By the Freudenthal theorem for standard homotopy we have that for $i \geq 0$, $\pi_q Y'_i \longrightarrow \pi_q \Omega SY'_i$ is an isomorphism for $q < 2n - 1$ and an epimorphism for $q = 2n - 1$. Therefore $\text{tow}\pi_q Y' \longrightarrow \text{tow}\pi_q \Omega SY'$ is an isomorphism for $q < 2n - 1$ and an epimorphism for $q = 2n - 1$.

Applying Lemma 3, we have that if $\dim X < 2n - 1$, $[X, Y'] \longrightarrow [X, \Omega SY']$ is an isomorphism and if $\dim X = 2n - 1$, then $[X, Y'] \longrightarrow [X, \Omega SY']$ is an epimorphism. This is equivalent to saying that if $\dim X < 2n - 1$, $[X, Y'] \longrightarrow [SX, SY']$ is an isomorphism and if $\dim X \leq 2n - 1$, then $[X, Y'] \longrightarrow [SX, SY']$ is an epimorphism.

Finally taking into account that f_* and $(Sf)_*$ are isomorphisms, we obtain the thesis of the theorem.

Lemma 5. *Let X be a tower of pointed CW -complexes. Then there is a tower X' of pointed CW -complexes such that the bonding morphisms of X' are cellular inclusions, $\bigcap_{i=0}^{\infty} X'_i = *$, $\dim X' = \dim X + 1$ and $X \cong X'$ in $\text{Ho}(\text{towTop}_*)$.*

Proof. Let $X = \{\dots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0\}$ be a tower of pointed CW -complexes. In order to have cellular bonding maps we can apply the cellular approximation theorem, so for each j there is a homotopy $F_j: X_j \otimes [0, 1] \rightarrow X_{j-1}$ such that $F_j \partial_0^{X_j} = X_{j-1}^j$ and $F_j \partial_1^{X_j} = \bar{X}_{j-1}^j$, where \bar{X}_{j-1}^j is a cellular map. By considering the commutative diagram

$$\begin{array}{ccccccc}
 \dots & \rightarrow & X_2 & \xrightarrow{X_1^2} & X_1 & \xrightarrow{X_0^1} & X_0 \\
 & & \partial_0^{X_2} \downarrow & & \partial_0^{X_1} \downarrow & & \partial_0^{X_0} \downarrow \\
 \dots & \rightarrow & X_2 \otimes [0, 1] & \xrightarrow{(F_2, pr_2)} & X_1 \otimes [0, 1] & \xrightarrow{(F_1, pr_1)} & X_0 \otimes [0, 1] \\
 & & \partial_1^{X_2} \uparrow & & \partial_1^{X_1} \uparrow & & \partial_1^{X_0} \uparrow \\
 \dots & \rightarrow & X_2 & \xrightarrow{\bar{X}_1^2} & X_1 & \xrightarrow{\bar{X}_0^1} & X_0
 \end{array}$$

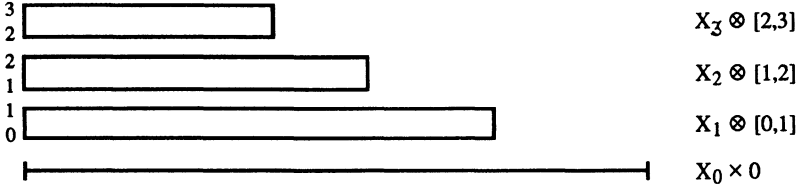
where $(F_j, pr_2)(x, t) = (F_j(x, t), t)$, we have that $\{\partial_0^{X_j}\}$ and $\{\partial_1^{X_j}\}$ are weak equivalences. Therefore $\{\dots \rightarrow X_2 \xrightarrow{X_1^2} X_1 \xrightarrow{X_0^1} X_0\}$ is isomorphic to $\{\dots \rightarrow X_2 \xrightarrow{\bar{X}_1^2} X_1 \xrightarrow{\bar{X}_0^1} X_0\}$ in $Ho(towTop_*)$, and now the bonding maps are cellular.

Recall that given a cellular map $f: X \rightarrow Y$ between pointed CW -complexes, the cylinder for f defined by the pushout

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \partial_0 \downarrow & & \downarrow \\
 X \otimes [0, 1] & \longrightarrow & (X \otimes [0, 1]) \cup_X Y
 \end{array}$$

admits a CW -complex structure.

Given a tower $X = \{\dots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0\}$ we can suppose that the bonding morphisms are cellular. By considering the cylinders for the bonding maps the following telescope TX can be constructed, see [10; page 115].



For $i \geq 0$, define X'_i as the quotient of

$$(X_i \times \{i\}) \cup \bigcup_{k=i+1}^{\infty} (X_k \otimes [k-1, k])$$

defined by the relations

$$(x, k) \sim (X_{k-1}^k x, k), \quad k \geq i+1, \quad x \in X_k$$

Notice that we have a natural sequence of cellular inclusions

$$\dots \rightarrow X'_2 \rightarrow X'_1 \rightarrow X'_0$$

such that $\bigcap_{i=0}^{+\infty} X'_i = *$. We also have the maps $in_i: X_i \rightarrow X'_i$, $in_i(x) = (x, i)$ and $pr_i: X'_i \rightarrow X_i$ defined by $pr_i(x, t) = X_i^t x$ if $x \in X_k$ ($k \geq i$).

It is easy to check that $in: X \rightarrow X'$ and $pr: X' \rightarrow X$ are levelwise morphisms, $pr \circ in = id_X$ and for each $i \geq 0$, $in_i, pr_i \simeq id_{X'_i}$. Therefore in and pr are weak equivalences in $topTop_*$. We also have, that if $dim X \leq n$, then $dim X' \leq n+1$.

Applying the last result and Theorem 1, we have the following version of the Freudenthal theorem for more general towers:

Theorem 2. *Let X, Y be towers of pointed CW-complexes such that $town\pi_q Y \cong 0$ if $q \leq n-1$ and $dim X < 2n-2$, then $S: [X, Y] \rightarrow [SX, SY]$ is an isomorphism. If $dim X \leq 2n-2$, S is an epimorphism.*

Notice that the dimension condition for a general tower of pointed CW-complexes is stronger than the condition for a tower of pointed CW-complexes in which the bonding morphisms are cellular inclusions and the inverse limit is trivial.

Examples. Let S^n be the standard n -dimensional sphere. We also denote by S^n the constant tower $\{\dots \rightarrow S^n \xrightarrow{id} S^n \xrightarrow{id} S^n\}$. The Grossman n -sphere Σ^n is the

tower defined by $\Sigma^n(k) = \bigvee_{i \geq k} S_i^n$, where $S_i^n = S^n$ and the bonding maps are given by inclusions.

From the comparison theorem of Edwards and Hastings, we have that

$$[S^n, \Sigma^n] \cong \lim_k^1 \pi_{n+1}(\bigvee_{i \geq k} S_i^n).$$

For $n = 1$, because $\pi_2(\bigvee_{i \geq k} S_i^1) = 0$, we have that $[S^1, \Sigma^1] = 0$. For $n = 2$, it is easy to check that $\bigoplus_{i \geq k} \pi_3(S_i^2)$ is a retract of $\pi_3(\bigvee_{i \geq k} S_i^2)$ and because the retractions commute with the bonding morphisms, it follows that $\{\bigoplus_{i \geq k} \pi_3(S_i^2)\}$ is a retract of $\{\pi_3(\bigvee_{i \geq k} S_i^2)\}$ in *topGps*. As a consequence of the functorial properties of \lim^1 we also have that $\lim_k^1 \bigoplus_{i \geq k} \pi_3(S_i^2)$ is a retract of $\lim_k^1 \pi_3(\bigvee_{i \geq k} S_i^2)$. On the other hand, it is easy to check that $\lim_k^1 \bigoplus_{i \geq k} \pi_3(S_i^2) \cong \prod_{i \in \mathbf{N}} \pi_3(S_i^2) / \bigoplus_{i \in \mathbf{N}} \pi_3(S_i^2) \neq 0$. Then $\lim_k^1 \pi_3(\bigvee_{i \geq k} S_i^2)$ is non trivial and $[S^2, \Sigma^2] \neq 0$.

For $n > 2$, it follows that $\bigoplus_{i \in \mathbf{N}} \pi_{n+1}(S_i^n)$ is isomorphic to $\pi_{n+1}(\bigvee_{i \in \mathbf{N}} S_i^n)$. Therefore we have that $[S^n, \Sigma^n] \cong \prod_{i \in \mathbf{N}} \pi_{n+1}(S_i^n) / \bigoplus_{i \in \mathbf{N}} \pi_{n+1}(S_i^n)$.

From these facts we can conclude that:

- 1) The suspension morphism $[S^1, \Sigma^1] \longrightarrow [S^2, \Sigma^2]$ is not surjective. ($n = 1$, $dim = 2n - 1$).
- 2) $[S^2, \Sigma^2] \longrightarrow [S^3, \Sigma^3]$ is a surjective map and it is not injective ($n = 2$, $dim = 2n - 2$). Notice that the suspension morphism $\pi_3(S^2) \longrightarrow \pi_4(S^3)$ induces the morphisms

$$\begin{aligned} \prod_{i \in \mathbf{N}} \pi_3(S_i^2) / \bigoplus_{i \in \mathbf{N}} \pi_3(S_i^2) &\xrightarrow{\varphi} \prod_{i \in \mathbf{N}} \pi_4(S_i^3) / \bigoplus_{i \in \mathbf{N}} \pi_4(S_i^3), \\ \lim_k^1 \pi_3(\bigvee_{i \geq k} S_i^2) &\xrightarrow{\psi} \lim_k^1 \pi_4(\bigvee_{i \geq k} S_i^3) \end{aligned}$$

Therefore $ker\varphi$ is a retract of $ker\psi$. Because $\pi_3(S^2) \longrightarrow \pi_4(S^3)$ has no zero kernel, it follows that $ker\psi \neq 0$.

- 3) For $n > 2$, $[S^n, \Sigma^n] \longrightarrow [S^{n+1}, \Sigma^{n+1}]$ is an isomorphism ($dim < 2n - 1$).

We also have the usual dimension condition if Y is a movable tower. Recall that a tower $Y = \{Y_i\}$ is movable if for each i there exists a $k > i$ such that for

each $l > k$ there exists a map $\varphi_{kl}: Y_k \rightarrow Y_l$ such that $\varphi_{li}\varphi'_{kl} = \varphi_{kl}$, where φ_{ij} are the bonding maps.

Proposition 1. *Let X, Y be towers of pointed CW-complexes and assume that Y is movable. If $\text{tow}\pi_q Y = 0$ for $q \leq n - 1$, and $\dim X < 2n - 1$ then $S: [X, Y] \rightarrow [SX, SY]$ is an isomorphism and if $\dim X \leq 2n - 1$, S is an epimorphism.*

Proof. If we apply the comparison theorem of Edwards and Hastings we have the following exact sequence

$$0 \longrightarrow \lim_i^1 \text{colim}_j [SX_j, Y_i] \longrightarrow [X, Y] \longrightarrow \text{towHo}(\text{Top}_*)(X, Y) \longrightarrow 0.$$

Since Y is a movable tower, it follows that $\{\text{colim}_j [SX_j, Y_i]\}_{i \in \mathbb{N}}$ is a movable tower of groups, then it is satisfied the Mittag-Leffler condition and $\lim_i^1 \text{colim}_j [SX_j, Y_i] = 0$. Therefore $[X, Y] \cong \lim_i \text{colim}_j [X_j, Y_i]$. Similarly, $[SX, SY] \cong \lim_i \text{colim}_j [SX_j, SY]$.

Now the Freudenthal suspension theorem for standard homotopy implies the Freudenthal theorem for towers under the conditions of Proposition 1.

Corollary 1. *Let X be a tower of pointed CW-complexes and let S^n be the constant tower ($S^n = \{\dots \rightarrow S^n \xrightarrow{\text{id}} S^n\}$). If $\dim X < 2n - 1$, then $S: [X, S^n] \rightarrow [SX, S^{n+1}]$ is an isomorphism and if $\dim X \leq 2n - 1$, S is an epimorphism.*

3. Applications.

We are going to see that the suspension theorem for towers of spaces implies suspension theorems for proper homotopy and strong shape theories.

a) *A suspension theorem on proper homotopy theory.*

The categories $(P_\sigma)_J$ and $((P_\sigma)_J)_\infty$, defined in section 1, have the structure of a cofibration category, see [2, 3, 7]. Therefore a suspension functor S can be defined as follows:

For a given well rayed space (X, α) of P_σ , consider the pushout

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & X \\ \alpha \downarrow & & \downarrow \\ X & \longrightarrow & X \cup_J X \end{array}$$

and the proper map $X \cup_J X \xrightarrow{id_X + id_X} X$. This map can be factored as the composite

$$X \cup_J X \xrightarrow{i} I_J X \xrightarrow{p} X$$

where i is a cofibration and p is a weak equivalence.

If $\alpha: J \rightarrow X$ is a cofibration and X is a Hausdorff, locally compact, σ -compact space, we can apply Uryshon's lemma to obtain a proper map $r: X \rightarrow J$ such that $r\alpha = id_J$. We are going to use this map r to define the proper suspension, however we also show that if we consider a different proper map $r': X \rightarrow J$ the proper suspension induced by r' has the same proper homotopy type as the suspension induced by r .

Given a well rayed space (X, α) the proper suspension, SX , is defined by considering the following composition of pushout diagrams

$$\begin{array}{ccccc} X \cup_J X & \xrightarrow{id+id} & X & \xrightarrow{r} & J \\ \downarrow & & \downarrow & & \downarrow \\ I_J X & \longrightarrow & S_J X & \longrightarrow & SX \end{array}$$

If we consider well rayed spaces, all vertical morphism of diagram above are cofibrations. Since $X \rightarrow S_J X$ is a cofibration and because two proper maps $r, r': X \rightarrow J$ are always properly homotopic, we have that the suspension induced by r is properly equivalent to the suspension induced by r' .

We are going to restrict ourselves to the full subcategory of $(P_\sigma)_J$ determined by well rayed spaces (X, α) such that X has a CW -decomposition, $\alpha(J)$ is a subcomplex and X has a sequence of subcomplexes $\{X_i \mid i \geq 0\}$ such that for each

$i \geq 0$, $cl(X - X_i)$ is compact, $X_{i+1} \subset intX_i$ and $\bigcap_{i=0}^{\infty} X_i = \emptyset$. We denote this full subcategory by $(CWP_{\sigma})_J$.

Consider the Edwards–Hastings embeddings

$$\begin{aligned} \varepsilon_g: \pi_0((CWP_{\sigma})_J) &\longrightarrow Ho_{Quillen}(towTop_{*}, Top_{*}), \\ \varepsilon_{\infty}: \pi_0((CWP_{\sigma})_J)_{\infty} &\longrightarrow Ho_{Quillen}(towTop_{*}) \end{aligned}$$

defined by $X \longrightarrow \{X_i/ray\}$, and recall the following result proved in [7].

Theorem 1. *For a given object X of $(CWP_{\sigma})_J$ there exist isomorphisms*

$$\begin{aligned} \varepsilon_g SX &\longrightarrow S\varepsilon_g X \\ \varepsilon_{\infty} SX &\longrightarrow S\varepsilon_{\infty} X \end{aligned}$$

in the categories $Ho(towTop_{}, Top_{*})$ and $Ho(towTop_{*})$, respectively.*

In 1975, Brown [6] defined the spheric object ${}^B S^n$ by attaching an n -sphere at each integer of the semiopen interval $[0, +\infty)$. The Brown proper homotopy groups are defined by

$$\begin{aligned} {}^B \pi_n^g(X) &= \pi_0((P_{\sigma})_J)({}^B S^n, X), \\ {}^B \pi_n^{\infty}(X) &= \pi_0(((P_{\sigma})_J)_{\infty})({}^B S^n, X) \end{aligned}$$

for a rayed space X . Brown also defined functors $towSet_{*} \xrightarrow{\mathcal{P}} Set_{*}$, $towGps \xrightarrow{\mathcal{P}} Gps$ such that

$$\mathcal{P}tow\pi_q\varepsilon_{\infty}X \cong {}^B \pi_q^{\infty}(X), \quad q \geq 0.$$

There also are \mathcal{P}_g functors defined for the global case that satisfy similar properties.

Grossman, proved that the \mathcal{P} functors reflect isomorphisms, therefore we have:

Lemma 1. *If X is a well rayed space the following conditions are equivalent*

- a) $tow\pi_q\varepsilon_{\infty}X \cong 0 \quad q \leq N$,
- b) ${}^B \pi_q^{\infty}X \cong 0 \quad q \leq N$.

There is a similar global lemma. Therefore we have the following definition:

Definition 1. A well rayed space X is said to be N -connected at infinity if $\text{tow} \pi_q \varepsilon X \cong 0$ for $q \leq N$ or equivalently ${}^B \pi_q^\infty(X) \cong 0$ for $q \leq N$. It is said to be N -connected if the corresponding global condition is satisfied.

Now we can establish the following proper suspension theorem, where we have denoted $\pi_0((P_\sigma)_J)_\infty(X, Y)$ by $[X, Y]_\infty$.

Theorem 2. Given X, Y spaces in $(CWP_\sigma)_J$ and suppose that Y is $(n - 1)$ -connected at infinity ($n \geq 1$). Then if $\dim X < 2n - 1$, the natural map

$$S: [X, Y]_\infty \longrightarrow [SX, SY]_\infty$$

is an isomorphism and if $\dim X = 2n - 1$, S is an epimorphism (Of course we also have a similar theorem for the global case).

Proof. We can consider the commutative diagram

$$\begin{array}{ccc} [X, Y]_\infty & \xrightarrow{\varepsilon_\infty} & [\varepsilon_\infty X, \varepsilon_\infty Y] \\ \downarrow s & & \downarrow s \\ [SX, SY]_\infty & \xrightarrow{\varepsilon_\infty} [\varepsilon_\infty SX, \varepsilon_\infty SY] \cong [S\varepsilon_\infty X, S\varepsilon_\infty Y] & \end{array}$$

By the Edwards–Hastings embedding, the maps of type ε_∞ are isomorphisms and by Theorem 1, $[\varepsilon_\infty SX, \varepsilon_\infty SY] \cong [S\varepsilon_\infty X, S\varepsilon_\infty Y]$. Because $\varepsilon_\infty X$ and $\varepsilon_\infty Y$ are under the conditions of Theorem 2.1, we have that $[\varepsilon_\infty X, \varepsilon_\infty Y] \longrightarrow [S\varepsilon_\infty X, S\varepsilon_\infty Y]$ is an isomorphism (or epimorphism). Therefore we conclude that $[X, Y]_\infty \longrightarrow [SX, SY]_\infty$ is an isomorphism (or epimorphism).

Corollary 1. Let Y be a rayed space in $(CWP_\sigma)_J$ and assume that Y is $(n - 1)$ -connected. Then the suspension ${}^B \pi_q^\infty(Y) \longrightarrow {}^B \pi_{q+1}^\infty(SY)$ is an isomorphism if $q < 2n - 1$ and an epimorphism for $q = 2n - 1$ (similarly for the global case).

Remark. Of particular interest are the epimorphisms

$${}^B \pi_1^\infty(BS^1) \longrightarrow {}^B \pi_2^\infty(BS^2),$$

$${}^B\pi_3^\infty({}^B S^2) \longrightarrow {}^B\pi_4^\infty({}^B S^3)$$

and the corresponding global versions

$${}^B\pi_1^g({}^B S^1) \longrightarrow {}^B\pi_2^g({}^B S^2),$$

$${}^B\pi_3^g({}^B S^2) \longrightarrow {}^B\pi_4^g({}^B S^3).$$

In the first case, ${}^B\pi_1^\infty({}^B S^1)$ has a natural near-ring structure, see [18], and ${}^B\pi_2^\infty({}^B S^2)$ a natural ring structure. The suspension is the natural near-ring morphism. The second case is the proper version of the standard morphism $\pi_3 S^2 \longrightarrow \pi_4 S^3$.

We also obtain a suspension theorem for the strong (or Steenrod, Čerin) proper homotopy groups, which are defined by using sphere objects of the form ${}^S S^q = S^q \times [0, \infty)$. Because $\dim {}^S S^q = q + 1$, we have:

Corollary 2. *Let Y be a rayed space in $(CWP_\sigma)_J$ and assume that Y is $(n - 1)$ -connected. Then the suspension morphism*

$$[{}^S S^q, Y]_\infty \longrightarrow [{}^S S^{q+1}, SY]_\infty$$

is an isomorphism if $q < 2n - 2$ and epimorphism if $q = 2n - 2$.

b) *The suspension theorem for strong shape theory.*

First we recall the definition of the Čech nerve $C(X)$ of a space and the Vietoris nerve $V(X)$. This second Vietoris functor was first introduced by Porter [20].

Given a pointed space X with a base point x we can consider the directed set $covX$. An element of $covX$ is an open covering \mathcal{U} of X with a distinguished $U \in \mathcal{U}$ such that $x \in U$. Given a pointed space X and a pointed open covering \mathcal{U} , $CX_{\mathcal{U}}$ denotes a pointed simplicial set such that a typical n -simplex is given by (U_0, \dots, U_n) where $U_0, \dots, U_n \in \mathcal{U}$ and $U_0 \cap \dots \cap U_n \neq \emptyset$. This defines a functor $C: Top_* \longrightarrow proHo(SS_*)$.

If \mathcal{U} is a pointed open covering of the pointed space X , the Vietoris nerve of \mathcal{U} , $VX_{\mathcal{U}}$ is the pointed simplicial set in which an n -simplex is an ordered $(n + 1)$ -tuple (x_0, \dots, x_{n+1}) of points contained in an open set $U \in \mathcal{U}$. One important difference with the Čech nerve is that if \mathcal{U}' refines \mathcal{U} there is a canonical map $VX_{\mathcal{U}'} \longrightarrow VX_{\mathcal{U}}$, in the case of the Čech nerve the corresponding map is only determined up to homotopy, i.e. in $Ho(SS_*)$.

Therefore we are going to consider the functors

$$C: Top_* \longrightarrow proHo(SS_*),$$

$$V: Top_* \longrightarrow proSS_*$$

and their compositions with the realization functor

$$RC: Top_* \longrightarrow proHo(Top_*),$$

$$RV: Top_* \longrightarrow proTop_*.$$

We shall use the Dowker theorem [10, page 251] that shows that for a (pointed) covering of X , $RCX_{\mathcal{U}}$ is canonically homotopy equivalent to $RVX_{\mathcal{U}}$.

The following notion of dimension, see Spanier [31], will be used. Given a space X , it is said to be $dim X \leq n$ if for every open covering \mathcal{U} of X there is an open covering \mathcal{U}' such that \mathcal{U}' refines \mathcal{U} and $CX_{\mathcal{U}'}$ is a simplicial set such that any nondegenerate simplex in $CX_{\mathcal{U}'}$ is of dimension at most n .

It is not difficult to check that if X is a compact metrisable space, then there is a cofinal sequence $\dots, \mathcal{U}_2, \mathcal{U}_1, \mathcal{U}_0$ of open coverings in $cov X$. Therefore $\{CX_{\mathcal{U}}\}$ is isomorphic to $\{CX_{\mathcal{U}_i}\}$ in $proHo(SS_*)$ and $\{VX_{\mathcal{U}}\}$ is isomorphic to $\{VX_{\mathcal{U}_i}\}$ in $proSS_*$.

We also have that if X is a compact metrisable space and $dim X \leq n$, then there is a cofinal sequence of open coverings $\dots, \mathcal{U}_2, \mathcal{U}_1, \mathcal{U}_0$ such that for $i \geq 0$ the simplicial set $CX_{\mathcal{U}_i}$ is of dimension at most n .

Therefore for compact metrisable pointed spaces we have functors

$$RC: CM_* \longrightarrow towHo(Top_*),$$

$$RV: CM_* \longrightarrow towTop_*$$

and we can define the pointed shape category and pointed strong category by considering the full embeddings

$$RC: Sh(CM_*) \longrightarrow towHo(Top_*),$$

$$RV: StSh(CM_*) \longrightarrow Ho(towTop_*).$$

Using the functor $Ho(towTop_*) \longrightarrow towHo(Top_*)$ and the Dowker Theorem, we have that for a compact metrisable space X , RCX and RVX are isomorphic in $towHo(Top_*)$. Applying Theorem 5.2.9 of [10] we also have that RCX and RVX are also isomorphic in $Ho(towTop_*)$.

Using this isomorphism we have that if X is a compact metrisable space and $\dim X \leq n$, then RVX is isomorphic to a tower of finite CW -complexes with dimensions less than or equal to n .

We also need Proposition 8.3.20 of [10] that asserts that the functor V commutes with the suspension.

For a given compact metrisable pointed space X , we have the natural functor $\text{tow}\pi_q RVX$ and the Quigley inward group ${}^Q\pi_q(X)$ that is obtained from $\text{tow}\pi_q RVX$ by the \mathcal{P} functor ${}^Q\pi_q^I(X) = \mathcal{P}\text{tow}\pi_q RVX$. It is said that X is shape n -connected if $\text{tow}\pi_q RVX \cong 0$ for $q \leq n$ or equivalently ${}^Q\pi_q^I(X) \cong 0$ for $q \leq n$.

Theorem 2. *Given X, Y compact metrisable pointed spaces and suppose that Y is shape $(n - 1)$ -connected. Then if $\dim X < 2n - 2$, the natural map*

$$S: StSh_*(X, Y) \longrightarrow StSh_*(SX, SY)$$

is an isomorphism and if $\dim X = 2n - 2$, S is an epimorphism.

Proof. Consider the following diagram

$$\begin{array}{ccc} StSh_*(X, Y) & \longrightarrow & Ho(\text{towTop}_*)(RVX, RVY) \\ \downarrow s & & \downarrow s \\ StSh_*(SX, SY) & \longrightarrow & Ho(\text{towTop}_*)(SRVX, SRVY) \end{array}$$

Because the realization functor and the Vietoris functor commute with the suspension, it follows that the diagram is commutative.

Since RVX is isomorphic in $Ho(\text{towTop}_*)$ to a tower of CW -complexes of dimension less than or equal to $\dim X$, we can apply Theorem 2.2, to obtain that S is an isomorphism if $\dim X < 2n - 2$ or an epimorphism if $\dim X = 2n - 2$.

References.

- [1] A. Artin and B. Mazur, "Étale Homotopy", Lect. Notes in Math., n 100, Springer, 1969.
- [2] H.J. Baues, "Algebraic homotopy", Cambridge Univ. Press, 1988.
- [3] H.J. Baues, "Foundations of proper homotopy theory", preprint 1992.
- [4] A.K. Bousfield and D.M. Kan, "Homotopy limits, Completions and Localizations", Lecture Notes in Maths. 304, Springer-Verlag, 1972.
- [5] K. Borsuk, "Concerning homotopy properties of compacta", Fund. Math. 62 (1968) 223-254.
- [6] E.M. Brown, "On the proper homotopy type of simplicial complexes" Lect. Notes in Math., n 375 (1975).
- [7] J.Cabeza, M.C. Elvira and L.J. Hernández, "Una categoría cofibrada para las aplicaciones propias", Actas del las XIV Jorn. Mat. Hispano-Lusas., sección Geometría y Topología (1989), Univ. de La Laguna.
- [8] Z. Čerin, "On various relative proper homotopy groups", Tsukuba J. Math., vol 4, 2 (1980) 177-202.
- [9] J.M. Cordier and T. Porter, "Shape Theory. Categorical methods of approximation", Ellis Horwood Series in Math. and its Appl., 1989.
- [10] D. Edwards and H. Hastings, "Čech and Steenrod homotopy theories with applications to Geometric Topology", Lect. Notes Math., 542, Springer, 1976.
- [11] C. Elvira, "Aplicaciones propias de X en \mathbb{R}^{n+1} ", Actas del IV Sem. de Topología, Bilbao (1988).
- [12] H. Freudenthal, "Über die Klassen der Sphärenabbildungen, Compositio Math., 5 (1937) 299-314.
- [13] P. Gabriel and M. Zisman, "Calculus of fractions and homotopy theory", Springer, Berlin (1966).
- [14] J.W. Grossman, "A homotopy theory of pro-spaces", Trans. Amer. Math. Soc., 201 (1975) 161-176.

- [15] J.W. Grossman, "Homotopy classes of maps between pro-spaces", *Michigan Math. J.* 21 (1974) 355-362.
- [16] J.W. Grossman, "Homotopy groups of Pro-spaces", *Illinois J. Math.* 20 (1976) 622-625.
- [17] A. Grothendieck, "Technique De Descente Et Théorèmes d'Existence En Géométrie Algèbrique I-IV", *Seminar Bourbaki, Exp.* 190, 195, 212, 221, 1959-60, 1960-61.
- [18] J.D.P. Meldrum, "Near-rings and their links with groups", *Pitman Publishing Inc*, 1985.
- [19] S. Mardešić and J. Segal, "Shape theory", *North-Holland*, 1982.
- [20] T. Porter, "Čech Homotopy I", *J. London Math. Soc.* 90 (1974) 8-26.
- [21] T. Porter, "Stability Results for Topological spaces", *Math. Z.* 140 (1974) 1-21.
- [22] T. Porter, "Abstract homotopy theory in procategories", *Cahiers de topologie et geometrie differentielle*, vol 17 (1976) 113-124.
- [23] T. Porter, "Coherent prohomotopical algebra", *Cahiers de topologie et geometrie differentielle*", vol 18 (1977) 139-179.
- [24] T. Porter, "Coherent pro-homotopy theory", *Cahiers de topologie et geometrie differentielle*", vol 19 n 1 (1978) 3-45.
- [25] T. Porter, "Čech and Steenrod homotopy and the Quigley exact couple in strong shape and proper homotopy Theory", *J. Pure Apl. Alg.* 24 (1983) 303-312.
- [26] T. Porter, "Homotopy groups for strong shape and proper homotopy", *Convegno di Topologia serie II n 4* (1984) 101-113.
- [27] T. Porter, "Proper homotopy, prohomotopy and coherence", *U.C.N.W. Pure Maths Preprint* 86.20.
- [28] T. Porter, "On the two definitions of $\text{Ho}(\text{pro}C)$ ", *Top. and its Appl.* 28 (1988) 289-293.
- [29] D. Quillen, "Homotopical Algebra", *Lect. Notes in Math*, 43, Springer, 1967.

- [30] J.B. Quigley, "An exact sequence from the n th to the $(n-1)$ -st fundamental group", *Fund. Math.* 77 (1973) 195-210.
- [31] E. Spanier, "Borsuk's cohomotopy groups", *Ann. of Math.*, 50 (1949) 203-245.
- [32] N. E. Steenrod, "Regular Cycles on Compact Metric Spaces", *Ann. of Math.* 41 (1940) 833-851.
- [33] Strøm, "The homotopy category is a homotopy category", *Arch. Math.* 23 (1973) 435-441.

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