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SUSAN B. NIEFIELD
SHU-HAO SUN

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ALGEBRAIC DE MORGAN'S LAWS FOR NON-COMMUTATIVE RINGS

by Susan B. NIEFIELD and Shu-Hao SUN

Résumé. Les analogues algébriques (dans un anneau commutatif) de principes logiques connus comme la loi forte et la deuxième loi de Morgan sont donnés par les équations entre idéaux $(I:J)+(J:I)=R$ et $Ann(I\cap J)=Ann(I)+Ann(J)$. Les propriétés des anneaux qui satisfont ces lois ont été étudiées par Niefield et Rosenthal [8], [9]. Dans le present article, cette étude est prolongée et de nouveaux résultats sont obtenus dans le cas non commutatif.

Introduction.

The logical principles

$$\neg(A\wedge B)=\neg A\vee\neg B \qquad \text{and} \qquad (A\Rightarrow B)\vee(B\Rightarrow A)=true$$

known as *second de Morgan's law* and *strong de Morgan law*, respectively, are not in general valid in intuitionistic logic. Johnstone [4] showed that in the topos $Sh(X)$ of set-valued sheaves on a topological space X (and hence in the locale $O(X)$ of open subsets of X) second de Morgan's law holds if and only if X is extremally disconnected, and strong de Morgan's law holds if and only if every closed subspace of X is extremally disconnected.

The algebraic analogues of these laws for ideals of a commutative ring R with identity were investigated by Niefield and Rosenthal in [8] and [9]. First, in [8], they proved that a Noetherian domain R satisfies the strong algebraic de Morgan's law

$$(I:J)+(J:I)=R$$

if and only if R is a Dedekind domain. Then, in [9], they showed that a reduced (i.e. with no nontrivial nilpotents) ring R satifies the algebraic de Morgan's law

$$Ann(I\cap J)=Ann(I)+Ann(J)$$

if and only if R is a Baer ring if and only if R satisfies the weak de Morgan's law

$$Ann(rs) = Ann(r) + Ann(s)$$

for all $r, s \in R$, and $Ann(I)$ is principal for all ideals I. Using these results, they characterized those R for which $Spcc(R)$ is extremally disconnected, and those for which every closed subspace of $Spec(R)$ is extremally disconnected.

Weak de Morgan's law has also been related to a weakened form of the definition of a Baer ring, which we will call a *weak Baer ring*. In [12], Simmons showed that a reduced commutative ring R with identity is a weak Baer ring if and only if the distributive lattice $L(R)$ generated by radicals of principal ideals is a Stonian lattice. Recently, Belluci [1] partially extended this result to the noncommutative case by showing that if R is a semiprime *qc* ring with unit (i.e. a ring for which $Spec(R)$ is a spectral space), then R is weak Baer if and only if $L(R)$ is a Stonian lattice.

The goal of this paper is to extend the main results of [8] and [9], as well as the Simmons/Belluce theorem, to the general noncommutative case.

1. Ring Theoretic Preliminaries.

Thoughout this paper R is a (not necessarily commutative) ring with identity, all ideals are 2-sided, $\langle r\rangle$ is the principal ideal generated by $r \in R$, and $Id(R)$ denotes the set of ideals of R. If I and J are ideals, then

$$I{:_l}\, J = \{r \in R | rJ \subseteq I\} \qquad \text{and} \qquad I{:_r}\, J = \{r \in R | Jr \subseteq I\}$$

are the usual *left* and *right residuated ideals*, respectively. It is not difficult to show that $K \subseteq I{:_l}\, J$ if and only if $KJ \subseteq I$, and $K \subseteq I{:_r}\, J$ if and only if $JK \subseteq I$. Note that if $I{:_l}\, J = I{:_r}\, J$, then we will write $I{:}\, J$ for their common value and say that $I{:}\, J$ is well-defined. In the special case where $I = 0$, $I{:_l}\, J$ and $I{:_r}\, J$ are the *left* and *right annihilator ideals* $Ann_l(J)$ and $Ann_r(J)$, respectively. As in the case of residuated ideals, we write or $Ann(J)$, if they are equal, and say that $Ann(J)$ is well-defined.

Recall that an ideal I of R is called *semiprime* if $J^2 \subseteq I$ implies $J \subseteq I$, for all ideals J. If R is commutative, then these are just the radical ideals. It is well know that an ideal is semiprime if and only if it is an intersection of prime ideals. Also, it is not difficult to show that the set $SId(R)$ of semiprime ideals of R is closed under arbitrary intersections. Thus, if I is any ideal of R, then the smallest semiprime ideal of R containing I is defined and is denoted by $\sqrt{I}$. Also, $SId(R)$ is a locale which is isomorphic to the open set lattice of $Spec(R)$, with $I \wedge J = I \cap J$ and $I \vee J = \sqrt{I+J}$. In fact, $SId(R)$ is the largest localic quotient of the quantale $Id(R)$ (c.f. Corollary 3.6 in [10]). As such, one obtains the following proposition (c.f. [10]), which can also be verified directly if the reader so desires.

Proposition 1.1. *If I is a semiprime ideal of R, then*

(a) $J_1 \cdots J_n \subseteq I$ iff $J_{\sigma(1)} \cdots J_{\sigma(n)} \subseteq I$, for all permutations σ and for all ideals $J_1 \ldots J_n$.

(b) $J_1 \cdots J_n \subseteq I$ iff $J_1 \cap \ldots \cap J_n \subseteq I$, for all ideals $J_1 \ldots J_n$.

(c) $I : J$ is a well-defined semiprime ideal, for all ideals J.

(d) $I : \sqrt{J} = I : J$, for all ideals J.

(e) $\sqrt{J} \to I = I : J$, for all ideals J, where $\to$ is the Heyting implication in $SId(R)$.

Recall that R is called *semiprime* if the zero ideal 0 is semiprime, i.e. R has no nontrivial idempotent ideals. If R is commutative, this says that R has no nontrivial nilpotent elements, i.e. R is reduced.

Proposition 1.2. *If R is semiprime then, for all ideals I,*

(a) $Ann(I)$ is a well-defined semiprime ideal.

(b) $\neg\sqrt{I} = Ann(I) = Ann(\sqrt{I}) = \sqrt{Ann(I)}$, where $\neg$ is the locale negation in $SId(R)$.

The following is easily established using Proposition 1.1.

Proposition 1.3. *Suppose R is semiprime. If $Ann(I_1) = Ann(I_2)$ and $Ann(J_1) = Ann(J_2)$, then $Ann(I_1 J_1) = Ann(I_2 J_2)$.*

Now, we turn to the definition of a Baer ring, of which there are many in the literature. Some pertain to annihilators of subsets of R (c.f. [5]), some to annihilators of ideals, some to annihilators of elements (c.f. [2]), and some to annihilators of principal ideals (c.f. [1]). For the purposes of this paper, we will consider only annihilators of ideals and use the adjective "weak" to distinguish between the two possible definitions. We begin with a proposition, part (i.e. (a)⇒(b)) of which is similar to one appearing in [5].

Proposition 1.4. *The following are equivalent for a ring R.*

(a) $Ann_l\langle r\rangle$ is a principal ideal generated by a central idempotent, for all $r \in R$.

(b) $Ann_r\langle r\rangle$ is a principal ideal generated by a central idempotent, for all $r \in R$.

(c) R is semiprime, and $Ann\langle r\rangle$ is a principal ideal generated by a central idempotent, for all $r \in R$.

Proof. We will show that (a)⇒(c). The converse is clear and the proof of (a)⇔(b) is similar. To show that R is semiprime, assume $I^2 = 0$ and $r \in I$. Then $Ann_l\langle r\rangle = \langle e\rangle$, for some central idempotent e. Since $r\langle r\rangle \subseteq I^2 = 0$, we know that $r \in Ann_l\langle r\rangle$, and so $r = er$. But, $er = 0$, since $e \in Ann\langle r\rangle$, and so $r = 0$. Therefore, $I = 0$, as desired.

A ring satisfying the equivalent conditions of Proposition 1.4 will be called a *weak Baer ring*. It is not difficult to show that this proposition remains valid if the annihilators of principal ideals are replace by annihilators of arbitrary ideals. A ring satisfying the equivalent conditions of this stronger version of the proposition will be called a *Baer ring*.

Next, we consider some properties of central idempotents, starting with their relationship to complemented ideals. Recall that an ideal I is called *complemented* in $Id(R)$ if there is an ideal J such that $I + J = R$ and $IJ = JI = 0$. Note that if I_1 and I_2 are complemented, with complements J_1 and J_2, then so are $I_1 + I_2$ and $I_1 I_2$, and their complements are given by $J_1 J_2$ and $J_1 + J_2$, respectively. Also, any direct summand of R is complemented since $IJ \subseteq I \cap J$.

Proposition 1.5. *I is complemented if and only if I is a principal ideal generated by a central idempotent.*

Proof. Suppose I is complemented and let J be its complement. Then $e + f = 1$, for some $e \in I$ and $f \in J$, and so clearly e and f are idempotent and $eR \subseteq I$. Given $r \in I$, we have $r = er + fr = er$ since $JI = 0$. Thus, $r \in eR$, and so $I = eR = \langle e\rangle$. Similarly, $I = Re$, and so $eR = Re$. It remains to show that $er = re$, for all $r \in R$. Since $er \in Re$, we know that $er = ere$ since e is idempotent. Similarly, $re = ere$, and so $er = re$, as desired.

For the converse, let $I = \langle e\rangle$, where e is central idempotent. Then $J = \langle 1 - e\rangle$ is clearly the complement of I.

Note that $I + J = R$ if and only if $\sqrt{I + J} = R$, and if R is semiprime, then $IJ = 0$ if and only if $I \cap J = 0$, by Proposition 1.1(b). Thus, for semiprime ideals, the above notion of complemented in $Id(R)$ agrees with the lattice theoretic definition of complemented element of $SId(R)$. Actually, we are really interested in complemented elements of a certain sublattice of $SId(R)$.

Let $L(R)$ denote the distributive lattice generated by $\{\sqrt{\langle r \rangle} \mid r \in R\}$. Then, using Proposition 1.1, it is not difficult to show that the typical element in $L(R)$ is of the form $\sqrt{I_1 + \ldots + I_n}$, where the I_j are finite products of principal ideals of R.

Lemma 1.6. *If e is a central idempotent of a semiprime ring R, then $\langle e \rangle$ is semiprime.*

Proof. Suppose e is a central idempotent and $I^2 \subseteq \langle e \rangle$. To show that $I \subseteq \langle e \rangle$, it suffices to show that $I(1 - e) = 0$, or equivalently $(I(1 - e))^2 = 0$, since R is semiprime. But, $(I(1 - e))^2 = I^2(1 - e) = 0$, since e is a central idempotent.

Proposition 1.7. *The following are equivalent for an ideal I of a semiprime ring R.*

(a) I is complemented in $L(R)$.

(b) I is complemented in $Id(R)$.

(c) I is a principal ideal generated by a central idempotent.

Proof. Since R is the top element of $L(R)$, (a)⇒(b) follows from the remark after Proposition 1.5. Also, (b)⇒(c) holds by Proposition 1.5. For (c)⇒(a), since the complement of $\langle e \rangle$ is $\langle 1 - e \rangle$ and $1 - e$ is a central idempotent whenever e is, it suffices to show that if e is idempotent, then $\langle e \rangle \in L(R)$. But, $\langle e \rangle \in L(R)$ since $\sqrt{\langle e \rangle} = \langle e \rangle$, by Lemma 1.6.

2. Algebraic de Morgan's Law.

In this section, we introduce second algebraic de Morgan's law (DML) and weak algebraic de Morgan's law (WDML), and relate them to Baer rings and weak Baer rings. We also show that, if R is semiprime, then $Spec(R)$ satisfies second de Morgan's law if and only if R satisfies DML.

Proposition 2.1. *The following are equivalent for a ring* R.

(a) $Ann_l(\langle r\rangle\langle s\rangle) = Ann_l\langle r\rangle + Ann_l\langle s\rangle$, *for all* $r, s \in R$.

(b) $Ann_r(\langle r\rangle\langle s\rangle) = Ann_r\langle r\rangle + Ann_r\langle s\rangle$, *for all* $r, s \in R$.

(c) R *is semiprime and* $Ann(\langle r\rangle\langle s\rangle) = Ann\langle r\rangle + Ann\langle s\rangle$, *for all* $r, s \in R$.

Proof. To prove (a)$\Rightarrow$(c), we will show that (a) implies that R is semiprime. The proof for (b) is similar, and the remaining implications are clear. Suppose (a) holds and $I^2 = 0$. Given $r \in R$, we know that $\langle r\rangle^2 \subseteq I^2 = 0$, and so $R = Ann_l(\langle r\rangle^2) = Ann_l\langle r\rangle$. Thus, $r = 0$, and so $I = 0$, as desired.

Definition 2.2. A ring R for which the equivalent conditions of 2.1 hold will be said to satisfy *weak algebraic de Morgan's law* (WDML).

Recall that a distributive lattice L is called *Stonian* if for every $a \in L$, there is a complemented $b \in L$ such that $x \leq b$ if and only if $x \wedge a = 0$.

Theorem 2.3. *The following are equivalent for a ring* R.

(a) R *satisfies WDML and* $Ann\langle r\rangle$ *is principal for all* $r \in R$.

(b) $Ann\langle r\rangle \oplus Ann(Ann\langle r\rangle) = R$, *for all* $r \in R$.

(c) R *is semiprime and* $L(R)$ *is a Stonian lattice.*

(d) R *is weak Baer.*

Proof. (a)$\Rightarrow$(b) R is semiprime by Proposition 2.1, and so $Ann\langle r\rangle$ is well-defined, for all $r \in R$. Since each $Ann\langle r\rangle$ is principal, we can write $Ann\langle r\rangle = \langle s\rangle$, for some $s \in R$, and it follows that $Ann\langle r\rangle + Ann(Ann\langle r\rangle) = Ann\langle r\rangle + Ann\langle s\rangle = Ann(\langle r\rangle\langle s\rangle) = Ann(\langle r\rangle Ann\langle r\rangle) = Ann(0) = R$. Since $Ann\langle r\rangle Ann(Ann\langle r\rangle) = 0$ and R is semiprime, it follows that $Ann\langle r\rangle \cap Ann(Ann\langle r\rangle) = 0$, and so $Ann\langle r\rangle \oplus Ann(Ann\langle r\rangle) = R$.

(b)$\Rightarrow$(c) Note that R is semiprime by Proposition 1.4, since $Ann\langle r\rangle$ is a direct summand of R, and hence a principal ideal generated by a central idempotent. Suppose $I \in L(R)$. Then $I = \sqrt{I_1 + \ldots + I_n}$, where each I_j is a finite product of principal ideals. Since R is semiprime, using Proposition 1.1, it is not difficult to show that $\sqrt{I} \cap \sqrt{J} = 0$ if and only if $J \subseteq Ann(I)$, and so it suffices to show that $Ann(I)$ is complemented in $L(R)$. Since each $Ann\langle r\rangle$ is a direct summand of R, it is a principal ideal generated by a central idempotent. By Proposition 1.3, the same is true for any finite product of principal ideals. Since $Ann(I_1 + \ldots + I_n) = Ann(I_1) \cap \ldots \cap Ann(I_n)$, using Proposition 1.2, it follows that $Ann(I)$ is complemented in $L(R)$.

(c)⇒(d) Given $r \in R$, since $\sqrt{\langle r\rangle} \in L(R)$ and $L(R)$ is Stonian, there exists a complemented $I \in L(R)$ such that $J \subseteq I$ if and only if $J \cap \sqrt{\langle r\rangle} = 0$, for all $J \in L(R)$, i.e. $I = Ann\langle r\rangle$. Since $Ann\langle r\rangle$ is complemented in $L(R)$, by Proposition 1.7, we know that $Ann\langle r\rangle = \langle e\rangle$, for some central idempotent e. Therefore, R is a weak Baer ring.

(d)⇒(a) Given $r, s \in R$, write $Ann\langle r\rangle = \langle e\rangle = Ann\langle 1-e\rangle$ and $Ann\langle s\rangle = \langle f\rangle = Ann\langle 1-f\rangle$, where e and f are central idempotents. Since R is semiprime, applying Proposition 1.3, we get $Ann(\langle r\rangle\langle s\rangle) = Ann(\langle 1-e\rangle\langle 1-f\rangle) = Ann\langle 1-e\rangle + Ann\langle 1-f\rangle = Ann\langle r\rangle + Ann\langle s\rangle$, as desired.

Note that the equivalence of (c) and (d) in the commutative case appeared in [12]. Bulluce [1] generalized the equivalence to the case where R is a qc ring, i.e. a ring for which $Spec(R)$ is a spectral space. In [14], Sun showed that the free ring R on two generators is not a qc ring. Since R is clearly a prime ring and hence weak Baer, it follows that Theorem 2.3 (c)⇔(d) is an extension of Belluce's result.

Also, using Theorem 2.3 and a characterization by Sun [13; Theorem 2.6], one can show that the minimal prime ideal space $MinSpec(L(R))$ is compact, whenever R is weak Baer. Since $L(R)$ is a distributive lattice, we know $MinSpec(L(R))$ is zero-dimensional, and so $MinSpec(L(R))$ is a Stone space. By a result given in Sun's forthcoming paper [15], we also have that, for a weak Baer ring R with no non-zero nilpotents, the minimal prime ideal space $MinSpec(R)$ of R is homeomorphic to $MinSpec(L(R))$, and hence is a Stone space.

Next, we consider second de Morgan's law. Unlike for WDML, R is not necessarily semiprime, and so one must distinguish between left and right annihilators. Our definition will be in terms of left annihilators. Similar ones can be made using right annihilators. Now, one can consider the two algebraic de Morgan's laws $Ann_l(I+J) = Ann_l(I) \cap Ann_l(J)$ and $Ann_l(I \cap J) = Ann_l(I) + Ann_l(J)$. As in the case of locales, the first de Morgan's law holds for all ideals I and J.

Definition 2.4. A ring R satifies *second algebraic de Morgan's law* (DML) if

$$Ann_l(I \cap J) = Ann_l(I) + Ann_l(J)$$

for all ideals I and J.

Theorem 2.5. *The following are equivalent for a ring R.*

(a) R is semiprime and satisfies DML

(b) $Ann(IJ) = Ann(I) + Ann(J)$, for all ideals I and J.

(c) $Ann(I) \oplus Ann(Ann(I)) = R$, for all ideals I.

(d) $Ann(I)$ is a direct summand of R, for all ideals I.

(e) R is a Baer ring.

(f) R is weak Baer and $Ann(I)$ is principal, for all ideals I.

(g) R satisfies WDML and $Ann(I)$ is principal, for all ideals I.

Proof. (a)$\Rightarrow$(b) since $Ann(IJ) = Ann(I \cap J)$ in a semiprime ring.

(b)$\Rightarrow$(c) Note that R is semiprime by Proposition 2.1, since (b) implies that WDML holds. Thus, $Ann(I) \cap Ann(Ann(I)) = Ann(I)Ann(Ann(I)) = 0$. Also, $Ann(I) + Ann(Ann(I)) = Ann(IAnn(I)) = Ann(0) = R$, as desired.

(c)$\Rightarrow$(d) is clear.

(d)$\Rightarrow$(e) by Proposition 1.5.

(e)$\Rightarrow$(f) is clear.

(f)$\Rightarrow$(g) by Theorem 2.3.

(g)$\Rightarrow$(a) Since R is semiprime (by Proposition 2.1), it suffices to show that for every ideal I, $Ann(I) = Ann\langle r\rangle$, for some $r \in R$. Since $Ann(I)$ is principal, we can write $Ann(I) = \langle s\rangle$ and $Ann\langle s\rangle = \langle r\rangle$, for some $r, s \in R$. Then using the fact that Ann^3=Ann, we get $Ann\langle r\rangle = Ann^2\langle s\rangle = Ann^3(I) = Ann(I)$, as desired.

Recall that open sets of $Spec(R)$ are of the form

$$D(I) = \{P \in Spec(R) | I \not\subseteq P\}$$

where $I \in Id(R)$. The complement of $D(I)$ is denoted by $V(I)$.

Proposition 2.6. $\overline{D(I)} = V(\sqrt{0}:I)$.

Proof. If P is prime and $I \not\subseteq P$, then $\sqrt{0}:I \subseteq P$, since $(\sqrt{0}:I)I \subseteq \sqrt{0} \subseteq P$. Thus $D(I) \subseteq V(\sqrt{0}:I)$ and so $\overline{D(I)} \subseteq V(\sqrt{0}:I)$. We will show that if $P \in V(\sqrt{0}:I)$, then every open neighbourhood of P meets $D(I)$. Let $D(J)$ be such a set, i.e. $J \not\subseteq P$. Since $J \not\subseteq P$ and $\sqrt{0}:I \subseteq P$, it follows that $J \not\subseteq \sqrt{0}:I$, and hence $IJ \not\subseteq \sqrt{0}$. Therefore $D(I) \cap D(J) \neq \emptyset$, as desired.

Corollary 2.7. *If R is semiprime, then* $\overline{D(I)} = V(Ann(I))$.

Recall that a space X is called *extremally disconnected* if the closure of every open subset of X is open.

Theorem 2.8. *If R is semiprime, then* $Spec(R)$ *is extremally disconnected if and only if R satisfies DML.*

Proof. Since every open set of $Spec(R)$ is of the form $D(I)$, for some ideal I, by Corollary 2.7, $Spec(R)$ is extremally disconnected iff $V(Ann(I))$ is open for all I. But, $V(J)$ is open for some ideal J if and only if J is a direct summand of R, and so the desired result follows from (a)⇔(d) of Theorem 2.5.

Since $Spec(R) \cong Spec(R/\sqrt{0})$, we get the following corollary.

Corollary 2.9. *The following are equivalent for any ring* R.

(a) $Spec(R)$ *is extremally disconnected.*

(b) $R/\sqrt{0}$ *satisfies DML.*

(c) $R/\sqrt{0}$ *is a Baer ring.*

Note that Niefield and Rosenthal [9] proved versions of Theorem 2.5, Corollary 2.7, Theorem 2.8, and Corollary 2.9 for commutative rings. On the other hand, the commutative version of Theorem 2.3 did not appear in [9].

3. Strong de Morgan's Law.

In this section we introduce algebraic strong de Morgan's law, and relate it to strong de Morgan's law for the locale $O(Spec(R))$. As in the case of DML, we will consider only left annihilators. Similar results can be obtained for right annihilators.

Definition 3.1. A ring R satisfies *algebraic strong de Morgan's law* (SDML) if

$$(I:_l J) + (J:_l I) = R$$

for all ideals I and J.

Proposition 3.2. *The following are equivalent for a ring* R.

(a) R *satisfies SDML.*

(b) $(I + J):_l K = (I:_l K) + (J:_l K)$, *for all ideals* I, J, K.

(c) $I{:_l}(J \cap K) = (I{:_l}J) + (I{:_l}K)$, *for all ideals* I, J, K.

Proof. (a)⇒(b) First, $(I{:_l}K) + (J{:_l}K) \subseteq (I + J){:_l}K$. For the reverse containment, since $(I+J){:_l}K = (I{:_l}J)[(I+J){:_l}K] + (J{:_l}I)[(I+J){:_l}K]$ by (a), it suffices to show that $(I{:_l}J)[(I+J){:_l}K] \subseteq (I{:_l}K)$ and $(J{:_l}I)[(I+J){:_l}K] \subseteq (J{:_l}K)$. But,

$$(I{:_l}J)[(I+J){:_l}K]K \subseteq (I{:_l}J)(I+J) \subseteq I + (I{:_l}J)J \subseteq I + I \subseteq I$$

and so $(I{:_l}J)[(I+J){:_l}K] \subseteq (I{:_l}K)$. Similarly, $(J{:_l}I)[(I+J){:_l}K] \subseteq (J{:_l}K)$, as desired.

(b)⇒(a) Clearly, $R = (I+J){:_l}(I+J) = (I{:_l}(I+J)) + (J{:_l}(I+J) = (I{:_l}J) + (J{:_l}I)$.

(a)⇒(c) First, $(I{:_l}J) + (I{:_l}K) \subseteq I{:_l}(J \cap K)$. For the reverse containment, since $I{:_l}(J \cap K) = [I{:_l}(J \cap K)](K{:_l}J) + [I{:_l}(J \cap K)](J{:_l}K)$, it suffices to show that $[I{:_l}(J \cap K)](K{:_l}J) \subseteq I{:_l}J$ and $[I{:_l}(J \cap K)](J{:_l}K) \subseteq I{:_l}K$. But,

$$[I{:_l}(J \cap K)](K{:_l}J)J \subseteq [I{:_l}(J \cap K)](J \cap K) \subseteq I$$

and so $[I{:_l}(J \cap K)](K{:_l}J) \subseteq I{:_l}J$. Similarly, $[I{:_l}(J \cap K)](J{:_l}K) \subseteq I{:_l}K$.

(c)⇒(a) Clearly, $R = (I \cap J){:_l}(I \cap J) = (I \cap J){:_l}I + (I \cap J)_l J = J{:_l}I + I{:_l}J$.

Note that (a)⇒(c) says that SDML implies DML. Also, the proof we have given is valid in an arbitrary quantale. The equivalence of (a) through (c) for commutative quantales was first proved by Ward and Dilworth in [16].

Theorem 3.3. *The following are equivalent for a ring* R.

(a) Every closed subspace of $Spec(R)$ *is extremally disconnected.*

(b) $O(Spec(R))$ *satisfies strong de Morgan's law.*

(c) $(\sqrt{I}:J) + (\sqrt{J}:I) = R$, *for all ideals* I *and* J.

(d) $(\sqrt{I} + \sqrt{J}):K = (\sqrt{I}:K) + (\sqrt{J}:K)$, *for all ideals* I, J, K.

(e) $\sqrt{I}:(J \cap K) = (\sqrt{I}:J) + (\sqrt{I}:K)$, *for all ideals* I, J, K.

(f) $R/\sqrt{I}$ *satisfies the algebraic second de Morgan's law, for all ideals* I.

Proof. (a)⇔(b) follows from a theorem in [4].

(b)⇒(c) follows from Proposition 1.1, since $(\sqrt{I}:J) + (\sqrt{J}:I) = R$ if and only if $\sqrt{(\sqrt{I}:J) + (\sqrt{J}:I)} = R$, (c)⇔(d)⇔(e) follows from Porposition 1.1(d) and the remark preceding this theorem.

(e)⇒(f) follows from that fact that every ideal of $R/\sqrt{I}$ is of the form $J/\sqrt{I}$, where J is an ideal of R containing $\sqrt{I}$, and the left annihilator of $J/\sqrt{I}$ in $R/\sqrt{I}$ is $(\sqrt{I}:_l J)/\sqrt{I}$.

(f)⇔(a) follows from the fact that every closed subspace of $Spec(R)$ is homeomorphic to $Spec(R/I)$, for some semiprime ideal I of R.

Note that Niefield and Rosenthal [8] proved a commutative version of Theorem 3.3.

Recall that the locale $Spec(R)$ is called *Noetherian* if the semiprime ideals of R satisfy the ascending chain condition. If R is a Noetherian ring, then $Spec(R)$ is Noetherian, but the converse does not hold (c.f [6]). On the other hand, as in the commutative case, if $Spec(R)$ is Noetherian, then the number of minimal primes is finite (cf. [3, Theorem 2.4]). Moreover, if $Spec(R)$ is Noetherian, so is $Spec(R/I)$, for every semiprime ideal I, and it follows that every semiprime ideal is a finite intersection of primes. Motivated by the characterization of Noetherian domains in [7], Niefield and Rosenthal [8] showed that if R is commutative and $Spec(R)$ is Noetherian, then the conditions of Theorem 3.3 are equivalent to the distributivity of $SId(R)$, as well as the property $P \subseteq Q$, $Q \subseteq P$, or $Q + P = R$, for all prime ideals P and Q. Although this result does not seem to generalize to the noncommutative case (if the two additional conditions are taken separately), we have the following theorem.

Theorem 3.4. *The following are equivalent for a ring R such that $Spec(R)$ is Noetherian.*

(a) Every closed subspace of $Spec(R)$ is extremally disconnected.

(b) $O(Spec(R))$ satisfies strong de Morgan's law.

(c) $(\sqrt{I}: J) + (\sqrt{J}: I) = R$, for all ideals I and J.

(d) $(\sqrt{I} + \sqrt{J}): K = (\sqrt{I}: K) + (\sqrt{J}: K)$, for all ideals I, J, K.

(e) $\sqrt{I}:(J \cap K) = (\sqrt{I}: J) + (\sqrt{I}: K)$, for all ideals I, J, K.

(f) $R/\sqrt{I}$ satisfies the algebraic second de Morgan's law, for all ideals I.

(g) The conjunction of:

(i) $I \cap (J + K) = (I \cap J) + (I \cap K)$, for all semiprime ideals I, J, K

(ii) $P \subseteq Q$, $Q \subseteq P$, or $Q + P = R$, for all prime ideals P and Q.

Proof. The equivalence of (a) through (f) is clearly Theorem 3.3. We will show that (b) is equivalent to (g).

(b)⇒(g) Since $O(Spec(R))$ satisfies strong de Morgan's law, we know

$$\sqrt{(I:J)+(J:I)}=R$$

or equivalently $(I:J)+(J:I)=R$, for all semiprime ideals I and J. Thus, if I, J, and K are semiprime ideals, then $I\cap(J+K)=(J:_l K)(I\cap(J+K))+(K:_l J)(I\cap(J+K))$, and so it suffices to show that $(J:_l K)(I\cap(J+K))\subseteq I\cap J$ and $(K:_l J)(I\cap(J+K))\subseteq I\cap K$. But, $(J:_l K)(J+K)\subseteq J+(J:_l K)K\subseteq J+J\subseteq J$ and so $(J:_l K)(I\cap(J+K))\subseteq I\cap J$. Similarly, $(K:_l J)(I\cap(J+K))\subseteq I\cap K$. Next, suppose P and Q are prime ideals of R such that $P\not\subseteq Q$ and $Q\not\subseteq P$. Then $(Q:_l P)\subseteq Q$, since $(Q:_l P)P\subseteq Q$. Similarly, $(P:_l Q)\subseteq P$. Thus, $R=(P:_l Q)+(Q:_l P)\subseteq P+Q$, as desired.

(g)⇒(b) Given $I,J\in SId(R)$, write $I=P_1\cap\ldots\cap P_n$ and $J=Q_1\cap\ldots\cap Q_m$, where all P_i and Q_j are primes. Then

$$\begin{aligned}(I:J)+(J:I)&=(P_1\cap\cdots\cap P_n:J)+(Q_1\cap\cdots\cap Q_m:I)\\&=((P_1:J)\cap\cdots\cap(P_n:J))+((Q_1:I)\cap\cdots\cap(Q_m:I))\\&=\bigcap_{i,j}((P_i:J)+(Q_j:I))\supseteq\bigcap_{i,j}((P_i:Q_j)+(Q_j:P_i))\end{aligned}$$

Thus, it suffices to show that $(P:Q)+(Q:P)=R$, for all primes P and Q of R. If $P\subseteq Q$ or $Q\subseteq P$, then we have $Q:P=R$ or $P:Q=R$, and the desired conclusion follows. If $P\not\subseteq Q$ and $Q\not\subseteq P$, then

$$R=P+Q\subseteq P:Q+Q:P$$

as desired.

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Susan B. Niefield
Department of Mathematics
Union College
Schenectady, N.Y. 12308
U.S.A.

Shu-Hao Sun
School of Mathematics F07
University of Sydney NSW 2006
Australia