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ON ACCUMULATION POINTS

by Petr SIMON

Résumé. On étudie les classes des espaces topologiques AC et AP introduites récemment par A. Pultr et A. Tozzi. On définit une autre classe, wAP, qui est comprise entre les deux, et on prouve que ces classes sont des sous-classes propres et distinguées de **Top**. Finalement on prouve que ni AP ni wAP ne sont stables par produit.

Recently, Aleš Pultr and Anna Tozzi introduced two classes of spaces, called AP and AC (see the definition below). The class AC arose naturally from lattice-theoretical investigations of quotient mappings, while AP is a subclass of AC and is easier to deal with. It turns out that both classes are interesting also on their own. The aim of the present paper is to present several examples, indicating additional properties of the classes in question.

Definition. [6] A topological space X satisfies the *condition of approximation by closed sets* (abbr. AC), if for each non-open set $M \subseteq X$ there are open sets $U, V \subseteq X$ and a closed set $F \subseteq X$ such that $U \cup (V \cap M) = U \cup (V \cap F)$ and the set $U \cup (V \cap F)$ is not open.

A topological space X has the *property of approximation by points* (abbr. AP), if for every $M \subseteq X$ and for every point $x \in M \setminus \operatorname{int} M$ there is a $C \subseteq X$ such that $C \cap M = \varnothing$ and $\overline{C} \cap M = \{x\}$.

Clearly, every AP space is AC, provided very mild separation is assumed. Consider X to be, say, T_1. Choose M not open, pick a point $x \in M \setminus \operatorname{int} M$ and let $C \subseteq X \setminus M$ satisfy $\overline{C} \cap M = \{x\}$; here we have used AP. Put $V = X$, $U = X \setminus \overline{C}$, $F = \{x\}$. Then U, V are open, F closed (because X is T_1) and $U \cup (V \cap M) = (X \setminus \overline{C}) \cup \{x\} = U \cup (V \cap F)$, which is not open. So X has AC. (In fact, less than T_1 may be assumed, see Prop. 5.2 from [6].)

Obviously, there was no need to use the full strength of AP in the above proof, which leads to the following definition.

Definition. A topological space X has the *weak property of approximation by points* (abbr. wAP), if for every not open $M \subseteq X$ there is a point $x \in M \setminus \operatorname{int} M$ and a set $C \subseteq X$ such that $C \cap M = \varnothing$ and $\overline{C} \cap M = \{x\}$.

The previous observation gives immediately

Proposition. *In the class of T_1 spaces, $AP \subseteq wAP \subseteq AC$.*

Notice that the class AP is a natural generalization of Fréchet spaces and the class wAP contains all sequential spaces. Indeed, if $M \subseteq X$ is not open, then, if X is Fréchet, for each $x \in M \setminus \operatorname{int} M$ there is a sequence $\langle x_n \rangle$ ranging in $X \setminus M$ and converging to x; and there is such a point and such a sequence provided X is sequential. For $C = \{x_n : n \in \omega\}$ one has $\overline{C} \cap M = \{x\}$ in both cases.

The next observation is trivial, however, it points on a rich source of AP spaces.

Observation. *Every space with a unique non-isolated point satisfies AP.*

Proof. Denote the space by X and the non-isolated point by x. If $M \subseteq X$ is not open, then $x \in M$ and every neighborhood of x meets the set $X \setminus M$. So it is enough to set $C = X \setminus M$. □

Proposition. *Every generalized ordered topological space has wAP.*

Proof. Recall that a generalized ordered space is a linearly ordered set, where neighborhood base at a point x is either $\{(y,x] : y < x\}$ or $\{[x,y) : x < y\}$ or $\{(y,z) : y < x < z\}$ depending on x; here $(y,x] = \{z \in X : y < z \leq x\}$, $[y,x) = \{z \in X : y \leq z < x\}$ and $(y,x) = \{z \in X : y < z < x\}$. If $M \subseteq X$ and $x \in M \setminus \operatorname{int} M$, then there is some cardinal τ and a (possibly transfinite) sequence $\langle x_\alpha : \alpha < \tau \rangle$ with each x_α belonging to the set $X \setminus M$ and converging to x. This is a clear consequence of the fact that the space is generalized ordered. Let $\tau(x)$ be the minimal such τ and let $\kappa = \min\{\tau(x) : x \in M \setminus \operatorname{int} M\}$. For $x \in M \setminus \operatorname{int} M$ with $\tau(x) = \kappa$ and for $C = \{x_\alpha : \alpha < \kappa\}$ we obviously have $C \subseteq X \setminus M$, however, $\overline{C} \cap M = \{x\}$ is true, too: If not, then there is some $y \in M$, $y \neq x$, $y \in \overline{C}$. So there must be some $\xi < \kappa$ with $y \in \overline{\{x_\alpha : \alpha < \xi\}}$, because $\langle x_\alpha : \alpha < \kappa \rangle$ converges to x. But this contradicts to the minimality of κ. □

In [6, Example 5.3], the authors showed that the ordinal space $\omega_1 + 1$ with standard topology does not have AP. Combined with the previous Proposition, we get

Theorem 1. *$AP \subsetneq wAP$.* □

We regret that we were unable to remove the set-theoretical assumption from the second half of the next theorem.

Theorem 2. *There are two spaces, both having AP, such that their product fails to have wAP. Hence the classes AP and wAP are not productive.*

Under the assumption of Contonuum Hypothesis, there are two countable spaces having AP such that their product does not satisfy AP.

Proof. Let X be the closed unit interval $[0,1]$ with the usual metric topology, let Y be the one-point Lindelöfication of a discrete space of size 2^ω. It is obvious that both spaces have AP.

In order to show that $X \times Y$ does not satisfy wAP, we shall represent for convenience the space Y as follows: $Y = [0,1] \cup \{\infty\}$, where the basic neighborhood of a point $x \in [0,1]$ is $\{x\}$, the point ∞ does not belong to $[0,1]$ and its neighborhood basis consists of all sets $\{\infty\} \cup (Y \setminus C)$, where C is at most countable subset of Y.

Let $M = X \times Y \setminus \{\langle x,x\rangle : x \in [0,1]\}$. The set M is not open: Let $x \in [0,1]$ be arbitrary and let V be its open neighborhood in X. Let $C \subseteq [0,1]$ be an arbitrary countable set. Since V is uncountable, there is some $y \in V \setminus C$. If $U = (Y \setminus C) \cup \{\infty\}$, then $\langle y,y\rangle \in V \times U$, so $\langle x,\infty\rangle \notin \operatorname{int} M$.

If $C \subseteq \{\langle x,x\rangle : x \in [0,1]\}$ is countable, then C is closed in $X \times Y$ and so $\overline{C}$ does not meet M. If C is uncountable, then one can find two distinct points $y, z \in X$ such that every neighborhood of y as well as every neighborhood of z meets the set $\{x : \langle x,x\rangle \in C\}$ in an uncountable subset. So $\overline{C}$ contains at least two points from M, namely $\langle y,\infty\rangle$ and $\langle z,\infty\rangle$, which shows that $X \times Y$ does not satisfy wAP.

Now, let us prove the second statement from the Theorem. In fact, we need much less than the full strength of CH. What we really need is only that there exist two selective ultrafilters p, q on ω, which are incomparable in Rudin-Keisler order. It is well known that this is a consequence also of MA and independent of it. However, it is also consistent that there are no selective ultrafilters at all [3, Th. 5.1].

For the reader's convenience, let us recall the definition and several basic properties of selective ultrafilters. An ultrafilter s on ω is called *selective*, if s is non-principal and whenever $\{R_n : n \in \omega\}$ is a partition of ω, then either there is some $n \in \omega$ with $R_n \in s$ or there is some $S \in s$ with $|S \cap R_n| \leq 1$ for all $n \in \omega$. Such a set is called a selector of the partition $\{R_n : n \in \omega\}$. An ultrafilter s is selective if and only if it is Ramsey [2, Theorem 9.6], which means that for every $n, k \in \omega$ and every $\varphi : [\omega]^n \longrightarrow \{0,1,\ldots,k\}$ there is some $S \in s$ such that $\varphi \restriction [S]^n$ is a constant mapping.

The Rudin-Keisler order $\leqslant_{RK}$ of ultrafilters is defined by $p \leqslant_{RK} q$ if there is a mapping $f : \omega \longrightarrow \omega$ such that for each $U \in q$ one has $f[U] \in p$; here it is not necessary to consider a mapping with domain ω, $\operatorname{dom} f \in q$ suffices. So two ultrafilters p, q are RK-incomparable, if for every $f : \omega \longrightarrow \omega$ there is some $U \in p$ with $f^{-1}[\omega \setminus U] \in q$ and some $V \in q$ with $f^{-1}[\omega \setminus V] \in p$.

We shall define $X = \omega \cup \{p\}$ and $Y = \omega \cup \{q\}$ with the topology of a subspace of $\beta\omega$, i.e., $U \subseteq X$ is open iff either $U \subseteq \omega$ or $p \in U$ and $U \cap \omega \in p$.

Both spaces have unique non-isolated point, hence both satisfy AP.

Before showing that $X \times Y$ does not have AP, we shall state two claims.

Claim 1. *Let s be a selective ultrafilter on ω, let $\mathcal{C}$ be a point-finite family of finite subsets of ω. Then there is an $S \in s$ such that for every two distinct $n, m \in S$, $St(n,\mathcal{C}) \cap St(m,\mathcal{C}) = \varnothing$. (Here $St(n,\mathcal{C})$ denotes the set $\bigcup\{C \in \mathcal{C} : n \in C\}$.)*

Define a mapping $\varphi : [\omega]^2 \longrightarrow \{0,1\}$ by $\varphi(\{n,m\}) = 0$ if $St(n,\mathcal{C}) \cap St(m,\mathcal{C}) \neq \varnothing$, $\varphi(\{n,m\}) = 1$ otherwise. Let $S \in s$ be homogeneous for φ. Since S is infinite, $\mathcal{C}$ is point-finite and consists of finite sets, the constant value of $\varphi \restriction [S]^2$ cannot be 0. Hence for distinct $n, m \in S$, $\varphi(\{n,m\}) = 1$, which means that $St(n,\mathcal{C}) \cap St(m,\mathcal{C}) = \varnothing$.

Claim 2. *Let p, s be RK-incomparable ultrafilters on ω with s selective. Let ϱ be a one-to-finite set-mapping from ω to ω such that for some $U \in p$, the family $\{\varrho(n) : n \in U\}$ is point-finite. Then there is a set $V \in p$ such that $\bigcup\{\varrho(n) : n \in V\} \notin s$.*

Consider the family $\mathcal{C} = \{\varrho(n) : n \in U\}$. Applying Claim 1, choose $S \in s$ such that $St(n, \mathcal{C}) \cap St(k, \mathcal{C}) = \emptyset$ for distinct $n, k \in S$. For $n \in U$, let $f(n)$ be the member of S which belongs to $\varrho(n)$, if there is any; if $S \cap \varrho(n) = \emptyset$, then $f(n)$ is undefined. If $\operatorname{dom} f \notin p$, then it is enough to set $V = U \setminus \operatorname{dom} f$ and clearly $\bigcup\{\varrho(n) : n \in V\} \cap S = \emptyset$. If $\operatorname{dom} f \in p$, then $f[\operatorname{dom} f] \subseteq S$, however, p and s are RK-incomparable, so there is some $T \in p$ with $f[T] \notin s$. Put $V = U \cap \operatorname{dom} f \cap T$. Now we have $f[V] \subseteq S$ and $S \setminus f[V] \in s$. By the choice of S, $S \setminus f[V] = S \setminus \bigcup\{\varrho(n) : n \in V\}$. So $\bigcup\{\varrho(n) : n \in V\} \notin s$, which was to be proved.

Now we are ready to show that $X \times Y$ has not AP. Conider the set $M = (X \times Y) \setminus (\omega \times \omega)$. The set M has empty interior, $M = \{p\} \times \omega \cup \omega \times \{q\} \cup \{\langle p, q\rangle\}$. Let $C \subseteq \omega \times \omega$ be arbitrary. Our aim is to show the following: There is no $C \subseteq \omega \times \omega$ with $\overline{C} \cap M = \{\langle p, q\rangle\}$. To this end, let $C = \omega \times \omega$ be arbitrary and suppose that $\overline{C}$ does not meet $\{p\} \times \omega \cup \omega \times \{q\}$.

For each $n \in \omega$ there are sets $U_n \in p$ and $V_n \in q$ such that $U_n \times \{n\} \cap C = \emptyset$, $\{n\} \times V_n \cap C = \emptyset$, because $\langle p, n\rangle \notin \overline{C}$ and $\langle n, q\rangle \notin \overline{C}$. By selectivity, there are sets $S \in p$ and $T \in q$ such that for each $n \in \omega$, $|S \cap \bigcap_{i<n} U_i \setminus \bigcap_{i<n+1} U_i| \leq 1$ and $|T \cap \bigcap_{i<n} V_i \setminus \bigcap_{i<n+1} V_i| \leq 1$. If $n \in \omega$ is arbitrary, $T \setminus V_n$ is finite, consequently $\{n\} \times T \cap C$ is finite, too. For $n \in \omega$ define $\varrho(n) = \{k \in \omega : \langle n, k\rangle \subset \{n\} \times T \cap C\}$. We have just observed that the set-mapping ϱ is one-to-finite.

By the choice of S, the set $S \times \{k\} \cap C$ is finite for each $k \in \omega$. This immediately implies that the family $\{\varrho(n) : n \in S\}$ is point-finite. Since q is selective and p, q are RK-incomparable, we are ready to use Claim 2. Let $V \in p$ and $W \in q$ be such that $W \cap \bigcup\{\varrho(n) : n \in V\} = \emptyset$. Then the set $(V \cup \{p\}) \times (W \cup \{q\})$ is a neighborhood of the point $\langle p, q\rangle$ in $X \times Y$, which is disjoint with C. So $\langle p, q\rangle \notin \overline{C}$, which was to be proved. □

Now, we shall find a compact Hausdorff space which satisfies AC and which has not wAP.

Theorem 3. *$wAP \subsetneq AC$.*

Proof. The space which satisfies AC and fails to have wAP is a special compactification of ω which was constructed for different purpose in [7] and [5]. For the present proof, we need to repeat the definition of the space. We shall adopt the standard notation when dealing with infinite subsets of ω: $A \subseteq^* B$ denotes that $A \setminus B$ is finite, $A =^* B$ stands for $(A \setminus B) \cup (B \setminus A)$ is finite, A, B are called almost disjoint, if $A \cap B$ is finite. $A \subsetneq^* B$ abbreviates $A \subseteq^* B$ and $B \setminus A$ is infinite.

There is a family $\mathcal{A} \subseteq [\omega]^\omega$ with the following properties: (a) For any pair A, B of members of $\mathcal{A}$ either A and B are almost disjoint, or $A \subseteq^* B$ or $B \subseteq^* A$, (b) for each $Z \in [\omega]^\omega$ there is some $A \in \mathcal{A}$ with $A \subseteq Z$, (c) $(\mathcal{A}, \supseteq^*)$ is a tree. The existence of such a family was proved in [1]. Let $\mathcal{B} \subseteq \mathcal{P}(\omega)$ be the Boolean algebra, generated by $\mathcal{A} \cup [\omega]^{<\omega}$. The desired space X is the Stone space of $\mathcal{B}$. Alternative description of X is as follows: $X = \beta\omega/\sim$, where the equivalence relation $\sim$ is defined by $x \sim y$ if either $x \in \omega$, $y \in \omega$ and $x = y$ or $x, y \in \beta\omega \setminus \omega$ and for all $A \in \mathcal{A}$, $A \in x$ if and only if $A \in y$.

The space X does not satisfy wAP: Let $M = X \setminus \omega$. Since the set ω is open and dense in X, $M \setminus \operatorname{int} M = M$. Let $C \subseteq X \setminus M$ be arbitrary. If C is finite, then $\overline{C} = C$, hence $C \cap M = \varnothing$. If C is infinite, choose two infinite disjoint subsets $C_1, C_2 \subseteq C$ and find two sets $A, B \in \mathcal{A}$ satisfying $A \subseteq C_1$, $B \subseteq C_2$. If p is an ultrafilter in $\mathcal{B}$ containing all cofinite subsets of ω and such that $A \in p$, and if q is an analogous one with $B \in q$, then $p \in \overline{C}$ and $q \in \overline{C}$, too. Therefore one cannot find a subset of $X \setminus M$, the closure of which would meet M in exactly one point, which violates wAP.

The space X satisfies AC: Let $M \subseteq X$ be an arbitrary non-empty non-open set. Consider its subset $M \setminus \omega$ in the remainder $X \setminus \omega$. We shall distinguish two cases:

Case 1. The set $M \setminus \omega$ is open in $X \setminus \omega$.

Since M is not open in X, there is a point $x \in M \setminus \operatorname{int} M$. Certainly $x \notin \omega$, since every point in ω is an open subset of X. According to the definition of X, there is some $B \in \mathcal{B}$ such that $x \in \overline{B}$ and $\overline{B} \setminus \omega \subseteq M \setminus \omega$. However, since $x \in \overline{X \setminus M}$ and $\overline{B}$ is a neighborhood of x and $x \notin \overline{(\overline{B} \setminus M) \setminus \omega} = \overline{\varnothing}$, $x \in \overline{B \setminus M}$.

Put $U = B \cap M$, $V = \overline{B}$ and $F = X \setminus \omega$. As a subset of an open discrete set ω, U is open. Since $B \in \mathcal{B}$ and X is the Stone space of $\mathcal{B}$, the set $\overline{B}$ is open in X, too. The set F is obviously closed. It is easy to check that $U \cup (V \cap M) = U \cup (V \cap F)$, because both sides meet ω in the set $B \cap M$ and both sides meet the remainder $X \setminus \omega$ in the set $\overline{B} \setminus \omega$. The point x is the witness that $U \cup (V \cap M)$ is not open.

Case 2. The set $M \setminus \omega$ is not open in $X \setminus \omega$.

Here we need a more close examination of the structure of $X \setminus \omega$. We claim the following: If a set Z is not closed in the space $X \setminus \omega$, $z \in X \setminus \omega$ and $z \in Z'$, then there is some ordinal τ and a (transfinite) sequence $\langle x_\alpha : \alpha < \tau \rangle$ ranging in Z and converging to z. (Here Z' denotes the set of all accumulation points of Z, that is, $Z' = \{x \in X : \text{Each neighborhood of } x \text{ contains at least 2 points from } Z\}$.)

To see this, notice that the system $\mathcal{A}$ is a tree under the order $\supseteq^*$. Thus one may assign to each member $A \in \mathcal{A}$ an ordinal $\alpha(A)$, the order type of the set $\{B \in \mathcal{A} : B \supseteq^* A\}$. As x may be viewed as an ultrafilter on $\mathcal{B}$ containing no finite set, we may put $\alpha(z) = \sup\{\alpha(A) + 1 : A \in \mathcal{A} \text{ and } A \in z\}$ (equivalently, $\alpha(z) = \sup\{\alpha(A)+1 : A \in \mathcal{A} \text{ and } x \in \overline{A}\}$). If $\alpha(z)$ is a successor ordinal, then by (a), there is a unique $A \in \mathcal{A}$ with $\alpha(A)+1 = \alpha(z)$ and for every $B \in \mathcal{A}$ with $\alpha(B) \geq \alpha(z)$ we have $\omega \setminus B \in z$. If $B \in \mathcal{A}$ is such that $\alpha(B) > \alpha(z)$, then there is some $C \in \mathcal{A}$ with $\alpha(C) = \alpha(z)$ and $C \supseteq^* B$, by (a) and (c). So $\{A\} \cup \bigcup\{\omega \setminus B : B \in \mathcal{A}, \alpha(B) = \alpha(z)\}$ is a subbasis of z. Pick $x_0 \in \overline{A} \cap Z$, $x_0 \neq z$ and find $B_0 \in \mathcal{A}$ with $x_0 \in \overline{B_0}$, $\alpha(B_0) = \alpha(z)$. There must be such a B_0, since $x_0 \neq z$. Then find $x_1 \in \overline{A \setminus B_0} \cap Z$ and $B_1 \in \mathcal{A}$ with $B_1 \subseteq^* A \setminus B_0$ and $x_1 \in \overline{B_1}$, $\alpha(B_1) = \alpha(z)$. Proceeding further, we obtain B_n and $x_n \in Z$ $(n \in \omega)$, such that B_n's are pairwise almost disjoint, $B_n \subseteq A$, $x_n \in \overline{B_n}$ and $\alpha(B_n) = \alpha(z)$. The sequence $\langle x_n : n \in \omega\rangle$ converges to z, since $\{\overline{A \setminus \bigcup_{n<k} B_n} \setminus L : k \in \omega, L \in [\omega]^{<\omega}\}$ is an open basis in z.

If $\alpha(z)$ is a limit ordinal and if there is an infinite set of those $B \in \mathcal{A}$ such that $\alpha(B) = \alpha(z)$, $\overline{B} \cap Z \neq \emptyset$ and for all $A \in \mathcal{A} \cap z$, $B \subseteq^* A$, then the same reasoning as for the successor case will produce an analogous countable convergent sequence. In the case which remains, we may w.l.o.g. assume that for every $B \in \mathcal{A}$, if $B \subseteq^* A$ for each $A \in z \cap \mathcal{A}$, then $\overline{B} \cap Z = \emptyset$. As $z \in Z'$, for every $A \in \mathcal{A} \cap z$ there is some $x_A \in Z$ with $x_A \in \overline{A}$, $x_A \neq z$ and such that there is a member $B \in \mathcal{A} \cap z$ with $x_a \notin \overline{B}$. Since the set $\mathcal{A} \cap z$ is well-ordered by $\supseteq^*$, it follows that the points x_A $(A \in \mathcal{A} \cap z)$ form a sequence of length not exceeding $\alpha(z)$ and converging to z.

Having proved the claim, let us conclude the proof. Since $M \setminus \omega$ is not open in $X \setminus \omega$, there is a point $x \in M$ and a sequence $\langle x_\xi : \xi < \tau\rangle$ ranging in $(X \setminus \omega) \setminus M$ such that $\langle x_\xi : \xi < \tau\rangle$ converges to x and τ is the minimal possible. Since every neighborhood of x contains a cofinal part of $\{x_\xi : \xi < \tau\}$ and since τ is minimal, $\overline{\{x_\xi : \xi < \alpha\}} \cap M = \emptyset$ for every $\alpha < \tau$. So $\overline{\{x_\xi : \xi < \tau\}} \cap M = \{x\}$. So it is enough to put $U = X \setminus \overline{\{x_\xi : \xi < \tau\}}$, $V = X$ and $F = \{x\}$ to get $M = U \cup (V \cap M) = U \cup (V \cap F)$. □

Remark. Notice that the space X from the above proof is a disjoint union of an open subspace ω and a closed subspace $X \setminus \omega$. The first subspace has clearly AP and our proof shows, in fact, that $X \setminus \omega$ satisfies wAP. The Proposition 5.5 from [6] says that if both subspaces satisfy AC, then the whole space does. Hence our example shows also that this proposition cannot hold in a stronger version replacing AC by wAP.

Next, we aim to show that there is a space not satisfying AC. The forthcoming Proposition shows a sufficient condition for it.

Proposition. *Let X be a regular T_1 space containing a set M such that for each $T \subseteq X$, if $T' \neq \emptyset$, then $T' \cap M \neq \emptyset \neq T' \setminus M$. Then X does not satisfy AC.*

Proof. Pick open sets U, V and a closed set F in X arbitrarily; we have to show that either $U \cup (V \cap M) \neq U \cup (V \cap F)$ or the set $U \cup (V \cap F)$ is open. We shall frequently use the following consequence of the fact that the space X is regular and T_1: If $T \subseteq X$ has an accumulation point, then $|T' \cap M| \geq 2$ and $|T' \setminus M| \geq 2$. Of course, by our assumption there is a point $x \in T' \cap M$ as well as a point $y \in T' \setminus M$. Choose disjoint closed neighborhoods G, H of x and y. Then $(T \cap G)' \neq \emptyset$, since $x \in (T \cap G)'$; similarly for $T \cap H$. So one point in $T' \cap M$ can be found in G, the second one in H, as well as the points from $T' \setminus M$.

If $V \cap F \subseteq U$, then $U \cup (V \cap F)$ is open. So for the rest of the proof we may assume that the set $K = (V \cap F) \setminus U$ is non-empty.

Case 1. There is a point $x \in V$, which is an accumulation point of K.

Pick an open set G such that $x \in G \subseteq \overline{G} \subseteq V$. Then x is also an accumulation point of $K \cap \overline{G}$. Therefore the set $K \cap \overline{G}$ contains also some accumulation point

$y \notin M$. We have obtained $U \cup (V \cap M) \neq U \cup (V \cap F)$, because the point y belongs to the right-hand side set only.

Case 2. No point in V is an accumulation point of K.

Choose for every $x \in K$ open sets G_x, O_x such that $x \in G_x \subseteq \overline{G_x} \subseteq O_x \subseteq \overline{O_x} \subseteq V$ and $O_x \cap K = \{x\}$.

Subcase 2a. There is a point $x \in K$ which is an accumulation point of $X \setminus U$.

Clearly, this x is also an accumulation point of the set $G_x \setminus U$. Thus there are at least two points in $(\overline{G_x} \setminus U) \cap M$. Therefore $O_x \cap F \setminus U \neq O_x \cap M \setminus U$, since the first set contains precisely one point, the second one at least two. The inequality $U \cup (V \cap M) \neq U \cup (V \cap F)$ follows.

Subcase 2b. No point in K is an accumulation point of the set $X \setminus U$.

In this subcase, each $x \in K$ has an open neighborhood H_x such that $H_x \subseteq O_x$ and $H_x \setminus U = \{x\}$. Since $K = (V \cap F) \setminus U$, we have: $U \cup (V \cap F) = U \cup K = U \cup \bigcup\{\{x\} : x \in K\} = U \cup \bigcup\{H_x \setminus U : x \in K\} = U \cup \bigcup\{H_x : x \in K\}$. Being the union of open sets, $U \cup (V \cap F)$ is open in this subcase.

All cases and subcases have been considered; the proof is complete. □

Proposition. *The space $\beta\mathbb{N}$, the Čech-Stone compactification of integers, satisfies the assumptions of the previous Proposition.*

Proof. We may restrict our attention to countable discrete subsets of $\beta\mathbb{N}$. There are 2^{2^ω} countable discrete subsets of $\beta\mathbb{N}$ and every countable discrete subset of $\beta\mathbb{N}$ has 2^{2^ω} accumulation points. By the disjoint refinement lemma [4], there is a pairwise disjoint family $\mathcal{R}$ such that each member of $\mathcal{R}$ is of the full size 2^{2^ω} and every set of accumulation points of a countable discrete set contains a member of $\mathcal{R}$. If $M \subseteq \beta\mathbb{N}$ is such that both M and $\beta\mathbb{N} \setminus M$ meet each member of $\mathcal{R}$, then M is as required. □

Thus we have proved

Theorem 4. *$AC \subsetneqq Top$.* □

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