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THE CATEGORY OF DISINTEGRATION

by **Michael WENDT**

Résumé. Nous présentons deux catégories pour établir un cadre pour les X-familles abstraites (X un espace de mesure finie). Les morphismes sont ceux qui réfléchissent les ensembles de mesure nulle et ceux que l'on nomme "désintégrations." La seconde catégorie est la meilleure parce qu'elle possède des propriétés d'auto-indexation et qu'elle encode, et le théorème de Fubini et la dérivée de Radon-Nikodym de façon essentielle.

1 Introduction

We present some categories of measure spaces appropriate for an understanding of the direct integral of Hilbert spaces (see, for example, [5]) in the context of indexed category theory (of [8]). Specifically, this paper grows out of two "directives." In [4], Breitsprecher suggested that (measure theoretic) disintegrations (see, for example, [10]) should be understood from the point of view of category theory. We address this directive in the last section. In [11], we sought a framework for the direct integral. That is, we sought to understand what X-families of Hilbert spaces should be for X a measure space (of finite measure; finiteness being forced on us for technical reasons). An appropriate morphism of measure spaces is required. We address this directive by describing two categories (in increasing order of utility) and inserting, from time to time, certain criteria required of an "appropriate category of measure spaces."

The first criterion is that the morphisms be measurable (or that a forgetful functor to measurable spaces exists). Measure theory is, in particular, "measurable theory." Mble denotes the category of measurable spaces (sets equipped with a σ-algebra of subsets), $(X,\mathcal{A})$, and measurable functions ($f:(X,\mathcal{A})\longrightarrow(Y,\mathcal{B})$; $f^{-1}(B)\in\mathcal{A}$ for all $B\in\mathcal{B}$). An excellent description of this category is given in [9].

Mble and Top (the category of topological spaces and continuous functions) are similar. Mble is both complete and cocomplete. The construction of limits, for example, is analogous (put the coarsest σ-algebra structure on the Set-limit to make the projections measurable). In fact, using the total opfibrations of [12], Mble is seen to be totally cocomplete (this is a consequence of the adjunctions $\mathit{Discrete} \dashv \mathit{Forget} \dashv \mathit{Indiscrete} : \mathbf{Mble} \leftrightarrow \mathbf{Set}$). Mble and Top are both topological over Set (see, for example, [1]).

The two categories are different, of course. The differences arise out of the arities of the operations (in particular, intersection and union) that σ-algebras and topologies are to be closed under. Our ultimate goal in [11] was to understand the differences in relation to abstract families (for a categorical treatment of indexing by topological spaces, see [7]).

The difference between Top and Mble becomes apparent when one naively translates topological notions into measure theory, encountering "mistakes" of triviality. For example, suppose we translate the notion of homotopy to measure theory by defining a "loop" as a measurable function $l : ([0,1],\mathcal{L}) \longrightarrow (X,\mathcal{A})$ such that $l(0) = l(1)$ and homotopy in an obvious way (in this paper, $(-,\mathcal{L},\lambda)$ denotes Lebesgue measure). Unfortunately, this definition makes the "fundamental groups" of the disc and the annulus the same ($=\mathbf{1}$). In essence, the difference between "continuous" and "measurable" is that we are allowed to measurably cut a loop but not continuously cut it.

This brings another similar example to mind. In the study of covering spaces, one has the nontrivial "spiral over the circle" example. Translated into measure theory, this example is trivial; it is simply a product of $\mathbf{Z}$ copies of the circle over the circle (we are allowed to measurably cut countably many times).

In each of the next two sections, we will introduce a category of measure spaces. Bringing measures into the indexing results in a whole new level of difficulty. "Nice" properties of Mble seem to disappear. The problem of major concern as far as indexed category theory is concerned is the disappearance of products. It turns out that, in some sense, the best we can hope for is a monoidal category. As we shall see in the sequel, we are forced to take a more complex approach to indexing by measure spaces and this complexity is the essence of the difference between continuous families and measurable families.

2 Measure Zero Reflecting Functions

The first category of measure spaces we introduce involves measure zero reflecting functions:

Definition 1 *A function* $f : (X,\mathcal{A},\mu) \longrightarrow (Y,\mathcal{B},\nu)$ *is said to be* measure zero reflecting *if it is measurable and if* $\nu(B) = 0 \Rightarrow \mu(f^{-1}B) = 0$. □

Remark: A comment to the term "reflecting": in analogy to "functor reflecting isomorphisms," one might consider measure zero reflecting as being $\nu(f(A)) = 0 \Rightarrow \mu(A) = 0$. For complete measure spaces, these two definitions are equivalent: assume the former and suppose $\nu(f(A)) = 0$. Then $\mu(A) \leq \mu(f^{-1}f(A)) = 0$. Conversely, assume the latter definition and suppose $\nu(B) = 0$. Then, since $ff^{-1}(B) \subseteq B$, $\nu(ff^{-1}(B)) = 0 \Rightarrow \mu(f^{-1}(B)) = 0$ as required. □

One might consider measure preserving functions between measure spaces as providing a good category. Measure preservation is too stringent a requirement for morphisms, however. These are simply too much like equivalences. The existence of a measure preserving morphism between X and Y says that they are more or less the same space (there may be many more $A \in \mathcal{A}$ than those of the form $f^{-1}(B)$ so the spaces aren't really indistinguishable). It is our contention that measure zero reflecting (abbreviated M0R) is the least requirement for a reasonable category of measure spaces.

In analysis, one often has the caveat "to within ϵ." Continuous functions, then, are the appropriate morphism for this caveat (of course, the opposite is the initial way of looking at the caveat; continuous functions force the "to within ϵ"). In measure theory, the caveat is often "almost everywhere" (which means "to within a set of measure zero"). And so, it seems appropriate to require that our morphisms be M0R. We wish to apply our constructions to measure theory.

M0R's are required whenever one considers the (Boolean) algebraic properties of $\mathcal{A}$ and $\mathcal{N}$, its ideal of measure zero sets (note: in general, we do not require our measure spaces to be complete so when we say $\mathcal{N}$ is downclosed, for example, this means $N \in \mathcal{N},\ A \in \mathcal{A},\ A \subseteq N \Rightarrow A \in \mathcal{N}$ from the monotonicity of the measure). For example, there is a well known metric on $\mathcal{A}/\mathcal{N}$ by $d([A],[B]) := \mu(A \triangle B)$ where $\triangle$ denotes the usual symmetric difference of sets. If $f : (X, \mathcal{A}, \mu) \longrightarrow (Y, \mathcal{B}, \nu)$ is M0R, then we have a map $f^{-1} \cdot \mathcal{B}/\mathcal{M} \longrightarrow \mathcal{A}/\mathcal{N}$. We see that measure zero reflecting is the least requirement for this map to be defined (after which, one may explore various properties of interest to metric space enthusiasts).

In practice, we will be interested in spaces of finite measure (as usual, referred to as "finite measure spaces"). The identity is measure zero reflecting and measure zero reflecting functions compose so:

Definition 2 <u>**M0R**</u> *is the category whose objects are finite measure spaces and whose morphisms are measure zero reflecting functions.* □

It is time for some examples. In some of the examples below, we will temporarily ignore the finiteness requirement.

Example 1: Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be two finite measure spaces. The projection $p : (X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu) \longrightarrow (X, \mathcal{A}, \mu)$ is a M0R for $\mu(A) = 0 \Rightarrow (\mu \otimes \nu)(p^{-1}A) = \mu(A) \cdot \nu(Y) = 0$ since $\nu(Y) < \infty$. If we use the common convention $\infty \cdot 0 = 0$, we can allow the spaces to have infinite measure. □

(Counter)example 2: $\delta : ([0,1], \mathcal{L}, \lambda) \longrightarrow ([0,1] \times [0,1], \mathcal{L}\overline{\otimes}\mathcal{L}, \lambda\overline{\otimes}\lambda)$ is not M0R. The diagonal has plane measure zero but nonzero length. □

Remarks: 1. $\mathcal{A} \otimes \mathcal{B}$ denotes the smallest σ-algebra containing the measurable rectangles. If μ and ν are complete, then $\mu \otimes \nu$ is not necessarily complete. $\mathcal{A}\overline{\otimes}\mathcal{B}$ and $\mu\overline{\otimes}\nu$ denote the completion. Each $D \in \mathcal{A}\overline{\otimes}\mathcal{B}$ is of the form $D = E \setminus F$ with $E \in \mathcal{R}_{\sigma\delta}$ (countable intersections of countable unions of measurable rectangles) and F of measure zero. We will use $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$ since $\mathcal{A} \otimes \mathcal{B}$ is the product in <u>**Mble**</u> and since we do not require our spaces to be complete.

2. $\delta[0,1]$ is, in fact, an $\mathcal{R}_{\sigma\delta}$ (take intersections of unions of "little squares" that cover the diagonal) so example 2 works for $\mathcal{L} \otimes \mathcal{L}$ and $\lambda\overline{\otimes}\lambda$ restricted to $\mathcal{R}_{\sigma\delta}$ subsets of the diagonal. Neither $\otimes$ nor $\overline{\otimes}$ gives the product in <u>**M0R**</u>. We will interpret $\otimes$ as a tensor below, however. □

Example 3: Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(A) = 0,\ \forall A \in \mathcal{A}$ (i.e. $\mu(X) = 0$). Then any measurable function out of X is M0R. □

Example 4: Let $(Y, \mathcal{B}, \nu)$ be a discrete space with counting measure. Then any measurable function into it is MOR since the only set of measure zero is the empty set. □

Example 5: A terminal object of <u>**MOR**</u> is $(1, \mathcal{I}, \iota)$ where $1 = \{\star\}$ is a one point set, $\mathcal{I} = \{\emptyset, \{\star\}\}$, and ι is the counting measure. This follows from example 4 and the fact that $(1, \mathcal{I})$ is a terminal object in <u>**Mble**</u>. □

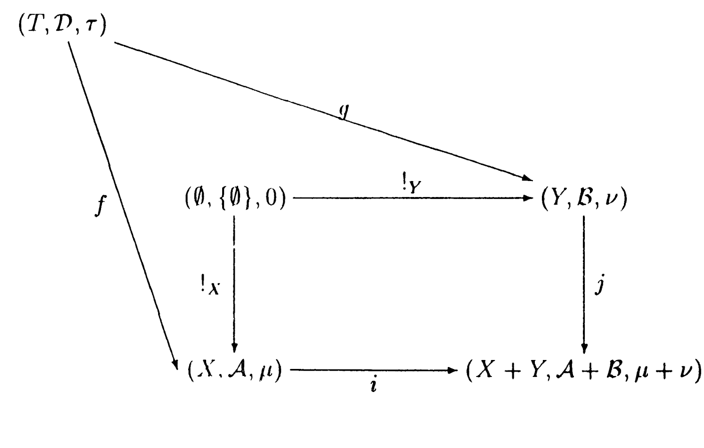
Example 6: As another "special case" of example 4, consider the measure space, $(\mathbf{N}, \mathcal{P}(\mathbf{N}), \text{counting})$, where $\mathbf{N}$ is the set of natural numbers. Now, this space is not finite (it is σ-finite) but any measurable function into it is MOR. In fact a MOR, $f : (X, \mathcal{A}, \mu) \longrightarrow (\mathbf{N}, \mathcal{P}(\mathbf{N}), \text{ counting})$, is the same as a measurable partition of X; $\langle f^{-1}(i) \rangle_{i \in N}$. □

Remark: From example 4, we see that, but for finiteness, we would have an adjunction $U \dashv D$ where D : <u>**Set**</u> $\longrightarrow$ <u>**MOR**</u> is the discrete space functor and U is the forgetful functor. Notice that, in example 3, we allowed an arbitrary measurable space structure so a left adjoint to the underlying functor does not exist. □

Colimits in <u>**MOR**</u> seem to be more well-behaved than limits:

Proposition 2.1 <u>**MOR**</u> *has (a) an initial object given by* $(\emptyset, \{\emptyset\}, 0)$, *(b) binary coproducts, and (c) these coproducts are disjoint.*

Proof: a): There is only one measurable function out of $(\emptyset, \{\emptyset\})$ and it is MOR.
b) The coproduct of $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ is $(X + Y, \mathcal{A} + \mathcal{B}, \mu + \nu)$; $X + Y$ is the disjoint union of X and Y, $\mathcal{A} + \mathcal{B}$ consists of sets of the form $A + B$ where $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $(\mu + \nu)(A + B) := \mu A + \nu B$. It is a simple matter to check that this does indeed define the coproduct. Notice, for example, that if $i : (X, \mathcal{A}, \mu) \longrightarrow (X + Y, \mathcal{A} + \mathcal{B}, \mu + \nu)$ denotes the injection and $(\mu + \nu)(A + B) = 0$, then $\mu(A) = \nu(B) = 0$ so $\mu(i^{-1}(A + B)) = 0$.
c) Consider the diagram:



Now, $j!_Y = i!_X = !_{X+Y}$. Since coproducts are disjoint in **Set**, there are no maps f, g satisfying $jg = if$ (and no map $(T, \mathcal{T}, \tau) \longrightarrow (\emptyset, \{\emptyset\}, 0)$), if $T \neq \emptyset$, and exactly one map, the identity, which is M0R, if $T = \emptyset$. Thus the square is a pullback square as required. ∎

(Counter)example 7: Constant functions are not, in general, M0R. □

(Counter)example 8: The "element of" function, $\lceil x \rceil : 1 \longrightarrow X$, is not, in general M0R (unless $x \in X$ is an atom; $\mu(\{x\}) > 0$). □

Finally, we note that **M0R** has an interesting continuity property.

Proposition 2.2 *Let $f : (X, \mathcal{A}, \mu) \longrightarrow (Y, \mathcal{B}, \nu) \in$ **M0R** and $\{B_n\}$ be a sequence of measurable sets with $B_{n+1} \subseteq B_n$. Then $\lim_{n\to\infty} \nu(B_n) = 0 \Rightarrow \lim_{n\to\infty} \mu(f^{-1}(B_n)) = 0$.*

Proof: $\lim_{n\to\infty} \mu(f^{-1}(B_n)) = \mu(\bigcap_{n=1}^{\infty} f^{-1}(B_n)) = \mu(f^{-1}(\bigcap_{n=1}^{\infty} B_n)) = 0$ since $f \in$ **M0R** and $\nu(\bigcap_{n=1}^{\infty} B_n) = \lim_{n\to\infty} \nu(B_n) = 0$. ∎

Remarks: 1. The proposition says that if the B_n's get small, the $f^{-1}(B_n)$'s get small. However, there is not necessarily a relationship between the rates of convergence to 0.

2. Interesting connections with the metric defined above ($d([A],[B]) = \mu(A \Delta B)$) will be explored in future work. □

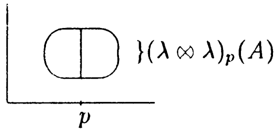
3 Disintegrations

In the introduction, we noted that Breitspecher [4] suggested that disintegrations should be studied from a categorical point of view. We now construct a category whose morphisms are "disintegration-like" (we employ the concept of disintegration in a new way). This turns out to be a useful category in the sense that a disintegration is like a family of measure spaces indexed by a measure space and, needless to say, (see [8]) this is a good thing as far as indexed category theory is concerned. As we shall see, disintegrations have a forgetful functor to **M0R**.

To motivate this category, let us recall a "naive theory of disintegrations." Let $(X, \mathcal{A}, \mu)$ be a measure space and $(P, \mathcal{P}, \rho)$ be another measure space, the *parameter space*. A *disintegration* of μ with respect to ρ is a collection of measures, μ_p, on X, indexed by $p \in P$, such that $\forall A \in \mathcal{A}$, $\mu_p(A)$ is a measurable function of p and $\int_P \mu_p(A) d\rho = \mu(A)$.

Example 1: Constant: Let $\rho(P) = 1$ and let $\mu_p(A) = \mu(A)$ $\forall A \in \mathcal{A}$. Then $\mu_p(A)$ is measurable (as a constant function) and $\int_P \mu_p(A) d\rho = \mu(A) \cdot 1 = \mu(A)$ (note: if $\rho(P) \neq 0$, then we can take $\mu_p(A) = \dfrac{\mu(A)}{\rho(P)}$). □

Example 2: Product: Let $X = (\mathbf{R} \times \mathbf{R}, \mathcal{L} \otimes \mathcal{L}, \lambda \otimes \lambda)$. Let $P = (\mathbf{R}, \mathcal{L}, \lambda)$. For a measurable $A \subseteq \mathbf{R} \times \mathbf{R}$, put $(\lambda \otimes \lambda)_p(A) := \lambda(\{y|(p, y) \in A\})$.



Now, by Fubini's theorem (applied to χ_A), $A_p := \{y|(p, y) \in A\}$ is a measurable subset of the real line and $\int_P \lambda_p(A) d\lambda = (\lambda \otimes \lambda)(A)$. □

Remarks: 1. In the space $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$, if $D \in \mathcal{A} \otimes \mathcal{B}$ then $D_x \in \mathcal{B}$, $\forall x \in X$ (fix $x \in X$, let $\mathcal{K}_x$ be the set of all $E \subseteq X \times Y$ such that $E_x \in \mathcal{B}$, then $\mathcal{K}_x$ contains the measurable rectangles and is a σ-algebra, hence contains $\mathcal{A} \otimes \mathcal{B}$, the smallest σ-algebra that contains the measurable rectangles).
2. We note that $(\mathbf{R} \times \mathbf{R}, \mathcal{L} \otimes \mathcal{L}, \lambda \otimes \lambda)$ is not the Lebesgue plane. Fubini's theorem says, for an $A \in \mathcal{L} \overline{\otimes} \mathcal{L}$, A_p is measurable for almost all $p \in \mathbf{R}$. "Slicing" by p, however, works for members of $\mathcal{R}_{\sigma\delta}$ (remark 1). The "almost all p" arises out of subsets of sets of measure zero (i.e. during the completion part of the process and not before) so $(\lambda \overline{\otimes} \lambda)_p(A) = \lambda(A_p)$ would provide an "almost everywhere" example of a disintegration. □

This is an important example for our purposes, as will be seen below. We will describe many more examples later. Given two measure spaces, one doesn't necessarily possess a disintegration with respect to the other. The main thrust of research in this field is to determine conditions for the existence of such. A definitive answer has not yet been given although there are some important existence theorems (see [10]).

An object of **Disint** is a finite measure space. We will use the projection from the product as suggested by example 2 above as the motivation for our notion of morphism.

Definition 3 *A morphism $(X, \mathcal{A}, \mu) \longrightarrow (Y, \mathcal{B}, \nu)$ in* **Disint** *consists of*

- $f : (X, \mathcal{A}) \longrightarrow (Y, \mathcal{B}) \in$ **Mble**
- *a family $(X_y, \mathcal{A}_y, \mu_y)_{y \in Y}$ of finite measure spaces, where $X_y := f^{-1}(y)$ and $\mathcal{A}_y = \{A \cap f^{-1}(y) \mid A \in \mathcal{A}\}$*

subject to the axioms:

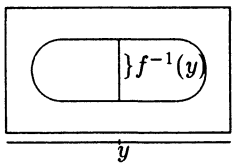
ax1: $\forall A \in \mathcal{A}$, the map $y \mapsto \mu_y(A \cap f^{-1}(y))$ is measurable and bounded

ax2: $\forall A \in \mathcal{A}$, $\mu(A) = \int_Y \mu_y(A \cap f^{-1}(y)) d\nu(y)$. □

Remarks: 1. $\mathcal{A}_y = \{A \cap f^{-1}(y) \mid A \in \mathcal{A}\}$ is a σ-algebra (this follows immediately from the fact that $\mathcal{A}$ is a σ-algebra).

2. We call these morphisms *disintegrations* as well and will refer to axiom 1 as "measure boundedness."
3. For boundedness, it is enough to say $\mu_y(X \cap f^{-1}(y)) \in L^\infty(Y)$ because of monotonicity of measures (of course, the measurability condition for all $A \in \mathcal{A}$ is still necessary).
4. Every disintegration has a "norm" via $||\mu_y(A \cap f^{-1}(y))||_\infty$. We will not explore this in this paper.
5. Each $\mu_y(X_y) < \infty$. Measure boundedness is a condition on the $\mu_y(X_y)$'s over $y \in Y$. □

Since the paradigm for a morphism of Disint is the product example above, we think of the fibres over the y's as slicing up A:



Notation: The fibre measurable spaces depend solely upon f. We write $(f, (\mu_y)_{y \in Y})$ or (f, μ_y) for a morphism in Disint. □

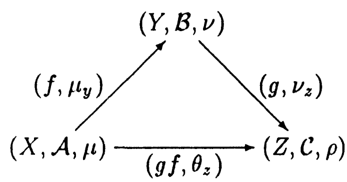
We make this into a category with the following definitions:

Identity: Define the identity in Disint as $(1_X, \iota_x) : (X, \mathcal{A}, \mu) \longrightarrow (X, \mathcal{A}, \mu)$ where 1_X is the identity function and ι_x is counting measure on $\mathcal{I}_x = \{A \cap 1^{-1}(x) \mid A \in \mathcal{A}\}$, the discrete σ-algebra on $\{x\}$.

Axiom 1: $x \mapsto \iota_x(A \cap \{x\})$ is measurable and bounded since it is just χ_A and A is a measurable set. □

Axiom 2: $\int_X \iota_x(A \cap \{x\}) d\mu(x) = \int_X \chi_A d\mu(x) = \mu(A)$ as required. □

Composition: Consider the diagram:



where θ_z is defined as:

$$\theta_z(E) := \int_{g^{-1}(z)} \mu_y(E \cap f^{-1}(y)) d\nu_z(y) \text{ for } E \in \mathcal{E}_z = \{A \cap f^{-1} g^{-1}(z) \mid A \in \mathcal{A}\}.$$

Note that ν_z is defined on $g^{-1}(z)$ and $f^{-1}g^{-1}(z) = \bigcup_{y \in g^{-1}(z)} f^{-1}(y)$, the union being disjoint and $A \cap f^{-1}g^{-1}(z) \cap f^{-1}(y) = \begin{cases} A \cap f^{-1}(y) & y \in g^{-1}(z) \\ \emptyset & y \notin g^{-1}(z). \end{cases}$

Axiom 1: We wish to show that $\theta_z(E) = \int_{g^{-1}(z)} \mu_y(E \cap f^{-1}(y))d\nu_z(y)$ is a measurable function of z. Before we do that, however, we must determine that the integral makes sense.

Proposition 3.1 *For each $z \in Z$ and for each $E \in \mathcal{A}$, $\mu_y(E \cap f^{-1}(y))$ is a ν_z-measurable function.*

Proof: $\mu_y(E \cap f^{-1}(y))$ is a ν-measurable function (by axiom 1 for the μ_y's). Let $\alpha \in \mathbf{R}$, then $B = \{y \in Y \mid \mu_y(E \cap f^{-1}(y)) < \alpha\} \in \mathcal{B}$ and $B \cap g^{-1}(z) = \{y \in g^{-1}(z) \mid \mu_y(E \cap f^{-1}(y)) < \alpha\} \in \mathcal{B}_z$ for all $z \in Z$. ∎

Proposition 3.2 *$z \mapsto \int_{g^{-1}(z)} k(y)d\nu_z(y)$ is a measurable function of z for $k(y)$ a non-negative ν-measurable function (in particular for $k(y) = \mu_y(E \cap f^{-1}(y))$).*

Proof: Case $k = \chi_E$, $E \in \mathcal{B}$: $z \mapsto \int_{g^{-1}(z)} \chi_E d\nu_z(y) = \nu_z(E \cap g^{-1}(z))$ which is z-measurable by axiom 1 for ν_z.
Case $k =$ a simple function: Apply the above case and linearity of the integral.
Case $k =$ a non-negative measurable function. Let $\langle \phi_n(y) \rangle$ be a sequence of simple functions increasing to k. Then $z \mapsto \int_{g^{-1}(z)} k(y)d\nu_z(y) = \int_{g^{-1}(z)} \lim \phi_n(y)d\nu_z(y)$ $= \lim \int_{g^{-1}(z)} \phi_n(y)d\nu_z(y)$ by the monotone convergence theorem. Each $z \mapsto \int_{g^{-1}(z)} \phi_n(y)d\nu_z$ is z-measurable by the above case and the limit of a sequence of measurable functions is measurable. ∎
Remark: The technique used in the above proposition is a very useful one. We will use the "build it up from simple functions" idea in many of our results. □

Proposition 3.3 *θ_z is a measure for each z.*

Proof: $\theta_z(\emptyset) = \int_{g^{-1}(z)} \mu_y(\emptyset \cap f^{-1}(y))d\nu_z(y) = \int_{g^{-1}(z)} 0 d\nu_z(y) = 0.$

$$\begin{aligned}
\theta_z(\bigcup_i A_i \cap f^{-1}g^{-1}(z)) &= \int_{g^{-1}(z)} \mu_y(\bigcup_i A_i \cap f^{-1}(y))d\nu_z(y) \\
&= \int_{g^{-1}(z)} \sum_i \mu_y(A_i \cap f^{-1}(y))d\nu_z(y) \\
&= \sum_i \int_{g^{-1}(z)} \mu_y(A_i \cap f^{-1}(y))d\nu_z(y) \\
&= \sum_i \theta_z(A_i \cap f^{-1}g^{-1}(z)). \quad ∎
\end{aligned}$$

Proposition 3.4 *θ_z is a bounded function (over $z \in Z$).*

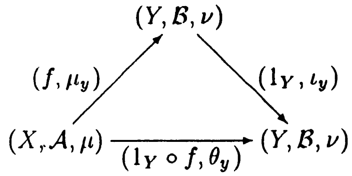
Proof: Certainly, $\theta_z(A \cap f^{-1}g^{-1}(z)) = \int_{g^{-1}(z)} \mu_y(A \cap f^{-1}(y))d\nu_z(y)$ is finite for all $z \in Z$ (since $\mu_y(A \cap f^{-1}(y))$ is bounded and ν_z is a finite measure). Furthermore, suppose ν_z and μ_y are bounded by K and M respectively, say. Then $\int_{g^{-1}(z)} \mu_y(A \cap f^{-1}(y))d\nu_z(y) \leq \int_{g^{-1}(z)} K d\nu_z(y) \leq M \cdot K < \infty$. ∎

Axiom 2:

Proposition 3.5 $\mu(A) = \int_Z \theta_z(A \cap f^{-1}g^{-1}(z))d\rho(z)$ *(axiom 2).*

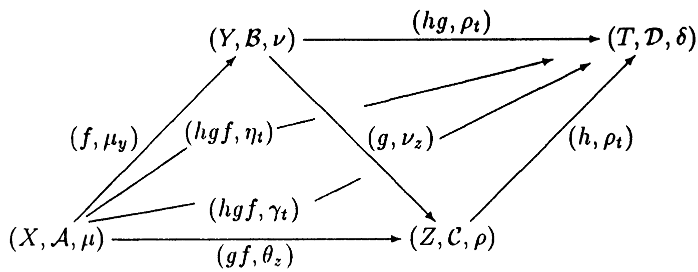
Proof: By axiom 2 for the μ_y's, $\mu(A) = \int_Y \mu_y(A \cap f^{-1}(y))d\nu(y) = (*)$. Now, $\int_Z \theta_z(A \cap f^{-1}g^{-1}(z))d\rho(z) = \int_Z \int_{g^{-1}(z)} \mu_y(A \cap f^{-1}(y))d\nu_z(y)d\rho(z) = (**)$
Thus, we must show $(*) = (**)$. We will show that $\int_Y k(y)d\nu(y)$
$= \int_Z \int_{g^{-1}(z)} k(y)d\nu_z(y)d\rho(z)$ for all (positive) measurable functions $k(y)$.
Case $k(y) = \chi_E$, $E \in \mathcal{B}$: $\int_Y \chi_E d\nu(y) = \nu(E)$ and $\int_Y \int_{g^{-1}(z)} \chi_E d\nu_z(y)d\rho(z)$
$= \int_Y \nu_z(E \cap g^{-1}(z))d\rho = \nu(E)$ by axiom 2 for ν_z.
Case $k(y)$ = a simple function: $\int = \int\int$ by linearity of the integral and the above case.
Case $k(y)$ = a positive measurable function: Let $\phi_n \uparrow k(y)$ be a sequence of simple functions increasing to k. Then $\int_Z \int_{g^{-1}(z)} k(y)d\nu_z(y)d\rho(z) =$
$\int_Z \int_{g^{-1}(z)} k(y)d\nu_z(y)d\rho(z) = \int_Z \int_{g^{-1}(z)} \lim \phi_n(y)d\nu_z(y)d\rho(z)$
$= \int_Z \lim \int_{g^{-1}(z)} \phi_n(y)d\nu_z(y)d\rho(z) = \lim \int_Z \int_{g^{-1}(z)} \phi_n(y)d\nu_z(y)d\rho(y) = \clubsuit$, by repeated application of the monotone convergence theorem. Now, by the above case, $\clubsuit = \lim \int_Y \phi_n(y)d\rho(z)$ and applying the monotone convergence theorem again, we have $\clubsuit = \int_Y k(y)d\nu$. ∎

Unit laws: Consider:



Now, $\theta_y(E \cap f^{-1}1^{-1}(y)) = \int_{1^{-1}(y)} \mu_y(E \cap f^{-1}(y))d\iota_y(x) = \mu_y(E \cap f^{-1}(y))$.
In a similar way, write $(f \circ 1_X, \gamma_y) = (f, \mu_y) \circ (1_X, \iota_x)$. Then $\gamma_y(E \cap 1^{-1}f^{-1}(y))$ $= \int_{f^{-1}(y)} \iota_x(E \cap 1^{-1}(x))d\mu_y(x) = \int_{f^{-1}(y)} \chi_E d\mu_y(x) = \mu_y(E \cap f^{-1}(y))$ as required. □

Associativity: Consider the diagram:



To prove associativity, we must show $\eta_t = \gamma_t$ for all $t \in T$. But,

$$\begin{aligned} \eta_t(F) &= \int_{h^{-1}(t)} \theta_z(F \cap f^{-1}g^{-1}(z))d\rho_t(z) \\ &= \int_{h^{-1}(t)} \int_{g^{-1}(z)} \mu_y(F \cap f^{-1}(y))d\nu_z(y)d\rho_t(z) \\ \text{and } \gamma_t(F) &= \int_{g^{-1}h^{-1}(t)} \mu_y(F \cap f^{-1}(y))d\beta_t(y). \end{aligned}$$

and we have:

Proposition 3.6 $\int_{g^{-1}h^{-1}(t)} k(y)d\beta_t(y) = \int_{h^{-1}(t)} \int_{g^{-1}(z)} k(y)d\nu_z(y)d\rho_t(z)$ *for all positive, measurable functions* $k(y)$.

Proof: Apply the proof of proposition 3.5 with $Y := g^{-1}h^{-1}(t)$, $Z := h^{-1}(t)$, $\nu := \beta_t$, and $\rho := \rho_t$. ∎

We next list some examples and basic properties.

Proposition 3.7 $(f,\mu_y):(X,\mathcal{A},\mu)\longrightarrow(Y,\mathcal{B},\nu)\in$ **Disint** $\Rightarrow f\in$ **MOR**

Proof: Let $\nu(B)=0$. Then we have:

$$\begin{aligned}\mu(f^{-1}B) &= \int_Y \mu_y(f^{-1}B\cap f^{-1}y)d\nu(y)\\ &= \int_Y \mu_y(f^{-1}(B\cap\{(y)\}))d\nu(y)\\ &= \int_B \mu_y(f^{-1}(y))d\nu(y)=0.\ \blacksquare\end{aligned}$$

Remark: In view of this proposition and counterexample 2 above, we see that the diagonal is not in **Disint**. □

Example 1: Let $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ be two finite measure spaces. Define $(p,(\mu\otimes\nu)_x):(X\times Y,\mathcal{A}\otimes\mathcal{B},\mu\otimes\nu)\longrightarrow(X,\mathcal{A},\mu)$ as follows: p is the projection to the first factor, $p^{-1}(x)=\{x\}\times Y$ and $(\mathcal{A}\otimes\mathcal{B})_x=\{D\cap p^{-1}(x)\mid D\in\mathcal{A}\otimes\mathcal{B}\}=\{x\}\times\mathcal{B}$ (certainly, we have $\supseteq$ (take $D=\{x\}\times B$); conversely, for $D=A\times B\in\mathcal{A}\otimes\mathcal{B}$, $D\cap p^{-1}(x)=\begin{cases}\{x\}\times B & x\in A\\ \emptyset & x\notin A\end{cases}$ both of which are in $\{x\}\times\mathcal{B}$ and since $\{x\}\times\mathcal{B}$ is a σ-algebra, we have $\subseteq$). So, define $(\mu\otimes\nu)_x(D\cap p^{-1}(x)):=\nu(D_x)$ where D_x is considered as an element of $\mathcal{B}$ (more precisely, we define $(\mu\otimes\nu)_x(\{x\}\times B):=\nu(B)$ but will, however, abuse notation on occasion).

We have already noted that the slices D_x are all measurable. One may exhibit axioms 1 and 2 in a similar manner. Let $\mathcal{D}$ be the collection of those D's (in $\mathcal{A}\otimes\mathcal{B}$) for which $x\mapsto\nu(D_x)$ is ($\mathcal{A}$-)measurable and for which $(\mu\otimes\nu)(D)=\int_X\nu(D_x)d\mu(x)$. It is straightforward to show that $\mathcal{D}$ contains all measurable rectangles and is a σ-algebra, whence $\mathcal{D}=\mathcal{A}\otimes\mathcal{B}$. Axiom 2 is a special case of Fubini's theorem. □

Example 2: Let A_0 be a measurable subset of $X=(X,\mathcal{A},\mu)$. We may interpret the inclusion $i:(A_0,\mathcal{A}_0,\mu_0)\longrightarrow(X,\mathcal{A},\mu)$, where $\mathcal{A}_0=\{A\subseteq A_0|A\in\mathcal{A}\}$ and $\mu_0(A)=\mu(A)$, as a disintegration. If $x\in A_0$, $\mathcal{I}_x=\{A\cap i^{-1}(x)\mid A\in\mathcal{A}\}=\{\emptyset,\{x\}\}$; put $\mu_{0x}=$ counting measure. If $x\notin A_0$, $\mathcal{I}_x=\{\emptyset\}$; put $\mu_{0x}=0$. So, $\mu_{0x}(A\cap i^{-1}(x))=\chi_{A\cap A_0}$. The proof that axioms 1 and 2 hold is exactly the same as that for the identity disintegration. □

Remark: This example does not "contradict" the fact that the diagonal is not a disintegration. Interpreting the diagonal as a *subspace* of the plane would give $(X,\mathcal{A},0)\longrightarrow(X\times X,\mathcal{A}\otimes\mathcal{A},\mu\otimes\mu)$. □

Example 3: Let $(X,\mathcal{A},\mu)$ be such that $\mu(A)=0$, $\forall A\in\mathcal{A}$. Then any measurable function $f:(X,\mathcal{A},\mu)\longrightarrow(Y,\mathcal{B},\nu)$ may be interpreted as a disintegration by defining $\mu_y(A\cap f^{-1}(y))=0$, for all $A\in\mathcal{A}$ and $y\in Y$. □

Example 4: A terminal object of **Disint** is $(1,2,$ counting$)$. The unique map, $!_X:(X,\mathcal{A},\mu)\longrightarrow(1,2,$ counting$)$ has $(X_\star,\mathcal{A}_\star)=(X,\mathcal{A})$ and $\mu_\star=\mu$. Suppose $(!,\beta_\star=\beta):(X,\mathcal{A},\mu)\longrightarrow(1,2,\text{counting})$ is another disintegration. By axiom 2 for $\beta_\star$, we have $\mu(A)=\int_\star\beta_\star(A\cap !^{-1}(\star))d(\text{counting})=\beta(A)$. □

Example 5: The initial object of <u>**Disint**</u> is $(\emptyset, \{\emptyset\}, 0)$, which is a special case of example 2. □

Proposition 3.8 <u>**Disint**</u> *has (a) binary coproducts and (b) these coproducts are disjoint.*

Proof: a) Referring to proposition 2.1, the injection is a disintegration $(i, \mu_t) : (X, \mathcal{A}, \mu) \longrightarrow (X + Y, \mathcal{A} + \mathcal{B}, \mu + \nu)$, with $\mathcal{A}_t = \{\emptyset\}$ and $\mu_t = 0$ if $t \in Y$ and $\mathcal{A}_t = \{\emptyset, \{t\}\}$ and $\mu_t(A \cap i^{-1}(t)) = \chi_A(t)$ if $t \in X$.
b) Again, referring to the diagram of proposition 2.1. If $T = \emptyset$, then we may insert the identity disintegration, $T \longrightarrow \emptyset$. If $T \neq \emptyset$, then there is no map, $T \longrightarrow \emptyset$ and no maps making the "outside" square commute. ∎

Example 6: Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two finite, discrete spaces. That is, X and Y are finite sets, $\mathcal{A} = 2^X$, $\mathcal{B} = 2^Y$, and μ and ν are counting measures. Every function, $f : X \longrightarrow Y$, is measurable. Let $(f, \mu_y) : (X, \mathcal{A}, \mu) \longrightarrow (Y, \mathcal{B}, \nu)$ be a disintegration. $\mathcal{A}_y = \{A \cap f^{-1}(y) \mid A \in \mathcal{A}\} = 2^{f^{-1}(y)}$ for all $y \in Y$. To satisfy axiom 2, μ_y must be counting measure. And, such will automatically satisfy axiom 1. Thus, every measurable function yields a unique disintegration. That is, there is a full functor $D : \underline{\mathbf{Set}}_f \longrightarrow \underline{\mathbf{Disint}}$. □

Example 7: Consider $1_X : (X, \mathcal{A}, \mu) \longrightarrow (X, \mathcal{B}, \nu)$. To say 1_X is measurable is to say $\mathcal{B} \subseteq \mathcal{A}$. To say $1_X \in$ <u>**M0R**</u> is to say, in addition, $\nu(B) = 0 \Rightarrow \mu(B) = 0$ or $\mu << \nu$ (μ restricted to $\mathcal{B}$).

Suppose we wish to put a disintegration structure on this. Each μ_x is a measure on $1^{-1}(x) = \{x\}$, i.e. a nonnegative real number, $m(x)$. For each $A \in \mathcal{A}$, $\mu_x(A \cap \{x\}) = \left\{ \begin{matrix} m(x) & x \in A \\ 0 & x \notin A \end{matrix} \right\} = m(x) \cdot \chi_A$. Now, $m(x)$ is measurable by axiom 1 (take $A = X$) and axiom 2 implies $\mu(A) = \int m(x) \cdot \chi_A d\nu(x) = \int_A m(x) d\nu(x)$.
Thus, $m(x) = \left[\frac{d\mu}{d\nu}\right]$ the Radon-Nikodym derivative.

Conversely, given $\mu << \nu$, there is a nonnegative measurable function $m(x)$ $= \left[\frac{d\mu}{d\nu}\right]$. Put $\mu_x(A \cap \{x\}) := m(x) \cdot \chi_A$ and we have a disintegration structure.

Thus, the identity function, if it is M0R, has an automatic disintegration structure. Of course, the Radon-Nikodym derivative is unique (up to a.e. equivalence), so there is essentially only one disintegration structure on the identity. □

Example 8: Let us expand on example 7 by computing isomorphisms. Consider $f : (X, \mathcal{A}, \mu) \rightleftarrows (Y, \mathcal{B}, \nu) : g$. To say f is an isomorphism in <u>**M0R**</u> is to say: 1. $fg = 1_Y$ and $gf = 1_X$ (i.e. f is a bijection), 2. $f(A) \in \mathcal{B}$ iff $A \in \mathcal{A}$, and 3. $\mu(A) = 0$ iff $\nu(f(A)) = 0$.

Again, let us put a disintegration structure on this: (f, μ_y), (g, ν_x), (gf, θ_x), and (fg, δ_y). We have $\chi_A = \iota_x(A \cap \{x\}) = \theta_x(A \cap f^{-1}g^{-1}(x))$
$= \int_{g^{-1}(X)} \mu_y(A \cap f^{-1}(y) d\nu_x(y)$. Now, $\mu_y(A \cap f^{-1}(y))$ determines a function, $m(y)$:

$\mu_y(A \cap f^{-1}(y)) = \left\{ \begin{array}{cc} m(y) & f^{-1}(y) \in A \\ 0 & \text{else} \end{array} \right\} = m(y) \cdot \chi_{f(A)}$. Axiom 1 says $m(y)$ is measurable and axiom 2 says $\mu(A) = \int_Y m(y) \cdot \chi_{f(A)} d\nu(y)$. Similarly, there is a nonnegative measurable function, $n(x)$, such that $\nu(B) = \int_X n(x) \cdot \chi_{g(B)} d\mu(x)$.

Proposition 3.9 *Define a measure by $\mu'(A) = \nu(f(A)$ (in the situation above). Then, for any positive measurable function, $m(y)$, we have*
$\int_Y m(y) \cdot \chi_{f(A)} d\nu(y) = \int_A m(f(x)) d\mu'(x)$.

Proof: We prove only the basic case $m(y) = \chi_B$: LHS $= \int_Y \chi_{B \cap f(A)} d\nu(y)$
$= \nu(B \cap f(A))$. RHS $= \int_A \chi_B(f(x)) d\mu'(x) = \int_A \chi_{f^{-1}(B)}(x) d\mu'(x)$
$= \mu'(A \cap f^{-1}(B)) = \nu(F(A) \cap f f^{-1}(B)) = \nu(B \cap f(A))$. ∎

And so, $\mu(A) = \int_Y m(y) \cdot \chi_{f(A)} d\nu(y) = \int_A n(f(x)) d\mu'(x)$ which yields a Radon-Nikodym derivative for $\mu << \mu'$ (and similarly for $\nu <<$ an appropriately defined ν').

Conversely, given f an isomorphism in **MOR**, $\mu << \mu' := \nu \circ f$ so there is a nonnegative measurable function $k(x)$ with $\mu(A) = \int_A k(x) d\mu'(x)$. Putting $m(y) := k(f^{-1}(y))$ and $\mu_y(A \cap f^{-1}(y)) := m(y) \cdot \chi_{f(A)}$ gives a disintegration structure (and likewise for ν).

Thus, an isomorphism in **MOR** determines a unique (up to a.e. equivalence) isomorphism in **Disint**. And so, to prove that two objects are isomorphic in **Disint**, it is enough to show they are isomorphic in **MOR**. □

As our final example, we note that not every MOR can be made into a disintegration.

Example 9: Let $(X, \mathcal{A}, \mu)$ be the region of the plane bounded by the positive axes, the curve $y = \frac{1}{\sqrt{x}}$, and the line $x = 1$ with $\mathcal{L} \otimes \mathcal{L}$ and $\lambda \otimes \lambda$ restricted. Let $(Y, \mathcal{B}, \nu) = ([0,1], \mathcal{L}, \lambda)$ and let f be projection. Then $f \in$ **MOR**. Suppose we have a disintegration structure (f, μ_y). Consider the set $A = [0, \frac{1}{n}] \times [0, \sqrt{n}] \subseteq X$. Then $\frac{1}{\sqrt{n}} = \frac{\sqrt{n}}{n} = \text{Area}(A) = \mu(A) = \int_0^{\frac{1}{n}} \mu_y(A \cap f^{-1}(y)) d\nu(y)$. Now, a disintegration has measure boundedness, so suppose $\mu_y(A \cap f^{-1}(y)) < K$ for all y so that the integral $< K \cdot \frac{1}{n}$, whence $\frac{1}{\sqrt{n}} < K \cdot \frac{1}{n}$. But, for any K, there is an n large enough so that this is false. □

Mble has products which make it into a monoidal category. The unit is a (fixed) terminal object $(1, 2)$. **Disint** is also monoidal. More precisely,

Proposition 3.10 *($\underline{\textbf{Disint}}, \otimes, (1, 2, counting)$) is a monoidal category. $\otimes$ denotes the usual product of (not necessarily complete) measure spaces. For $(f, \mu_y) : (X, \mathcal{A}, \mu) \longrightarrow (Y, \mathcal{B}, \nu)$ and $(g, \rho_t) : (Z, \mathcal{C}, \rho) \longrightarrow (T, \mathcal{D}, \delta)$, the product $f \times g : (X \times Z, \mathcal{A} \otimes \mathcal{C}, \mu \otimes \rho) \longrightarrow (Y \times T, \mathcal{B} \otimes \mathcal{D}, \nu \otimes \delta)$ is a disintegration as follows: $(\mathcal{A} \otimes \mathcal{C})_{(y,t)} = \{D \cap f^{-1}(y) \times g^{-1}(t) \mid D \in \mathcal{A} \otimes \mathcal{C}\} = \mathcal{A}_y \otimes \mathcal{C}_t$, so define $(\mu \otimes \rho)_{(y,t)}(D) = (\mu_y \otimes \rho_t)(D)$ with D considered as an element of $\mathcal{A}_y \times \mathcal{C}_t$.*

Lemma 3.1 *$\mu_y \otimes \rho_t$ satisfies axioms 1 and 2.*

Proof: **Axiom 1:** $k(y,t) = (\mu_y \otimes \rho_t)(D \cap f^{-1}(y) \times g^{-1}(t))$ is measurable and bounded:

If $D = A \times C$ is a measurable rectangle, then $k(y,t) = \mu_y(A \cap f^{-1}(y)) \cdot \rho_t(C \cap g^{-1}(t))$ is measurable and bounded since it is a product of two such (axiom 1 for μ_y and ρ_t). Furthermore, $k(y,t) \leq (\mu_y \otimes \rho_t)(X \times Z \cap f^{-1}(y) \times g^{-1}(t)) < \infty$. That is, k is bounded for any D. We need only check that it is measurable.

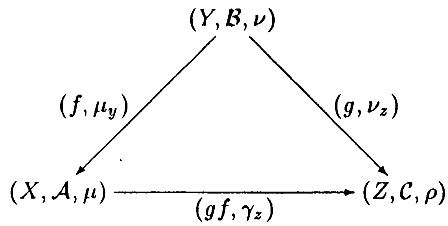
If $D = \bigcup_{i=1}^{\infty} A_i \times C_i$ is a disjoint union of rectangles, then $k(y,t)$ $= \sum_{i=1}^{\infty} \mu_y(A_i \cap f^{-1}(y)) \cdot \rho_t(C_i \cap g^{-1}(t))$ is a sum of measurable functions so is measurable. Now an arbitrary (countable) union can be written as a disjoint (countable) union (for example, for $\bigcup_{i=1}^{\infty} A_i$, put $B_i = A_i \setminus \bigcup_{j<i} A_j$ then $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$ and the B_i's are disjoint). For finite intersections, use, for γ a finite measure, $\gamma(D_1 \cap D_2) = \gamma(D_1) + \gamma(D_2) - \gamma(D_1 \cup D_2)$. Finally, for countable intersections, use $\gamma(\bigcap_{i=1}^{\infty} D_i) = \lim_{n\to\infty} \gamma(\bigcap_{i=1}^{n} D_i)$ (with $\gamma(D_1) < \infty$).

Axiom 2: Again, the process is exactly as for axiom 1 (disjoint unions use additivity; increasing limits and sums pull out of integrals by the monotone convergence theorem). We only check the basic case, $D = A \times C$:

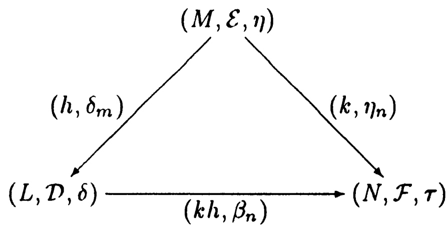
$$
\begin{aligned}
&\int (\mu_y \otimes \rho_t)(A \times C \cap f^{-1}(y) \times g^{-1}(t)) d(\nu \otimes \delta)(y,t) \\
&= \int \mu_y(A \cap f^{-1}(y)) d\nu(y) \cdot \int \rho_t(C \cap g^{-1}(t)) d\delta(t) \\
&= \mu(A) \cdot \rho(C) \\
&= (\mu \otimes \rho)(A \times C) \quad \blacksquare
\end{aligned}
$$

Proof: (of proposition 3.10): It is straightforward to check that $(1, 2, \text{counting})$ is a unit. We exhibit bifunctoriality of $\otimes$.

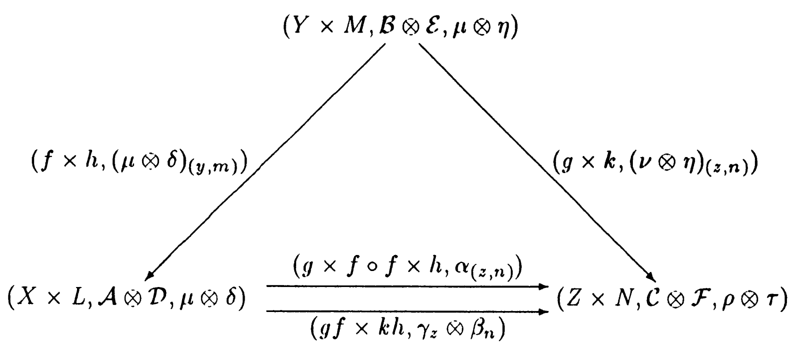
In $(1 \times 1, (\iota \otimes \iota)_{(x,y)}) : (X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \otimes \mu) \longrightarrow (X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \otimes \mu)$, we have $(\iota \otimes \iota)_{(x,y)} = \iota_x \otimes \iota_y = \iota_{(x,y)}$. Now, suppose



and



denote two compositions in **<u>Disint</u>** and consider:



where $\alpha_{(z,n)} = (\nu \otimes \eta)_{(z,n)} \circ (\mu \otimes \delta)_{(y,m)}$. We show that $\alpha_{(z,n)} = \gamma_z \otimes \beta_n$ on the generators of $(\mathcal{A} \otimes \mathcal{D})_{(z,n)} = \mathcal{A}_z \otimes \mathcal{D}_n$.

$$
\begin{aligned}
&\alpha_{(z,n)}(A \times D \cap (g \times k \circ f \times h)^{-1}(z,n) \\
&= \int_{(g\times k)^{-1}(z,n)} (\mu \otimes \delta)_{(y,m)}(A \times D \cap (f \times h)^{-1}(y,m))d(\nu \otimes \eta)_{(z,n)}(y,m) \\
&= \int_{g^{-1}(z)} \int_{k^{-1}(n)} (\mu \otimes \delta)_{(y,m)}(A \times D \cap (f \times h)^{-1}(y,m))d\nu_z(y)d\eta_n(m)
\end{aligned}
$$

$$\begin{aligned}
&= \int_{g^{-1}(z)} \mu_y(A \cap f^{-1}(y))d\nu_z(y) \cdot \int_{k^{-1}(n)} \delta_m(D \cap h^{-1}(m))d\eta_n(m) \\
&= \gamma_z(A \cap f^{-1}g^{-1}(z)) \cdot \beta_n(D \cap h^{-1}k^{-1}(n)) \\
&= (\gamma_z \otimes \beta_n)(A \cap f^{-1}g^{-1}(z) \times D \cap h^{-1}k^{-1}(n)). \quad \blacksquare
\end{aligned}$$

<u>**MOR**</u> is also monoidal. Our proof, however, is quite complicated and uses "disintegration machinery" (part of our ongoing investigation is to find a more elementary and elegant proof). Here we prove the special case: $f \in \underline{\mathbf{MOR}} \Rightarrow f \times 1_Z \in \underline{\mathbf{MOR}}$ (after which, it is straightforward to exhibit <u>**MOR**</u> as monoidal). Consider the square:

$$\begin{array}{ccc}
(X \times Z, \mathcal{A} \otimes \mathcal{C}, \mu \otimes \rho) & \xrightarrow{f \times 1_Z} & (Y \times Z, \mathcal{B} \otimes \mathcal{C}, \nu \otimes \rho) \\
\Big\downarrow (q, (\mu \otimes \rho)_y) & & \Big\downarrow (p, (\nu \otimes \rho)_y) \\
(X, \mathcal{A}, \mu) & \xrightarrow[f]{} & (Y, \mathcal{B}, \nu)
\end{array}$$

with p and q projections (this square is a special case of a pull-back-like construction (not universal) which we will consider in the sequel).

By Fubini's theorem (or, see example 1), for $E \in \mathcal{A} \otimes \mathcal{C}$, $(\mu \otimes \rho)(E) = \int_X (\mu \otimes \rho)_x(E \cap q^{-1}(x))d\mu(x)$. Thus, for $K \in \mathcal{B} \otimes \mathcal{C}$, $(\mu \otimes \rho)((f \times 1_Z)^{-1}(K)) = \int_X (\mu \otimes \rho)_x((f \times 1_Z)^{-1}(K) \cap q^{-1}(x))d\mu(x) = (\heartsuit)$. Now, $(f \times 1_Z)^{-1}(K) \cap q^{-1}(x) = \{(x_0, z_0) \mid (f(x_0), z_0) \in K, x_0 = x\} = \{x\} \times \{z \mid z \in K_{f(x)}\}$. There is an alternate description: $K \cap p^{-1}f(x) = \{(y_0, z_0) \mid y_0 = f(x), (y_0, z_0) \in K\} = \{f(x)\} \times K_{f(x)}$. $(\nu \otimes \rho)_y$ is the constantly ρ measure (example 1 of a disintegration) so $(\heartsuit) = \int_X (\nu \otimes \rho)_{f(x)}(K \cap p^{-1}f(x)))d\mu(x)$.

Next, suppose $(\nu \otimes \rho)(K) = 0$. We wish to show $(\clubsuit) = (\mu \otimes \rho)((f \times 1_Z)^{-1}(K)) = \int_X (\nu \otimes \rho)_{f(x)}(K \cap p^{-1}f(x)))d\mu(x) = 0$. We have a disintegration, so $(\spadesuit) = \int_Y (\nu \otimes \rho)_y(K \cap p^{-1}(y))d\nu(y) = (\nu \otimes \rho)(K) = 0$. The integrand in $(\clubsuit)$ is the composite of the integrand in $(\spadesuit)$ (which is, in particular, a positive, measurable function) and $f(x)$ so our result will be established if we prove the following:

Proposition 3.11 *If* $X \xrightarrow{f} Y \xrightarrow{t} \mathbf{R}^{\geq 0}$ *with* $f \in \underline{\mathbf{MOR}}$ *and* $\int_Y t(y)d\nu(y) = 0$, *then* $\int_X (t \circ f)(x)d\mu(x) = 0$.

Proof: As usual, we proceed in steps:

Case $t = \chi_B$, $\nu(B) = 0$: $t \circ f(x) = \left\{ \begin{array}{ll} 1 & f(x) \in B \\ 0 & f(x) \notin B \end{array} \right\} = \left\{ \begin{array}{ll} 1 & x \in f^{-1}(B) \\ 0 & xnotinf^{-1}(B) \end{array} \right.$ so $\int_X t \circ f(x) d\mu(x) = \int_{f^{-1}(B)} 1 d\mu(x) = \mu(f^{-1}(B)) = 0$.

Case $t = a\chi_{B_1} + b\chi_{B_2}$, $B_1 \cap B_2 = \emptyset$ and both of measure zero: $\int_X t \circ f(x) d\mu(x)$ $= a \cdot 0 + b \cdot 0 = 0$ (note: $a, b \geq 0$).

Case t a nonnegative measurable function: let $s_n \uparrow t$ be a sequence of simple functions. Then $\int_X t \circ f(x) d\mu(x) = \int_X \lim s_n \circ f(x) d\mu(x) = \lim \int_X s_n \circ t(x) d\mu(x)$ $= \lim 0 = 0$. ∎

Remark: Finiteness of measure is important for consider the following example: let $([0,1], \mathcal{A}, \mu)$ denote the Lebesgue measurable sets but with $\mu(A) = \left\{ \begin{array}{ll} 0 & \lambda(A) = 0 \\ \infty & \lambda(A) \neq 0 \end{array} \right.$ and form $1 \times 1 = 1 : ([0,1] \times [0,1], \mathcal{A} \otimes \mathcal{A}, \mu \otimes \mu) \longrightarrow ([0,1] \times [0,1], \mathcal{L} \otimes \mathcal{L}, \lambda \otimes \lambda)$. The diagonal has $\lambda \otimes \lambda$-measure zero. But, if we cover it in $\mathcal{A} \otimes \mathcal{A}$ by any collection of rectangles (of nonempty interior say), the measure of such a cover is infinite;, whence $(\mu \otimes \mu)(1^{-1}(\text{diagonal})) \not\to 0$. □

Epilogue: We consider the work here as providing a background for our framework for X-families (of Hilbert spaces, for example, but ultimately, of many other types of objects as well). Indeed, in the sequel, we will interpret **<u>MOR</u>** and **<u>Disint</u>** in the context of fibrations (for Bénabou style indexed category theory, [2]). The material presented in this paper also stands alone, however. We have constructed two monoidal categories. With **<u>Disint</u>**, we have addressed a suggestion given long ago by Breitsprecher. **<u>Disint</u>** (more precisely, examples 7 and 8) seems to encode the Radon-Nikodym derivative and Fubini's theorem (example 1) in a categorical way (and in a way that is different from [3] and [6]). □

References

1. Adamek, J., Herrlich, H., Strecker, G., **Abstract and Concrete Categories**, Wiley (U.S.A., 1990)

2. Bénabou, J., "Fibered Categories and the Foundations of Naive Category Theory," J. Symb. Logic (1985), pp.10-37.

3. Börger, R., "Integration Over Sequential Boolean Algebras," preprint (1989).

4. Breitsprecher, S., "On the Concept of a Measurable Space: I," in Springer Lecture Notes 753, Springer (New York, 1979), pp. 157-168.

5. Dixmier, J., **C^*-Algebras**, North-Holland Mathematical Library 15, North Holland (New York, 1982).

6. Linton, F. E., **The Functorial Foundations of Measure Theory**, Ph. D. Thesis, (Columbia, 1963).

7. Lever, D., **Continuous Families, Topological Categories, and Toposes**, Ph. D. Thesis, (Dalhousie, 1981).

8. Paré, R., Schumacher, D., "Abstract Families and the Adjoint Functor Theorem," in Springer Lecture Notes 661, Springer (New York, 1978), pp. 1-125.

9. Schiopu, I., **A Categorical Approach to Probability Theory**, M. Sc. Thesis, (Dalhousie, 1973).

10. Ionescu Tulcea, A., Ionescu Tulcea, C., **Topics in the Theory of Lifting**, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 48, Springer Verlag, (Berlin, 1969).

11. Wendt, M., **On Measurably Indexed Families of Hilbert Spaces**, Ph. D. Thesis, (Dalhousie, 1993).

12. Wood, R.J., "Some remarks on total categories", J. Algebra **75** (1982), 538–545.

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