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# REPRESENTATIONS OF MODULES AND CAUCHY COMPLETENESS

*by SHU HAO SUN and R.F.C. WALTERS*

Soit $R$ un anneau, non nécessairement commutatif, et soit $M$ un $R$-module à droite. Soit $\mathcal{V}$ la catégorie monoidale des idéaux faibles à droite de $R$. Nous décrivons une $\mathcal{V}$-catégorie Cauchy-complète construite à partir de $M$, dont les objets sont les éléments de $M$.

## 1. Introduction

Recently, Borceux and Van den Bossche presented in [1] an interesting 'generic' sheaf representation of commutative rings from a quantale theoretic point of view. But it does not seems to be very clear how this is connected with classical sheaves.

On the other hand, F.W. Lawvere introduced in [3] the notion of *cauchy complete $\mathcal{V}$-category*, a notion which generalizes that of cauchy complete metric space. In [4], [3], R.F.C. Walters showed that sheaves on a space could be regarded as cauchy complete categories on the distributive bicategory of relations of the space.

The authors believe that this language of cauchy complete categories based on a distributive bicategory provides a flexible and precise language for analysing sheaf-like representations of algebraic structures. The general methods of enriched category theory provide an excellent guide to this particular situation.

We intend a series of investigations, of which this is the first, into different bicategories suitable for analysing different aspects of the representation of $R - mod$, $R$ a non-commutative ring.

Firstly, if we choose the base bicategory to be the monoidal category Id ($R$), whose objects are ideals, whose arrows are inclusions, and whose tensor product is the product of ideals (i.e., the bicategory is a quantale), then it is not too hard to show that each $R$-module $M$ occurs as a symmetric skeletal cauchy complete $\mathcal{V}$-category $L(M)$, whose objects are the elements of $M$ (more details will be given in Section 2).

However, in the light of papers [3] and [4], to obtain representations which are more sheaf-like, we prefer to choose the bicategory to be a locale or a category having the form Rel(C) (the definition see [3]) with a small category $C$.

For this reason we introduce the notion of *weak right ideals* of a *non-commutative* ring $R$: weak right ideals are subsets of $R$ closed under right-multiplication by elements of the ring and *"weak addition"*. The poset of weak right ideals is then a locale. We consider in this paper the monoidal category $\mathcal{V}$ whose objects are *weak right ideals* of a *non-commutative* ring $R$, whose arrows are inclusions, and whose tensor products is intersection. Given a right-$R$ module (or left-$R$ module) $M$ we construct a symmetric skeletal cauchy complete $\mathcal{V}$-category $L(M)$, whose objects are the elements of $M$.

In our future paper, we will consider such representations whose base bicategory has the form Rel(C) with $C$ a small category.
The notation of this paper is that of [3] and [4], with the simplification that the base category is a monoidal category $\mathcal{V}$ rather than a bicategory $\mathcal{B}$.
After this paper was written we became aware of further work of Borceux, with Cruciani, extending [1] to the non-commutative case. Their main contribution is to present a notion of sheaf over a quantale, but again without our enriched-categòrical setting.

## 2. The base bicategory is a quantale

In this short section, we sketch how the results in [1] are connected with the theory of enriched category, in particular, with the theory of cauchy completion. Our main results will be given in next two sections.
Let $R$ be a commutative ring with an identity and $\mathcal{V} = Id(R)$ the distributive monoidal category of ideals of $R$.

**Definition 2.1.** *If $M$ is a right-$R$-module, then $\mathcal{V}$-category $L(M)$ is defined by*
*(i) the set of objects of $L(M)$ is $M$;*
*(ii) if $m, m' \in M$ then the hom $d(m, m')$ is $Ann(m - m')$,*
*where $Ann(m) = \{r \in R | 0 = mr\}$.*

**Lemma 2.1.** *Each $L(M)$ is a symmetric skeletal $\mathcal{V}$-category.*

*Proof.* See section 4. □

**Lemma 2.2.** *For any adjoint pair of $\mathcal{V}$-modules*

$$\phi : \{*\} \longrightarrow\!\!\!\!\!\!+\!\!\!> M \qquad \text{and} \qquad \psi : M \longrightarrow\!\!\!\!\!\!+\!\!\!> \{*\},$$

*it is the case that $\phi(*, m) = \psi(m, *)$ for all $m \in M$.*

*Proof.* Note that $R$ is commutative. □

**Lemma 2.3.** *Each $L(M)$ is cauchy complete.*

*Proof.* For the details see section 4. □

Now let $f : M_1 \to M_2$ be a right $R$-module morphism and let $Lf = f$. Then we have,

**Lemma 2.4.** *$Lf$ is a $\mathcal{V}$-functor from $\mathcal{V}$-category $\mathcal{V}_1$ to the $\mathcal{V}$-category $\mathcal{V}_2$.*

*Proof.* It suffices to show that $d_1(m, m') \leq d_2(fm, fm')$ but this is obvious. □

We have thus proved the following

**Theorem 2.1.** *$L$ is a functor from the category MOD-$R$ to the category $\mathcal{V} - Cat_{cc}$ of symmetric skeletal cauchy complete $\mathcal{V}$-categories and $\mathcal{V}$-functors.*

After this paper was written we became aware of further work of Borceux, with Cruciani, extending [1] to the non-commutative case. Their proof and ours in the next sections suggest that the above statements remain true if a commutative ring $R$ is replaced by a non-commutative one.
However, our main contributions are contained in the next two sections.

## 3. The locale of weak right ideals

Let $R$ be a not-necessarily commutative ring with identity.

**Definition 3.1.** A subset $I$ of $R$ is called a weak right ideal of $R$ if

(i) $IR \subseteq I$;

(ii) for all $a \in R$ and central elements $u_1, u_2, \ldots, u_n \in R$ with $\sum_{i=1}^{n} u_i = 1$, if $au_i \in I$ for $i = 1, \cdots, n$, then $a \in I$.

For convenience, if $I$ satisfies $(ii)$, we will say that $I$ is closed under *weak addition*. It is clear that each right ideal is a weak right ideal.

**Example 3.1.** Let $R$ be a (not necessarily commutative) ring without central elements except 0 and invertible elements. Then any union of right ideals of $R$ is a weak right ideal. The free ring $R$ with more than one generator, over a skew-field, is such an example. Moreover, any simple ring (i.e., a ring with no non-trivial two sided ideals) is such a ring. A particular example is the ring of $n \times n$ matrices over a skew-field.

**Example 3.2.** On the other hand, however, in the Euclidean domain, **Z**, the fact that $I$ is closed under weak addition implies that $I$ is closed under addition; that is that $I$ is an ideal. The proof is as follows. If $I$ is a weak ideal and $m$, $n$ lie in $I$ and $d = HCF(m,n)$ then write $m = dm'$, $n = dn'$ and solve $u_1 + u_2 = m'x + n'y = 1$. Since $du_1 = mx \in I$ and $du_2 = ny \in I$ then closure under weak addition implies that $d \in I$. It is then trivial that $m + n = d(m' + n') \in I$.

**Lemma 3.1.** *The set $WRId(R)$ of all weak right ideals of $R$ is closed under intersections and hence is a complete lattice.*

*Proof.* Easy □

**Lemma 3.2.** *The suprema $\bigvee J_i$ of $J_i \in WRId(R)$ is calculated as follows: $\bigvee J_i = \{a \in R | \exists$ central elements $u_l$ with $\sum u_l = 1$, such that $u_i a \in J_{i(l)}$ for some $J_{i(l)}\}$*

*Proof.* First check that $\bigvee J_i$ is a weak right ideal:

(1) If $a \in \bigvee J_i$, then there are central elements $\{u_k\}$ with $\sum u_k = 1$ and $u_k a \in J_{i(k)}$. Thus for any $x \in R$, one has $u_k ax \in J_{i(k)}$ and hence $ax \in \bigvee J_i$.

(2) If there are central elements $z_k$ with $\sum z_k = 1$ so that $z_k a \in \bigvee J_i$, then there are centrals $u_{k,l}$ with $\sum_l u_{k,l} = 1$ such that $u_{k,l} z_k a \in J_{i(k,l)}$. Note that $u_{k,l} z_k$ are central elements with $\sum_{k,l} u_{k,l} z_K = 1$, so that $a \in \bigvee J_i$.

Then we note that $\cup J_i \subseteq \bigvee J_i$ and it is easy to check $\bigvee J_i$ is the least upper bound of $J_i' s$. □

**Theorem 3.1.** *$WRId(R)$ is a locale.*

*Proof.* It suffices to prove that $J \cap \bigvee I_i \subseteq \bigvee (J \cap I_i)$ for any $J, I_i \in WId(R)$. Let $x \in J \cap \bigvee I_i$. Then $x \in J$ and there are centrals $u_k$ with $\sum u_k = 1$ such that $u_i x \in J_{i(k)}$. Moreover we have $u_i x = x u_i \in J$ since $u_i$ are central and $JR \subseteq J$. Thus $u_i x \in J \cap I_{i(k)}$ and hence $x \in \bigvee J \cap I_i$. □

Note that a principal right ideal $aR$ is compact in the lattice $WRId(R)$ in the usual sense and that each weak right ideal can be expressed as a join of $aR$'s. So $WRId(R)$ is in fact an algebraic locale, and hence is spatial. Thus we have

**Theorem 3.2.** *For a general ring $R$ with identity, $WRId(R)$ is a spatial locale.*

**Remark 3.1.** Note that the lattice of all right ideals of a ring $R$ is not necessarily a locale, even for commutative ring $R$. For example, Let $R$ be the polynomial (commutative) ring $k[X,Y]$, with two generators $X$ and $Y$, over a field $k$. Then $R$ is Noetherian, but is not Dedekind, since non-zero prime ideals are not necessarily maximal (for example, the prime ideal $(X)$). Thus Id$R$ is not distributive by the fact that a Noetherian domain is Dedekind iff Id $(R)$ is distributive.

## 4. The construction

Let $R$ be a not-necessarily commutative ring with an identity and $\mathcal{V} = WRIdR$ the distributive monoidal category of weak right ideals of $R$.

**Definition 4.1.** If $M$ is a right-$R$-module, then $\mathcal{V}$-category $L(M)$ is defined by

(1) the set of objects of $L(M)$ is $M$;

(2) if $m, m' \in M$ then the hom $d(m,m')$ is $ann_r(m - m')$,

where $ann_r(m) = \{r \in R | 0 = mr\}$ is a right ideal and hence is a weak right ideal.

**Lemma 4.1.** *Each $L(M)$ is a symmetric skeletal $\mathcal{V}$-category.*

*Proof.* The symmetry is obvious. If $d(m,m') = R$, then trivially $m = m'$ since $R$ has an identity; so that $L(M)$ is skeletal. It remains to show that

$$ann_r(m' - m'') \cap ann_r(m - m') \subseteq ann_r(m - m'').$$

Let $x \in ann_r(m' - m'') \cap ann_r(m - m')$. Then

$$(m' - m'')x = 0 = (m - m')x$$

and hence $(m - m'')x = 0$; i.e., $x \in ann_r(m - m'')$. □

**Lemma 4.2.** *For any adjoint pair of $\mathcal{V}$-modules*

$$\phi : \{*\} \longrightarrow\!\!\!\!\!\!\mapsto M \qquad and \qquad \psi : M \longrightarrow\!\!\!\!\!\!\mapsto \{*\},$$

*it is the case that $\phi(*,m) = \psi(m,*)$ for all $m \in M$.*

*Proof.* First for any $m, m' \in M$, we have

$$\phi(*,m) \cap \phi(*,m') = R \cap \phi(*,m) \cap \phi(*,m') = \Big(\bigvee_{m'' \in M} \psi(m'',*)\Big) \cap \phi(*,m) \cap \phi(*,m')$$

$$= (\bigvee_{m''\in M} \psi(m'',*)\cap\phi(*,m))\cap\psi(m'',*)\cap\phi(*,m') \le \bigvee_{m''\in M} ann_r(m''-m)\cap ann_r(m''-m')$$

$$\le ann_r(m'-m).$$

Then

$$\phi(*,m) = R\cap\phi(*,m) = \bigvee_{m'}\phi(*,m')\cap\psi(m',*)\cap\phi(*,m)$$

$$= \bigvee_{m'}(\phi(*,m')\cap\phi(*,m))\cap\psi(m',*) \le ann_r(m-m')\cap\psi(m',*) \le \psi(m,*).$$

Here we use Theorem 3.1. Similarly we have $\psi(m,*) \le \phi(*,m)$. □

**Lemma 4.3.** *Each $L(M)$ is cauchy complete.*

*Proof.* Consider adjoint pair of $\mathcal{V}$-modules

$$\phi : \{*\} \longrightarrow\!\!\!\!\!\!\mapsto M \qquad \text{and} \qquad \psi : M \longrightarrow\!\!\!\!\!\!\mapsto \{*\}.$$

That is, $\phi(*,m), \psi(m,*), m\in M$ are objects of $\mathcal{V}$ satisfying the following:

(1) $d(m',m)\cap\phi(*m') \le \phi(*,m)$;
(2) $\psi(m',*)\cap d(m,m') \le \psi(m,*)$;
(3) $R \le \bigvee_m \psi(m,*)\cap\phi(*,m)$;
(4) $\phi(*,m')\cap\psi(m,*) \le d(m,m')$.

By (3) we have

$$R = \bigvee_m \psi(m,*)\cap\phi(*,m)$$

So there are central elements $\epsilon_i \in \phi(*,m_i)\cap\psi(m_i,*)$ such that $\sum_{i=1}^n \epsilon_i = 1$ by Lemma 3.2.
In particular, we have

$$R = \bigvee_i \psi(m_i,*)\cap\phi(*,m_i).$$

Let $m_0 = \sum_{i=1}^n m_i\epsilon_i$. We shall prove that

$$\psi(m,*) \le ann_r(m_0-m) \le \phi(*,m), \text{for all} \quad m\in M.$$

To this end, we want to show

$$\psi(m_i,*) \le ann_r(m_0-m_i) \quad \text{for all} \quad i.$$

In fact, for any $t\in\psi(m_i,*)$, we have

$$(m_0-m_i)t = (\sum_j^n m_j\epsilon_j - \sum_j^n m_i\epsilon_j)t = \sum_{j=1}^n (m_j-m_i)\epsilon_j t = 0$$

since $\epsilon_j$ is central, and hence $\epsilon_j t\in\phi(*,m_j)\cap\psi(m_i,*) \le ann_r(m_j-m_i)$. So $\psi(m_i,*) \le ann_r(m_0-m_i)$.

Moreover, we have, by (2)

$$\phi(*, m_0) \geq \bigvee_i \phi(*, m_i) \cap ann_r(m_0 - m_i) \geq \bigvee_i \phi(*, m_i) \cap \psi(m_i, *) = R.$$

Thus we have, by (2) again,

$$\phi(*, m_0) \cap ann_r(m - m_0) \leq \phi(*, m)$$

and hence

$$ann_r(m - m_0) \leq \phi(*, m).$$

On the other hand, by (4), we have

$$\phi(*, m) \cap \psi(m_0, *) \leq ann_r(m_0 - m)$$

and hence

$$\psi(m, *) \leq ann_r(m_0 - m) \leq \phi(*, m)$$

for all $m \in M$.
Similarly, we have

$$\phi(m, *) \leq ann_r(m_0 - m) \leq \psi(*, m)$$

(it also follows from Lemma 4.2). Thus we finally have

$$\phi(*, m) = ann_r(m_0 - m) = \psi(m, *), \qquad \text{for all} \qquad m \in M.$$

and so that this adjoint pair is induced from the point $m_0 \in M$. □

Now let $f : M_1 \longrightarrow M_2$ for a left $R$- and right $R$-module morphism and let $Lf = f$. Then we have,

**Lemma 4.4.** *$Lf$ is a $\mathcal{V}$-functor from $\mathcal{V}$-category $\mathcal{V}_1$ to the $\mathcal{V}$-category $\mathcal{V}_2$.*

*Proof.* It suffices to show that $d_1(m, m') \leq d_2(fm, fm')$ but this is obvious. □

We have thus proved the following

**Theorem 4.1.** *$L$ is a functor from the category MOD-$R$ to the category $\mathcal{V} - Cat_{cc}$ of symmetric skeletal cauchy complete $\mathcal{V}$-categories and $\mathcal{V}$-functors.*

## 5. Remarks

**Remark 5.1.** The questions of how characterize those symmetric skeletal cauchy complete $\mathcal{V}$-categories which is isomorphic to some $L(M)$,and of the existence of a left adjoint of the functor $L$, will be considered in the first author's forthcoming paper.

**Remark 5.2.** It is possible to establish a counterpart of the main results in §3 and §4 for the case that weak right ideals are replaced by weak ideals (i.e., those weak right ideals $I$ satisfying $RI \subseteq I$). However, the proofs are similar to the previous sections and left to the readers.

**Lemma 5.1.** *The set $WId(R)$ of all weak ideals of $R$ is closed under intersections and hence is a complete lattice.*

**Theorem 5.1.** *The suprema $\bigvee J_i$ of $J_i \in WId(R)$ is calculated as follows: $\bigvee J_i = \{a \in R \mid \exists$ central elements $u_l$ with $\sum u_l = 1$, such that $u_i a \in J_{i(l)}$ for some $J_{i(l)}\}$ and hence $WId(R)$ is a locale.*

Let $R$ be a not-necessarily commutative ring with an identity and $\mathcal{W} = WIdR$ the distributive monoidal category of weak ideals of $R$.

**Definition 5.1.** If $M$ is a right-$R$-module, then $\mathcal{W}$-category $L(M)$ is defined by

(1) the set of objects of $L(M)$ is $M$;

(2) if $m, m' \in M$ then the hom $d(m, m')$ is $Ann_r(m - m')$,

where $Ann_r(m) = \{r \in R \mid \{0\} = mRr\}$ is a two sided ideal and hence is a weak ideal.

**Theorem 5.2.** *Each $L(M)$ is a symmetric skeletal cauchy complete $\mathcal{W}$-category.*

Now let $f : M_1 \to M_2$ for a left $R$- and right $R$-module morphism and let $Lf = f$. Then we have,

**Theorem 5.3.** *$L$ is a functor from the category MOD-R to the category $\mathcal{W} - Cat_{cc}$ of symmetric skeletal cauchy complete $\mathcal{W}$-categories and $\mathcal{W}$-functors.*

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