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AGEBRAS DETERMIED BY THEIR ENDOMORPHISM MONOIDS

by **V. *KOUBEK* and *H. RADOVANSKÁ***

Dedicated to the memory of Jan Reiterman

Résumé. Deux objets A, B d'une catégorie K son dits *équimorphes* si leurs monoïdes des endomorphismes sont isomorphes. Si la cardinalité de toute famille d'objets de K deux-à-deux équimorphes mais non isomorphes est inférieure à un cardinal α on dira que la catégorie K est α-déterminée.

Notre but est de jeter les bases d'une théorie de α-déterminisme pour les catégories additives et les catégories sur les relations. Comme conséquences de cette théorie générale nous obtenons les résultats suivants:

a) une description des catégories 3-déterminés de treillis généralisant les résultats connus de B. M. Schein, R. Ribenboim, R. McKenzie et C. Tsinakis;

b) une nouvelle preuve du fait que la variété B_2 des p-algèbres distributives est 3-déterminée;

c) certaines variétés finiment engendrées d'algèbres de Heyting qui sont 3-déterminées;

d) pour les groupes Abéliens avec base l'équimorphisme entraîne l'isomorphisme.

INTRODUCTION

Let $\mathcal{K}$ be a category. The endomorphism monoid of an $\mathcal{K}$-object A will be denoted by $End_{\mathcal{K}}(A)$ (or $End(A)$, if the category $\mathcal{K}$ is apparent from the context). Numerous papers studied various properties of $End(A)$ in a given category $\mathcal{K}$. For example, many familiar categories are universal, and hence monoid universal, that is, such that every monoid is isomorphic to $End(A)$ for some object A – see the monograph by Pultr and Trnková [22].

The present paper aims to study how $End_{\mathcal{K}}(A)$ determines the object A within a given category $\mathcal{K}$. In any universal category $\mathcal{K}$ for any monoid M there is a proper class of non-isomorphic objects A of $\mathcal{K}$ with $End(A) \cong M$, see [22]. Thus we shall deal with categories whose properties are diametrally opposite to universality.

We say that objects A, B in a category $\mathcal{K}$ are *equimorphic* if $End(A)$ and $End(B)$ are isomorphic and we write $End(A) \simeq End(B)$. Isomorphic objects are always

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equimorphic but the converse does not hold. We shall study categories in which at least a partial converse is true.

Let α be a cardinal. We say that a category $\mathcal{K}$ is *α-determined* if every set of non-isomorphic equimorphic objects of $\mathcal{K}$ has a cardinality smaller than α. For example, the following categories $\mathcal{K}$ are

(1) 2-determined, i.e. equimorphic implies isomorphic:
$\mathcal{K}$ =boolean algebras, see – Maxson [17], Magill [16], and Schein [25];
$\mathcal{K}$ =distributive (0)-lattices - see Ribenboim [24];
$\mathcal{K}$ =median algebras, see Bandelt [5];
$\mathcal{K}$ =Stone algebras, see [2];
$\mathcal{K}$ =principal Brouwerian semilattices – see Köhler [12], and Tsinakis [29].

(2) 3-determined:
$\mathcal{K}$ =posets – see Gluskin [10], and Schein [25];
$\mathcal{K}$ =distributive lattices – see Schein [25];
$\mathcal{K}$ =distributive (0,1)-lattices – see McKenzie and Tsinakis [18];
$\mathcal{K}$ =normal bands - see Schein [26];
$\mathcal{K}$ =variety of distributive p-algebras generated by the four element Boolean algebra with adjoined a new 1, see Adams, Koubek, and Sichler [2];
(in fact, in the first three examples equimorphic objects are either isomorphic or anti-isomorphic).

(3) 5-determined:
$\mathcal{K}$ =left or right regular bands - see Demlová and Koubek [7].

Moreover, another variety of distributive p-algebras is not α-determined for any cardinal α, see [2].

The correlation between endomorphism monoids and clone algebras in universal algebra was investigated by Adams and Clark [1], and analogous problems were studied also by Trnková [30] and Taylor [28].

The present paper has five sections. The first one introduces definitions and basic facts about α-determinacy in general. The second section deals with a general theory of α-determined subcategories of n-ary relations and its consequences for certain categories of posets and topological posets. In addition, we show that equimorphic lattices with a prime ideal or $(0,1)$-equimorphic $(0,1)$-lattices with a three-element chain of prime ideals are either isomorphic or anti-isomorphic, while 0-equimorphic 0-lattices with a two-element chain of prime ideals are isomorphic. The third section is devoted to varieties of distributive p-algebras, and it contains a new proof of the determinacy results of [2]. The fourth section exhibits some 2-determined and 3-determined varieties of Heyting algebras. In the last section we investigate categories with zero in general, and α-determined subcategories of Abelian groups; we show that equimorphic Abelian groups with a basis are isomorphic.

1. Basic definitions and facts

For any mapping $f : X \longrightarrow Y$ denote by $Ker(f)$ the equivalence on X with $(x, y) \in Ker(f)$ if and only if $f(x) = f(y)$, and $Im(f)$ the subset of Y with $Im(f) = \{y \in Y; \exists x \in X, f(x) = y\}$.

For a cardinal α denote by α^+ the cardinal successor of α.

The definitions of standard semigroup notions (for example *left* (or *right*) *zero*, *Green relations, left* (or *right) divisor, left (*or *right) ideal*) using here can be found in the monograph of Clifford and Preston see [6].

We say that a property $\mathcal{P}$ of elements, (or n-tuples of elements, or subsets, or family of subsets) is an *isoproperty* if for every semigroup isomorphism $f : S \longrightarrow T$ and any element s of S (or any n-tuple of elements of S, or any subset of S, or any family of subsets of S, respectively), s has the property $\mathcal{P}$ in S if and only if $f(s)$ has the property $\mathcal{P}$ in T. We say that a property $\mathcal{P}$ is an element property (or n-tuple property, or set property, or family of sets property) if $\mathcal{P}$ concerns of elements (or n-tuples, or subsets, or family of subsets, respectively) of a given semigroup.

The study of α-determinacy in concrete categories is based on transformation monoids. A *transformation monoid* is a pair (X, M) where X is a set and M is a set of mappings of X into itself closed under composition and containing the identity mapping. The set M with the operation of composition and the identity mapping is a monoid. Let $(X, M), (Y, N)$ be transformation monoids then an isomorphism φ from M to N is called *strong* if there exists a bijection $g : X \longrightarrow Y$ with $g \circ f = \varphi(f) \circ g$ for every $f \in M$. The bijection g is called a *carrier* of the isomorphism φ. For a submonoid M' of M and for $x \in X$ denote by $Stab(M', x) = \{f \in M'; f(x) = x\}$. Clearly, $Stab(M', x)$ is a submonoid of M. For $A, B \subseteq M$ we shall write $A \circ B = \{f \circ g; f \in A, g \in B\}$. Thus $M \circ f$ for $f \in M$ is a right ideal in M generated by f. For a subset $A \subseteq M$ define an equivalence $\cong_A$ as the smallest equivalence such that $f \cong_A f \circ g$ for every $f \in M$, $g \in A$. A right ideal $Q \subseteq M$ is called *left* 1*-transitive* if there exists a left congruence $\sim$ on Q such that for every $y \in X$ there exists exactly one class Q_y of $\sim$ on Q such that for $f \in Q$ we have $f(x) = y$ if and only if $f \circ Q_x \subseteq Q_y$. We say that $\sim$ is *associated with* Q. For a right ideal Q if there exists $x \in X$ – which is called a *source* – such that for every $y \in X$ there exists $f \in Q$ with $f(x) = y$ then Q is left 1-transitive where the associated congruence $\sim$ on Q is defined as follows: $f \sim g$ just when $f(x) = g(x)$. If there exists a source $x \in X$ with $\sim = \cong_{Stab(Q,x)}$ then we say that Q is 1*-transitive* and if $\sim$ is identical then Q is *strictly* 1*-transitive*. First we give some elementary properties.

Lemma 1.1. *Let (X, M) be a transformation monoid. Then the following hold*

(1) *For every $A \subseteq M$, the equivalence $\cong_A$ is a left congruence;*
(2) *For every 1-transitive ideal Q, if $Stab(Q, x) = \{f\}$ then f is idempotent and Q is strictly 1-transitive.*
(3) *For every idempotent $g \in M$, and every $f \in M$ we have $Im(f) \subseteq Im(g)$ if*

and only if $g \circ f = f$.

□

Lemma 1.2. *Let* $Q \subseteq M$ *be a 1-transitive right ideal in a transformation monoid* (X, M) *with a source* $x \in X$. *Then for every* $f \in M$ *hold*

(1) $f(y) = z$ *if and only if* $f \circ h \in Q_z$ *for every* $h \in Q_y$;
(2) $Im(f) = \{y \in X; f \circ h \in Q_y$ *for some* $h \in M\}$;
(3) $(z, y) \in Ker(f)$ *if and only if for every* $h \in Q_z$, $g \in Q_y$ *we have* $f \circ g \cong_{Stab(Q,x)} f \circ h$;
(4) *for every* $y \in X$, $f^{-1}(y) = \{z \in X; f \circ h \in Q_y$ *for* $h \in Q_z\}$.

Proof. Since Q is a right ideal we obtain that $f \circ h \in Q$ for every $h \in Q_y$, $y \in X$. Since $f \circ h(x) = f(y)$ we conclude that $f \circ h \in Q_{f(y)}$ and (1) is proved. The rest is a consequence of (1). □

In this paper every considered category $\mathcal{K}$ will have a factorization system $(\mathcal{E}, M)$. Thus every morphism $f \in \mathcal{K}$ has a unique decomposition, up to isomorphism, into $f = h \circ g$ where $g \in \mathcal{E}$, $h \in \mathcal{M}$. We shall write $g = \mathcal{E}(f)$, $h = \mathcal{M}(f)$. The range object of $\mathcal{E}(f)$ will be denoted by O(f). Thus, if $f : A \longrightarrow B \in \mathcal{K}$ is an idempotent then $\mathcal{E}(f) : A \longrightarrow O(f) \in \mathcal{E}$, $\mathcal{M}(f) : O(f) \longrightarrow B \in \mathcal{M}$ and $f = \mathcal{M}(f) \circ \mathcal{E}(f)$. Note, if $f : A \longrightarrow A \in \mathcal{K}$ then $\mathcal{E}(f) \circ \mathcal{M}(f) = 1_{O(f)}$. We recall the diagonalization property of a factorization system which we will often apply without reference, if $\epsilon : A \longrightarrow B \in \mathcal{E}$, $g : B \longrightarrow D$, $f : A \longrightarrow C$, $\mu : C \longrightarrow D \in \mathcal{M}$ are $\mathcal{K}$-morphisms with $g \circ \epsilon = \mu \circ f$ then there exists a $\mathcal{K}$-morphism $h : B \longrightarrow C$ with $h \circ \epsilon = f$, $\mu \circ h = g$.

We show two basic facts about the correlation between the endomorphism monoid of an object and endomorphism monoids of its subobjects. First denote by $Id_{\mathcal{K}}(A) = \{f \in End_{\mathcal{K}}(A); f \circ f = f\} \subseteq End_{\mathcal{K}}(A)$ for any $\mathcal{K}$-object A. For a concrete category $\mathcal{K}$ and for every $\mathcal{K}$-object A define $Fin_{\mathcal{K}}(A) = \{f \in End_{\mathcal{K}}(A); Im(f)$ is finite$\}$. If the category $\mathcal{K}$ is clear we omit the index $\mathcal{K}$.

Lemma 1.3. *Let* A *be a* $\mathcal{K}$*-object and let* $f \in Id(A)$ *then* $End(O(f))$ *and* $f \circ End(A) \circ f$ *are isomorphic monoids.*

Proof. Let $\Phi : End(O(f)) \longrightarrow End(A)$ be a mapping such that $\Phi(g) = \mathcal{M}(f) \circ g \circ \mathcal{E}(f)$ for any $g \in End(O(f))$. Since $\mathcal{E}(f)$ is an epimorphism and $\mathcal{M}(f)$ is a monomorphism we conclude that Φ is injective. Since $\mathcal{M}(f) \circ \mathcal{E}(f) \circ \mathcal{M}(f) = \mathcal{M}(f)$ and $\mathcal{E}(f) \circ \mathcal{M}(f) \circ \mathcal{E}(f) = \mathcal{E}(f)$ we conclude for every $g \in End(O(f))$ that $f \circ \Phi(g) \circ f = \Phi(g)$ and for every $g \in End(A)$ that $\Phi(h) = f \circ g \circ f$ for $h = \mathcal{E}(f) \circ g \circ \mathcal{M}(f) \in End(O(f))$. Hence $Im(\Phi) = f \circ End(A) \circ f$. It remains to show that Φ is a homomorphism. It is clear because $\Phi(g) \circ \Phi(h) = \mathcal{M}(f) \circ g \circ \mathcal{E}(f) \circ \mathcal{M}(f) \circ h \circ \mathcal{E}(f) = \mathcal{M}(f) \circ g \circ h \circ \mathcal{E}(f) = \Phi(g \circ h)$ for every $g, h \in End(O(f))$. □

Lemma 1.4. *Let A be an object of $\mathcal{K}$ and let $f, g \in Id(A)$ then $O(f)$ and $O(g)$ are isomorphic if and only if there exist $k, h \in End(A)$ with $h \circ g = f \circ h = h$, $k \circ f = g \circ k = k$, $k \circ h = g$, $h \circ k = f$.*

Proof. If $O(f)$ and $O(g)$ are isomorphic then there exist $m : O(f) \longrightarrow O(g), n : O(g) \longrightarrow O(f)$ with $n \circ m = 1_{O(f)}, m \circ n = 1_{O(g)}$. Define $k = \mathcal{M}(g) \circ m \circ \mathcal{E}(f)$, $h = \mathcal{M}(f) \circ n \circ \mathcal{E}(g)$. By a direct calculation we obtain that h and k satisfy the required conditions. On the other hand if there exist $h, k \in End(A)$ satisfying the required conditions then by a diagonalization property of a factorization system there exist $\phi : O(k) \longrightarrow O(g)$ with $\mathcal{M}(g) \circ \phi = \mathcal{M}(k)$, $\psi : O(g) \longrightarrow O(h)$ with $\psi \circ \mathcal{E}(g) = \mathcal{E}(h)$, $\omega : O(f) \longrightarrow O(k)$ with $\omega \circ \mathcal{E}(f) = \mathcal{E}(k)$, $\nu : O(h) \longrightarrow O(f)$ with $\mathcal{M}(f) \circ \nu = \mathcal{M}(h)$, $\sigma : O(h) \longrightarrow O(g)$ with $\mathcal{E}(g) = \sigma \circ \mathcal{E}(h)$, $\tau : O(g) \longrightarrow O(k)$ with $\mathcal{M}(k) \circ \tau = \mathcal{M}(g)$, $\eta : O(k) \longrightarrow O(f)$ with $\mathcal{E}(f) = \eta \circ \mathcal{E}(k)$, $\theta : O(f) \longrightarrow O(h)$ with $\mathcal{M}(f) = \mathcal{M}(h) \circ \theta$. Hence $\mathcal{E}(h) = \psi \circ \sigma \circ \mathcal{E}(h)$, $\mathcal{E}(g) = \sigma \circ \psi \circ \mathcal{E}(g)$, $\mathcal{E}(k) = \omega \circ \eta \circ \mathcal{E}(k)$, $\mathcal{E}(f) = \eta \circ \omega \circ \mathcal{E}(f)$ and thus $\psi \circ \sigma = 1_{O(h)}$, $\sigma \circ \psi = 1_{O(g)}$, $\omega \circ \eta = 1_{O(k)}$, $\eta \circ \omega = 1_{O(f)}$. Therefore $\psi^{-1} = \sigma$, $\omega^{-1} = \eta$, and analogously $\tau = \phi^{-1}$, $\psi = \nu^{-1}$. Thus $O(f)$ and $O(g)$ are isomorphic. □

The categories of main interest in this paper will be concrete categories, i.e. category $\mathcal{K}$ with a forgetful functor $| - | : \mathcal{K} \longrightarrow SET$ where SET is the category of all sets and mappings. Then every endomorphism monoid $End(A)$ corresponds to a transformation monoid on the set $|A|$ with the set $\{|f|; f \in End(A)\}$ of mappings. If the misunderstanding cannot occur then we will identify $End(A)$ and the transformation monoid corresponding to $End(A)$.

A subcategory $\mathcal{L}$ of $\mathcal{K}$ is called *isomorphism-full* if any pair A, B of $\mathcal{L}$-objects is isomorphic in $\mathcal{L}$ if and only if it is isomorphic in $\mathcal{K}$. We say that a concrete category $\mathcal{K}$ is *amenable* if for every $\mathcal{K}$-object A and for every bijection $f : |A| \longrightarrow X$ where X is a set there exist an $\mathcal{K}$-object B with $|B| = X$ and an $\mathcal{K}$-isomorphism $\varphi : A \longrightarrow B$ with $|\varphi| = f$. A concrete category $\mathcal{K}$ has a *unique empty object* if there exists at most one $\mathcal{K}$-object A, up to isomorphism, with $|A| = \emptyset$, then it is called an *empty object* and the other objects are *non-empty*.

Two $\mathcal{K}$-objects A, B are called *strongly equimorphic* if $End(A) = End(B)$ (as transformation monoids, not only isomorphic). Clearly, strongly equimorphic objects are equimorphic, the following easy proposition gives a partial converse of this fact.

Proposition 1.5. *Let $\mathcal{K}$ be an amenable concrete category. If $f : End(A) \longrightarrow End(B)$ is a strong isomorphism where $A, B \in \mathcal{K}$ then there exists a $\mathcal{K}$-object C isomorphic to B such that A and C are strongly equimorphic.*

Proof. Let $g : |A| \longrightarrow |B|$ be a carrier of f. Since $\mathcal{K}$ is amenable there exists a $\mathcal{K}$-object C isomorphic with B such that g^{-1} is an underlying mapping of an isomorphism between B and C. Then g^{-1} is a carrier of the isomorphism between $End(B)$ and $End(C)$ and thus $End(A) = End(C)$. □

A pair $(\mathcal{P}_0, \mathcal{P}_1)$ of isoproperties is called a *coordination property* for a concrete category $\mathcal{K}$ if $\mathcal{K}$ has a unique empty object, for every non-empty $\mathcal{K}$-object A there exists $Q \subseteq End(A)$ satisfying $\mathcal{P}_0$, and if $Q \subseteq End(A)$ satisfies $\mathcal{P}_0$ where A is a non-empty $\mathcal{K}$-object then Q is a left 1-transitive right ideal in $End(A)$. A subset $R \subseteq Q \times Q$ satisfies $\mathcal{P}_1$ if and only if R is a left congruence associated with Q. If Q is 1-transitive then the isoproperty $\mathcal{P}_1$ is the set property and $R \subseteq Q$ satisfies $\mathcal{P}_1$ if and only if $R = Stab(Q, x)$ for some source $x \in |A|$. If Q is strictly 1-transitive then $\mathcal{P}_1$ is omitted.

Theorem 1.6. *Assume that a concrete category $\mathcal{K}$ has a coordination property. Then every isomorphism $\varphi : End(A) \longrightarrow End(B)$ between $\mathcal{K}$-objects A, B is strong.*

Proof. Let $\varphi : End(A) \longrightarrow End(B)$ be an isomorphism. If $|A| = \emptyset$ then φ is strong because $\mathcal{K}$ has a unique empty object. Assume that $|A| \neq \emptyset$. Since $\mathcal{K}$ has a coordination property $(\mathcal{P}_0, \mathcal{P}_1)$ there exists $Q \subseteq End(A)$ satisfying $\mathcal{P}_0$ and Q is a left 1-transitive right ideal in $End(A)$. Then $\varphi(Q)$ satisfies $\mathcal{P}_0$ and thus $\varphi(Q)$ is a left 1-transitive right ideal in $End(B)$. Further there exists $R \subseteq Q \times Q$ satisfying $\mathcal{P}_1$ and $R =\sim$ is a left congruence associated with Q. Then $\varphi(R) \subseteq \varphi(Q) \times \varphi(Q)$ satisfies $\mathcal{P}_1$ because φ is an isomorphism and thus $\varphi(R) =\sim_1$ is associated with $\varphi(Q)$. Since Q is a left 1-transitive right ideal with the associated left congruence $\sim$ there exists a surjection $\phi : Q \longrightarrow X$ with $\phi(f) = \phi(g)$ for $f, g \in Q$ just when $f \sim g$ and $\phi(f \circ g) = f(\phi(g))$ for every $g \in Q$, $f \in End(A)$. Analogously, there exists a surjection $\phi' : \varphi(Q) \longrightarrow B$ with $\phi'(f) = \phi'(g)$ for $f, g \in \varphi(Q)$ just when $f \sim_1 g$ and $\phi'(f \circ g) = f(\phi'(g))$ for every $g \in \varphi(Q)$, $f \in End(B)$. Define a mapping $h : |A| \longrightarrow |B|$ such that $h(u) = \phi'(\varphi(g))$ where $g \in Q$ with $\phi(g) = u$. We prove that h is correctly defined and that h is a bijection. If $f, g \in Q$ with $f \sim g$ then $\varphi(f) \sim_1 \varphi(g)$ and we conclude that $\phi'(\varphi(f)) = \phi'(\varphi(g))$ and thus h is correctly defined. Since φ is an isomorphism we have for $f, g \in Q$ that $f \sim g$ if and only if $\varphi(f) \sim_1 \varphi(g)$ and thus h is injective. Since ϕ and ϕ' are surjections we conclude that h is a surjection and whence h is a bijection. It remains to show that $\varphi(k) \circ h = h \circ k$ for every $k \in End(A)$. For any $u \in |A|$ there exists $g \in Q$ with $\phi(g) = u$. Then $h \circ k(u) = h(k \circ \phi(g)) = \phi'(\varphi(k \circ g)) = \varphi(k)(\phi'(\varphi(g)) = \varphi(k)(h(u))$ because $k \circ g \in Q$. Thus h is a carrier of φ. □

Note that if a concrete category $\mathcal{K}$ has a coordination property then the following ones are isoproperties:

(1) $f \in End(A)$ is one-to-one for $a \in \mathcal{K}$;
(2) $card(Im(f)) = n$ for $f \in End(A)$, $A \in \mathcal{K}$, and a natural number n;
(3) $card(f^{-1}(x) \cap Im(f)) = n$ for some $x \in Im(f)$, $f \in Id(A)$ $A \in \mathcal{K}$, and a natural number n.

We give two sufficient conditions for the existence of a coordination property in a concrete category $\mathcal{K}$.

Proposition 1.7. *Let $\mathcal{K}$ be a concrete category with a unique empty object such* *that for every non-empty object A of $\mathcal{K}$ any constant mapping of $|A|$ is an underlying mapping of an endomorphism of A. Then the isoproperty $\mathcal{P}_0$ such that*

Q is the set of all left zeros of $End(A)$;

is a coordination property.

Proof. Obviously, if a transformation monoid (X, M) contains all constants of X then the set Q of all left zeros is strictly 1-transitive right ideal. Since $f \in M$ is a constant if and only if f is a left zero in M the proof is complete. □

For a concrete category $\mathcal{K}$ and a non-empty $\mathcal{K}$-object A denote by $Kernel_{\mathcal{K}}(A)$ the smallest non-empty both-sided ideal in $End(A)$ if it exists. We can omit the index $\mathcal{K}$ if the misunderstanding cannot occur. If $Kernel(A)$ exists then the smallest subset $U \subseteq |A|$ such that $Im(f) \subseteq U$ for any $f \in Kernel(A)$ and $f(U) \subseteq U$ for every $f \in End(A)$ is denoted by A_{Ker}. We say that $\mathcal{K}$ *has kernels* if it has a unique empty object and $Kernel(A)$ exists for every non-empty $\mathcal{K}$-object A. We say that an element isoproperty $\mathcal{P}$ *coordinatizes kernels* in $\mathcal{K}$ if for every $\mathcal{K}$-object A there exists $f \in Id(A)$ satisfying $\mathcal{P}$ and if $|A| = A_{Ker}$ then every $f \in Id(A)$ satisfying $\mathcal{P}$ generates the strictly 1-transitive right ideal and f belongs to $Kernel(A)$ and if $|A| \neq A_{Ker}$ then no $f \in Id(A)$ satisfies both $\mathcal{P}$ and $Im(f) \subseteq A_{Ker}$.

Theorem 1.8. *Let $\mathcal{K}$ be a concrete category with kernels. Let $\mathcal{P}$ be a property coordinatizing kernels such that every $f \in Id(A)$ satisfying both $\mathcal{P}$ and $Im(f) \setminus A_{Ker} = \{x\}$ for $x \in |A| \setminus A_{Ker}$ fulfils*

(1) $End(A) \circ f$ is 1-transitive with a source x;

(2) $Stab(End(A) \circ f, x)$ is a group;

(3) $g \in Kernel(A)$ whenever $g \in End(A) \circ f \circ End(A)$ and $Im(g) \subseteq A_{Ker}$.

Moreover, if $|A| \neq A_{Ker}$ then there exists $f \in Id(A)$ satisfying both $\mathcal{P}$ and $card(Im(f) \setminus A_{Ker}) = 1$. Then $\mathcal{K}$ has a coordination property.

Proof. We must show that there exists an element isoproperty $\mathcal{P}'$ such that $f \in Id(A)$ satisfying $\mathcal{P}'$ satisfies $\mathcal{P}$ and if $Ker_A \neq |A|$ then $card(Im(f) \setminus A_{Ker}) = 1$ (i.e. $f \notin Kernel(A)$) for every non-empty $\mathcal{K}$-object A. Consider the property $\mathcal{P}'$ such that:

$f \in Id(A)$ satisfies $\mathcal{P}$ and for every $h \in Id(A)$ satisfying $\mathcal{P}$ either $f \circ k \circ h \in Kernel(A)$ for every $k \in End(A)$ or there exist $k, k_1 \in End(A)$ with $f \circ k \circ h \circ k_1 \circ f = f$.

Clearly, $\mathcal{P}'$ is an isoproperty. Let A be a non-empty $\mathcal{K}$-object. If $A_{Ker} = |A|$ then there exists $f \in Id(A)$ satisfying $\mathcal{P}$ because $\mathcal{P}$ coordinatizes kernels and $f \in Kernel(A)$. Thus f satisfies $\mathcal{P}'$ because $f \circ k \in Kernel(A)$ for every $k \in End(A)$.

Assume that $|A| \neq A_{Ker}$ and $f \in Id(A)$ satisfies $\mathcal{P}'$. Then $Im(f) \not\subseteq A_{Ker}$ and thus there exists $y \in |A|$ with $f(y) \notin A_{Ker}$. By the assumption on $\mathcal{K}$ there exists $g \in Id(A)$ satisfying $\mathcal{P}$ and $card(Im(g) \setminus A_{Ker}) = 1$. Since $End(A) \circ g$ is 1-transitive for a source $x \in |A|$, there exists $h \in End(A) \circ g$ with $h(x) = y$ and hence $f \circ h \notin Kernel(A)$. Therefore there exist $k, k_1 \in End(A)$ with $f \circ k \circ g \circ k_1 \circ f = f$.

Since $l(A_{Ker}) \subseteq A_{Ker}$ for every $l \in End(A)$ and $card(Im(g) \setminus A_{Ker}) = 1$ we conclude that $card(Im(f) \setminus A_{Ker}) \leq 1$ and thus $card(Im(f) \setminus A_{Ker}) = 1$.

Conversely, assume that a $f \in Id(A)$ satisfies $\mathcal{P}$ and $Im(f) \setminus A_{Ker} = \{x\}$. Let $g \in Id(A)$ satisfy $\mathcal{P}$ and assume that there exists $h \in End(A)$ with $f \circ h \circ g \notin Kernel(A)$. Then there exists $z \in |A|$ with $x = f \circ h \circ g(z) \notin A_{Ker}$. Since $End(A) \circ f$ is 1-transitive there exists $h_1 \in End(A)$ with $x = f \circ h \circ g \circ h_1 \circ f(x)$ and thus $f \circ h \circ g \circ h_1 \circ f \in Stab(End(A) \circ f, x)$. Since $Stab(End(A) \circ f, x)$ is a group and $f \in Stab(End(A) \circ f, x)$ is an idempotent we conclude that f is the unity of $Stab(End(A) \circ f, x)$ and therefore there exists $h_2 \in Stab(End(A) \circ f, x)$ with $h_2 \circ h \circ g \circ h_1 \circ f = f = f \circ h_2 \circ h \circ g \circ h_1 \circ f$ and thus f satisfies $\mathcal{P}'$.

It remains to find an isoproperty characterizing $Stab(End(A) \circ f, x)$ if $f \in Id(A)$ satisfies $\mathcal{P}'$ and $Im(f) \setminus A_{Ker} = \{x\}$. Consider a maximal subgroup $G \subseteq End(A)$ of $End(A)$ containing f. Then for every $g \in G$ we have $g \circ f = g$ and therefore $G \subseteq End(A) \circ f$. Further $f \circ g = g$ implies that $Im(g) \subseteq Im(f)$ and since $f = h \circ g$ for some $h \in G$ we conclude that $g(x) \notin A_{Ker}$ and thus $g(x) = x$. Hence $G \subseteq Stab(End(A) \circ f, x)$ and $Stab(End(A) \circ f, x)$ is the $\mathcal{H}$-class containing f – this is an isoproperty describing $Stab(End(A) \circ f, x)$. □

The notion of α-determinacy can be strengthened for concrete categories. We say that a concrete category $\mathcal{K}$ is *strongly α-determined*, where α is a cardinal, if every set of non-isomorphic strongly equimorphic $\mathcal{K}$-objects has a cardinality smaller than α. The following theorem shows that for suitable concrete categories the notion of α-determinacy coincides with strong α-determinacy.

Theorem 1.9. *Let $\mathcal{K}$ be a concrete amenable category such that any isomorphism between $End(A)$ and $End(B)$ for $\mathcal{K}$-objects A, B is strong. Then $\mathcal{K}$ is α-determined if and only if $\mathcal{K}$ is strongly α-determined.*

Proof. Clearly, every α-determined category is also strongly α-determined. Conversely, assume that $\mathcal{K}$ is strongly α-determined. Let $\{A_i; i \in I\}$ be a set of non-isomorphic equimorphic $\mathcal{K}$-objects. Choose $i_0 \in I$. Since for every $i \in I$ an isomorphism between $End(A_i)$ and $End(A_{i_0})$ is strong we obtain by Proposition 1.5 that for every $i \in I \setminus \{i_0\}$ there exists a $\mathcal{K}$-object B_i isomorphic with A_i on the set $|A_{i_0}|$ such that $End(B_i) = End(A_{i_0})$. Set $B_{i_0} = A_{i_0}$, then $\{B_i; i \in I\}$ is the set of non-isomorphic strongly equimorphic $\mathcal{K}$-objects and hence $|I| < \alpha$. Thus $\mathcal{K}$ is α-determined. □

Corollary 1.10. *Let $\mathcal{K}$ be a concrete amenable category with a coordination property. Then $\mathcal{K}$ is α-determined if and only if $\mathcal{K}$ is strongly α-determined.* □

2. Subcategories of relations

Denote by *POSET* the category of all posets and order preserving mappings. Gluskin proved

Theorem 2.1. [10] *If two posets P_0, P_1 are equimorphic then either P_0 and P_1 are isomorphic or antiisomorphic.* □

The method of the proof was generalized by Schein [25]. He defined a sufficient subsemigroup and by this notion generalized Theorem 2.1 for semilattices and distributive lattices. We attempt to generalize Gluskin's idea by another way. We generalize his method for subcategories of n-ary relations over a concrete category. Let $\mathcal{L}$ be a concrete category. Objects of the category $REL_n(\mathcal{L})$ of n-ary relations over $\mathcal{L}$ are pairs (A, R) where A is an $\mathcal{L}$-object and R is an n-ary relation over $|A|$, morphisms from (A, R) into (B, S) are all $\mathcal{L}$-morphisms $f : A \longrightarrow B$ with $f^n(R) \subseteq S$ (i.e. $|f|$ is a compatible mapping of relations). If $\mathcal{L} = SET$ then we shall write only REL_n. Then $POSET$ is a full subcategory of REL_2. For a relation $(A, R) \in REL_n(\mathcal{L})$ and for an arc $\alpha =< x_1, x_2, ..., x_n >\in R$ denote by $d(\alpha) = \{x_1, x_2, ..., x_n\}$, and for a subset $Q \subseteq R$ denote by $d(Q) = \{d(\alpha); \alpha \in Q\}$.

As concrete application of general theorems we shall investigate relations over SET or over TOP - the category of topological spaces and continuous mappings. Denote by SEM the category of all semilattices and semilattice homomorphisms, SEM_c the full subcategory of SEM formed by all semilattices with at least one pair of incomparable elements, Lat_1 (or $0 - Lat_1$) the category of all lattices (or 0-lattices i.e. lattices with 0) having a prime ideal and lattice homomorphisms (0-homomorphisms, respectively), $0 - Lat_2$ (or $(0,1) - Lat_2$) is the category of 0-lattices (or (0,1)-lattices, i.e. lattices with 0,1) having two distinct prime ideals I_0, I_1 with $I_0 \subseteq I_1$ and lattice 0-homomorphisms, ((0,1)-homomorphisms, respectively), $(0,1) - Lat_3$ is the full subcategory of $(0,1) - Lat_2$ formed by all lattices having three distinct prime ideals I_0, I_1, I_2 with $I_0 \subseteq I_1 \subseteq I_2$. The categories SEM, SEM_c, Lat_1, $0 - Lat_1$, $0 - Lat_2$, $(0,1) - Lat_2$, $(0,1) - Lat_3$ are amenable isomorphism-full subcategories of REL_2. Moreover, any semilattice S can be identified with a ternary relation $(|S|, R)$ where $R = \{(x, y, x \wedge y); x, y \in |S|\}$ (we assume that every semilattice is a meet-semilattice) then SEM and SEM_c are amenable full subcategories of REL_3. Clearly, every distributive lattice belongs to Lat_1.

Denote by $PRIEST$ the category of all Priestley spaces – *Priestley space* is a triple $(X, \leq, \tau)$ where X is a set, $\leq$ is an ordering on X, τ is a compact topology on X such that for every $x \not\leq y$ there exists a *clopen* (i.e. closed and open) decreasing set U with $y \in U$, $x \notin U$, and morphisms are all continuous order preserving mappings (a set U is *decreasing* if $u \in U$, $v \leq u$ imply $v \in U$, the dual notion is an *increasing* set, for a set U denote by $[U)$ the smallest increasing set containing U, $(U]$ the smallest decreasing containing U). We recall that by the standard topological arguments we obtain that for closed disjoint sets $Z, Y \subseteq X$ such that Z is decreasing there exists a clopen decreasing set $U \subseteq X$ with $Z \subseteq U$ and $U \cap Y = \emptyset$. Clearly, $PRIEST$, is an amenable isomorphism-full subcategory of $REL_2(TOP)$. Every constant mapping is a morphism of $POSET$, SEM, Lat_1, $PRIEST$ thus by Proposition 1.7 the categories $POSET$, SEM, SEM_c, Lat_1, $PRIEST$ have a coordination property. Priestley proved

Theorem 2.2. [19] *The category PRIEST is dually isomorphic to the category of distributive (0,1)-lattices and lattice (0,1)-homomorphisms.* □

Let $\mathcal{K}$ be a subcategory of $REL_n(\mathcal{L})$ and let $(A,R) \in \mathcal{K}$. Denote by $R^t = \{\alpha \in R;$ for every $(A,R') \in \mathcal{K}, \alpha \in R'\}$ and $R^r = R \setminus R^t$. A subset $S \subseteq R$ is called *weak $\mathcal{K}$-origin* (or shortly weak origin) if for every $\rho \in R^r$ there exist $\sigma \in S$ and $f \in End_{\mathcal{K}}(A,R)$ with $f(\sigma) = \rho$. A weak $\mathcal{K}$-origin is called *$\mathcal{K}$-origin* if for every $\sigma \in S$ and $\tau \in R$ with $d(\tau) = d(\sigma)$ we have $\tau \in S$, and for every pair $\sigma_1, \sigma_2 \in S$ there exist a finite sequence $\alpha_1, \alpha_2, ..., \alpha_m$ of elements of S and finite sequences $f_1, f_2, ..., f_{m-1}$, $g_1, g_2, ..., g_{m-1}$ of endomorphisms of (A,R) in $\mathcal{K}$ with $\sigma_1 = \alpha_1$, $\sigma_2 = \alpha_m$, $|f_i|$ is one-to-one on $d(\alpha_i)$, $|g_i|$ is one-to-one on $d(\alpha_{i+1})$, and $|f_i|(\alpha_i) = |g_i|(\alpha_{i+1})$ for every $i = 1, 2, ..., m-1$. We say that $\mathcal{K}$ *has weak origins* (or *origins*) if every $\mathcal{K}$-object has a weak origin (or origin, respectively). An element semigroup property $\mathcal{P}$ is called *arc-determining* in $\mathcal{K}$ if for every $\mathcal{K}$-object (A,R) and every $f \in End_{\mathcal{K}}(A,R)$ satisfying $\mathcal{P}$ a subset $\mathcal{P}(f) \subseteq A$ is determined such that there exists an arc $\alpha \in R$ with $\mathcal{P}(f) = d(\alpha)$. We say that a set semigroup property $\mathcal{P}$ is *subset-determining* in $\mathcal{K}$ if there exists a natural number $s(\mathcal{P})$ such that $card(Q) \leq s(\mathcal{P})$ for every $Q \subseteq End_{\mathcal{K}}(A,R)$ satisfying $\mathcal{P}$ and every $\mathcal{K}$-object (A,R), and for every $f \in Q$ a unique arc-determining property $\mathcal{P}_f$ is given. Denote by $\mathcal{P}(Q) = \{\mathcal{P}_f(f); f \in Q\}$. A subset semigroup isoproperty $\mathcal{P}$ is called *determining $\mathcal{K}$-origin* (or *determining weak $\mathcal{K}$-origin*) if it is subset-determining and for every $\mathcal{K}$-object (A,R) if a subset $Q \subseteq End_{\mathcal{K}}(A,R)$ satisfies $\mathcal{P}$ then there is a origin S (or a weak origin) of (X,R) with $d(S) = \mathcal{P}(Q)$, and if (X,R) has a origin (or weak origin, respectively) then there exists $Q \subseteq End_{\mathcal{K}}(A,R)$ satisfying $\mathcal{P}$. Let $\mathcal{P}$ be a determining weak $\mathcal{K}$-origin property. For any $\mathcal{K}$-object (A,R), any subset $Q \subseteq End(A,R)$, and a weak origin S with $d(S) = \mathcal{P}(Q)$, denote by $s_{Q,(A,R)}$ the number of $f \in Q$ such that there exists a strongly equimorphic $\mathcal{K}$-object (A,R') with (A,R) having a weak origin T with $d(T) = \mathcal{P}(Q)$ and $\{\alpha \in S; d(\alpha) = d(f)\} \neq \{\beta \in T; d(\beta) = d(f)\}$. Set $s_{\mathcal{P}} = max\{s_{Q,(A,R)}; (A,R)$ is a $\mathcal{K}$-object, $Q \subseteq End(A,R)$ satisfies $\mathcal{P}\}$.

Example. Assume $POSET$, SEM, $PRIEST$ as categories of binary relations. If $A = (X,R)$ is a poset, or a semilattice, or a Priestley space then $R^r = \{(x,y); x \leq y, x \neq y\}$. Assume that there exist $x, y \in |A|$ with $x < y$ then $\{(x,y)\}$ is an origin of A – indeed if $u \leq v$ is another pair in A, consider in the case $POSET$, or SEM the mapping $f : |A| \longrightarrow |A|$ such that $f(z) = v$ for $z \geq y$, $f(z) = x$ otherwise. Clearly, $f \in End(A)$. In the case of Priestley space choose a clopen decreasing set $Z \subseteq A$ with $x \in Z$, $y \notin Z$ and define $f : |A| \longrightarrow |A|$ such that $f(z) = u$ for $z \in Z$, $f(z) = v$ otherwise, then $f \in End(A)$. If such a pair in A does not exist then the empty set is an origin of A thus $POSET$, SEM, $PRIEST$ as binary relations have origins. If $L \in Lat_1$ then again $R^r = \{(x,y); x \leq y, x \neq y\}$. Choose $x < y$ such that $x \in I$, $y \notin I$ for a prime ideal I and for an arbitrary pair $u \leq v$ in L we define a mapping f such that $f(z) = u$ if $z \in I$, $f(z) = v$ otherwise. Thus Lat_1 has origins. Consider SEM_c as a subcategory of REL_3. Let S be a semilattice then $R^r = \{(x, y, x \wedge y); x, y \in |S|, x \neq y\}$. If there exist incomparable elements

$x, y \in S$ such that no element $u \in S$ satisfies $u \geq x, y$ then $\{(x, y, x \wedge y\}$ is a origin of S. Indeed, for every $u, v \in S$ define $f : S \longrightarrow S$ such that $f(z) = u$ if $z \geq x$, $f(z) = v$ if $z \geq y$, $f(z) = u \wedge v$ otherwise, then $f \in End(S)$. If for every pair of incomparable elements $x, y \in S$ there exists $z \in S$ with $z \geq x, y$ then $\{(x, y, x \wedge y)\}$ is an origin whenever x and y are incomparable. Indeed, for $u, v \in S$ choose $w \in S$ with $w \geq u, v$ (by the assumption such w exists) and define $f : S \longrightarrow S$ such that $f(z) = w$ if $z \geq x, y$, $f(z) = u$ if $z \geq x$ and $z \not\geq y$, $f(z) = v$ if $z \geq y$ and $z \not\geq x$, $f(z) = u \wedge v$ otherwise, then $f \in End(S)$. Hence SEM_c as ternary relations has origins.

Example. Consider $POSET$, SEM, Lat_1, $PRIEST$ as binary relations. Let A be a poset, or semilattices, or lattice with a prime ideal, or Priestley space. There exists a pair $\{x, y\}$ of elements of A such that $\{x, y\}$ is an origin from the foregoing example if and only if there exists $f \in Id(A)$ with $Im(f) = \{x, y\}$ and $card(f \circ End(A) \circ f) = 3$. Thus by note after Proposition 1.5 there exists a determining origin property $\mathcal{P}$ in $POSET$, SEM, Lat_1, $PRIEST$. Consider SEM_c as a subcategory of REL_3. Let $S \in SEM_c$. If $f \in Id(S)$ with $card(Im(f)) = 3$ and such that $Im(f)$ is not a chain, then over $Im(f)$ there exists an origin from the foregoing example. By easy calculation we obtain that $Im(f)$ is not chain if and only if $card(f \circ End(S) \circ f) = 9$. On the other hand if $\{(x, y, x \wedge y)\}$ is an origin such that no $z \in S$ satisfies $z \geq x, y$ then such endomorphism exists. If for every $f \in Id(S)$ with $card(Im(f)) = 3$ we have that $Im(f)$ is a chain then for every $f \in Id(S)$ such that $card(Im(f)) = 4$ and $Im(f)$ is not chain there is an origin on $Im(f) \setminus \{u\}$ where u is the greatest element of $Im(f)$. Since for $f \in Id(S)$ with $card(Im(f)) = 4$ we easily obtain that $Im(f)$ is not chain if and only if $card(f \circ End(S) \circ f) = 25$ it suffices to recognize the set $Im(f) \setminus \{u\}$ – it is the unique 3-element subset $Z \subseteq Im(f)$ such that there exists $g \in f \circ End(S) \circ f$ with $card(g(Z)) = 1$. Thus by Lemma 1.2 and Proposition 1.5 we conclude that there exists a determining SEM_c-origin property $\mathcal{P}$.

Let $\mathcal{K}$ be a category of n-ary relations over $\mathcal{L}$. A permutation φ of the set $\{1, 2, \dots, n\}$ is called a *$\mathcal{K}$-permutation* if there exist $\mathcal{K}$-objects (A, R), (A, R') with

$$R' = \{(x_{\varphi(1)}, x_{\varphi(2)}, \dots, x_{\varphi(n)}); (x_1, x_2, \dots, x_n) \in R\},$$

$$End_{\mathcal{K}}(A, R) = End_{\mathcal{K}}(A, R'),$$

and $R \neq R'$. We say that a $\mathcal{K}$-object (B, S) is *φ-isomorphic* to (A, R) if (B, S) and (A, R') are isomorphic in $\mathcal{K}$. For example, antiisomorphic posets or lattices are φ-isomorphic where φ is a permutation of $\{1, 2\}$ with $\varphi(1) = 2$. Let $\rho_{\mathcal{K}}$ be the number of $\mathcal{K}$-permutations. If the following categories are considered as binary relations then $\rho_{POSET} = \rho_{SEM} = \rho_{Lat_1} = \rho_{PRIEST} = \rho_{(0,1)-Lat_1} = 1$, $\rho_{0-Lat_1} = 0$ (because we cannot exchange pair $(0, x)$ for $x \neq 0$). If we consider SEM_c as ternary relations then $\rho_{SEM_c} = 0$. Indeed, assume that φ is a SEM_c-permutation of the set $\{1,2,3\}$. Since every semilattice $S \in SEM_c$ contains incomparable elements we conclude that $\varphi(3) = 3$. Since semilattices are commutative by the exchange of 1 and 2 we obtain the same object – a contradiction.

For a category $\mathcal{K}$ of relations denote by

$$m_{\mathcal{K}} = max\{k; \exists (A, R) \in \mathcal{K} \text{ and there exist } \alpha_1, \alpha_2, \dots, \alpha_k \in R$$

$$\text{with } d(\alpha_1) = d(\alpha_2) = \dots = d(\alpha_k)\}.$$

Obviously,

$$m_{POSET} = m_{SEM} = m_{Lat_1} = m_{0-Lat_1} = m_{(0,1)-Lat_1} = m_{PRIEST} = 1$$

where all categories are taken as binary relations. If SEM is taken as ternary relations then $m_{SEM} = 2$.

For natural numbers n,m with $1 \le m \le n$ denote by $t(n,m) = \Sigma\{\binom{n}{j}; 1 \le j \le m\}$. Obviously $t(n,m) \le \binom{n+m}{m}$.

Theorem 2.3. *Let $\mathcal{K}$ be an amenable isomorphism-full subcategory of REL_n $(\mathcal{L})$ such that $\mathcal{K}$ has weak origins and a determining weak $\mathcal{K}$-origin property $\mathcal{P}$. Moreover, for $\mathcal{L}$-objects A, B assume that $A = B$ whenever $End_{\mathcal{K}}(A,R) = End_{\mathcal{K}}(B,S)$ for some $\mathcal{K}$-objects $(A,R), (B,S)$. Then $\mathcal{K}$ is strongly $(t(n!, m_{\mathcal{K}})s_{\mathcal{P}} + 1)$-determined. If $\mathcal{K}$ has a coordination property then $\mathcal{K}$ is $(t(n!, m_{\mathcal{K}})s_{\mathcal{P}} + 1)$-determined.*

Proof. Let $\{\mathcal{A}_i = (A_i, R_i); i \in I\}$ be a family of non-isomorphic strongly equimorphic $\mathcal{K}$-objects. By the assumption on $\mathcal{K}$ and $\mathcal{L}$ we obtain that $\mathcal{A}_i = (A, R_i)$. Assume that $\mathcal{P}$ is a determining weak $\mathcal{K}$-origin property. Since every $\mathcal{A}_i$ has a weak origin there exists $Q \subseteq End_{\mathcal{K}}(\mathcal{A}_i)$ having $\mathcal{P}$ and for every $i \in I$ there exists a weak origin S_i of $\mathcal{A}_i$ with $d(S_i) = \mathcal{P}(Q)$. Since $R_i = \{|f|(\sigma); f \in End_{\mathcal{K}}(\mathcal{A}_i), \sigma \in S_i\} \cup R_i^t$, $R_i^t = R_j^t$, and $End_{\mathcal{K}}(\mathcal{A}_i) = End_{\mathcal{K}}(\mathcal{A}_j)$ we obtain that $S_i = S_j$ implies $R_i = R_j$ thus for $i \ne j$ we have $S_i \ne S_j$. Therefore for given $Q \subseteq End_{\mathcal{K}}(\mathcal{A}_i)$ having $\mathcal{P}$ we compute the number of distinct weak origins S with $d(S) = \mathcal{P}(Q)$. For any $f \in Q$ we have $card(\{\sigma \in S_i; d(\sigma) = \mathcal{P}_f(f)\}) = q \le m_{\mathcal{K}}$ and for given q there exist $\binom{n!}{q}$ such sets. Thus we conclude that for given $f \in Q$ there are at most $t(n!, m_{\mathcal{K}})$ sets $\{\sigma \in S_i; d(\sigma) = \mathcal{P}_f(f)\}$. Since the number of $f \in Q$ such that f distinguishes distinct origins is at most $s_{\mathcal{P}}$ we obtain $card(I) \le t(n!, m_{\mathcal{K}})s_{\mathcal{P}}$ and hence $\mathcal{K}$ is strongly $(t(n!, m_{\mathcal{K}})s_{\mathcal{P}} + 1)$-determined. If $\mathcal{K}$ has a coordination property then we apply Corollary 1.10 and we obtain that $\mathcal{K}$ is $(t(n!, m_{\mathcal{K}})s_{\mathcal{P}} + 1)$-determined. ☐

We say that a category $\mathcal{K}$ is *max-uniform* if for every $\mathcal{K}$-object (A,R) and for every $\alpha \in R$ such that the cardinality of $d(\alpha)$ is the greatest in R we have $card(\{\beta \in R; d(\beta) = d(\alpha)\}) = m_{\mathcal{K}}$.

Theorem 2.4. *Assume that $\mathcal{K}$ is an amenable max-uniform isomorphism-full subcategory of REL_n ($\mathcal{L}$) such that $\mathcal{K}$ has origins and a determining $\mathcal{K}$-origin property $\mathcal{P}$. Assume that $A = B$ for $\mathcal{L}$-objects A, B whenever $End_{\mathcal{K}}(A, R) = End_{\mathcal{K}}(B, S)$ for some $\mathcal{K}$-objects $(A, R), (B, S)$. Then $\mathcal{K}$ is strongly $(\rho_{\mathcal{K}} + 2)$-determined and every strongly equimorphic objects are either isomorphic or φ-isomorphic for a $\mathcal{K}$-permutation φ. If $m_{\mathcal{K}} = 1$ then $\mathcal{K}$ is max-uniform. If $\mathcal{K}$ has a coordination property then $\mathcal{K}$ is $(\rho_{\mathcal{K}} + 2)$-determined and every equimorphic objects are either isomorphic or φ-isomorphic for a $\mathcal{K}$-permutation φ.*

Proof. As in the proof of Theorem 2.3 let $\{\mathcal{A}_i = (A, R_i); i \in I\}$ be a family of non-isomorphic strongly equimorphic $\mathcal{K}$-objects with origins S_i such that $d(S_i) = d(S_j)$ for $i, j \in I$. Then $S_i \neq S_j$ for $i \neq j$. We prove that $\mathcal{A}_i$ and $\mathcal{A}_j$ are φ-isomorphic for some $\mathcal{K}$-permutation φ whenever $i \neq j$. Choose $\sigma_0 \in S_i$, $\tau_0 \in S_j$ with $d(\sigma_0) = d(\tau_0)$. We shall define a mapping $\psi : S_i \longrightarrow S_j$ by induction: $\psi(\sigma_0) = \tau_0$. Assume that $\sigma_1, \sigma_2 \in S_i$ and that there exist $f, g \in End_{\mathcal{K}}(\mathcal{A}_i)$ such that f is one-to-one on $d(\sigma_1)$, g is one-to-one on $d(\sigma_2)$, and $f(\sigma_1) = g(\sigma_2)$. If $\psi(\sigma_1)$ is defined then $d(\sigma_1) = d(\psi(\sigma_1))$ and by the assumptions on $\mathcal{K}$ there exists exactly one $\tau \in S_j$ with $f(\psi(\sigma_1)) = g(\tau)$ and $d(\sigma_2) = d(\tau)$ because $card(\{\beta \in S_j; d(\beta) = d(\sigma_1)\}) = card(\{\beta \in S_j; d(\beta) = d(\sigma_2)\}) = card(\{\beta \in R_i; d(\beta) = d(f(\psi(\sigma_1)))\}) = m_{\mathcal{K}}$. Further if φ is a permutation of $\{1, 2, .., n\}$ such that for $\sigma_1 =< x_1, x_2, ..., x_n >$ we have $\psi(\sigma_1) =< x_{\varphi(1)}, x_{\varphi(2)}, ..., x_{\varphi(n)} >$ then for $\sigma_2 =< y_1, y_2, ..., y_n >$ we have $\tau =< y_{\varphi(1)}, y_{\varphi(2)}, ..., y_{\varphi(n)} >$. Define $\psi(\sigma_2) = \tau$. Then ψ is a bijection and there exists a permutation φ of $\{1, 2, ..., n\}$ such that for every $\sigma_1 =< x_1, x_2, ..., x_n >\in S_i$ we have $\psi(\sigma) =< x_{\varphi(1)}, x_{\varphi(2)}, ..., x_{\varphi(n)} >$. From the definition of the origin we conclude that $R_j = \{(x_{\varphi(1)}, x_{\varphi(2)}, ..., x_{\varphi(n)}); (x_1, x_2, ..., x_n) \in R_i\}$. Therefore φ is a $\mathcal{K}$-permutation and $\mathcal{A}_i$ and $\mathcal{A}_j$ are φ-isomorphic. If $\mathcal{K}$ has a coordination property we apply Corollary 1.10. The rest is obvious. □

For the case of $\mathcal{L} = SET$ the implication that $X = Y$ whenever $End_{\mathcal{K}}(X, R) = End_{\mathcal{K}}(Y, S)$ is obvious because X is an underlying set of $End_{\mathcal{K}}(X, R)$ and Y is an underlying set of $End_{\mathcal{K}}(Y, S)$. In the case $\mathcal{L} = TOP$ we shall use the following folklore lemma.

Lemma 2.5. *Let (X, τ_i) be a topological T_1 space with a subbase $\mathcal{B}_i$ for $i = 1, 2$. If $Q \subseteq Fin(X, \tau_1) \cap Fin(X, \tau_2)$ such that $\mathcal{B}_1 \cup \mathcal{B}_2$ is contained in the Boolean closure of the sets $\{f^{-1}(x); f \in Q, x \in Im(f)\}$ then $\tau_1 = \tau_2$.* □

Since the category $POSET$ satisfies the assumption of Theorem 2.4, $m_{\mathcal{K}} = 1$ and $\rho_{\mathcal{K}} = 0$ we obtain Theorem 2.1 as a consequence of Theorem 2.4. Also SEM as a subcategory of binary relations and SEM_c as a subcategory of ternary relations satisfy the assumption of Theorem 2.4 ($m_{SEM} = 1$, $\rho_{SEM} = 1$, $m_{SEM_c} = 2$, $\rho_{SEM_c} = 0$ and SEM_c is max-uniform) we obtain a theorem proved originally by Schein [25]

Theorem 2.6. [25] *Equimorphic semilattices are either isomorphic or antiisomorphic chains.* □

Corollary 2.7. *Lat_I is 3-determined, moreover two equimorphic lattices with a prime ideal are either isomorphic or antiisomorphic.*

Proof. Apply Theorem 2.4, $m_{Lat_1} = 1$, $\rho_{Lat_1} = 1$. □

Corollary 2.8. *$PRIEST$ are 3-determined. Equimorphic Priestley spaces are either isomorphic or antiisomorphic.*

Proof. If $(X, \leq, \tau)$ is a Priestley space, then the set of all clopen decreasing and clopen increasing subsets of X is a subbase of τ. For $x < y$ and for a clopen decreasing set $U \subseteq X$ define $f : X \longrightarrow X$ such that $f(z) = x$ for $z \in U$, $f(z) = y$ otherwise – f is continuous order preserving. We apply Lemma 2.5 (if $\leq$ is discrete then the assumptions of Lemma 2.5 are also satisfied) and we obtain that the assumptions of Theorem 2.4 are satisfied. Since $m_{PRIEST} = 1$, $\rho_{PRIEST} = 1$ the proof is complete. □

The dual form of Corollary 2.8 was proved by McKenzie and Tsinakis [18] – the equimorphic distributive (0,1)-lattices are either isomorphic or antiisomorphic.

Lemma 2.9. $0 - Lat_1$ *has a coordination property,* $0 - Lat_2$ *has origins and a determining origin property.* $(0,1) - Lat_2$ *has a coordination property,* $(0,1) - Lat_3$ *has origins and a determining origin property.*

Proof. Let $L \in 0 - Lat_1$ then the constant mapping to 0 is an endomorphism which is a zero of $End(L)$. Thus $Kernel(L)$ consists of the constant to 0, and $L_{Ker} = \{0\}$. Since L has a prime ideal we conclude that $card(L) > 1$ and thus $L \neq L_{Ker}$. Hence the isoproperty $\mathcal{P} = \{f \notin Kernel(L)\}$ coordinatizes kernels and $f \in End(L)$ satisfies $\mathcal{P}$ and $card(Im(f) \setminus L_{Ker}) = 1$ if and only if $card(Im(f)) = 2$. Further there exists $f \in Id(L)$ with $card(Im(f)) = 2$. Consider $f \in Id(L)$ with $Im(f) = \{0, y\}$, $y \neq 0$. Then $I = f^{-1}(0)$ is a prime ideal in L and for $x \in L$ define $f_x : L \longrightarrow L$ such that $f_x(z) = 0$ for $z \in I$, $f_x(z) = x$ otherwise. Clearly, $f_x \in End(L)$ and $\{f_x; x \in L\}$ is a strictly 1-transitive right ideal in $End(L)$ generated by f with a source y, $Stab(End(L) \circ f, y) = \{f\}$, and $f_x \in Kernel(L)$ if and only if $Im(f_x) \subseteq L_{Ker}$. By Theorem 1.8 $0 - Lat_1$ has a coordination property.

Let $L \in (0,1) - Lat_2$ then $Kernel(L) = \{f \in End(L); card(Im(f)) = 2\}$ which is the set of all right zeros of $End(L)$. Further $L_{Ker} = \{0, 1\}$. Since there exist distinct prime ideals I, J with $I \subseteq J \subseteq L$ we conclude that there exists $f \in Id(L)$ with $card(Im(f)) = 3$ and thus $L \neq L_{Ker}$. The isoproperty $\mathcal{P} = \{f \notin Kernel(L)\}$ coordinatizes kernels and $f \in End(L)$ satisfies $\mathcal{P}$ and $card(Im(f) \setminus L_{Ker}) = 1$ if and only if $card(Im(f)) = 3$. Let $f \in Id(L)$ with $Im(f) = \{0, y, 1\}$ where $0 \neq y \neq 1$ then $f^{-1}(0) = I$, $f^{-1}(\{0, y\}) = J$ are distinct prime ideals with $I \subseteq J \subseteq L$. For every $x \in L$ define a mapping $f_x : L \longrightarrow L$ such that $f_x(z) = 0$ if $z \in I$, $f_x(z) = x$ if $z \in J \setminus I$, $f_x(z) = 1$ if $z \in L \setminus J$. Clearly, $f_x \in End(L)$ and $\{f_x; x \in L\}$ is a strictly 1-transitive right ideal generated by f with a source y, $Stab(End(L) \circ f, y) = \{f\}$, and $f_x \in Kernel(L)$ if and only if $x \in L_{Ker}$. By Theorem 1.8 $(0,1) - Lat_2$ has a coordination property.

Let $L \in 0 - Lat_2$ with distinct prime ideals $I \subseteq J \subseteq L$. Choose $x \in J \setminus I$, $y \in L \setminus J$ with $x \leq y$. We show that $\{(x, y)\}$ is an origin and there exists $f \in Id(L)$ with $Im(f) = \{0, x, y\}$. Let $u \leq v$ be elements of L. Define $f : L \longrightarrow L$ such that $f(z) = 0$ for $z \in I$, $f(z) = u$ for $z \in J \setminus I$, $f(z) = v$ for $z \in L \setminus J$. Since $f \in End(L)$ we conclude that $\{(x, y)\}$ is an origin and there exists $f \in Id(L)$ with $Im(f) = \{0, x, y\}$. If $f \in Id(L)$ with $card(Im(f)) = 3$ then $Im(f)$ is a chain. Assume $Im(f) = \{0, x, y\}$ and $x \leq y$ then $f^{-1}(0)$, $f^{-1}(\{0, x\})$ are prime ideals and $\{(x, y)\}$ is an origin. Since $0 - Lat_2$ has a coordination property we conclude that there exists a determining $0 - Lat_2$-origin property.

Let L be a (0,1)-lattice with distinct prime ideals $I \subseteq J \subseteq K \subseteq L$. Choose $x \in J \setminus I$, $y \in K \setminus J$ with $x \leq y$. We prove that $\{(x, y)\}$ is an origin of L and there exists an idempotent $f \in End(L)$ with $Im(f) = \{0, x, y, 1\}$. For $u \leq v$ in L define a mapping $f : L \longrightarrow L$ with $f(z) = 0$ for $z \in I$, $f(z) = u$ for $z \in J \setminus I$, $f(z) = v$ for $z \in K \setminus J$, $f(z) = 1$ for $z \in L \setminus K$. Since $f \in End(L)$ we obtain that $\{(x, y)\}$ is an origin of L. Let $f \in Id(L)$ such that $card(Im(f)) = 4$ and $Im(f)$ is a chain. Assume that $Im(f) = \{0, x, y, 1\}$ and $x < y$ then $f^{-1}(0)$, $f^{-1}(\{0, x\})$, $f^{-1}(\{0, x, y\})$ are distinct prime ideals and $\{(x, y)\}$ is an origin of L. Since for $f \in Id(L)$ with $card(Im(f)) = 4$ we have that $Im(f)$ is a chain if and only if $card(f \circ End(L) \circ f) = 10$ we conclude that $(0, 1) - Lat_3$ has a determining origin property because $(0, 1) - Lat_3$ has a coordination property. □

Theorem 2.10. *Equimorphic lattices in $0 - Lat_2$ are isomorphic. The category $0 - Lat_2$ is 2-determined. Equimorphic lattices in $(0, 1) - Lat_3$ are either isomorphic or antiisomorphic. The category $(0, 1) - Lat_3$ is 3-determined.*

Proof. By Lemma 2.9 $0 - Lat_2$ and $(0, 1) - Lat_3$ satisfy the assumptions of Theorem 2.4. Since $m_{0-Lat_2} = 1$, $\rho_{0-Lat_2} = 0$, $m_{(0,1)-Lat_3} = 1$, $\rho_{(0,1)-Lat_3} = 1$ statements follow from Theorem 2.4. □

3. Distributive p-algebras

We recall that a distributive (0,1)-lattice with added unary operation * such that $a \wedge b = 0$ if and only if $b \leq a^*$ is called a *distributive p-algebra*. Ribenboim proved that distributive p-algebras form a variety, see [24]. Denote by B_n the distributive p-algebra obtained from the 2^n-element Boolean algebra with adjoined new 1 and let L_n be a variety of distributive p-algebras generated by B_n.

For an investigation of distributive p-algebras we shall exploit the Priestley duality. A Priestley space $A = (X, \leq, \tau)$ is called a *p-space* if for every clopen decreasing set $U \subseteq X$ the set $[U)$ is also clopen and a mapping $f : X \longrightarrow Y$ is called *p-mapping* from $(X, \leq, \tau)$ to $(Y, \leq, \sigma)$ if it is continuous, order preserving mapping, and for every $x \in X$ we have $f(Min(x)) = Min(f(x))$ where $Min(x) = \{y; y \leq x$ and y is a minimal element of $X\}$. We recall that in every p-space the set of all minimal elements is closed. The subcategory of $PRIEST$ formed by all p-spaces and p-mappings is denoted by $P - SP$. Priestley proved

Theorem 3.1. [20] *The category $P-SP$ is dually isomorphic to the variety of all distributive p-algebras.* □

For a natural number $n \geq 1$, denote by $P-SP_n$ the full subcategory of $P-SP$ formed by all p-spaces fulfiling $card(Min(x)) \leq n$ for every element $x \in X$. Denote by $P-SP^-$ the full subcategory of $P-SP_2$ formed by all p-spaces in $P-SP_2$ with non-discrete ordering, and $P-SP^+$ the full subcategory of $P-SP^-$ formed by all p-spaces $A=(X,\leq,\tau)$ such that either there exists $x \in X$ which is not minimal and $card(Min(x))=1$ or every chain in X has length ≤ 1. The following statement was proved by Lee:

Theorem 3.2. [14] *For every $n \geq 1$, the category $P-SP_n$ is dually isomorphic to the variety L_n. Boolean algebras and the variety L_n, $n \geq 1$ are unique proper non-trivial subvarieties of distributive p-algebras.* □

Let $A=(X,\leq,\tau)$ be a non-empty p-space. A constant mapping $f: X \longrightarrow X$ is an endomorphism of A if and only if f is a constant mapping to a minimal element. Hence $Kernel(A)$ exists and it consists of all left zeros, and A_{Ker} is the set of all minimal elements (and it is closed). Moreover, $X=A_{Ker}$ if and only if $\leq$ is discrete. For any $f \in End(A)$ denote by $M(f)=card(\{f \circ g \in Kernel(A); g \in Kernel(A)\})$ then $M(f)=card(Im(f) \cap A_{Ker})$. Clearly, $M(f)=n$ is an element isoproperty and $x \in Im(f) \cap A_{Ker}$ if and only if there exists $h \in End(A)$ such that $f \circ h$ is a constant mapping to x.

First we give an easy lemma of the existence of special p-mappings from a p-space $A \in P-SP^-$ into itself. An endomorphism $f \in End(A)$ is called *x-spanning* where $x \in X$ if $Im(f)=\{x\} \cup Min(x)$.

Lemma 3.3. *Let $A=(X,\leq,\tau)$ be a p-space from $P-SP^-$.*

(1) *If there exist distinct $x,y \in X$ with $card(Min(x))=2$ and $x \leq y$ then for every $u,v \in X$ with $u \leq v$ and $Min(u)=Min(v)$ there exists $f \in End(A)$ with $f(x)=u$, $f(y)=v$, and $Im(f)=\{u,v\} \cup Min(u)$;*
(2) *If there exist $x,y \in X \setminus A_{Ker}$ with $card(Min(x))=1$, $card(Min(y))=2$, and $x \leq y$ then for every $u,v \in X$ with $u \leq v$ and $card(Min(u))=1$ there exists $f \in End(A)$ with $f(x)=u$, $f(y)=v$, and $Im(f)=\{u,v\} \cup Min(v)$;*
(3) *If there exist distinct $x,y \in X \setminus A_{Ker}$ with $card(Min(y))=1$ and $x \leq y$ then for every $u,v \in X$ with $u \leq v$ and $card(Min(v))=1$ there exists $f \in End(A)$ with $f(x)=u$, $f(y)=v$, and $Im(f)=\{u,v\} \cup Min(v)$;*
(4) *For every clopen decreasing set $U \subseteq X$ there exists $f \in Fin(A)$ with $[U)=f^{-1}(V)$ for some $V \subseteq Im(f)$;*
(5) *If there exists $x \in X \setminus A_{Ker}$ with $card(Min(x))=1$ then for every clopen increasing set $U \subseteq X \setminus A_{Ker}$ there exist $f \in Fin(A)$ and $x \in Im(f)$ with $f^{-1}(x)=U$;*
(6) *If there exist distinct $x,y \in X \setminus A_{Ker}$ with $Min(x)=Min(y)$ then for every clopen increasing set $U \subseteq X \setminus A_{Ker}$ and for every $u \in U$ there exist $f \in Fin(A)$ and $v \in Im(f)$ with $u \in f^{-1}(v) \subseteq U$;*

(7) *For every $x \in X$ there exists x-spanning $f \in Id(A)$;*
(8) *If $f \in Id(A)$ is x-spanning for some $x \in X$ then $g \in End(A) \circ f$ if and only if g is v-spanning for some $v \in X$ and there exists $k : \{x\} \cup Min(x) \longrightarrow \{v\} \cup Min(v)$ with $k(x) = v$, $k(Min(x)) = Min(v)$ and $g(z) = k(f(z))$ for every $z \in X$.*

Proof. Assume that $s, t \le x \le y$, $s, t \in A_{Ker}$, $s \ne t$, $x \ne y$. Then there exist clopen decreasing sets U_0, V_0 with $s \in U_0$, $t \notin U_0$, $x \in V_0$, $y \notin V_0$. Since A is a p-space the following sets are clopen $U = [U_0) \setminus [A_{Ker} \setminus U_0)$, $W = [A_{Ker} \setminus U_0) \setminus [U_0)$, $V = [A_{Ker} \setminus U_0) \cap [U_0) \cap V_0$, $T = ([A_{Ker} \setminus U_0) \cap [U_0)) \setminus V_0$. Moreover, U, W are decreasing, T is increasing, $A_{Ker} \subseteq U \cup W$, $s \in U$, $t \in W$, $x \in V$, $y \in T$ and $\{U, W, V, T\}$ is a decomposition of X. For $u, v \in X$ with $u \le v$, $Min(u) = Min(v) = \{w_1, w_2\}$ define $f : X \longrightarrow X$ such that $f(z) = w_1$ for $z \in U$, $f(z) = w_2$ for $z \in W$, $f(z) = u$ for $z \in V$, $f(z) = v$ for $z \in T$. Then $f \in End(A)$ and $f(x) = u$, $f(y) = v$, $Im(f) = \{u, v, w_1, w_2\}$. (1) is proved.

Assume that $s \le x \le y \ge t$, $s, t \in A_{Ker}$, $x \ne s \ne t$. Then there exists a clopen decreasing set U_0 with $s \in U_0$, $t, x \notin U_0$. Since A is a p-space the following sets are clopen $U = U_0$, $V = [U_0) \setminus ([A_{Ker} \setminus U_0) \cup U_0)$, $W = [A_{Ker} \setminus U_0) \setminus [U_0)$, $T = [A_{Ker} \setminus U_0) \cap [U_0)$. Moreover, U, W are decreasing, T is increasing, $A_{Ker} \subseteq U \cup W$, $s \in U$, $t \in W$, $x \in V$, $y \in T$ and $\{U, W, V, T\}$ is a decomposition of X. For $u, v \in X$ with $u \le v$, $Min(u) = \{w_1\}$, $Min(v) = \{w_1, w_2\}$ define $f : X \longrightarrow X$ such that $f(z) = w_1$ for $z \in U$, $f(z) = w_2$ for $z \in W$, $f(z) = u$ for $z \in V$, $f(z) = v$ for $z \in T$. Then $f \in End(A)$ and $f(x) = u$, $f(y) = v$, $Im(f) = \{u, v, w_1, w_2\}$. (2) is proved.

Assume that $s \le x \le y$, $s \in A_{Ker}$, $s \ne x \ne y$. Then there exist clopen decreasing sets U_0, V_0 with $s \in U_0$, $x \notin U_0$, $x \in V_0$, $y \notin V_0$ and $A_{Ker} \subseteq U_0$. Since A is a p-space the following sets are clopen $U = U_0$, $V = (X \setminus U_0) \cap V_0$, $T = X \setminus (U_0 \cup V_0)$ moreover, U is decreasing, T is increasing, $A_{Ker} \subseteq U$, $s \in U$, $x \in V$, $y \in T$ and $\{U, V, T\}$ is a decomposition of X. For $u, v \in X$ with $u \le v$, $Min(u) = Min(v) = \{w\}$ define $f : X \longrightarrow X$ such that $f(z) = w$ for $z \in U$, $f(z) = u$ for $z \in V$, $f(z) = v$ for $z \in T$. Then $f \in End(A)$ and $f(x) = u$, $f(y) = v$, $Im(f) = \{w, u, v\}$. (3) is proved.

If $x \in A_{Ker}$ then the constant mapping to x is the x-spanning idempotent. Assume that $x \in X \setminus A_{Ker}$. If $Min(x) = \{y\}$ then choose an arbitrary increasing clopen set $T \subseteq X$ with $x \in T$, $T \cap A_{Ker} = \emptyset$ and define $f : X \longrightarrow X$ with $f(z) = x$ for $z \in T$, $f(z) = y$ for $z \in X \setminus T$. Clearly, $f \in Id(A)$ is x-spanning. If $Min(x) = \{y_1, y_2\}$ with $y_1 \ne y_2$ then choose a clopen decreasing set $U_0 \subseteq X$ with $y_1 \in U_0$, $y_2 \notin U_0$. The following sets are clopen $U = [U_0) \setminus [A_{Ker} \setminus U_0)$, $W = [A_{Ker} \setminus U_0) \setminus [U_0)$, $V = [U_0) \cap [A_{Ker} \setminus U_0)$. Further U, W are decreasing, V is increasing, $y_1 \in U$, $y_2 \in W$, $x \in V$, and $\{U, W, V\}$ is a decomposition of X. Define $f : X \longrightarrow X$ such that $f(z) = y_1$ for $z \in U$, $f(z) = y_2$ for $z \in W$, $f(z) = x$ for $z \in V$. Clearly, $f \in Id(A)$ is x-spanning. (7) is proved.

If $[U)$ is not decreasing then there exists $x \in [U)$ with $Min(x) = \{y_1, y_2\}$, $y_1 \in U$, $y_2 \notin U$ and by the foregoing part of the proof there exists an x-spanning $f \in Id(A)$ with $f \in Fin(A)$ and $U = f^{-1}(\{x, y_1\})$. If $[U)$ is decreasing and $[U) \ne X$ then choose $x \in [U) \cap A_{Ker}$, $y \in A_{Ker} \setminus [U)$ and define $f : X \longrightarrow X$ such that $f(z) = x$ for $z \in [U)$, $f(z) = y$ for $z \in X \setminus [U)$. Obviously, $f \in Fin(A)$ and $f^{-1}(x) = [U)$. If $[U) = X$ then for any constant f to a minimal element $x \in A_{Ker}$ we have $f^{-1}(x) = [U) = X$. (4) is proved.

Assume that there exists $x \in X \setminus A_{Ker}$ with $Min(x) = \{y\}$. Then for every clopen increasing set $T \subseteq X \setminus A_{Ker}$ define $f : X \longrightarrow X$ such that $f(z) = x$ for $z \in T$, $f(z) = y$ for $z \in X \setminus T$. Obviously, $f \in Fin(A)$ and $T = f^{-1}(x)$. (5) is proved.

Assume that there exist distinct $x, y \in X$ with $Min(x) = Min(y)$. Let $T_0 \subseteq X \setminus A_{Ker}$ be a clopen increasing set. If there exists $u \in X \setminus A_{Ker}$ with $card(Min(u)) = 1$ then by (5) there exist $f \in Fin(A)$, $v \in Im(f)$ with $f^{-1}(v) = T_0$. Assume that $card(Min(v)) = 2$ for every $v \in X \setminus A_{Ker}$. Choose $u \in T_0$ and assume $Min(u) = \{v_1, v_2\}$. There exists a clopen decreasing set U_0 with $v_1 \in U_0$, $v_2 \notin U_0$. Since A is a p-space the following sets are clopen $U = [U_0) \setminus [A_{Ker} \setminus U_0)$, $W = [A_{Ker} \setminus U_0) \setminus [U_0)$, $T = [A_{Ker} \setminus U_0) \cap [U_0) \cap T_0$, $V = ([A_{Ker} \setminus U_0) \cap [U_0)) \setminus T_0$ moreover, U, W are decreasing, T is increasing, $A_{Ker} \subseteq U \cup W$, $v_1 \in U$, $v_2 \in W$, $u \in T$, $T \subseteq T_0$, and $\{U, W, V, T\}$ is a decomposition of X. Let $x, y \in X \setminus A_{Ker}$ be distinct with $Min(x) = \{w_1, w_2\}$ such that $x \leq y$ whenever there exists a chain in A of length > 1. Define $f : X \longrightarrow X$ such that $f(z) = w_1$ for $z \in U$, $f(z) = w_2$ for $z \in W$, $f(z) = x$ for $z \in V$, $f(z) = y$ for $z \in T$. Since V is increasing whenever every chain of A has length ≤ 1 we obtain that $f \in Fin(A)$ and $u \in f^{-1}(y) \subseteq T_0$. (6) is proved.

If $f \in Id(A)$ is x-spanning for some $x \in X$ then every $g \in End(A) \circ f$ is $g(x)$-spanning and $g(Min(x)) = Min(g(x))$. On the other hand if k is a mapping from $\{x\} \cup Min(x)$ onto $\{y\} \cup Min(y)$ with $k(x) = y$ and $k(Min(x)) = Min(y)$ then a mapping $g : X \longrightarrow X$ such that $g(z) = k(f(z))$ for every $z \in X$ belongs to $End(A) \circ f$. (8) is proved. □

Corollary 3.4. *Let $A = (X, \leq, \tau) \in P - SP^-$ with $card(Min(x)) = 2$ for some $x \in X$. Then for every x-spanning $f \in Id(A)$, the right ideal $End(A) \circ f$ is 1-transitive and $Stab(End(A) \circ f, x)$ is a group.*

Let $A = (X, \leq, \tau) \in P - SP_1$ with $x \in X \setminus A_{Ker}$. Then for every x-spanning $f \in Id(A)$, the right ideal $End(A) \circ f$ is 1-transitive and $Stab(End(A) \circ f, x)$ is a group.

Let $A = (X, \leq, \tau) \in P - SP_1$ then there exists $x \in X \setminus A_{Ker}$ if and only if there exists $f \in Id(A)$ with $M(f) = 1$ and $f \notin Kernel(A)$.

Proof. Let $A = (X, \leq, \tau) \in P - SP^-$ and let $f \in Id(A)$ be x-spanning for some $x \in X$ with $card(Min(x)) = 2$. By Lemma 3.3 (8) $Stab(End(A) \circ f, x)$ contains two elements creating a group and if $y \in X$ then $\{g \in End(A) \circ f; g(x) = y\} = Stab(End(A) \circ f, x) \circ h$ for any $h \in End(A) \circ f$ with $h(x) = y$. Thus $End(A) \circ f$ is 1-transitive.

The remaining statements follow immediately from Lemma 3.3 (8). □

Lemma 3.5. *Let $A = (X, \leq, \tau) \in P - SP^-$. Then*

(1) *There exist distinct comparable $x, y \in X$ with*

$$card(Min(x)) = card(Min(y)) = 2$$

if and only if there exists $f \in Id(A)$ with $M(f) = 2$, $card(Im(f)) = 4$, $card(f \circ End(A) \circ f) = 8$ such that for every $h \in f \circ End(A) \circ f$ we have $h \in Kernel(A)$ whenever $M(h) = 1$, and $\{x, y\} = Im(f) \setminus A_{Ker}$.

(2) *There exist $x \in X \setminus A_{Ker}$, $y \in X$ with*

$$x \leq y, card(Min(x)) = 1, card(Min(y)) = 2$$

if and only if there exists $f \in Id(A)$ with $M(f) = 2$, $card(Im(f)) = 4$ such that there exists exactly one $h \in f \circ End(A) \circ f$ with $h \notin Kernel(A)$, $h \in Id(A)$, and $M(h) = 1$, and $\{x, y\} = Im(f) \setminus A_{Ker}$.

(3) *There are distinct comparable elements $x, y \in X \setminus A_{Ker}$ with*

$$card(Min(y)) = card(Min(x)) = 1$$

just when there exists $f \in Id(A)$ with $M(f) = 1$, $card(Im(f)) = 3$, $card(f \circ End(A) \circ f) = 6$, and $\{x, y\} = Im(f) \setminus A_{Ker}$.

Proof. If there exist distinct comparable $x, y \in X$ such that $card(Min(x)) = card(Min(y)) = 2$ then by Lemma 3.3 (1) there exists $f \in Id(A)$ with $Im(f) = \{x, y\} \cup Min(x)$. By a direct calculation we obtain that f satisfies the required conditions. Conversely, assume that $f \in Id(A)$ satisfies the required conditions. Then $card(Im(f) \cap A_{Ker}) = 2$ and because every $h \in f \circ End(A) \circ f$ with $M(h) = 1$ is in $Kernel(A)$ we conclude that for every $u \in Im(f) \setminus A_{Ker}$ we have $Min(u) = Im(f) \cap A_{Ker}$. Obviously, $Im(f)$ is a p-space with 10 endomorphisms if elements of $Im(f) \setminus A_{Ker}$ are incomparable, and with 8 endomorphisms if they are comparable. Lemma 1.3 completes the proof.

If there exist $x \in X \setminus A_{Ker}$, $y \in X$ with $x \leq y$, $card(Min(x)) = 1$, and $card(Min(y)) = 2$ then by Lemma 3.3 (2) there exists $f \in Id(A)$ with $Im(f) = \{x, y\} \cup Min(y)$. By a direct calculation we obtain that f satisfies the required conditions. Let $f \in Id(A)$ fulfil the required conditions. Then $card(Im(f) \cap A_{Ker}) = 2$ because $M(f) = 2$. Since there exists exactly one $h \in f \circ End(A) \circ f \cap Id(A)$ with $M(h) = 1$ and $h \notin Kernel(A)$ we conclude that there exists exactly one $u \in Im(f) \setminus A_{Ker}$ with $card(Min(u)) = 1$. Then for $v \in Im(f) \setminus (A_{Ker} \cup \{u\})$ we have $Min(v) = Im(f) \cap A_{Ker}$ and moreover $v \geq u$ (else there exist two $h \in (f \circ End(A) \circ f \cap Id(A)) \setminus Kernel(A)$ with $M(h) = 1$). The proof is complete.

Let $x, y \in X$ be distinct comparable with $card(Min(x)) = card(Min(y)) = 1$ then by Lemma 3.3 (3) there exists $f \in Id(A)$ with $Im(f) = \{x, y\} \cup Min(x)$. By a direct calculation we obtain that f satisfies the required conditions. Conversely, assume that $f \in Id(A)$ satisfies the required conditions. Then $card(Im(f) \cap A_{Ker}) = 1$ because $M(h) = 1$ and we conclude that for every $u \in Im(f) \setminus A_{Ker}$ we have $Min(u) = Im(f) \cap A_{Ker}$. Obviously, $Im(f)$ is a p-space with 9 endomorphisms if elements of $Im(f) \setminus A_{Ker}$ are incomparable, and with 6 endomorphisms if they are comparable. Lemma 1.3 completes the proof. □

Theorem 3.6. *Let $A = (X, \leq, \tau) \in P-SP_2$. Then Boolean closure $\mathcal{B}$ of the family $\mathcal{C} = \{f^{-1}(x); f \in Fin(A), x \in Im(f)\}$ of sets consists of the all clopen sets.*

Proof. Since every set in $\mathcal{C}$ is clopen we conclude that every set in $\mathcal{B}$ is clopen. The family of all decreasing clopen sets and of all increasing clopen sets is a subbase of τ thus it suffices to show that every clopen increasing set is in $\mathcal{B}$. If $\leq$ is discrete then it holds. If there exists $x \in X \setminus A_{Ker}$ with $card(Min(x)) = 1$ then by Lemma 3.3 (5) for every increasing set $T \subseteq X \setminus A_{Ker}$ we have $T \in \mathcal{C}$. If there exist two distinct elements $u, v \in X$ with $Min(u) = Min(v)$ then by Lemma 3.3 (6) for every increasing set $T \subseteq X \setminus A_{Ker}$ and every $t \in T$ there exists a set $U \in \mathcal{C}$ with $t \in U \subseteq T$. Since T is compact we conclude that $T \in \mathcal{B}$. If $T \subseteq X$ is increasing then there exists a clopen decreasing set $V \subseteq T$ with $T \cap A_{Ker} \subseteq V$ (because A_{Ker} is closed). Then by Lemma 3.3 (4) $[V) \in \mathcal{B}$ and because $T = [V) \cup T \setminus V$ and $T \setminus V \cap A_{Ker} = \emptyset$ we obtain that $T \in \mathcal{B}$ and thus $\mathcal{B}$ consists of the all clopen sets. It remains to investigate the case that every $x \in X \setminus A_{Ker}$ satisfies $card(Min(x)) = 2$ and for $x, y \in X$ we have $Min(x) = Min(y)$ if and only if $x = y$. If we prove that $\mathcal{B}$ separates elements of X, then $\mathcal{B}$ is a subbase of τ and hence $\mathcal{B}$ consists of all clopen sets. Let $x, y \in X$ be distinct elements. If $x, y \in A_{Ker}$ then there exists a clopen decreasing set $U \subseteq X$ with $x \in U$, $y \notin U$, then $x \in [U)$, $y \notin [U)$ and by Lemma 3.3 (4) $[U) \in \mathcal{B}$. If $x \in A_{Ker}$, $y \notin A_{Ker}$ then there exists $v \in Min(y)$, $v \neq x$ and by the foregoing part there exists a clopen decreasing set $U \subseteq X$ with $[U) \in \mathcal{B}$, $x \notin [U)$, $v \in [U)$ and hence $y \in [U)$. Finally, assume that $x, y \notin A_{Ker}$. Then there exists $v \in Min(y)$ with $v \notin Min(x)$. Thus there exists a clopen decreasing $U \subseteq X$ with $v \in U$, $U \cap Min(x) = \emptyset$. Then $[U) \in \mathcal{B}$, $x \notin [U)$ (since $[U) = [U \cap A_{Ker}))$ and $y \in [U)$ because $v \in U$. Thus $\mathcal{B}$ separates elements of X and the proof is complete. □

Define isoproperties $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, and $\mathcal{P}_4$ such that

$f \in End(A)$ satisfies $\mathcal{P}_1$ if and only if

$f \in Id(A)$, $M(f) = 2$ and there exist distinct $h_1, h_2 \in Kernel(A)$ such that for any $h \in End(A)$ if $h \circ f$ is a right divisor of h_1 then $h \circ f$ is not a right divisor of h_2.

$f \in End(A)$ satisfies $\mathcal{P}_2$ if and only if

$f \in Id(A)$, $M(f) = 2$, and for $h \in Id(A)$ we have $h \in Kernel(A)$ whenever $M(h) = 1$ and $f \circ h = h$.

$f \in End(A)$ satisfies $\mathcal{P}_3$ if and only if

$f \in Id(A)$ and for every $h \in Id(A) \setminus Kernel(A)$ with $M(h) = 1$ there exists $k \in End(A)$ with $k \circ f \notin Kernel(A)$ and $h \circ k \circ f = k \circ f$

$f \in End(A)$ satisfies $\mathcal{P}_4$ if and only if

$M(f) = 2$, $f \in Id(A)$ satisfies $\mathcal{P}_2$ and $\mathcal{P}_3$, and f satisfies $\mathcal{P}_1$ whenever there exists $g \in End(A)$ satisfying $\mathcal{P}_1$

Lemma 3.7. *For $A = (X, \leq, \tau) \in P - SP^-$ the following statements hold:*

(1) *$f \in Id(A)$ satisfies $\mathcal{P}_1$ if and only if $card(Im(f) \cap A_{Ker}) = 2$, there exists $x \in Im(f)$ with $card(Min(x)) = 2$, and there exist distinct $u_1, u_2 \in A_{Ker}$ such that $Min(u) = \{u_1, u_2\}$ for no $u \in X$;*

(2) *$f \in Id(A)$ satisfies $\mathcal{P}_2$ if and only if $card(Im(f) \cap A_{Ker}) = 2$ and there exists no $x \in Im(f) \setminus A_{Ker}$ with $card(Min(x)) = 1$;*

(3) *$f \in Id(A)$ satisfies $\mathcal{P}_3$ if and only if $Im(f) \setminus A_{Ker} \neq \emptyset$ whenever there exists $x \in X \setminus A_{Ker}$ with $card(Min(x)) = 1$;*

(4) *$f \in Id(A)$ satisfies $\mathcal{P}_4$ if and only if $card(Im(f) \cap A_{Ker}) = 2$, $Im(f) \setminus A_{Ker} \neq \emptyset$ and $card(Min(x)) = 2$ for every $x \in Im(f) \setminus A_{ker}$.*

Proof. If $f \in Id(A)$ satisfies $\mathcal{P}_1$ then $card(Im(f) \cap A_{Ker}) = 2$ because $M(f) = 2$. Let $g \in Id(A)$ be such that $Im(g) \cap A_{Ker} = \{v_1, v_2\}$, $v_1 \neq v_2$ and there exists no $v \in Im(g)$ with $Min(v) = \{v_1, v_2\}$. For every pair u_1, u_2 of distinct elements of A_{Ker} there exists $h \in End(A)$ with $Im(h \circ g) = \{u_1, u_2\}$. Indeed, define $h(z) = u_1$ if $g(z) \geq v_1$, $h(z) = u_2$ if $g(z) \geq v_2$. For $i = 1, 2$ the set $\{z \in Im(g); z \geq v_i\} = Im(g) \cap [v_i)$ is closed because it is the meet of two closed sets. Hence we conclude that $(g)^{-1}(\{z \in Im(g); z \geq v_i\}) = h^{-1}(u_i)$, $i = 1, 2$ are clopen and $h \in End(A)$. Thus g does not satisfy $\mathcal{P}_1$ and therefore for every $f \in Id(A)$ satisfying $\mathcal{P}_1$ there exists $v \in Im(f)$ with $Min(v) = Im(f) \cap A_{Ker}$. By Lemma 3.3 (7) there exists v-spanning $g \in Id(A)$ and then $g \circ f$ is also v-spanning. If for every pair of distinct $u_1, u_2 \in A_{Ker}$ there exists $u \in X$ with $Min(u) = \{u_1, u_2\}$ then by Lemma 3.3 (8) $\mathcal{P}_1$ is not satisfied for f. Thus if $f \in Id(A)$ satisfies $\mathcal{P}_1$ then $card(Im(f) \cap A_{Ker}) = 2$, there exists $v \in Im(f)$ with $Min(v) = Im(f) \cap A_{Ker}$, and there exist distinct $u_1, u_2 \in A_{Ker}$ with $Min(u) = \{u_1, u_2\}$ for no $u \in X$. If $f \in Id(A)$ satisfies these conditions then from the definition of a p-mapping we obtain that f satisfies $\mathcal{P}_1$.

If $f \in Id(A)$ satisfies $\mathcal{P}_2$ then $card(Im(f) \cap A_{Ker}) = 2$. Assume that there exists $x \in Im(f) \setminus A_{Ker}$ with $card(Min(x)) = 1$. By Lemma 3.3 (7) there exists x-spanning $h \in Id(A)$. Then $M(h) = 1$, $f \circ h = h$ because $Im(h) \subseteq Im(f)$, and $h \notin Kernel(A)$ because $x \in Im(h) \setminus A_{Ker}$ – this is a contradiction, and therefore $card(Min(x)) = 2$ for every $x \in Im(f) \setminus A_{Ker}$. The converse follows from the definition of a p-mapping.

Let $f \in Id(A)$ satisfy $\mathcal{P}_3$. If there exists $x \in X \setminus A_{Ker}$ with $card(Min(x)) = 1$ then by Lemma 3.3 (7) there exists x-spanning $h \in Id(A)$. Thus $h \circ k \circ f = k \circ f$ for some $k \in End(A)$ and $k \circ f \notin Kernel(A)$. Hence $Im(k \circ f) \subseteq Im(h)$ and we conclude that $Im(k \circ f) \setminus A_{Ker} \neq \emptyset$. Then $Im(f) \setminus A_{Ker} \neq \emptyset$. Conversely, if for any $x \in X \setminus A_{Ker}$ we have $card(Min(x)) = 2$ then $h \in Kernel(A)$ for every

$h \in End(A)$ with $M(h) = 1$ and thus every $f \in Id(A)$ satisfies $\mathcal{P}_3$. Assume that $y \in Im(f) \setminus A_{Ker}$ and there exists $h \in End(A) \setminus Kernel(A)$ with $M(h) = 1$ then there exists $u \in Im(h) \setminus A_{Ker}$. By Lemma 3.3 (7) there exists y-spanning $g \in Id(A)$. Then $g \circ f$ is also y-spanning. By Lemma 3.3 (8) there exists $k \in End(A)$ with $k(y) = u$. Then $k \circ g \circ f$ is u-spanning and thus $Im(k \circ g \circ f) \subseteq Im(h)$ – therefore $k \circ g \circ f \notin Kernel(A)$ and $h \circ k \circ g \circ f = k \circ g \circ f$. Thus f satisfies $\mathcal{P}_3$.

If $f \in End(A)$ satisfies $\mathcal{P}_4$ then $card(Im(f) \cap A_{Ker}) = 2$ and there exists no $x \in Im(f) \setminus A_{Ker}$ with $card(Min(x)) = 1$ (by $\mathcal{P}_2$). If there exists $x \in X \setminus A_{Ker}$ with $card(Min(x)) = 1$ then $Im(f) \setminus A_{Ker} \neq \emptyset$ (by $\mathcal{P}_3$). Assume that $card(Min(x)) = 2$ for every $x \in X \setminus A_{Ker}$. If there exist distinct $u_1, u_2 \in A_{Ker}$ with $[u_1) \cap [u_2) = \emptyset$ then there exists $g \in End(A)$ satisfying $\mathcal{P}_1$ and hence f satisfies $\mathcal{P}_1$ and $Im(f) \setminus A_{Ker} \neq \emptyset$. If for every distinct $u_1, u_2 \in A_{Ker}$ there exists $u \in X$ with $Min(u) = \{u_1, u_2\}$ then $f(x) \in Im(f) \setminus A_{Ker}$ for $x \in X$ with $Min(x) = Im(f) \cap A_{Ker}$. Conversely, if $f \in Id(A)$, $card(Im(f) \cap A_{Ker}) = 2$, $Im(f) \setminus A_{Ker} \neq \emptyset$, and $card(Min(x)) = 2$ for every $x \in Im(f) \setminus A_{Ker}$ then $M(f) = 2$, f satisfies $\mathcal{P}_2$ and $\mathcal{P}_3$ and if some $g \in Id(A)$ satisfies $\mathcal{P}_1$ then f also satisfies $\mathcal{P}_1$, thus f satisfies $\mathcal{P}_4$. □

Corollary 3.8. *The categories $P - SP_1$ and $P - SP^-$ have a coordinatization property.*

Proof. Let $A = (X, \leq, \tau) \in P - SP_1$. If we show that there exists an isoproperty $\mathcal{P}$ coordinatizes kernel in $P - SP_1$ then by Corollary 3.4 (2) we can apply Theorem 1.8. Consider property $\mathcal{P}$ such that

$f \in Id(A)$, $M(f) = 1$ and $f \notin Kernel(A)$ whenever there exists $g \in End(A) \setminus Kernel(A)$ with $M(g) = 1$.

If f satisfies $\mathcal{P}$ then f is constant if and only if $X = A_{Ker}$ and hence if $X \neq A_{Ker}$ then $Im(f) \not\subseteq A_{Ker}$. Theorem 1.8 implies that $P - SP_1$ has a coordination property.

Let $A = (X, \leq, \tau) \in P - SP^-$. To use Theorem 1.8 we must find an isoproperty $\mathcal{P}$ such that if $f \in End(A)$ satisfies $\mathcal{P}$ then $Im(f) \setminus A_{Ker} \neq \emptyset$ and if there exists $x \in X$ with $card(Min(x)) = 2$ then there exists $y \in Im(f)$ with $card(Min(y)) = 2$. Consider $\mathcal{P}$ such that

$f \in Id(A)$ and if there exists $g \in End(A)$ satisfying $\mathcal{P}_4$ then f satisfies $\mathcal{P}_4$ else $M(f) = 1$ and $f \notin Kernel(A)$.

Lemma 3.7 implies that $\mathcal{P}$ has the required properties. By Corollary 3.4 (1) and (2) the assumptions of Theorem 1.8 are fulfiled and thus $P - SP^-$ has a coordination property. □

Lemma 3.9. *The categories $P - SP_1$ and $P - SP^+$ have origins and determining origin properties. The category $P - SP^-$ has a weak origin and determining weak origin property $\mathcal{P}$ with $s_{\mathcal{P}} = 1$.*

Proof. Assume that $A = (X, \leq \tau) \in P - SP^-$. Consider the following cases:

If there exist $x, y, u, v \in X \setminus A_{Ker}$ with $x \leq y$, $u \leq v$, $x \neq y$, $card(Min(x)) = card(Min(y)) = card(Min(v)) = 2$, $card(Min(u)) = 1$ then $\{(x, y), (u, v)\}$ is an origin by Lemma 3.3 (1) and (2). Moreover by Lemma 3.5 (1) and (2) there exists a determining origin property for this case.

Assume that for every $t, z \in X$ with $t \leq z$, $card(Min(z)) = 2$, $card(Min(t)) = 1$ we have $t \in A_{Ker}$. If there exist distinct $x, y \in X$ with $x \leq y$, $card(Min(x)) = card(Min(y)) = 2$ and there exists $v \in X \setminus A_{Ker}$ with $card(Min(v)) = 1$ then by Lemma 3.3 (1) and (8) $\{(w, x), (x, y)\}$ is an origin where let $w \in Min(x)$. By Lemmas 3.5 and 3.7 there exists a determining origin property in this case.

If there exist distinct $x, y \in X$ with $x \leq y$, $card(Min(x)) = card(Min(y)) = 2$ and for every $z \in X \setminus A_{Ker}$ we have $card(Min(z)) = 2$ then by Lemma 3.3 (1) and (8) $\{(w, x), (x, y)\}$ is a weak origin where $w \in Min(x)$. By Lemmas 3.5 and 3.7 there exists a determining weak origin property. Moreover, only $\{x, y\}$ can be permuted, because $w \in A_{Ker}$.

Assume that for every $u, v \in X$ with $u \leq v$, $u \neq v$ we have $card(Min(u) = 1$. If there exist $x, y \in X \setminus A_{Ker}$ with $x \leq y$, $card(Min(x)) = 1$, $card(Min(y)) = 2$ then by Lemma 3.3 (2) $\{(x, y)\}$ is a origin and by Lemma 3.5 (2) there exists a determining origin property in this case.

Assume that for distinct $t, z \in X$ with $t \leq z$ and $card(Min(z)) = 2$ we have $t \in A_{Ker}$. If there exist distinct $x, y \in X \setminus A_{Ker}$ with $x \leq y$ and $card(Min(y)) = card(Min(x)) = 1$ and there exists $v \in X$ with $card(Min(v)) = 2$ then by Lemma 3.3 (3) and (8) $\{(u, v), (x, y)\}$ is an origin where $u \in Min(v)$. By Lemma 3.5 (3) and Lemma 3.7 there exists a determining origin property in this case.

Assume that $card(Min(z)) = 1$ for every $z \in X$. If there exist distinct $x, y \in X \setminus A_{Ker}$ with $x \leq y$ and $card(Min(y)) = card(Min(x)) = 1$ then by Lemma 3.3 (3) $\{(x, y)\}$ is an origin and by Lemma 3.5 (3) there exists a determining origin property in this case.

If there exists $x \in X$ with $card(Min(x)) = 2$ and every chain in A has length ≤ 1 then by Lemma 3.3 (8) $\{(z, x)\}$ is an origin where $z \in Min(x)$. By Lemma 3.7 there exists a determining origin property in this case.

If every chain in A has length ≤ 1, $card(Min(z)) = 1$ for every $z \in X$, and there exists $x \in X \setminus A_{Ker}$ then by Lemma 3.3 (8) $\{(z, x)\}$ is an origin where $z \in Min(x)$. Obviously, there exists a determining origin property in this case.

If $\leq$ is discrete then the empty set is an origin and determining origin property exists in this case.

If we summarize the discussion we obtain that $P - SP_1$ and $P - SP^+$ have origins and determining origin properties, and $P - SP^-$ has weak origins and a weak determining origin property $\mathcal{P}$. Moreover, we conclude that $s_{\mathcal{P}} = 1$. □

Theorem 3.10. [2] *The equimorphic p-spaces in $P - SP_1$ or $P - SP^+$ are isomorphic. Thus $P - SP_1$ and $P - SP^+$ are 2-determined. $P - SP_2$ is 3-determined.*

Proof. By Lemma 3.7 and Lemma 2.5 we can apply Theorems 2.3 and 2.4. According to Theorem 2.4 we obtain that equimorphic *p*-spaces from $P - SP_1$ or $P - SP^+$ are isomorphic because $m_{P-SP_1} = m_{P-SP^+} = 1$ and $\rho_{P-SP_1} = \rho_{P-SP^+} = 0$. By Theorem 2.3 we obtain that $P - SP^-$ is 3-determined because $m_{P-SP^-} = 1$ and

$s_{\mathcal{P}} = 1$ for a determining weak origin property $\mathcal{P}$ and $P - SP^-$ is considered as a subcategory of binary relations.

Finally consider the category $P - SP_2$. If $A = (X, \leq, \tau) \in P - SP_2$ and $A \notin P - SP^-$ then $\leq$ is discrete. Let $\mathcal{Z}(A) = (X', \leq, \sigma) \in P - SP^-$ such that $X' = X \cup \{(x, y); x, y \in X, x \neq y\}$, for every $x, y \in X$ define $(x, y) \geq x, y$ and σ is the extension of τ on X'. Such extension is unique and by an easy calculation we obtain $End(A) \cong End(\mathcal{Z}(A))$. Obviously for $A, B \in P - SP_2$ with $A, B \notin P - SP^-$ we have that $\mathcal{Z}(A)$ is isomorphic to $\mathcal{Z}(B)$ if and only if A is isomorphic to B. Moreover, $\mathcal{Z}(A)$ is equimorphic with some p-space B in $P - SP^-$ if and only if they $\mathcal{Z}(A)$ and B are isomorphic because $\mathcal{Z}(A) \in P - SP^+$. Hence we obtain that $P - SP_2$ is 3-determined. □

Remark. The isomorphism between $End(A)$ and $End(\mathcal{Z}(A))$ is not strong. Therefore $P - SP_2$ has not a coordination property. Note that the full subcategory $P - SP_2$ formed by all p-spaces distinct from $\mathcal{Z}(A)$ for some A with discrete ordering has also a coordination property.

By Priestley duality we obtain that equimorphic Stone algebras (i.e. p-algebras in the variety L_1) are isomorphic and the variety L_2 is 3-determined. This result was originally proved by Adams, Koubek and Sichler – see [2]. Adams, Koubek and Sichler [3] showed that L_3 is not determined in any sense, in precise:

Theorem 3.11. [3] *For every monoid M denote by M_c a monoid obtained from M by adjoined countable many left zeros. Then there exists a proper class of non-isomorphic p-algebras in L_3 with endomorphism monoid isomorphic to M_c.* □

4. Heyting algebras

Recall that an algebra $(H, \vee, \wedge, \rightarrow, 0, 1)$ of type (2,2,2,0,0) is called a *Heyting algebra* if $(H, \vee, \wedge, 0, 1)$ is a distributive (0,1)-lattice with an added operation $\rightarrow$ of relative pseudocomplementation defined by $z \leq x \rightarrow y$ just when $x \wedge z \leq y$. The class of all Heyting algebras with its homomorphisms (i.e. mappings preserving all five operations) is a variety, see H. Rasiova and R. Sikorski [23].

For a Priestley space $A = (X, \leq, \tau)$ a subset $W \subseteq X$ is called *convex* if it is a meet of an increasing set and a decreasing set. We say that A is an *$h - space$* if for every clopen convex set $U \subseteq X$ the set $[U)$ is clopen. A mapping $f : X \longrightarrow Y$ is an *h-mapping* from an h-space $A = (X, \leq, \tau)$ into an h-space $B = (Y, \leq, \sigma)$ if f is continuous, order preserving and $f((x]) = (f(x)]$ for every $x \in X$. A subcategory of $PRIEST$ formed by all h-spaces and h-mappings is denoted by $H - SP$. Then it holds

Theorem 4.1. [20] *The category $H - SP$ is dually isomorphic to the variety of all Heyting algebras and their homomorphisms.* □

We recall that a constant mapping is an h-mapping between h-spaces if and only if it is a constant mapping to a minimal element. Hence for a non-empty h-space

$A = (X, \leq, \tau)$ the $Kernel(A)$ is the set of all constant mappings to a minimal element and it is the set of all left zeros in $End(A)$ and A_{Ker} is the set of all minimal elements of A. The set A_{Ker} is closed. Let $\{f_i : A_i \longrightarrow A; i \in I\}$ be a family of injective h-mappings such that X is the closure of $\cup\{Im(f_i); i \in I\}$ then the dual algebra of A is a subdirect power of dual algebras of $A_i, i \in I$, see [13]. Hence we immediately obtain the following folklore statement:

Proposition 4.2. *An h-space $A = (X, \leq, \tau)$ is a dual of a subdirectly irreducible Heyting algebra if and only if X contains the open greatest element. Let V be a variety of Heyting algebras. An h-space $A = (X, \leq, \tau)$ is a dual of an algebra from V just when for every $x \in X$, the h-space $(x]$ is a dual of an algebra in V. If, moreover, V is finitely generated and $A \in V$ then $(x]$ is a dual of a subdirectly irreducible algebra in V for every $x \in X$.* □

Let $A = (X, \leq, \tau)$ be an h-space. For $x \in X$ denote by $\lambda(x)$ the supremum of length of all chains in $(x]$ and $\lambda(A) = sup\{\lambda(x); x \in X\}$. For an element $x \in X$ denote by $p(x) = \{y \in (x]; \lambda(y) + 1 = \lambda(x)\}$. We say that $f \in End(A)$ is *x-spanning* if $Im(f) = (x]$ for some $x \in X$. We say that a finite h-space $A = (X, \leq, \tau)$ is an *e-space* if every independent subset $Z \subseteq X$ has at most two elements, all maximal chains in X have the same length, and for $x, y \in X$ with $\lambda(x) = \lambda(y)$ we have $p(x) \cap p(y) \neq \emptyset$, e.g. every finite chain is an e-space. Denote by E_∞ the full subcategory of $H - SP$ formed by all h-spaces $A = (X, \leq, \tau)$ such that $\lambda(A)$ is finite and $(x]$ is an e-space for every $x \in X$. Denote by C_∞ the full subcategory of E_∞ formed by all h-spaces $A = (X, \leq, \tau)$ such that either there exists $x \in X$ with $|p(x)| = 1$ or for any pair of distinct elements $x, y \in X$ with $\lambda(x) = \lambda(A) = \lambda(y)$ we have that $(x] \setminus \{x\} \neq (y] \setminus \{y\}$.

Lemma 4.3. *Let $A = (X, \leq, \tau)$ be an h-space then for every $x \in X$ such that $(x]$ is an e-space and there exists $z' \in (x]$ with $(z'] = (u] \cup (v] \cup \{z'\}$ for every $u, v \in (x]$ covered by an element $z \in X$ there exists an x-spanning $f \in Id(X)$. Moreover, for $v \in X$ with $(x] \setminus ((v] \cup \{x\}) \neq \emptyset$ we can assume that $f(v) \neq x$.*

Proof. Let $A = (X, \leq, \tau)$. For any e-subspace $Y \subseteq X$ we shall prove by induction over $\lambda(x)$, that for every $x \in Y$ and for every $v \in X$ with $(x] \setminus ((v] \cup \{x\}) \neq \emptyset$ there exists an x-spanning $f \in Id(A)$ such that $f(x) \neq f(v)$. If $\lambda(x) = 0$ then $x \in A_{Ker}$ and the constant mapping f to x is x-spanning and $f \in Id(A)$. Assume that the statement holds for all $y \in Y$ with $\lambda(y) < n$ and $\lambda(x) = n$ for $x \in Y$. Choose $y \in (x]$ with $\lambda(y) = n - 1$ and $|(x] \setminus (y]| \leq 2$ – clearly such y exists. Then by the induction assumption there exists y-spanning $g \in Id(A)$. Consider two cases – there exists $z \in p(x)$ with $z \neq y$ or $p(x) = \{y\}$. First, assume that $(x] \setminus (y] = \{x, z\}$ for some $z \neq x$. Then one of the following three possibilities occurs:

(a) $\{u \in (x]; \lambda(u) = n - 2\} = \{t\}$ then $t \leq z, y$;
(b) $\{u \in (x]; \lambda(u) = n - 2\} = \{t, w\}$ and $t, w \leq y, z$;
(c) $\{u \in (x]; \lambda(u) = n - 2\} = \{t, w\}$ and $w \not\leq z$, then $t \leq z$ and $t, w \leq y$.

If (a) or (b) holds then set $U = g^{-1}(y)$. Clearly, U is clopen increasing and $y, z \in U$. Thus $B = (U, \leq, \tau)$ is an h-space. Choose clopen decreasing sets $Z, Y \subseteq U$ with $z \in Z$, $y \in Y$, $B_{Ker} \subseteq Z \cup Y$, and $Z \cap Y = \emptyset$. Define $f : X \longrightarrow X$ such that $f(u) = g(u)$ for $u \in X \setminus U$, $f(u) = y$ for $u \in [Y) \setminus [Z)$, $f(u) = z$ for $u \in [Z) \setminus [Y)$, and $f(u) = x$ for $u \in [Y) \cap [Z)$. Since g is an idempotent h-map we obtain by a routine calculation that f is an idempotent x-spanning h-map.

Assume that (c) holds. Set $V = [g^{-1}(t))$, clearly, V is clopen increasing and $z, y \in V$, thus $C = (V, \leq, \tau)$ is an h-space. Further $g^{-1}(y) = [g^{-1}(w)) \cap V$ and hence $[g^{-1}(w)) \cap C_{Ker} = \emptyset$ because g is an h-map. There exists a clopen increasing set $U_0 \subseteq V$ with $C_{Ker} \cap U_0 = \emptyset$, $[g^{-1}(w)) \cap V \subseteq U_0$, and $y, z \in U_0$. Set $Z_0 = U_0 \setminus [g^{-1}(w))$, $Y_0 = g^{-1}(y) \setminus [Z_0)$, and $W_0 = g^{-1}(y) \cap [Z_0)$. Then Z_0, Y_0, W_0 are clopen and they form a partition of U_0. Hence $W_0 \setminus [Y_0)$ is clopen and thus $(W_0 \setminus [Y_0)] \cap Z_0$ is closed. By the assumption on X we obtain $z \notin (W_0 \setminus [Y_0)]$. Hence there exists a clopen decreasing set $Z_1 \subseteq Z_0$ with $(W_0 \setminus [Y_0)] \cap Z_0 \subseteq Z_1$ and $z \notin Z_1$. Set $U = U_0 \setminus Z_1$, $Z = Z_0 \setminus Z_1$, $Y = g^{-1}(y) \setminus [Z)$, and $W = [Z) \cap [Y)$. Define $f : X \longrightarrow X$ such that $f(u) = g(u)$ for $u \in X \setminus U$, $f(u) = y$ for $u \in Y$, $f(u) = z$ for $u \in Z$, $f(u) = x$ for $u \in W$. Since g is an idempotent h-map we immediately obtain that f is a continuous, order preserving idempotent map with $f((u]) = (f(u)]$ for every $u \in X \setminus W$. Clearly, $(x] \setminus \{y\} \subseteq f(u)$ for every $u \in W$ and by a choice of U there exists $w \in (u]$ with $g(w) \in Y$, in contrary $g(u) \in W_0 \setminus [Y_0)$. This is a contradiction because $W_0 \setminus [Y_0) \subseteq Y$. Thus f is an idempotent x-spanning h-map. Moreover, if $v \in U$ then we can assume that $v \in Z \cap Y$ and hence $f(v) \neq f(x)$.

Assume that $p(x) = \{y\}$. Set $U = g^{-1}(y)$, since g is continuous, order preserving we conclude that U is clopen increasing and thus $B = (U, \leq, \tau)$ is an h-space. There exists an increasing clopen set $T \subseteq U$ such that $x \in T$, $v, y \notin T$, $T \cap B_{Ker} = \emptyset$. Define $f : X \longrightarrow X$ such that $f(u) = g(u)$ if $g(u) \neq y$, $f(u) = y$ if $g(u) = y$ and $u \in U \setminus T$, $f(u) = x$ if $g(u) = y$ and $u \in T$. Obviously, $f \in Id(A)$ is x-spanning $f(v) \neq x$. The proof is complete. □

For an $f \in Id(A)$ denote by P_f the poset $(Id(A) \cap f \circ End(A) \circ f, \preceq)/\equiv$ where $h \preceq g$ if and only if $g \circ h = h$ and $g \equiv h$ if and only if $g \preceq h \preceq g$. The class of $\equiv$ containing h will be denoted by $[h]$. Denote by $PId(f) = Id(A) \cap f \circ End(A) \circ f$. Define $\lambda(f)$ as the supremum of length of all chains in P_f.

Lemma 4.4. *Let $A = (X, \leq, \tau) \in E_\infty$ then*

(1) *if $f \in End(A)$ is x-spanning for some $x \in X$ then $g \circ f$ is $g(x)$-spanning for every $g \in End(A)$;*
(2) *if $f \in Id(A)$ is x-spanning then $(x]$ is isomorphic to the poset P_f;*
(3) *if $f_i \in End(A)$ is x_i-spanning for $x_i \in X$ and $i = 1, 2$ then $x_1 = x_2$ if and only if for every y-spanning $g \in Id(A)$ we have $g \circ f_1 = f_1$ just when $g \circ f_2 = f_2$;*
(4) *if $f \in Id(A)$ is not x-spanning for any $x \in X$ and P_f is an e-space then $Im(f)$ is an e-space such that $Im(f) \not\subseteq (z]$ for any $z \in X$ and P_f is isomorphic to $Im(f)$ with adjoined the greatest element.*

Proof. Since $(g(x)] = g((x])$ we immediately obtain (1).

Let $f \in Id(A)$ be x-spanning. By Lemma 1.1, if $g, h \in PId(f)$ then $g \equiv h$ if and only if $Im(g) = Im(h)$. By (1) if $g \in PId(f)$ then g is $g(x)$-spanning and Lemma 4.3 completes the proof of (2).

We prove (3). If $x_1 = x_2$ then by Lemma 1.1 for every $g \in Id(A)$ we have $g \circ f_1 = f_1$ just when $g \circ f_2 = f_2$ because $Im(f_1) = Im(f_2)$. Conversely, if $x_1 \neq x_2$ then there exists $y \in X$ with either $x_1 \in (y]$ and $x_2 \notin (y]$ or $x_1 \notin (y]$ and $x_2 \in (y]$. Then Lemmas 4.3 and 1.1 complete the proof.

Assume that P_f is an e-space. If $Im(f)$ is not an e-space then by Lemma 4.3 we obtain that $Im(f)$ is isomorphic to a subposet of P_f and thus P_f is not an e-space – a contradiction. Assume that $g \in PId(f)$ such that g is not x-spanning for any $x \in Im(f)$. Then there exist two maximal elements $x, y \in Im(g)$ and because $Im(g) = \cup\{(z]; z \in Im(g)\}$ we obtain $Im(g) = (x] \cup (y]$. Assume that $\lambda(x) > \lambda(y)$, then there exists $z \in Im(f)$ with $\lambda(z) = \lambda(y) + 1$ and $z > y$ because maximal chains in $Im(f)$ have the same length. Let $u \in (x]$ with $\lambda(u) = \lambda(z)$, then there exists $v \in p(u) \cap p(z)$, hence $v \in (x]$ and $g(v) = v$, $g(y) = y$ imply $g(z) = z$ – a contradiction with the maximality y in $Im(g)$. Hence $\lambda(x) = \lambda(y)$. If there exists $z \in X$ with $z > x, y$ then $g(z) = z$ for some $z \in Im(f)$ with $z > x, y$ because $g(x) = x$, $g(y) = y$ – a contradiction. Hence $Im(g) = Im(f)$, P_f is isomorphic to $Im(f)$ with adjoined a new greatest element, and $Im(f) \not\subseteq (z]$ for any $z \in X$. □

Lemma 4.5. *Let $A = (X, \leq, r) \in C_\infty$. For $f \in Id(A)$ we have that f is u-spanning for some $u \in X$ if and only if P_f is an e-space and one of the following conditions holds*

1. *there exist $g \in Id(A)$, $g', h \in PId(g)$, and $h' \in End(A)$ such that P_g is an e-space, $p([g']) = \{[h]\}$, and for every $k \in PId(f)$ with $[k] \neq [f]$ we have $h \circ h' \circ k = h' \circ k$ and $h \circ h' \circ f \neq h' \circ f = g' \circ h' \circ f$;*
2. *there exist $g \in Id(A)$, $h, k \in End(A)$ such that P_g is an e-space, $\lambda(g) > \lambda(f)$, $g \circ h \circ f = h \circ f$, and $k \circ h \circ f = f$;*
3. *for every $g \in Id(A)$ such that P_g is an e-space we have that $\lambda(g) \leq \lambda(f)$ and $|p([h])| = 2$ for every $h \in PId(g)$, and if $\lambda(g) = \lambda(f)$ then either there exist $h, k \in End(A)$ with $g \circ h \circ f = h \circ f$ and $k \circ h \circ f = f$ or for every $h \in End(A)$ with $g \circ h \circ f = h \circ f$ there exists $k \in PId(g)$ with $[k] \neq [g]$ and $k \circ h \circ f = h \circ f$.*

Proof. Assume that $f \in Id(A)$ is u-spanning for $u \in X$ then by Lemma 4.4 (2) P_f is an e-space and one of the following occurs:

1. There exists $x \in X$ with $p(x) = \{y\}$ and $\lambda(y) < \lambda(u)$;
2. For every $v \in (u]$ we have $|p(v)| = 2$ and there exists $x \in X$ with $\lambda(x) > \lambda(u)$ such that $|p(y)| = 2$ for every $y \in (x]$;
3. $|p(x)| = 2$ for every $x \in X$ and $\lambda(u) = \lambda(A)$.

In the first case we assume that $\lambda(y)$ is the smallest possible. Choose x-spanning $g \in Id(A)$ – by Lemma 4.3 g exists. By Lemma 4.4 (2) there exists an y-spanning

$h \in PId(g)$ with $p(g) = \{[h]\}$. Since $\lambda(y)$ is the smallest there exists $h' \in End(A) \circ f$ with $h'(Im(f) \setminus \{u\}) = (y]$, $h'(u) = x$. Since for every $k \in PId(f)$ with $k \neq f$ we have that $Im(k) \subseteq (u] \setminus \{u\}$ we conclude by Lemma 1.1 that $h \circ h' \circ k = h' \circ k$ but $h \circ h' \circ f \neq h' \circ f$ and (1) holds.

In the second case let $g \in Id(A)$ be x-spanning. Clearly, there exist $h \in End(A) \circ f$, $k \in End(A) \circ g$ such that h on $Im(f)$ is injective, $h(u) \in (x] \setminus \{x\}$ and $k(x) = u$. Then by Lemma 1.1 we obtain $g \circ h \circ f = h \circ f$ and $k \circ h \circ f = f$ because k is injective on $Im(h \circ f)$ and (2) holds.

Investigate the last case. Let $g \in Id(A)$ such that P_g is an e-space. By Lemma 4.4 (2) and (4) $Im(g)$ is an e-space, $\{[h]; h \in PId(g)$ is x-spanning$\}$ is a subspace of P_g isomorphic to $Im(g)$, and if $h \in PId(g)$ is not x-spanning for any $x \in Im(g)$ then $Im(h) = Im(g)$. Thus for a maximal element $x \in Im(g)$ either $\lambda(g) = \lambda(x)$ if g is x-spanning or $\lambda(g) = \lambda(x) + 1$ if g is not x-spanning. From this follows that $|p([h])| = 2$ for every $h \in PId(g)$ and $\lambda(g) \leq \lambda(f)$. Assume that $\lambda(g) = \lambda(f)$. If g is x-spanning then there exist $h \in End(A) \circ f$, $k \in End(A) \circ g$ such that $h(u) = x$, $k(x) = u$ and by Lemma 1.1 (3) we obtain $g \circ h \circ f = h \circ f$ and $k \circ h \circ f = f$ because k is injective on $Im(h \circ f)$. If g is not x-spanning for any $x \in X$ then by Lemma 4.4 (1) is $h \circ f$ $h(u)$-spanning and for $h(u)$-spanning $k \in PId(g)$ we have $[k] \neq [g]$ and $k \circ h \circ f = h \circ f$. Thus (3) is proved.

Assume that f is not u-spanning for any $u \in X$, and P_f is an e-space then by Lemma 4.4 (4) $Im(f)$ is an e-space and there exist exactly two maximal elements $v, w \in Im(f)$ such that $v, w < t$ for no $t \in X$. If (1) holds then for v-spanning $k \in PId(f)$ we have $h \circ h' \circ k = h' \circ k$ and by Lemma 1.1 $Im(h' \circ k) \subseteq Im(h)$. The same holds for w-spanning $k' \in PId(f)$ but $Im(f) = Im(k) \cup Im(k')$ and thus $Im(h' \circ f) \subseteq Im(h)$. Hence $h \circ h' \circ f = h' \circ f$ – a contradiction with (1). Assume that f satisfies (2). Then we can assume that $k \in End(A) \circ g$ and by Lemma 4.4 (1) we conclude that k is y-spanning for some $y \in X$ and $Im(f) \subseteq Im(k)$ – a contradiction with the property of v and w. Assume that f satisfies (3). By Lemma 4.4 (2) for every x-spanning $g \in Id(A)$, $x \in X$ we have that P_g is an e-space isomorphic to $(x]$ and thus $|p(y)| = 2$ for every $y \in (x]$. Choose $x \in X$ with $\lambda(x) = \lambda(A)$. Since $A \in C_\infty$ we conclude that $\lambda(x) > \lambda(v) = \lambda(w)$. Hence for x-spanning $g \in Id(A)$ we obtain $\lambda(g) \geq \lambda(f)$ and therefore $\lambda(g) = \lambda(f)$. The existence $h, k \in End(A)$ with $g \circ h \circ f = h \circ f$, $k \circ h \circ f = f$ implies $Im(f) \subseteq Im(k \circ g)$ because $k \circ g \circ h \circ f = f$ and by Lemma 4.4 (1) $k \circ g$ is $k(x)$-spanning – a contradiction with the property of v and w. Thus for every $h \in End(A)$ with $g \circ h \circ f = h \circ f$ there exists $k \in Id(A) \cap g \circ End(A) \circ g$ with $k \neq g$ and $k \circ h \circ f = h \circ f$. From the assumptions we obtain that there exists an injective h-mapping $k' : Im(f) \longrightarrow (x]$ such that $k'(Im(f)) = (x] \setminus \{x\}$. Define $h : X \longrightarrow X$ such that $h(z) = k'(f(z))$ for every $z \in X$, then $h \in End(A)$ and $g \circ h \circ f = h \circ f = h$. For every $k \in PId(g)$ with $k \circ h \circ f = h$ we conclude that $(x] \setminus \{x\} \subseteq Im(k)$. Let $p(x) = \{y_1, y_2\}$ then $k(y_i) = y_i$ for $i = 1, 2$ and $k(y_1) = y_1, k(y_2) = y_2 \leq k(x)$ and thus $k(x) = x$. Hence $Im(k) = (x]$ and $[k] = [g]$ – a contradiction with (3). □

According to Lemma 4.5 there exists the isoproperty determining x-spanning $f \in Id(A)$ for $A \in C_\infty$. Thus there exists the isoproperty determining the right ideal Q generated by all x-spanning $f \in Id(A)$. From Lemma 4.3 we obtain that Q is left 1-transitive where the associated congruence $\sim$ is defined such that $f \sim g$ if $Im(f) = Im(g) = (x]$ for some $x \in X$. Since by Lemmas 4.4 (3) and 4.5 there exists the isoproperty determining the left congruence $\sim$ we conclude

Corollary 4.6. *The category C_∞ has a coordination property.* □

Lemma 4.7. *For every $A = (X, \leq, \tau) \in E_\infty$ the Boolean closure of the family $\{f^{-1}(x); f \in Fin(A), x \in Im(f)\}$ is the set of all clopen sets.*

Proof. Since every set in $\{f^{-1}(x); f \in Fin(A), x \in Im(f)\}$ is clopen we conclude that every set in the Boolean closure is clopen. If we prove that the family $\{f^{-1}(x); f \in Fin(A), x \in Im(f)\}$ separates elements of X then the proof will be complete. Assume that $x, y \in X$ are distinct. If $x \notin A_{Ker}$ and $(x]\setminus((y]\cup\{x\}) \neq \emptyset$ or $x, y \in (z]$ for some $z \in X$ then we apply Lemma 4.3. Thus it suffices to investigate the case that $(x] \setminus \{x\} = (y] \setminus \{y\}$ and $x, y \in (z]$ for no $z \in X$. Let $f \in Id(A)$ be x-spanning. Assume that there exist $h \in End(A) \circ f$ and $v \in X$ such that $h(u) \neq h(x)$ for every $u \in (x] \setminus \{x\}$, $h(x) \in (v]$, and $(w] = ((h(x)] \setminus \{h(x)\}) \cup \{w\}$ for some $w \in (v]$. Then either $h(x) \neq h(y)$ or there exist clopen decreasing disjoint sets $U, V \subseteq f^{-1}(x)$ such that $x \in U$, $y \in V$ and $U \cup V$ contains all minimal element of $f^{-1}(x)$. Define $g : X \longrightarrow X$ such that $g(z) = h(z)$ for every $z \in X$ with $h(z) \neq h(x)$, $g(z) = h(x)$ if $z \in [U) \setminus [V)$, $g(z) = w$ if $z \in [V) \setminus [U)$, $g(z) = t$ if $z \in [U) \cap [V)$ where $t \in (v]$ is a minimal element with $h(x), w \leq t$ – such t exists because $h(x), w \leq v$. Obviously, $g \in Fin(A)$ and $g(x) \neq g(y)$. Assume that there exists $v \in X$ with $|p(v)| = 1$ and $\lambda(v) \leq \lambda(x)$ then we can assume that v has the smallest $\lambda(v)$. In this case there exists $h \in End(A) \circ f$ with $h(x) = v$ and $h(u) \neq v$ for any $u \in (x] \setminus \{x\}$. Assume that $p(v) = \{w\}$ then w has the required property. Thus we can assume that for every $v \in X$ with $\lambda(v) \leq \lambda(x)$ we have $|p(v)| = 2$. If there exists $v \in X$ with $\lambda(v) = \lambda(x) + 1$ and $p(v) = \{w, u\}$ then there exists $h \in End(A) \circ f$ such that $h(x) = u$ and h is injective on $(x]$. Then w has the required property and hence we can assume that $|p(v)| = 1$ for every $v \in X$ with $\lambda(v) = \lambda(x) + 1$. Choose a decreasing clopen set $U \subseteq f^{-1}(x)$ with $x \in U$, $y \notin U$. Then $[U)$ and $f^{-1}(x) \setminus [U)$ are disjoint clopen increasing, $y \in f^{-1}(x) \setminus [U)$ and we can define $g : X \longrightarrow X$ such that $g(z) = f(z)$ if $f(z) \neq x$, $g(z) = x$ if $z \in [U)$, $g(z) = y$ if $z \in f^{-1}(x) \setminus [U)$. Obviously, $g \in Fin(A)$ and $g(x) \neq g(y)$. □

Theorem 4.8. *The strong equimorphic h-spaces from C_∞ are the same.*

Proof. Let $A = (X, \leq, \tau), B = (X, \leq, \sigma) \in C_\infty$ be strongly equimorphic. Since for $x, y \in X$ we have by Lemma 1.1 $x \leq y$ in A if and only if there exist x-spanning $f \in Id(A)$ and y-spanning $g \in Id(A)$ with $g \circ f = f$ and this is just when for every x-spanning $f \in Id(A)$ and for every y-spanning $g \in Id(A)$ we have $g \circ f = f$ we conclude that $x \leq y$ in A if and only if $x \leq y$ in B. By Lemma 4.7 $\sigma = \tau$ and the proof is complete. □

Corollary 4.9. *Equimorphic h-spaces in C_∞ are isomorphic. Thus C_∞ is 2-determined.*

Proof. Combine Corollaries 4.6 and 1.10 and Theorem 4.8. □

Denote by K_n the variety determined by the identity $(x_1 \to x_2) \vee (x_2 \to x_3) \vee \ldots \vee (x_n \to x_{n+1}) = 1$. As proved Hecht and Katriňák see [11], K_n is generated by the n-element chain and therefore a dual $A = (X, \leq, \tau)$ of some algebra in K_n belongs to C_∞ because $(x]$ is at most $n-1$ element chain for every $x \in X$. As a consequence of Corollary 4.9 we immediately obtain

Corollary 4.10. *Equimorphic algebras in $\cup\{K_n; n \geq 1\}$ are isomorphic.* □

More generally, we say that a finite subdirectly irreducible algebra A satisfies (e1) if the poset of join irreducible elements of A is an e-space and there exists distinct join irreducible elements $a, b \in A$ with $b < a$ such that for every join irreducible element $c \in A$ we have $c < a$ just when $c \leq b$. Note that then the dual of A belongs to C_∞. We say that a finitely generated variety of Heyting algebras V satisfies (e1) if every subdirectly irreducible algebra in V satisfies (e1). If V satisfies (e1) then dual of any algebra in V belongs to C_∞, thus

Corollary 4.11. *Equimorphic algebras in*

$$\cup\{V; V \text{ is the variety of Heyting algebras satisfying (e1)}\}$$

are isomorphic. □

Finally, we investigate h-spaces $A = (X, \leq, \tau) \in E_\infty \setminus C_\infty$. We say that an h-space $A = (X, \leq, \tau)$ satisfies (s1) if $|p(x)| = 2$ for every $x \in X$. Consider that any $A \in E_\infty \setminus C_\infty$ satisfies (s1). Let A satisfy (s1). Denote by $P(A) = \{\{x, y\}; x, y \in X, x \neq y, (x] \setminus \{x\} = (y] \setminus \{y\}, \lambda(x) = \lambda(y) = \lambda(A)\}$. For every $\{x, y\} \in P(A)$ choose a new element $z_{x,y}$ and define $E(X) = X \cup \{z_{x,y}; \{x, y\} \in P(A)\}$. We extend the ordering $\leq$ from X to $E(X)$ such that $x, y \leq z_{x,y}$ for every $\{x, y\} \in P(A)$. For every clopen decreasing set $U \subseteq X$ define $E(U) = U \cup \{z_{x,y}; \{x, y\} \in P(A), x, y \in U\}$ then $E(U)$ is decreasing and for every $x, y \in E(X)$ with $x \not\leq y$ there exists a clopen decreasing set $U \subseteq X$ with $y \in E(U)$ and $x \notin E(U)$ because A is a Priestley space. Let σ be the smallest topology on $E(X)$ such that $E(U)$ is clopen whenever $U \subseteq X$ is clopen decreasing. The restriction of σ on X coincides with τ. Moreover, for every clopen convex $V \subseteq X$ we have that $E(X \setminus [V)) = E(X) \setminus [V)$ is clopen decreasing. If we prove that σ is compact we obtain that $E(A) = (E(X), \leq, \sigma) \in C_\infty$ and $X = E(X) \setminus \{y \in E(X); \lambda(y) = \lambda(E(A))\}$ implies that $E(A)$ and $E(B)$ are isomorphic if and only if A and B are isomorphic. We prove:

Proposition 4.12. *If $A = (X, \leq, \tau) \in E_\infty \setminus C_\infty$ then A satisfies (s1). If A satisfies (s1) then $E(A) \in C_\infty$ and $End(E(A))$ is isomorphic with $End(A)$. Moreover, if $A, B \in E_\infty \setminus C_\infty$ then A is isomorphic to B if and only if $E(A)$ is isomorphic to $E(B)$.*

Proof. From the above discussion to prove that $E(A)$ is an h-space it suffices to show that σ is compact. Assume that $A = (X, \leq, \tau)$ satisfies (s1), set $PM(A) = \{y \in X; \exists x \in X, \{x, y\} \in P(A)\}$ and denote by $M(A)$ the set of all maximal elements of A. Define $Q(X) = \{(u, x); u \in (x], x \in M(A) \setminus PM(A))\} \cup \{(u, (x, y)); u \in (\{x, y\}], \{x, y\} \in P(A)\}$, $R(X) = Q(X) \cup \{(z_{x,y}, (x, y)); \{x, y\} \in P(A)\}$. Define the ordering $\leq$ on $Q(X)$ and $R(X)$ such that $(u, x) \leq (v, x)$ whenever $u \leq v$. For a given set Y denote by $\beta(Y)$ the set of all ultrafilters on Y and let β be the topology on $\beta(Y)$ being the β-compactification of (Y, δ) where δ is the discrete topology. Koubek and Sichler [13] proved that $\beta(Q(A)) = (\beta(Q(X)), \leq, \beta)$ with the natural ordering $\leq$ (two ultrafilters F, G satisfy $F \leq G$ if for every $U \in G$ there exists $V \in F$ with $U \subseteq [V)$) is a free Priestley compactification of $Q(A) = (Q(X), \leq, \delta)$ where δ is the discrete topology and $\beta(R(A)) = (\beta(R(X)), \leq, \beta)$ with the natural ordering is a free Priestley compactification of $R(A) = (R(X), \leq, \delta)$ where δ is the discrete topology δ. Moreover, it is easy to see that both $\beta(Q(A))$ and $\beta(R(A))$ are h-spaces and $\beta(Q(A))$ is the clopen decreasing subspace of $\beta(R(A))$. Hence there exists an h-mapping $f : \beta(Q(A)) \longrightarrow A$ such that $f(u, x) = u$ for every $(u, x) \in Q(A)$. Set $V = \{y \in \beta(R(X)); \lambda(y) = \lambda(A) + 1\} = \beta(R(X)) \setminus \beta(Q(X))$. To extend f to an h-mapping $g : \beta(R(A)) \longrightarrow E(A)$ we must define g on V. For $v \in V$ let $p(v) = \{v_0, v_1\} \subseteq \beta(R(X))$. Define $g(v) = z_{x,y}$ if $g(\{v_0, v_1\}) = \{x, y\}$, $g(v) = g(v_0)$ if $g(v_0) = g(v_1)$. It is easy to verify that $\beta(R(A))$ satisfies (s1) and thus either $g(v_0) = g(v_1)$ or $\{g(v_0), g(v_1)\} \in P(A)$ and the definition of g is correct. Obviously, g is surjective, preserves ordering and $g((y]) = (g(y)]$ for every element y of $\beta(R(X))$. To prove that g is continuous it suffices to show that $g^{-1}(E(U))$ is clopen for every clopen decreasing set $U \subseteq X$. By the definition of g we have $g^{-1}(E(U)) = f^{-1}(U) \cup \{v \in V; g(v) = g(v_0) \in U \text{ or } g(v) = z_{x,y}, g(\{v_0, v_1\}) = \{x, y\} \subseteq U\} = f^{-1}(U) \cup \{v \in V; g(\{v_0, v_1\}) \subseteq f^{-1}(U)\}$ The set $f^{-1}(U)$ is clopen in $\beta(R(X))$ because U is clopen, f is continuous, and $\beta(Q(X))$ is a clopen subspace of $\beta(R(X))$. Since $\{v_0; v \in V\}$, $\{v_1; v \in V\}$ are homeomorphic clopen subsets of $\beta(R(X))$ we conclude that $U_i = f^{-1}(U) \cap \{v_i; v \in V\}$ is clopen for $i = 0, 1$. Hence $U_2 = (U_0 \cap \{v_0; v_1 \in U_1\}) \cup (U_1 \cap \{v_1; v_0 \in U_0\})$ is clopen and convex. Thus $U_3 = [U_2) \setminus U_2$ is clopen and by a direct calculation we obtain that $U_3 = \{v \in V; g(\{v_0, v_1\}) \subseteq f^{-1}(U)\}$. Therefore $g^{-1}(E(U))$ is clopen. Since β is compact on $\beta(R(X))$ and g is surjective continuous we obtain that σ is compact.

It remains to show that $End(A)$ and $End(E(A))$ are isomorphic. Let $h \in End(A)$, define an extension $\varphi(h) : E(A) \longrightarrow E(A)$ of h such that $\varphi(h)(z_{x,y}) = z_{h(x),h(y)}$ if $\{h(x), h(y)\} \in P(A)$, $\varphi(h)(z_{x,y}) = h(x)$ if $h(x) = h(y)$. By a direct calculation we obtain that $\varphi(h)$ preserves the ordering and $\varphi(h)((y]) = (\varphi(h)(y)]$ for every $y \in E(A)$. Since for a clopen decreasing set $U \subseteq X$ we have $\varphi(h)^{-1}(E(U)) =$

$E(h^{-1}(U))$ we conclude that $\varphi(h) \in End(E(A))$ and $\varphi : End(A) \longrightarrow End(E(A))$ is an isomorphism. □

Theorem 4.13. *The category E_∞ is 3-determined.*

Proof. Combine Corollary 4.9 and Proposition 4.12. □

A variety V of Heyting algebras is called an *e-variety* if V is finitely generated and for every subdirectly irreducible algebra $A \in V$ the set of join irreducible elements in A is an e-space.

Theorem 4.14. *Every e-variety V of Heyting algebras is 3-determined.*

Proof. Since a dual A of every finite subdirect irreducible algebra in V is an e-space we conclude that $A \in E_\infty$. Since the lattice of congruences is distributive we conclude that V has only finitely many subdirectly irreducible algebras and every subdirectly irreducible algebra in V is finite. Thus by Theorem 4.2 we conclude that a dual of any algebra in V belongs to E_∞. Apply Theorem 4.13. □

As it was shown in [2] the variety of all Heyting algebras is not determined and this solved the problem given by McKenzie and Tsinakis in [18]. This result was strengthened by Adams, Koubek and Sichler in [4]. They proved that the variety of Heyting algebras cannot be determined in no sense.

Theorem 4.15. [4] *For every monoid M there exists a proper class of non-isomorphic Heyting algebras such that their endomorphism monoid is isomorphic to the monoid M with adjoined a new zero.* □

5. Abelian groups

The part is devoted to study an α-determinacy by general categorical methods. We will apply obtained results on Abelian groups. First we give some conventions.

Assume that $\mathcal{K}$ is a category. For a family $\{f_i : B \longrightarrow A_i; i \in I\}$ of $\mathcal{K}$-morphisms such that the product $\Pi\{A_i; i \in I\}$ exists the canonical morphism from B to $\Pi\{A_i; i \in I\}$ is denoted by $p(f_i; i \in I)$. Dually, for a family $\{f_i : A_i \longrightarrow B; i \in I\}$ of $\mathcal{K}$-morphisms such that the coproduct $\Sigma\{A_i; i \in I\}$ exists the canonical morphism from $\Sigma\{A_i; i \in I\}$ to B is denoted by $s(f_i; i \in I)$. If $\mathcal{K}$ is a category with a *zero* then the zero morphism from A to B in $\mathcal{K}$ is denoted by $c_{A,B} : A \longrightarrow B$. Let $\mathcal{K}$ be a category with zero and let A be a coproduct of B, C with coproduct injections $\sigma_B : B \longrightarrow A$, $\sigma_C : C \longrightarrow A$. The endomorphism $f = s(\sigma_B, c_{C,A})$ of A is called a ***summand corresponding*** to σ_B. We say that an endomorphism $f \in End(A)$ is a ***summand*** if f is a summand corresponding to a coproduct injection σ. A family $\{f_i; i \in I\}$ of $\mathcal{K}$-endomorphisms of an object A is ***isomorphic to*** $f \in End(A)$ if a coproduct $\Sigma\{O(f_i); i \in I\}$ exists and it is isomorphic to $O(f)$ and $\mathcal{M}(f) = s(\mathcal{M}(f_i); i \in I)$. We say that a family $\{f_i; i \in I\}$ of endomorphisms of A is ***isomorphic to a summand*** of A if there exists a summand $f \in End(A)$ such that $\{f_i; i \in I\}$ is isomorphic to f. Two endomorphisms $f, g \in End(A)$ are called *perpendicular* if $f \circ g = g \circ f = c_{A,A}$. By an easy calculation we obtain:

Lemma 5.1. *Let $\mathcal{K}$ be a category with a zero. For a family $\{f_i : B \longrightarrow A_i; i \in I\}$ of $\mathcal{K}$-morphisms such that a product of $\{A_i; i \in I\}$ exists we have that $p(f_i; i \in I) = c_{B,A}$ if and only if $f_i = c_{B,A_i}$ for every $i \in I$. Dually, for a family $\{f_i : A_i \longrightarrow B; i \in I\}$ of $\mathcal{K}$-morphisms such that a coproduct of $\{A_i; i \in I\}$ exists we have that $s(f_i; i \in I) = c_{A,B}$ if and only if $f_i = c_{A_i,B}$ for every $i \in I$.* □

Lemma 5.2. *Let $\mathcal{K}$ be a category with zero. Then for every $\mathcal{K}$-object A we have*

(1) *Every summand is an idempotent endomorphism;*
(2) *If $f, g \in End(A)$ are perpendicular and g is an automorphism then $f = c_{A,A}$;*
(3) *For a coproduct A of $\{B_i; i \in I\}$ the summands $\{f_i; i \in I\}$ corresponding to the coproduct injections $\sigma_i : B_i \longrightarrow A$ are pairwise perpendicular.*

Proof. If $A = B \vee C$ and $f = s(\sigma_B, c_{C,A})$ where $\sigma_B : B \longrightarrow B \vee C$ is the coproduct injection then $f \circ f \circ \sigma_B = f \circ \sigma_B = \sigma_B$ and $f \circ f \circ \sigma_C = f \circ c_{C,A} = c_{C,A}$, hence $f \circ f = f$ and (1) is proved.

If f, g are perpendicular and g is an automorphism then $f = f \circ 1_A = f \circ g \circ g^{-1} = c_{A,A} \circ g^{-1} = c_{A,A}$ and (2) is proved.

Let $\sigma_i : B_i \longrightarrow A$, $i \in I$ be the coproduct injections. Choose distinct $i, j \in I$. Then for every $k \in I \setminus \{j\}$ we have $f_i \circ f_j \circ \sigma_k = f_i \circ c_{B_k,A} = c_{B_k,A}$, and $f_i \circ f_j \circ \sigma_j = f_i \circ \sigma_j = c_{B_j,A}$. By Lemma 5.1 we obtain $f_i \circ f_j = c_{A,A}$. Therefore $\{f_i; i \in I\}$ are pairwise perpendicular. (3) is proved. □

We say that a category $\mathcal{K}$ has *conditional coproducts* if for every family $\{A_i; i \in I\}$ of $\mathcal{K}$-objects with a coproduct the family $\{A_i; i \in I'\}$ has also a coproduct for every $I' \subseteq I$. A class $\mathcal{C}$ of non-isomorphic $\mathcal{K}$-objects is called a *coproduct generator* if every $\mathcal{K}$-object is isomorphic to a coproduct of a family of objects in $\mathcal{C}$. Denote by $\beta(\mathcal{C})$ the number of all one-to-one mappings $f : \mathcal{C} \longrightarrow \mathcal{C}$ such that C and $f(C)$ are equimorphic for every $C \in \mathcal{C}$.

Theorem 5.3. *Let $\mathcal{K}$ be a category with zero, conditional coproducts, and a coproduct generator $\mathcal{C}$. Assume that*

(1) *There exists an isoproperty $\mathcal{P}_1$ such that for every $\mathcal{K}$-object A, $f \in Id(A)$ is a summand with $O(f) \in \mathcal{C}$ if and only if f satisfies $\mathcal{P}_1$;*
(2) *There exists a set isoproperty $\mathcal{P}_2$ such that for every $\mathcal{K}$-object A, $F \subseteq End(A)$ satisfies $\mathcal{P}_2$ if and only if F is a set of pairwise perpendicular idempotent endomorphisms satisfying $\mathcal{P}_1$ such that $\{f; f \in F\}$ is isomorphic to a summand of A.*

Then K is $\beta(\mathcal{C})^+$-determined.

Proof. First we prove that $F \subseteq End(A)$ is a maximal set satisfying $\mathcal{P}_2$ if and only if $\{f; f \in F\}$ is isomorphic to 1_A. Indeed, there exists a family $\{B_i; i \in I\}$ of $\mathcal{K}$-objects from $\mathcal{C}$ such that $A = \Sigma\{B_i; i \in I\}$. For every $i \in I$ let f_i be a summand corresponding to the coproduct injection $\sigma_i : B_i \longrightarrow A$. By Lemma 5.2 (1) and (3) $F = \{f_i; i \in I\}$ is a set of idempotent pairwise perpendicular endomorphisms and because $B_i \in \mathcal{C}$, f_i satisfies $\mathcal{P}_1$. Thus F satisfies also $\mathcal{P}_2$, and $\{f_i; i \in I\}$ is isomorphic to 1_A. If F is not maximal then $F \cup \{h\}$ satisfies $\mathcal{P}_2$, thus h is perpendicular to every f_i. Lemma 5.1 implies $1_A \circ h = c_{A,A}$ and by Lemma 5.2 (2) we obtain $h = c_{A,A}$. Thus F is maximal. Conversely, let $F \subseteq End(A)$ be a maximal subset satisfying $\mathcal{P}_2$. Set $B = \Sigma\{O(f); f \in F\}$, then $A = B \vee B'$ for some $\mathcal{K}$-object B'. Thus there exists a summand h of B' with $O(h) \in \mathcal{C}$. Then $F \cup \{h\}$ also satisfies $\mathcal{P}_2$ and therefore $h = c_{A,A}$ and $B = A$.

Let $\{A_i; i \in I\}$ be a class of equimorphic objects with A and let $\Phi_i : End(A) \longrightarrow End(A_i)$ be an isomorphism for every $i \in I$. If $F \subseteq End(A)$ is a maximal set satisfying $\mathcal{P}_2$ then so is $\Phi_i(F) \subseteq End(A_i)$. Define $\Psi : F \longrightarrow \mathcal{C}$, $\Psi(f) = O(f)$ for every $f \in F$, and $\Psi_i : F \longrightarrow \mathcal{C}$, $\Psi_i(f) = O(\Phi_i(f))$ for every $f \in F$. Since $O(f)$ is isomorphic with $O(f')$ for $f, f' \in F$ if and only if $O(\Phi_i(f))$ is isomorphic with $O(\Phi_i(f'))$ – see Lemma 1.4, we conclude that $Ker(\Psi) = Ker(\Psi_i)$ for every $i \in I$ and that $\Psi(f)$ and $\Psi_i(f)$ are equimorphic for every $f \in F$. If A and A_i are non-isomorphic then Ψ and Ψ_i are distinct. Define an equivalence $\cong$ such that $f \cong f'$ if $\Psi(f) = \Psi(f')$. Set $F' = F/_{\cong}$. The number of pairwise non-isomorphic objects equimorphic with A is less or equal to the number of one-to-one mappings Λ from F' to $\mathcal{C}$ such that $O(f)$ and $\Lambda([f])$ are equimorphic for every $f \in [f] \in F'$. Therefore $card(I) \leq \beta(\mathcal{C})$. □

Corollary 5.4. *Let $\mathcal{K}$ be a category with zero, conditional coproducts, and a coproduct generator $\mathcal{C}$ such that objects in $\mathcal{C}$ are non-equimorphic. Assume that*

1. *There exists an isoproperty $\mathcal{P}_1$ such that for every $\mathcal{K}$-object A, $f \in Id(A)$ is a summand with $O(f) \in \mathcal{C}$ if and only if f satisfies $\mathcal{P}_1$;*
2. *There exists a set isoproperty $\mathcal{P}_2$ such that for every $\mathcal{K}$-object A, $F \subseteq End(A)$ satisfies $\mathcal{P}_2$ if and only if F is a set of pairwise perpendicular idempotent endomorphisms satisfying $\mathcal{P}_1$ such that $\{f; f \in F\}$ is isomorphic to a summand of A.*

Then equimorphic $\mathcal{K}$-objects are isomorphic. □

We apply the foregoing result to Abelian groups. It is well known that the category of Abelian groups and their homomorphisms is a category with zero, see [15]. First we recall several conventions and definitions for Abelian groups. We shall use an additive notation for Abelian groups. A subset A of an Abelian group G is called a *base* if A generates G and for every finite family of distinct elements $\{a_i; i \in I\}$ of A if $\Sigma\{m_i a_i; i \in I\} = 0$ then $m_i a_i = 0$ for every $i \in I$. We say that a base A is a *p-base* whenever order of every element $a \in A$ of A is either 0 or a power of a prime. Denote by $CYCL_1$ the class of all cyclic groups G such that either G is infinite or the order of G is a power of a prime, and $CYCL_2$ the class of all quasicyclic groups and the group of rational numbers with the addition. Set $CYCL = CYCL_1 \cup CYCL_2$. The following easy lemma is folklore.

Lemma 5.5. *For every Abelian group G the following are equivalent:*

1. *G is a coproduct of groups from $CYCL_1$;*

(2) *G has a base;*
(3) *G has a p-base.* □

We recall

Proposition 5.6. [9] *Every divisible Abelian group is a coproduct of Abelian groups from $CYCL_2$.* □

Let AB be a category of Abelian groups and their homomorphisms. Denote by AB_1 the full subcategory of AB formed by all groups G which are a coproduct of a divisible Abelian group and an Abelian group with a base, and AB_2 is a full subcategory of AB formed by all Abelian groups with a base. Clearly, AB_1 and AB_2 have zero and conditional coproducts. Moreover, $CYCL$ is a coproduct generator of AB_1, and $CYCL_1$ is a coproduct generator for AB_2.

Let G be an Abelian group. The zero morphism $c_{G,G}$ is a constant mapping to 0. For $f \in Id(G)$ we have that G is a coproduct of $f^{-1}(0)$ and $Im(f)$ and thus $f \in End(G)$ is a summand if and only if f is an idempotent. We say that $f \in Id(A)$ is 0-*minimal* if for every $g \in Id(G)$ with $Im(g) \subseteq Im(f)$ we have either $g = f$ or $g = c_{G,G}$. The following is an easy observation:

Lemma 5.7. *Let G be an Abelian group, then $f \in Id(G)$ is 0-minimal if and only if for every $g \in Id(G)$ with $g = f \circ g$ we have either $g = f$ or $g = c_{G,G}$. If $G \in AB_1$ then $f \in Id(G)$ is 0-minimal if and only if $Im(f) \in CYCL$.* □

Hence "f is 0-minimal" is an isoproperty satisfying the conditions of $\mathcal{P}_1$ in Theorem 5.3 for AB_1 and AB_2.

We recall a well known and useful statement characterizing coproducts in Abelian groups.

Proposition 5.8. *Let G be an Abelian group and let $\{H_i; i \in I\}$ be a family of subgroup of G. Then G is a coproduct of $\{H_i; i \in I\}$ such that the inclusions are coproduct injections if and only if for every element $g \in G$ there exists exactly one family $\{h_i; i \in I\}$ of elements of G such that $g = \Sigma\{h_i; i \in I\}$ and $h_i \in H_i$ for every $i \in I$ (if I is infinite then $h_i \neq 0$ for only finitely many $i \in I$).* □

Lemma 5.9. *Let G be an Abelian group which is a coproduct of n groups from $CYCL$ for finite n. For every family $F \subseteq Id(G)$ of 0-minimal, pairwise perpendicular endomorphisms we have $card(F) \leq n$.*

Proof. For simplicity every natural number n we identify with the set $\{0, 1, ..., n-1\}$. We prove the statement by induction over n. For $n = 1$ the statement is true. Assume that it holds for $n-1$ and let G be a coproduct of $\{A_i; i \in n\}$ of subgroups where $A_i \in CYCL$ for every $i \in n$. For every $a \in G$ by Proposition 5.8 there exists exactly one family $\{a_i; i \in n\}$ with $a_i \in A_i$ and $a = \Sigma\{a_i; i \in n\}$. In the following for $a \in G$, a_i denotes the corresponding element of A_i. Let $F \subseteq Id(A)$ be a family of 0-minimal, pairwise perpendicular endomorphisms. Choose $f_0 \in F$, denote by $B = Im(f_0)$, $D = f_0^{-1}(0)$. Then $G = D \vee B = \Sigma\{A_i; i \in n\}$. First we prove that we can exchange B and some A_i. Since f_0 is 0-minimal (thus $Im(f_0) \in CYCL$) there exists $i \in n$ such that for distinct $a, b \in Im(f_0)$ we have $a_i \neq b_i$, - without loss of generality we can assume that $i = n-1$ - and $\{a_{n-1}; a \in B\} = A_{n-1}$. Let $a \in G$. For a_n there exists exactly one $\eta(a_n) \in B$ with $\eta(a_n)_n = a_n$. Then $\Sigma\{(a_i - \eta(a_n)_i); i \in n-1\} + \eta(a_n) = (a - a_n) - (\eta(a_n) - \eta(a_n)_n) + \eta(a_n) = a$ Hence $\{A_i; i \in n-1\}$ and B generates G. Let $a = \Sigma\{c(i); i \in n-1\} + c = \Sigma\{d(i); i \in n-1\} + d$ where $c(i), d(i) \in A_i$ for $i \in n-1$, $c, d \in B$. Then $\Sigma\{c(i) - d(i); i \in n-1\} + (c-d) = 0$. Hence $c_n = d_n$ and we obtain that $c = d$ and thus $c(i) = d(i)$ because $G = \Sigma\{A_i; i \in n\}$. We conclude by Proposition 5.8 that G is isomorphic to a coproduct of $\{A_i; i \in n-1\} \cup \{B\}$. Thus if we rename elements of G we can assume $B = A_n$. Since $G = D \vee A_n$ there exists exactly one $g \in Id(G)$ with $Im(g) = D$, $g^{-1}(0) = B$. Set $D_i = g(A_i)$ for $i \in n-1$. We show that D is isomorphic to a coproduct of $\{D_i; i \in n-1\}$. Let $d \in D$ then $d = \Sigma\{d_i; i \in n\} = \Sigma\{g(d_i); i \in n-1\} + \Sigma\{f_0(d_i); i \in n-1\} + d_n$. Since $\Sigma\{g(d_i); i \in n-1\} \in D$, $\Sigma\{f_0(d_i); i \in n-1\} \in B$ we conclude that $\Sigma\{f_0(d_i); i \in n-1\} = -d_n$ and $\Sigma\{g(d_i); i \in n-1\} = d$. Whence $\{D_i; i \in n-1\}$ generates D. Assume that $d = \Sigma\{d(i); i \in n-1\} = \Sigma\{c(i); i \in n-1\}$ where $d(i), c(i) \in D_i$ for $i \in n-1$. Choose $a(i), b(i) \in A_i$ for $i \in n-1$ with $g(a(i)) = d(i)$, $g(b(i)) = c(i)$. Then $\Sigma\{a(i) - b(i); i \in n-1\} = \Sigma\{d(i) - c(i); i \in n-1\} + \Sigma\{f_0(a(i)) - f_0(b(i)); i \in n-1\} = \Sigma\{f_0(a(i)) - f_0(b(i)); i \in n-1\}$. Since $\Sigma\{f_0(a(i)) - f_0(b(i)); i \in n-1\} \in A_n$ we conclude that $\Sigma\{f_0(a(i)) - f_0(b(i)); i \in n-1\} = 0$ therefore $a(i) = b(i)$ and thus $d(i) = c(i)$ for every $i \in n-1$. Hence by Proposition 5.8 D is isomorphic to a coproduct of $\{D_i; i \in n-1\}$. Consider $F' = \{f \circ \sigma_D; f \in F \setminus \{f_0\}\}$ where $\sigma_D : D \longrightarrow G$ is the inclusion. Since for every $f \in F \setminus \{f_0\}$ we have that $f_B \circ f = c_{G,G}$ we conclude that $Im(f) \subseteq D$ and therefore $f \circ \sigma_D \neq c_{D,D}$ because $f \neq c_{G,G}$. Thus $F' \subseteq Id(D)$ is the set of 0-minimal pairwise perpendicular endomorphisms and by induction assumptions $card(F') \leq n-1$. Whence $card(F) \leq n$. □

Let $F \subseteq End(G)$ where $G = \Sigma\{A_j; j \in J\}$. If for every $j \in J$ the set $\{f \in F; f(A_j) \neq \{0\}\}$ is finite we can define an endomorphism $\Sigma F = \Sigma\{f; f \in F\}$ such that $\Sigma F(x) = \Sigma\{f(x); f \in F\}$ because for every $x \in G$ there exist only finitely many $f \in F$ with $f(x) \neq 0$. Define a set property $\mathcal{P}$ such that

$F \subseteq End(G)$ satisfies $\mathcal{P}$ if

endomorphisms in F are 0-minimal, idempotent, and pairwise perpendicular, and for every subgroup $H \subseteq G$ which is a finite coproduct of groups from $CYCL$ the set $\{f \in F; f(H) \neq \{0\}\}$ is finite.

Corollary 5.10. *Let $G \in AB_1$. If $F \subseteq End(A)$ satisfies $\mathcal{P}$ then $\{f; f \in F\}$ is isomorphic to ΣF which is a summand of G. In particular, there exists a coproduct of $\{O(f); f \in F\}$.*

Proof. By a direct calculation we obtain that $\Sigma F \in Id(G)$ and hence $\{f; f \in F\}$ is isomorphic to ΣF. The rest is clear. □

The following folklore lemma describes $End(G)$ of cyclic groups, quasicyclic groups, and the group of rational numbers.

Lemma 5.11. *Let G be a cyclic group of order n or the group of integers with addition or the group of rational numbers with addition. Then $End(G)$ is isomorphic to the multiplicative semigroup of integers modulo n or the multiplicative semigroup of integers or the multiplicative semigroup of rational numbers. Let G be a p-quasicyclic group for a prime p, then $End(G)$ is isomorphic to the multiplicative semigroup of p-adic numbers.* □

Theorem 5.12. *The category AB_1 is $(2^{\aleph_0})^+$-determined, the category AB_2 is 2-determined.*

Proof. From Lemma 5.10 and Corollary 5.11 follows that the property $\mathcal{P}$ is an isoproperty and satisfies the conditions of the property $\mathcal{P}_2$ in Theorem 5.3. Since $card(CYCL) = \aleph_0$ we obtain the first statement as a consequence of Theorem 5.3. By Lemma 5.11 groups in $CYCL_1$ are equimorphic if and only if they are isomorphic and thus $\beta(CYCL_1) = 1$. According to Proposition 5.5 $CYCL_1$ is a coproduct generator of AB_2 and the second statement follows from Theorem 5.3. □

Corollary 5.13. *Every pair of equimorphic bounded Abelian groups is isomorphic, every pair of equimorphic finitely generated Abelian groups is isomorphic.*

Proof. By Prüfer theorem [21] every bounded Abelian group is a coproduct of cyclic groups, and also every finitely generated Abelian group is a coproduct of cyclic groups [9]. □

As proved S. Shelah [27] the category AB is not α-determined for any cardinal α:

Theorem 5.14. [27] *There exists a proper class of non-isomorphic Abelian groups G such that $End(G)$ is isomorphic to multiplicative semigroup of integers.* □

Conclusion

On the end we give several open problems.

Problem 1. Let V be a variety. We say that V is a monoid decidable (or group decidable) if there exists an algorithm which for a given finite monoid M (or a finite group G) decides whether there exists an algebra $A \in V$ with $End(A) \cong M$ (or $Aut(A) \cong G$). Which varieties are monoid decidable or group decidable? The only known non-trivial results are for finite monoid universal variety or finite group universal variety – in which case for every monoid (group) there exists a required algebra. Foldes and Sabidussy showed [8] that it is undecidable whether a variety is monoid or group universal. Is it undecidable whether a variety is monoid decidable or group decidable? Or is it undecidable whether a variety is finite monoid (group) universal? We can restrict ourselves on subvarieties of a given variety – here the problem can be decidable even the general problem will be undecidable.

Problem 2. Are there two non-isomorphic quasi-cyclic groups which are equimorphic? If equimorphic quasi-cyclic groups are isomorphic then we can strengthen Theorem 5.10 such that equimorphic groups in AB_1 are isomorphic. It is well known, see Lemma 5.11 that the endomorphism monoid of p-quasi-cyclic group for some prime p is isomorphic to the endomorphism monoid of q-quasi-cyclic group for some prime q if and only if the multiplicative semigroup of p-adic numbers is isomorphic to the multiplicative semigroup of q-adic numbers and it is equivalent to that the multiplicative group of invertible p-adic numbers is isomorphic to the multiplicative group of invertible q-adic numbers.

Problem 3. Let V be a variety of 0-lattices (or (0,1)-lattices) such that each non-trivial lattice has a prime ideal. Are there two equimorphic lattices in V which are not isomorphic nor antiisomorphic? Theorem 2.11 gives an answer only for the subclass of such varieties.

Problem 4. Denote by K the variety of Heyting algebras generated by all chains. It is well known that K is a supremum of K_n where n is taken over all natural numbers – see [11]. Are there non-isomorphic equimorphic algebras in K?

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