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FINITELY GENERATED UNIVERSAL VARIETIES OF DISTRIBUTIVE DOUBLE p-ALGEBRAS

by V. KOUBEK and J. SICHLER

Dedicated to the memory of Jan Reiterman

Résumé: Une catégorie **C** s'appelle universelle si la catégorie des graphes est pleinement plongeable dans **C**. On se propose ici de caractériser parmi les variétés finiment engendrées de doubles algèbres distributives (DAD) celles qui sont universelles. On montre que pour une variété de DAD finiment engendrée **V** les conditions suivantes sont équivalentes: **V** est universelle; **V** est monoïde-universelle (i.e. tout monoïde est isomorphe au monoïde des endomorphismes d'une algèbre appartenant à **V**); **V** admet une algèbre rigide infinie; tout groupe est isomorphe au monoïde des endomorphismes d'une algèbre appartenant à **V**; **V** contient une algèbre nucléaire A admettant trois éléments sup-irréductibles et comparables qui ne sont ni minimaux ni maximaux et au plus trois d'autres éléments sup-irréductibles non minimaux et non maximaux, et de plus telle que tout endomorphisme de A fixant ces éléments soit identité. On obtient comme corollaire que toute variété universelle finiment engendrée de DAD admet une sous variété universelle engendrée par au plus six algèbres finies sous-directement irréductibles, et qu'aucune variété de DAD engendrée par une seule algèbre finie et sous-directement irréductible n'est universelle. On montre cependant qu'il existe une variété universelle de DAD engendrée par deux algèbres finies sous-directement irréductibles.

1. Introduction

An algebra $A = (L, \vee, \wedge, ^*, ^+, 0, 1)$ of type $(2,2,1,1,0,0)$ is a distributive double p-algebra whenever $(L, \vee, \wedge, 0, 1)$ is a distributive (0,1)-lattice in which * and $^+$ are the respective unary operations of pseudocomplementation and dual pseudocomplementation: the operation * is determined by the requirement that $x \leq a^*$ be equivalent to $x \wedge a = 0$, while $y \geq a^+$ exactly when $y \vee a = 1$.

As shown in [6], the category of all distributive double p-algebras and all their homomorphisms is universal, that is, it contains a copy of the category of graphs, and hence also a copy of any category of algebras as a full subcategory, see [12]. This fact implies that every monoid is isomorphic to the endomorphism monoid of some distributive double p-algebra larger than a given cardinal and, in particular,

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the existence of a proper class of nonisomorphic rigid algebras, that is, algebras with no nontrivial endomorphisms, cf. [12]. We recall that [6] presented an example of a finitely generated universal variety of distributive double p-algebras and asked for a description of all such varieties.

The present paper fully characterizes finitely generated universal varieties of distributive double p-algebras in structural terms.

To formulate our main result, we define the rudiment $Rud(A)$ of a distributive double p-algebra A as the smallest sublattice of A containing all pseudocomplements and dual pseudocomplements of A which is closed also under relative complementation. We say that an algebra A is rudimentary if $Rud(A) = A$, and call it a nucleus whenever $Rud(A) = A$ is directly indecomposable. Finally, for any finite distributive double p-algebra A, we write $Mid(A)$ for the set of all its join irreducible elements which are neither maximal nor minimal.

Our aim is to prove the result below.

Theorem 1.1. *The following eight properties are equivalent for any finitely generated variety* $\mathbf{V}$ *of distributive double p-algebras:*

(1) $\mathbf{V}$ *is universal;*
(2) $\mathbf{V}$ *contains a proper class of non-isomorphic rigid algebras;*
(3) $\mathbf{V}$ *contains an infinite rigid algebra;*
(4) $\mathbf{V}$ *contains a rigid algebra which is not rudimentary;*
(5) *every finite monoid is isomorphic to the endomorphism monoid of some algebra from* $\mathbf{V}$*;*
(6) *every prime order cyclic group is isomorphic to the endomorphism monoid of some algebra from* $\mathbf{V}$*;*
(7) $\mathbf{V}$ *contains a finite nucleus F such that the poset $Mid(F)$ has an order component C with more than two elements, and such that the identity is the only endomorphism of F whose fixpoints include $Mid(F)$;*
(8) $\mathbf{V}$ *contains a finite nucleus G such that $Mid(G)$ has a three-element order component C and at most three other elements, and such that the identity is the only endomorphism of G whose fixpoints include C.*

Let A be a distributive double p-algebra. For any $a \in A$ and $n \geq 0$, set $a^{0(+*)} = a^{0(*+)} = a$, and recursively define $a^{(n+1)(+*)} = a^{n(+*)+*}$ and $a^{(n+1)(*+)} = a^{n(*+)*+}$. Recall that A is of range n if and only if it satisfies the identity $x^{(n+1)(+*)} = x^{n(+*)}$ or its equivalent dual form $x^{(n+1)(*+)} = x^{n(*+)}$. Thus the variety of Boolean algebras, which is not universal [8], consists of all distributive double p-algebras of range zero.

Following Beazer [1], we let Φ_A stand for the determination congruence of a distributive double p-algebra A, that is, the congruence consisting of all pairs $(a, b) \in A^2$ for which $a^* = b^*$ and $a^+ = b^+$. For any directly indecomposable algebra A of finite range, the algebra A/Φ_A is simple [1]. From Davey's description [3] of duals of finite subdirectly irreducibles it follows that the determination congruence Φ_A is the least nontrivial congruence – the monolith – of any finite non-simple subdirectly irreducible algebra.

Corollary 1.2. *If* $\mathbf{V}$ *is a universal finitely generated variety of distributive double p-algebras, then*

(1) $\mathbf{V}$ *contains a universal subvariety* $\mathbf{W}$ *generated by a set of no more than six nonisomorphic subdirectly irreducible generators with a common monolith quotient, and*
(2) $\mathbf{V}$ *must have at least two nonisomorphic subdirectly irreducible algebras which are not simple and have a common monolith quotient.*

In the concluding section we give an example of a universal variety of range one generated by a pair of finite subdirectly irreducibles with a common monolith quotient.

Double p-algebras whose determination congruence is trivial form the variety $\mathbf{R}$ of regular distributive double p-algebras; this variety is universal [7]. It may be of some interest to recall that, in fact, [7] demonstrates the universality of a regular variety generated by finitely many subdirectly irreducibles, none of which has a finite range.

To prove our main result, Theorem 1.1, we proceed as follows.

Since any universal category satisfies 1.1(2) and 1.1(5), see Pultr and Trnková [12], it follows that (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (5). Implications (2) $\Rightarrow$ (3) and (5) $\Rightarrow$ (6) are trivial, while (3) $\Rightarrow$ (4) will easily obtain once we show, in Section 3, that all rudimentary rigid algebras in $\mathbf{V}$ are finite. The fourth section demonstrates that (4) $\Rightarrow$ (7) and (6) $\Rightarrow$ (7), and the two subsequent sections contain respective proofs of (7) $\Rightarrow$ (8) and (8) $\Rightarrow$ (1).

Throughout the paper, we use Priestley's duality for distributive (0,1)-lattices and its restriction to distributive double p-algebras.

2. Preliminaries

We begin with a brief review of the essentials of Priestley's duality.

Let $(X,\tau,\leq)$ be an ordered topological space, that is, let (X,τ) be a topological space and $(X,\leq)$ a partially ordered set. For any $Z\subseteq X$ denote

$$(Z] = \{x\in X \mid \exists z\in Z \quad x\leq z\} \quad \text{and} \quad [Z) = \{x\in X \mid \exists z\in Z \quad z\leq x\}.$$

A subset Z of X is decreasing if $(Z]=Z$, increasing if $[Z)=Z$, and clopen if it is both τ-open and τ-closed. Any compact ordered topological space $(X,\tau,\leq)$ possessing a clopen decreasing set D such that $x\in D$ and $y\notin D$ for any $x,y\in X$ with $x\not\geq y$ is called a Priestley space. Let $\mathbf{P}$ denote the category of all Priestley spaces and all their continuous order preserving mappings.

Clopen decreasing sets of any Priestley space form a distributive (0,1)-lattice, and the inverse image map f^{-1} of any $\mathbf{P}$-morphism f is a (0,1)-homomorphism of these lattices. This gives rise to a contravariant functor $D : \mathbf{P} \longrightarrow \mathbf{D}$ into the category $\mathbf{D}$ of all distributive (0,1)-lattices and all their (0,1)-homomorphisms.

Conversely, for any lattice $L \in \mathbf{D}$, let $P(L) = (F(L), \tau, \leq)$ be the ordered topological space for which $(F(L), \leq)$ is the set $F(L)$ of all prime filters of L ordered by the reversed inclusion, and such that all sets $\{x \in F(L) \mid A \in x\}$ and $\{x \in F(L) \mid A \notin x\}$ with $A \in L$ form an open subbasis of τ. If $h : L \longrightarrow L'$ is a morphism in $\mathbf{D}$ then h^{-1} maps $P(L')$ into $P(L)$ and, according to [9], this determines a contravariant functor $P : \mathbf{D} \longrightarrow \mathbf{P}$.

Theorem 2.1. *(Priestley [9], [10]). The two composite functors $P \circ D : \mathbf{P} \longrightarrow \mathbf{P}$ and $D \circ P : \mathbf{D} \longrightarrow \mathbf{D}$ are naturally equivalent to the identity functors of their respective domains. Therefore $\mathbf{D}$ is a category dually isomorphic to $\mathbf{P}$.*

A morphism $f : L \longrightarrow L'$ of $\mathbf{D}$ is surjective if and only if $P(f)$ is a both a homeomorphism and an order isomorphism of $P(L')$ onto a closed order subspace of $P(L)$, and it is one-to-one just when $P(f)$ is surjective. □

Following is a useful separation property of Priestley spaces.

Proposition 2.2. *For any closed disjoint subsets Y_0 and Y_1 of any Priestley space $(X, \tau, \leq)$ there exists a clopen set C containing Y_1 and disjoint from Y_0. If, in addition, $Y_0 \cap (Y_1] = \emptyset$, then C may be chosen to be a clopen decreasing set.* □

Let $Min(X)$ and $Max(X)$ respectively denote the sets of all minimal and maximal elements of an ordered topological space $(X, \tau, \leq)$. For any $Y \subseteq X$, write $Min(Y) = (Y] \cap Min(X)$, $Max(Y) = [Y) \cap Max(X)$ and $Ext(Y) = Min(Y) \cup Max(Y)$. When $Y = \{y\}$, we write $Min(y)$ instead of $Min(\{y\})$, and similarly for Max and Ext. If $(X, \tau, \leq)$ is a Priestley space and $y \in X$, then the sets $Min(y)$ and $Max(y)$, and hence also their union $Ext(y)$ are nonvoid and closed.

Theorem 2.3. *(Priestley [11]). Let $P : \mathbf{D} \longrightarrow \mathbf{P}$ be the functor assigning Priestley spaces to distributive (0,1)-lattices, and let $f : L \longrightarrow L'$ be a morphism in $\mathbf{D}$. Then:*

- (a) *L is a distributive double p-algebra if and only if $(Y]$ is clopen for every clopen increasing subset Y of $P(L)$ and $[W)$ is clopen for any clopen decreasing set W; if this is the case, then $W^* = P(L) \backslash [W) = P(L) \backslash [Min(W))$ and $W^+ = (P(L) \backslash W] = (Max(P(L)) \backslash W]$ for any clopen decreasing subset W of $P(L)$;*
- (b) *f is a double p-algebra homomorphism iff $P(f)(Min(x)) = Min(P(f)(x))$ and $P(f)(Max(x)) = Max(P(f)(x))$ for every $x \in P(L')$;*
- (c) *the sets $Min(P(A))$ and $Max(P(A))$ are closed for any distributive double p-algebra A.* □

The Priestley space $P(A)$ of a distributive double p-algebra A will be called a *dp*-space, and the dual of a double p-algebra homomorphism a *dp*-map. We note that a *dp*-map $P(f) : P(B) \longrightarrow P(A)$ is the Priestley dual of an injective homomorphism $f : A \longrightarrow B$ exactly when it is surjective. A double p-algebra homomorphism

$f : A \longrightarrow B$ is surjective if and only if $P(f) : P(B) \longrightarrow P(A)$ is a homeomorphism and order isomorphism of $P(B)$ onto a closed order subspace $Z \subseteq P(A)$ satisfying $Ext(Z) \subseteq Z$. Any such subspace Z will be called a closed c-set. The kernel $Ker(f)$ of f then consists of all pairs $(d, e) \in A^2$ such that $D \cap Z = E \cap Z$ for the clopen decreasing sets D, E respectively representing $d, e \in A$. It follows that congruences of A are in one-to-one order-reversing correspondence with closed c-sets in $P(A)$.

For any distributive double p-algebra A, let $Cen(A)$ be the center of A, the set of all complemented elements of A.

For any filter F of $Cen(A)$, let $\Theta(F)$ be the least congruence of A that collapses F. According to Beazer [2], the congruence $\Theta(F)$ consists of all $(x, y) \in A^2$ satisfying $x \wedge f = y \wedge f$ for some $f \in F$.

If $c \in Cen(A) \backslash \{0, 1\}$, then the complement $X_1 = P(A) \backslash X_0$ of the nonvoid clopen decreasing set $X_0 \subseteq P(A)$ representing c is also nonvoid, clopen and decreasing, and hence it represents the complement c' of c. But then every clopen decreasing $E \subseteq P(A)$ is a disjoint union of clopen decreasing sets $E \cap X_0$ and $E \cap X_1$. Furthermore, if $E_i \subseteq X_i$ are decreasing and clopen in X_i for $i = 0, 1$, then $E = E_0 \cup E_1$ is decreasing and clopen in $P(A)$. Hence the algebra A is isomorphic to the product $D(X_0) \times D(X_1)$ whose factors $D(X_0) \cong A/\Theta([c))$ and $D(X_1) \cong A/\Theta([c'))$ are nontrivial.

Conversely, if algebras A_0 and A_1 are nontrivial, and if $A = A_0 \times A_1$, then the dp-space $P(A)$ is a disjoint union of nonvoid clopen decreasing sets $X_i = P(A_i)$ with $i = 0, 1$. But then X_0 represents some $c \in Cen(A)$ whose complement c' is represented by X_1 and, for $i = 0, 1$, the closed c-set X_i represents the kernel of the projection $A \longrightarrow A_i$. If $d, e \in A$ are respectively represented by clopen decreasing sets $D, E \subseteq P(A)$, then $D \cap X_0 = E \cap X_0$ exactly when $(d, e) \in \Theta([c))$, so that the closed c-set X_0 is the Priestley dual of $\Theta([c))$, and X_1 similarly represents $\Theta([c'))$.

Altogether, nontrivial direct decompositions of a distributive double p-algebra A are in one-to-one correspondence with elements $c \in Cen(A) \setminus \{0, 1\}$. For any such c, we have $A \cong A/\Theta([c)) \times A/\Theta([c'))$. Furthermore, a distributive double p-algebra A is directly indecomposable exactly when $Cen(A) = \{0, 1\}$.

Let $f : A \longrightarrow D$ be a homomorphism from A to a directly indecomposable algebra D. Since $Cen(D) = \{0, 1\}$, the set $Q = f^{-1}\{1\} \cap Cen(A)$ is a prime filter of the Boolean algebra $Cen(A)$ and, because $f(q) = f(1)$ for every $q \in Q$, the kernel of f contains the least congruence $\Theta(Q)$ of A collapsing Q. If $\alpha_Q : A \longrightarrow A/\Theta(Q)$ is the homomorphism with $Ker(\alpha_Q) = \Theta(Q)$, then $f = f' \circ \alpha_Q$ for some $f' : A/\Theta(Q) \longrightarrow D$. For any prime filter Q of $Cen(A)$, the algebra $A_Q = A/\Theta(Q)$, called a component of A, is then a maximal directly indecomposable quotient of A (a rationale for this terminology will soon become apparent).

If $f : A \longrightarrow B$ is a homomorphism and $\beta_R : B \longrightarrow B_R$ is the natural homomorphism onto a component B_R of B, then $Q = f^{-1}(R) \cap Cen(A)$ is a prime filter of $Cen(A)$ and $\beta_R(f(q)) = 1$ for every $q \in Q$. Thus $Ker(\beta_R \circ f) \supseteq \Theta(Q) = Ker(\alpha_Q)$, and there exists a homomorphism $f' : A_Q \longrightarrow B_R$ such that $f' \circ \alpha_Q = \beta_R \circ f$. The claim below partially complements this observation.

Lemma 2.4. *If $f : A \longrightarrow B$ is a homomorphism of distributive double p-algebras which is one-to-one on $Cen(A)$, then for every component A_Q of A there exist a component B_R of B and a homomorphism $f' : A_Q \longrightarrow B_R$ such that $f' \circ \alpha_Q = \beta_R \circ f$ for the natural surjections $\alpha_Q : A \longrightarrow A_Q$ and $\beta_R : B \longrightarrow B_R$.*

Proof. Since the restriction of f to $Cen(A)$ is a one-to-one homomorphism of $Cen(A)$ into $Cen(B)$, the congruence extension property of Boolean algebras implies the existence of a prime filter $R \subseteq Cen(B)$ with $f^{-1}(R) = Q$. But then $\beta_R(f(q)) = 1$ for all $q \in Q$, that is, $Ker(\alpha_Q) = \Theta(Q) \subseteq Ker(\beta_R \circ f)$, and the claim follows. $\square$

Next we intend to note that components of a distributive double p-algebra A from a variety $\mathbf{V}$ of finite range correspond to order components of its dp-space $P(A)$.

We recall that elements x and y of a poset $(X, \leq)$ are <u>order connected</u> whenever there exists a sequence $x = x_0, x_1, ..., x_k = y$ in which x_i is comparable to x_{i+1} for $i = 0, 1, \ldots, k-1$. The classes of the equivalence ε formed by all pairs of connected elements, that is, the maximal connected subsets in $(X, \leq)$, are the <u>order components</u> of $(X, \leq)$. It is clear that C is a component of a dp-space $(X, \tau, \leq)$ just when $Ext(C)$ is a component of its closed order subspace $Ext(X)$.

Any finitely generated variety $\mathbf{V}$ of distributive double p-algebras clearly satisfies, for some integer $n \geq 0$, the identity $x^{(n+1)(+*)} = x^{n(+*)}$ and its equivalent $x^{(n+1)(*+)} = x^{n(*+)}$, that is, any finitely generated variety $\mathbf{V}$ consists of algebras of range n. According to [7], all order components of dp-spaces of such algebras are closed. If C_0 and C_1 are distinct components of the dp-space X of some algebra $A \in \mathbf{V}$ then, by 2.2, there exists a clopen decreasing set $B \supseteq C_1$ such that $C_0 \subseteq X \setminus B$. Since $B^{*+} = ([B)]$ is clopen by 2.3 and because A is of range n, the clopen set $B^{n(*+)} \supseteq C_1$ is both increasing and decreasing, and disjoint from C_0. Any two components of X can be thus separated by complementary clopen decreasing sets and, as a result, the quotient space X/ε obtained by collapsing all order components of X is the Priestley dual of $Cen(A)$.

It follows that the dp-space $X = P(A)$ of any algebra A of finite range is the disjoint union of its order components $C_Q = P(A_Q)$ indexed by $Q \in P(Cen(A))$. In algebraic terms, any algebra A of finite range is subdirect in the product $\Pi\{A_Q \mid Q \in P(Cen(A))\}$ of its 'algebraic' components $A_Q = D(C_Q)$.

Next we describe dp-spaces of subdirectly irreducible algebras of finite range.

If $Y = P(B)$ is the dual of a subdirectly irreducible algebra B of a finite range then Y must be connected, and hence the dp-space of any nontrivial quotient algebra B' of B must contain the connected closed set $Ext(Y)$. Since Y has a unique maximal closed order subspace Z satisfying $Ext(Z) \subseteq Z$ that represents the monolith of B, and because all points of $Y \setminus Ext(Y)$ are closed, either $Y = Ext(Y)$ and B is simple, or $Y \setminus Ext(Y)$ is a singleton which is clopen because $Ext(Y)$ is closed in Y. Since the converse is clear, following Davey [3] we conclude that B is simple if and only if $Y = Ext(Y)$ is connected, and that B is subdirectly irreducible but not simple just when Y is connected and $Y \setminus Ext(Y)$ is a clopen singleton; in the latter case, $Ext(Y)$ is the Priestley dual of the quotient of B modulo its monolith.

For any algebra A of finite range and any $x \in X = P(A)$, let $K(x)$ denote the component of X containing x. The subposet $E(x) = \{x\} \cup Ext(K(x))$ of X is then closed in X and $Ext(E(x)) = Ext(K(x)) \subseteq E(x)$, so that $E(x)$ is the *dp*-space of a subdirectly irreducible quotient of A. Since X is the union of all its subspaces $E(x)$ with $x \in X$, the algebra A is a subdirect product of subdirectly irreducible algebras $D(E(x))$ with $x \in X$.

3. Rudimentary algebras and nuclei

For any double p-algebra A, let $L(A)$ be the sublattice of A generated by the set

$$Q(A) = \{a^* \mid a \in A\} \cup \{a^+ \mid a \in A\}$$

of all pseudocomplements and dual pseudocomplements of A. Clearly, any sublattice of A containing $Q(A)$ is a subalgebra of A. Recall that <u>rudiment</u> of A is the least sublattice of A containing $Q(A)$ and closed under relative complementation. Thus $L(A) \subseteq Rud(A)$, and $Rud(A)$ is the subalgebra of A obtained by intersecting all sublattices $S \subseteq A$ that contain $L(A)$ and include every $a \in A$ for which there is an $s \in S$ with $a \vee s \in S$ and $a \wedge s \in S$.

When $Rud(A) = A$, we say that A is <u>rudimentary</u>. Any directly indecomposable and rudimentary algebra is called a <u>nucleus</u>.

The set $R(A)$ consisting of all $a \in A$ such that $g(a) = h(a)$ for any two homomorphisms $g, h : A \longrightarrow B$ that coincide on $Q(A)$ is a subalgebra of A containing $L(A)$. From the distributive law it immediately follows that $R(A)$ is closed under relative complementation; hence $Rud(A) \subseteq R(A)$.

Lemma 3.1. *If A is a distributive double p-algebra, then:*

1. *$R(A) = Rud(A)$; that is, $R(A)$ is the least sublattice of A that contains $Q(A)$ and is closed under relative complementation;*
2. *the dual h of the inclusion $Rud(A) \subseteq A$ satisfies $h(x) = h(y)$ just when $Ext(x) = Ext(y)$ in $P(A)$;*
3. *the order of the Priestley dual of $Rud(A)$ is the least partial order containing all pairs $(h(x), h(y))$ for which $x \leq y$ in the dual of A;*
4. *A is rudimentary if and only if $Ext(x) \neq Ext(y)$ for any two distinct elements x, y of its Priestley dual.*

Proof. Let X denote the *dp*-space of A. Then, for any $x \in X$ and any clopen decreasing $D \subseteq X$, we have $x \in X \setminus [D) = D^*$ just when $Min(x) \cap D = \emptyset$, while $x \in (X \setminus D] = D^+$ if and only if $Max(x) \setminus D \neq \emptyset$. It easily follows that any two prime ideals x, y of A satisfy $x \cap Q(A) = y \cap Q(A)$ just when $Ext(x) = Ext(y)$ in X. But then clearly $x \cap L(A) = y \cap L(A)$ and, because x, y are prime ideals, $x \cap Rud(A) = y \cap Rud(A)$. Therefore x, y coincide on $Rud(A)$ if and only if $Ext(x) = Ext(y)$ in the dp-space $X = P(A)$ of A. This demonstrates (2).

For any $d \in A \setminus Rud(A)$ there are prime ideals $x, y \in X$ such that $d \in x$, $d \notin y$ and $x \cap Rud(A) = y \cap Rud(A)$, see, for instance, p. 141 of [4]. In the dp-space X of A, this means the existence of a clopen decreasing set $D \subseteq X$ and $x, y \in X$ with $Ext(x) = Ext(y)$ for which $y \in D$ and $x \notin D$. To show that $D \in A \setminus R(A)$, let $Y = Ext(X) \cup \{e\}$ be a proper extension of the closed subspace $Ext(X)$ of X by a clopen singleton $\{e\}$ for which $Ext(e) = Ext(x) = Ext(y)$. Then Y is a dp-space. For $z \in \{x, y\}$, let $f_z : Y \longrightarrow X$ be the extension of the identity mapping of $Ext(X)$ determined by $f_z(e) = z$. Then $D(f_x)$ and $D(f_y)$ agree on $Q(A)$, by 2.3. Since f_x and f_y are obviously dp-maps such that $e \notin f_x^{-1}(D)$ and $e \in f_y^{-1}(D)$, it follows that $d \in A \setminus R(A)$. Therefore (1) holds.

Since $Rud(A)$ is the largest subalgebra of A whose dual is carried by the quotient space $h(X)$, the least order induced by h defines a dp-space of $Rud(A)$, so that (3) holds. Claim (4) is obviously true. □

Remark 3.2. The following three claims are easily established:

(a) The dual h of the inclusion $Rud(A) \subseteq A$ is one-to-one on $Max(X)$ and on $Min(X)$, but not on $Ext(X)$: in fact, it collapses those (and only those) components C of X for which $Max(C)$ and $Min(C)$ are singletons.

(b) With any Priestley space we may associate its <u>extremal pre-order</u> $\leq_E$ defined by $x \leq_E y$ just when $Min(x) \subseteq Min(y)$ and $Max(x) \supseteq Max(y)$. Then $Ext(x) = Ext(y)$ is equivalent to $x \leq_E y \leq_E x$ for $x, y \in X$. The quotient of $\leq_E$ on $h(X)$ is the partial order which, together with the quotient topology of $h(X)$, determines the dp-space of the subalgebra $L(A)$.

(c) Any sublattice S of A satisfying $L(A) \subseteq S \subseteq Rud(A)$ is a double p-algebra such that $L(S) = L(A)$; by the congruence extension property, the algebra S is also rudimentary.

Lemma 3.3. *Let $f : A \longrightarrow B$ be a homomorphism of distributive double p-algebras. Then $f(Rud(A)) \subseteq Rud(B)$. Moreover, if A is rudimentary and f is surjective, then B is rudimentary. In particular, every component of a rudimentary algebra is a nucleus.*

Proof. To prove the first claim, let $g_0, g_1 : B \longrightarrow C$ be any two homomorphisms for which $g_0 \restriction L(B) = g_1 \restriction L(B)$. Since $f(L(A)) \subseteq L(B)$, the homomorphisms $g_0 \circ f$ and $g_1 \circ f$ coincide on $L(A)$, and hence also on $Rud(A)$. But then g_0 coincides with g_1 on $f(Rud(A))$, and $f(Rud(A)) \subseteq Rud(B)$ follows from the definition of $Rud(B)$.

Secondly, if $f : A \longrightarrow B$ is surjective and $A = Rud(A)$, then $B = f(A) = f(Rud(A)) \subseteq Rud(B)$ as required. □

We say that a double p-algebra A is <u>uniform</u> if all its components have isomorphic rudiments. For an algebra A of a finite range, this means that its quotients –

represented by the (closed) order components C of $X = P(A)$ – have isomorphic rudiments.

Any finitely generated variety $\mathbf{V}$ of distributive double p-algebras contains only finitely many nonisomorphic nuclei, all of them finite. The claim below shows that the existence of homomorphisms determines a partial order on isomorphism classes of finite nuclei.

Lemma 3.4. ***Any endomorphism of a finite nucleus is invertible. Consequently, if F and G are nuclei for which there are homomorphisms $F \longrightarrow G$ and $G \longrightarrow F$, then $F \cong G$.***

Proof. **Write $X = P(F)$. Let $h : X \longrightarrow X$ be a dp-map. Then $h(Ext(X)) = Ext(X)$ because $Ext(X)$ is connected; since $Ext(X)$ is finite, h permutes members of $Ext(X)$. If $h(x) = h(y)$, then $h(Ext(x)) = Ext(h(x)) = Ext(h(y)) = h(Ext(y))$. But then $Ext(x) = Ext(y)$ because $h \restriction Ext(X)$ is one-to-one; since F is rudimentary, we obtain $x = y$. Therefore h is invertible.**

With f, g as above, the composites $g \circ f$ and $f \circ g$ are automorphisms, so that $(g \circ f)^k = id_F$ and $(f \circ g)^k = id_G$ for some integer $k \geq 1$. Hence f is an isomorphism of F onto G. $\square$

Proposition 3.5. ***Any distributive double p-algebra A from a finitely generated variety $\mathbf{V}$ has a uniform direct factor.***

Proof. **Let K denote the finite set of non-isomorphic nuclei that occur as rudiments of maximal directly indecomposable factors of the algebra A. By 3.4, there exists a maximal $M \in K$ in the sense that there is no homomorphism from M to any $N \in K \setminus \{M\}$. We aim to exhibit a direct factor B of A such that $Rud(B/\Theta(R)) \cong M$ for every prime filter R of $Cen(B)$.**

For any prime filter $Q \subseteq Cen(A)$ of the center of A we have $A_Q = A/\Theta(Q)$ and $Rud(A_Q) \in K$. Write

$$I = \{Q \in P(Cen(A)) \mid Rud(A_Q) \not\cong M\}, \text{ and}$$
$$J = \{Q \in P(Cen(A)) \mid Rud(A_Q) \cong M\}.$$

It is clear that I, J form a decomposition of $P = P(Cen(A))$. Suppose that $I \neq \emptyset$, for else there is nothing to prove.

If D is a component of $S_I = \Pi\{Rud(A_Q) \mid Q \in I\}$, then $D = S_I/\Theta(U)$ for some prime filter U of the center $Cen(S_I) = 2^I$. But then U is an ultrafilter on the set I, and hence $D = S_I/\Theta(U)$ is an ultraproduct of finite algebras isomorphic to members of the finite set $K \setminus \{M\}$. It follows that every component of S_I is isomorphic to a member of $K \setminus \{M\}$.

Let $\alpha_Q : A \longrightarrow A_Q$ be the surjective homomorphism with $Ker(\alpha_Q) = \Theta(Q)$, and let a homomorphism $f : Rud(A) \longrightarrow S_I$ be defined by $f(s)(Q) = \alpha_Q(s)$ for all $s \in Rud(A)$ and $Q \in I$. For any $c \in Cen(Rud(A))$ with $f(c) = 1$ we then have $(c, 1) \in Ker(\alpha_Q) = \Theta(Q)$ for every $Q \in I$, and hence also $c \in \bigcap I$. Should

$\bigcap I = \{1\}$, the homomorphism f would be one-to-one on $Cen(Rud(A))$ and, by 2.4, any component of $Rud(A)$ isomorphic to M would have a homomorphism to a component of S_I isomorphic to a member of $K \setminus \{M\}$. The choice of M makes this impossible, however, and we must have $\bigcap I \neq \{1\}$.

Select any $c \in Cen(Rud(A)) = Cen(A)$ with the complement $c' \in \bigcap I \setminus \{1\}$. Then $B = A/\Theta([c))$ is a nontrivial direct factor of A; let $k : A \longrightarrow B$ be the homomorphism with $Ker(k) = \Theta([c))$. To show that B is uniform, choose any prime filter $R \subseteq Cen(B)$, and let $\beta_R : B \longrightarrow B_R$ be the surjective homomorphism with $Ker(\beta_R) = \Theta(R)$. Then $Q = k^{-1}(R) \cap Cen(A)$ is a prime filter of $Cen(A)$, and $c \in Q$. Should $Q \in I$, then $0 = c \wedge c' \in Q$, a contradiction. Thus $Q \in J$ and $Rud(A_Q) \cong M$.

From $c \in Q$ it follows that $\Theta(Q) = Ker(\alpha_Q) \subseteq Ker(\beta_R \circ k)$, and we need only justify the reverse inclusion. To do this, choose any $x, y \in Ker(\beta_R \circ k)$. Since $Ker(\beta_R) = \Theta(R)$, there exists an $r \in R$ such that $k(x) \wedge r = k(y) \wedge r$. The homomorphism k is surjective, so that $k(a) = r$ for some $a \in A$. But A is of finite range, so that $q = a^{n(+*)} \in Cen(A)$ for some $n \geq 1$; clearly $k(q) = r$. But then $q \in Q$ and $k(x \wedge q) = k(y \wedge q)$. From $Ker(k) = \Theta([c))$ we obtain $x \wedge (q \wedge c) = y \wedge (q \wedge c)$, and then $(x, y) \in \Theta(Q)$ because $q \wedge c \in Q$.

Altogether, $B/\Theta(R) \cong A_Q$ and hence also $Rud(B/\Theta(R)) \cong Rud(A_Q) \cong M$, as was to be shown. □

4. Nuclei in universal varieties

In this section we show that every finitely generated variety $\mathbf{V}$ of distributive double p-algebras which satisfies 1.1(4) or 1.1(6) contains a nucleus F (which is, of course, finite) such that

(X1) there are three distinct non-extremal order connected join irreducible elements of F, and

(X2) the identity map is the only endomorphism of F which fixes every non-extremal join irreducible element of F.

Assume that $\mathbf{V}$ either contains algebras with endomorphism monoids isomorphic to arbitrarily large prime order cyclic groups, or a rigid algebra which is not a product of finitely many nuclei. Since any finitely generated variety $\mathbf{V}$ contains only finitely many non-isomorphic nuclei, the latter requirement is satisfied by any infinite rigid algebra in $\mathbf{V}$.

We shall consider dp-spaces rather than the algebras themselves, and extend all algebraic terminology to corresponding dp-spaces.

In terms of dp-spaces, we aim to exhibit a finite connected poset $(X, \leq)$ dual to some algebra in $\mathbf{V}$ in which $Ext(x) = Ext(y)$ only when $x = y$, and such that

(P1) $Mid(X) = X \setminus Ext(X)$ has a component with at least three elements, and

(P2) the identity of X is the only *dp*-map $f : X \longrightarrow X$ whose fixpoints include $Mid(X)$.

Extending our earlier notation, for any subset U of a *dp*-space X, we write $K(U)$ for the set of all components of X intersecting U. If X is dual to an algebra of finite range then $K(U)$ is a union of components; furthermore,

$K(U)$ is closed whenever U is, and
$K(U)$ is clopen for any clopen $U \subseteq X$ which is increasing or decreasing.

The lemma below is of central importance.

Lemma 4.1. *If X is the dp-space of a uniform algebra from a finitely generated variety, and if Y is the dp-space of the (finite) nucleus isomorphic to the rudiment of every component of X, then there exists a surjective dp-map $h : X \longrightarrow Y$.*

Proof. With no loss of generality, we may assume that the space X is rudimentary.

We aim to show that every component C of the rudimentary uniform space X is contained in a clopen union X_C of components of X for which there exists a surjective *dp*-map $h_C : X_C \longrightarrow C$. The existence of the *dp*-map $h : X \longrightarrow Y$ then follows immediately from the compactness of X and the fact that C is isomorphic to Y.

Let C be an arbitrary component of X. Since there is nothing to prove when C is a singleton, we shall assume that $Max(C) \cap Min(C) = \emptyset$.

(a) First we construct a clopen union $D \supseteq C$ of components of X and a family of natural *dp*-maps $f_{C'} : Ext(C) \longrightarrow Ext(C')$ indexed by components $C' \subseteq D$.

Since C is finite and $Max(X)$ is closed, for every $z \in Min(C)$ there exists a clopen decreasing set $dA_z \subseteq X$ such that $dA_z \cap C = \{z\}$ and $dA_z \cap Max(X) = \emptyset$. Then the set $\bigcup\{dA_z \mid z \in Min(C)\}$ is clopen decreasing and disjoint from $Max(X)$. Consequently, for each $u \in Max(C)$ there is a clopen increasing set iA_u such that $iA_u \cap C = \{u\}$ and $iA_u \cap dA_z = \emptyset$ for all $z \in Min(C)$. For each $z \in Min(C)$ and $u \in Max(C)$, respectively, set

$dX_z = dA_z \setminus (\bigcup\{[dA_t) \mid t \in Min(C) \setminus \{z\}\})$, and
$iX_u = iA_u \setminus (\bigcup\{(iA_v] \mid v \in Max(C) \setminus \{u\}\})$;

furthermore, denote

$dB_z = dX_z \cap (\bigcap\{(iX_u] \mid z < u\})$, and
$iB_u = iX_u \cap (\bigcap\{[dX_z) \mid z < u\})$.

Since $(dA_t] = dA_t$ for all $t \in Min(C)$ and $[iA_v) = iA_v$ for all $v \in Max(C)$, the finiteness of C and the fact that X is a *dp*-space imply that every dB_z is clopen decreasing, and that every iB_u is clopen increasing. Moreover, members of the family

$$B = \{dB_z \mid z \in Min(C)\} \cup \{iB_u \mid u \in Max(C)\}$$

are nonvoid and pairwise disjoint.

The set $D = \bigcap\{K(W) \mid W \in B\} \supseteq C$ is a clopen union of components of X. Thus each set $dD_z = dB_z \cap D$ with $z \in Min(C)$ is nonvoid, clopen and decreasing, while $iD_u = iB_u \cap D$ is nonvoid, clopen and increasing for each $u \in Max(C)$. All of these sets are pairwise disjoint.

Let $C' \subseteq D$ be any component of X. Since $Min(C')$ is finite and bijective to $Min(C)$, and because each dD_z is decreasing, the set $dD_z \cap Min(C')$ is nonvoid for every $z \in Min(C)$; having recalled that the sets dD_z are pairwise disjoint, we conclude that every $dD_z \cap Min(C')$ must be a singleton. A similar observation applies to $iD_u \cap Max(C')$. Hence there is a bijection $f_{C'} : Ext(C) \longrightarrow Ext(C')$ with $f_{C'}(Max(C)) = Max(C')$ and $f_{C'}(Min(C)) = Min(C')$ such that

$\{f_{C'}(z)\} = dD_z \cap Min(C')$ for all $z \in Min(C)$, and
$\{f_{C'}(u)\} = iD_u \cap Max(C')$ for all $u \in Max(C)$.

An analogous argument shows that $\{f_{C'}(z)\} = dX_z \cap Min(C') = dB_z \cap Min(C')$ for all $z \in Min(C)$ and $\{f_{C'}(u)\} = iX_u \cap Max(C') = iB_u \cap Max(C')$ for all $u \in Max(C)$. If $z \leq u$ in $Ext(C)$, then $f_{C'}(z) \leq f_{C'}(u)$ by the definition of dB_z; since $Ext(C')$ is isomorphic to $Ext(C)$ and both are finite, we conclude that the bijection $f_{C'}$ is an order isomorphism – and hence a dp-map – of $Ext(C)$ onto $Ext(C')$.

For any $p \in dB_z \cap C'$, the set $Min(p)$ is nonvoid and $Min(p) \subseteq dB_z \cap Min(C') = \{f_{C'}(z)\}$; moreover, $Max(p) \supseteq \{f_{C'}(u) \mid u \in Max(z)\} = Max(f_{C'}(z))$. Hence $f_{C'}(z) \leq p$ and $p \leq_E f_{C'}(z)$ in the extremal order $\leq_E$ of the rudimentary poset C', and this is possible only when $p = f_{C'}(z)$. Therefore, for any component $C' \subseteq D$, we have

$\{f_{C'}(z)\} = dD_z \cap C'$ for all $z \in Min(C)$, and
$\{f_{C'}(u)\} = iD_u \cap C'$ for all $u \in Max(C)$,

where the second claim follows dually from the definition of iB_u. Consequently,

$$dD_z \subseteq Min(D) \text{ and } iD_u \subseteq Max(D).$$

Together with the finiteness of $Ext(C)$, this implies that $Min(D)$ and $Max(D)$ are clopen sets.

(b) Next we exhibit clopen sets needed in considerations of partial maps between components of the clopen set D.

For any $Z \subseteq Min(C)$ and $U \subseteq Max(C)$, write $dD_Z = \bigcup\{dD_z \mid z \in Z\}$ and $iD_U = \bigcup\{iD_u \mid u \in U\}$. The following equalities are easily verified:

$$\{y \in D \mid Min(y) \subseteq dD_Z\} = D \setminus [dD_{Min(C)\setminus Z}),$$
$$\{y \in D \mid Min(y) \supseteq dD_Z \cap K(y)\} = \bigcap\{[dD_z) \mid z \in Z\},$$

$$\begin{aligned} P(Z) &= \{y \in D \mid Min(y) = dD_Z \cap K(y)\} \\ &= (\bigcap\{[dD_z) \mid z \in Z\}) \cap (D \setminus [dD_{Min(C)\setminus Z})) \end{aligned}$$

and, dually,

$$\{y \in D \mid Max(y) \subseteq iD_U\} = D \setminus (iD_{Max(C)\setminus U}],$$
$$\{y \in D \mid Max(y) \supseteq iD_U \cap K(y)\} = \bigcap\{(iD_u] \mid u \in U\},$$

$$R(U) = \{y \in D \mid Max(y) = iD_U \cap K(y)\}$$
$$= (\bigcap\{(iD_u] \mid u \in U\}) \cap (D \setminus (iD_{Max(C)\setminus U}]).$$

Since $Ext(C)$ is finite and all sets dD_z and iD_u are clopen in the clopen dp-space D, all right hand sides of the above six equalities define sets that are clopen in X. Thus the set $S(Z,U) = P(Z) \cap R(U)$ below is also clopen:

$$S(Z,U) = \{y \in D \mid Min(y) = dD_Z \cap K(y) \text{ and } Max(y) = iD_U \cap K(y)\}.$$

For any $x \in D$, write $Mn(x) = f^{-1}_{K(x)}(Min(x))$ and $Mx(x) = f^{-1}_{K(x)}(Max(x))$, where $f_{K(x)}$ is the bijection of $Ext(C)$ onto $Ext(K(x))$ defined in (a). Clearly, the sets $Mn(x) \subseteq Min(C)$ and $Mx(x) \subseteq Max(C)$ satisfy

$dD_{Mn(x)} \cap K(x) = f_{K(x)}(Mn(x)) = Min(x)$ and
$iD_{Mx(x)} \cap K(x) = f_{K(x)}(Mx(x)) = Max(x)$.

Finally, for each $x \in D$ we now write $M(x) = S(Mn(x), Mx(x))$. The set $M(x)$ is clopen, $x \in M(x)$ and

$$M(x) = \{y \in D \mid Ext(y) = f_{K(y)}(Mn(x) \cup Mx(x))\};$$

equivalently, $y \in M(x)$ exactly when $f_{K(y)}(f^{-1}_{K(x)}(Ext(x))) = Ext(y)$.

Since D is rudimentary, the set $M(x) \cap C'$ has at most one element for any component $C' \subseteq D$. Moreover, since each $f_{C'} : Ext(C) \longrightarrow Ext(C')$ is a dp-map and because C' is rudimentary, in fact we have $M(x) \cap C' = \{f_{C'}(f^{-1}_{K(x)}\{x\})\}$ for every $x \in Ext(D)$.

(c) For each dp-map $g : Ext(C) \longrightarrow Ext(C)$ we define a set P_g to be the union of all components $C' \subseteq D$ for which $f_{C'} \circ g$ extends to a dp-map $k : C \longrightarrow C'$.

Next we show that the finitely many sets P_g form a decomposition of D.

By the hypothesis, for any component C' of D there exists an isomorphism $\alpha : C \longrightarrow C'$. The composite $g = f^{-1}_{C'} \circ (\alpha \restriction Ext(C))$ is, clearly, an automorphism g of $Ext(C)$ such that $f_{C'} \circ g = \alpha \restriction Ext(C)$ extends to the dp-map $\alpha : C \longrightarrow C'$. This shows that every component C' of D lies in some P_g.

If $C' \subseteq P_g \cap P_{g'}$ and $C'' \subseteq P_g$ then there exist dp-maps h, h' and k such that
$h \restriction Ext(C) = f_{C'} \circ g$, $h' \restriction Ext(C) = f_{C'} \circ g'$ and $k \restriction Ext(C) = f_{C''} \circ g$.

Since the composite dp-map $k \circ h^{-1} \circ h' : C \longrightarrow C''$ extends the isomorphism $f_{C''} \circ f^{-1}_{C'} \circ f_{C'} \circ g' = f_{C''} \circ g'$, we have $C'' \subseteq P_{g'}$. Hence $P_g \subseteq P_{g'}$ and, by symmetry, $P_{g'} = P_g$. Therefore, the sets P_g form a finite decomposition of D as claimed.

(d) Next we show that every set P_g is closed; from (c) it then follows that every P_g is, in fact, clopen. To this end, for any $x, y \in D$ set

$$T(x, y) = (M(x)] \cap M(y).$$

Since $M(x)$ and $M(y)$ are closed, so is $T(x, y)$, and hence the union $K(T(x, y))$ of all components intersecting $T(x, y)$ is closed as well. Our claim will thus be proved once we show that, for a component $C' \subseteq P_g$,

$$P_g = \bigcap\{K(T(x, y)) \mid x, y \in C' \text{ and } x > y\}.$$

Let C' and C'' be components contained in P_g and let $x > y$ in C'. If $k' : C \longrightarrow C'$ and $k'' : C \longrightarrow C''$ are dp-maps such that $k' \restriction Ext(C) = f_{C'} \circ g$ and $k'' \restriction Ext(C) = f_{C''} \circ g$, then $k = k'' \circ (k')^{-1} : C' \longrightarrow C''$ is a dp-map extending $f_{C''} \circ f_{C'}^{-1}$. But then $Ext(k(x)) = k(Ext(x)) = (f_{C''} \circ f_{C'}^{-1})(Ext(x)) = f_{C''}(Mn(x) \cup Mx(x))$, so that $k(x) \in M(x)$; similarly we find that $k(y) \in M(y)$. In addition, $k(x) \geq k(y)$ because k preserves order, so that $k(y) \in (M(x)] \cap M(y) = T(x, y)$ and, consequently, $C'' \subseteq K(T(x, y))$.

To prove the reverse inclusion, we need to show that for any component $C'' \subseteq P_{g'}$ with $P_{g'} \cap P_g = \emptyset$ there exists a pair $x > y$ in C' such that $T(x, y) \cap C'' = \emptyset$.

Recall that, for any $x \in C'$, either $M(x) \cap C'' = \emptyset$ or $M(x) \cap C'' = x''$ with $Ext(x'') = f_{C''}(Mn(x) \cup Mx(x)) = f_{C''}(f_{C'}^{-1}(Ext(x)))$. This fact allows us to define a <u>partial</u> mapping $\kappa : C' \longrightarrow C''$ by setting $\kappa(x) = M(x) \cap C''$ whenever the latter set is nonvoid. Should κ be a dp-map, then the dp-map $\kappa \circ f_{C'} : Ext(C) \longrightarrow Ext(C'')$ would coincide with $f_{C''}$. If $k' : C \longrightarrow C'$ is the dp-map extending $f_{C'} \circ g$, then $(\kappa \circ k') \restriction Ext(C) = (\kappa \circ f_{C'} \circ g) \restriction Ext(C) = (f_{C''} \circ g) \restriction Ext(C)$, that is, the composite $\kappa \circ k'$ extends $f_{C''} \circ g$ in a contradiction to the choice of C''. Hence κ cannot be a dp-map.

If the domain of κ does not include all of C', then $M(x) \cap C'' = \emptyset$ for some $x \in C'$ and, by definition, $T(x, y) \cap C'' = T(z, x) \cap C'' = \emptyset$ whenever $x > y$ or $z > x$ in C'. The existence of such y or z follows from the fact that C' is not a singleton.

Suppose that $\kappa : C' \longrightarrow C''$ is defined on all of C', so that $\{\kappa(x)\} = M(x) \cap C''$ and $Ext(\kappa(x)) = (f_{C''} \circ f_{C'}^{-1})(Ext(x))$ for all $x \in C'$. Since C' and C'' are isomorphic finite nuclei, the mapping κ is a bijection of C' onto C'' that maps $Ext(C')$ isomorphically onto $Ext(C'')$. Since κ is not a dp-map, there must exist a pair $y < x$ in C' such that $\kappa(y) \not\leq \kappa(x)$ in C''. But then $T(x, y) \cap C'' = (\kappa(x)] \cap \kappa(y) = \emptyset$ again.

Every set P_g is thus clopen in D, and hence also in the original dp-space X.

(e) Select $g = id_{Ext(C)} = id$. Then $X_C = P_{id}$ is a clopen union of components of X, and for every component $C' \subseteq X_C$ there exists a dp-map $k_{C'}$ of C onto C' that extends $f_{C'}$. For every $x \in C$ we thus have $M(x) \cap C' = \{k_{C'}(x)\}$. Since each $M(x)$ is clopen, a mapping $h_C : X_C \longrightarrow C$ defined by $h_C^{-1}\{x\} = M(x)$ for all $x \in C$

is continuous; it is a dp-map because its restriction to any component C' of X_C is the inverse of the dp-map $k_{C'}$.

Together with the initial remarks, this completes the proof. $\square$

In algebraic terms, Lemma 4.1 says that any uniform algebra from a finitely generated variety $\mathbf{V}$ contains an isomorphic copy of its nucleus F. If S is a rudimentary uniform algebra, then F is also a homomorphic image of S and, consequently, every rudimentary uniform rigid algebra must be a nucleus.

We say that the dp-space $X = P(A)$ of an algebra $A \in \mathbf{V}$ is $\mathbf{V}$-cyclic whenever the endomorphism monoid $End(A)$ is isomorphic to the cyclic group C_p of an odd prime order $p > | Ext(P(F)) |$ for any nucleus $F \in \mathbf{V}$. The space X is said to be $\mathbf{V}$-rigid whenever it is rigid and non-rudimentary.

Lemma 4.2. *Any dp-map $f : X \longrightarrow X$ of a $\mathbf{V}$-cyclic space maps $Ext(X)$ identically onto itself and, consequently, every $\mathbf{V}$-cyclic space is non-rudimentary.*

Proof. Let $f : X \longrightarrow X$ be any non-identity dp-map. Suppose that there exists a component C of X for which $f(C) \cap C = \emptyset$. Since f is invertible, $f(C)$ is another component of X and, by 2.2, there exists a clopen decreasing set B such that $C \subseteq B$ and $f(C) \subseteq X \setminus B$. Since X represents an algebra of a finite range n, the clopen decreasing set $W = B^{n(+*)}$ is also increasing, contains C, and is disjoint from $f(C)$. Therefore $A = \bigcap\{W \setminus f^i(W) \mid i \in \{1, 2, \dots p-1\}\}$ is again clopen, both decreasing and increasing, and contains C; furthermore, $f^i(A)$ intersects $f^j(A)$ only when $i = j \in \{0, 1, \dots, p-1\}$.

It is now routine to verify that the mapping $g : X \longrightarrow X$ defined as the identity on $X \setminus (A \cup f(A))$ extended by $g \restriction A = f \restriction A$ and $g \restriction (f(A)) = f^{-1} \restriction (f(A))$ is an invertible dp-map of order two, in contradiction to the hypothesis.

Therefore $f(C) = C$, and hence also $f(Ext(C)) = Ext(C)$ for every component C of X; since the prime order p of f exceeds $| Ext(C) |$, all orbits of f on $Ext(C)$ must be trivial. Finally, if X were rudimentary then, by 3.1, f would have to be the identity on X. $\square$

We say that S is a set of mutually rigid objects if the identity morphisms are the only morphisms between members of S.

Lemma 4.3. *Any rigid dp-space X with $D(X) \in \mathbf{V}$ which is not a finite disjoint union of mutually rigid nuclei must contain a $\mathbf{V}$-rigid uniform subspace Y representing a direct factor of $D(X)$.*

Proof. Recall that 4.1 implies that any rudimentary uniform rigid algebra from $\mathbf{V}$ is a nucleus.

Let X be a rigid dp-space with $D(X) \in \mathbf{V}$. By 3.5, the space X contains a uniform subspace H_0 representing a direct factor of $D(X)$. The rigidity of X implies that both H_0 and $X_1 = X \setminus H_0$ are rigid. If H_0 is not rudimentary, then $Y = H_0$ is $\mathbf{V}$-rigid, and the conclusion follows. If H_0 is rudimentary, and hence a nucleus, we apply 3.5 to the rigid dp-space $X_1 = X \setminus H_0$ to obtain a uniform subspace H_1

of X_1 representing a direct factor of $D(X_1)$. Again, if H_1 is not rudimentary, then $Y = H_1$ is $\mathbf{V}$-rigid, and we are done. Else H_1 is a nucleus and, because X is rigid, there are no *dp*-maps between the rigid nuclei H_0 and H_1. An inductive extension of this argument completes the proof because $\mathbf{V}$ can contain only finitely many mutually rigid nuclei. $\square$

Lemma 4.4. *Any* $\mathbf{V}$*-cyclic dp-space* X *contains a uniform subspace* Y *representing a direct factor of* $D(X)$ *that is either* $\mathbf{V}$*-cyclic or* $\mathbf{V}$*-rigid.*

Proof. As in 4.3, we note that every rigid rudimentary algebra in $\mathbf{V}$ is a nucleus.

If the $\mathbf{V}$-cyclic space X is not uniform then, according to 3.5, it must contain a uniform subspace H_0 representing a direct factor of $D(X)$. Since the respective endomorphism monoids satisfy $End(X) \cong End(H_0) \times End(X \setminus H_0)$, and because $End(X)$ is isomorphic to a prime order cyclic group, one of the subspaces H_0, $X \setminus H_0$ must be rigid and the other $\mathbf{V}$-cyclic. There is nothing to prove if H_0 is $\mathbf{V}$-cyclic or $\mathbf{V}$-rigid. In the remaining case, the space H_0 is rigid and rudimentary – and hence a nucleus – while $X_1 = X \setminus H_0$ must be $\mathbf{V}$-cyclic. Applying the above argument to X_1 instead of X and then extending it inductively, we find that this procedure terminates after finitely many steps because the variety $\mathbf{V}$ contains only a finite number of nuclei. But then the terminal step supplies a uniform subspace Y of X representing a uniform direct factor of $D(X)$ that is either $\mathbf{V}$-cyclic or $\mathbf{V}$-rigid. $\square$

Thus, in particular, any finitely generated universal variety contains a non-rudimentary uniform algebra A for which $End(A)$ is a finite group.

Lemma 4.5. *Let* X *be a uniform dp-space of an algebra from* $\mathbf{V}$ *such that all endomorphisms of* X *are invertible. Then either* X *is a chain with at most two elements or else* $Ext(e) \neq Ext(x)$ *for all* $e \in Ext(X)$ *and* $x \in X \setminus \{e\}$.

Proof. If X has a singleton component $\{e\}$, then the constant map $k : X \longrightarrow X$ with $k(X) = \{e\}$ is an idempotent endomorphism of X, and k is invertible only when $X = \{e\}$. Secondly, assume that X has no singleton components and that $\{c, d\}$ with $c < d$ is a component of $Ext(X)$. Since $Max(X)$ and $Min(X)$ are compact monotone disjoint sets, 2.2 supplies a clopen decreasing set C containing $Min(X)$ and disjoint from $Max(X)$. But then the mapping $f : X \longrightarrow X$ given by $f^{-1}\{c\} = C$ and $f^{-1}\{d\} = X \setminus C$ is an idempotent *dp*-map that is invertible only when $X = \{c, d\}$.

Suppose that all components of $Ext(X)$ have more than two elements, so that $Ext(d) \neq Ext(e)$ whenever $d, e \in Ext(X)$ are distinct. Let $h : X \longrightarrow S$ be the surjective *dp*-map dual to the inclusion homomorphism of the rudiment $D(S) = Rud(D(X))$ into $D(X)$. Then h is bijective on $Ext(X)$, so that we may replace $Ext(S) \subseteq S$ by $Ext(X)$. For any $e \in Ext(X)$, the set $K_e = \{x \in C \mid Ext(x) = Ext(e)\} = h^{-1}\{e\}$ is closed because h is continuous, and these sets are pairwise disjoint. Define $g : X \longrightarrow X$ by $g(x) = e$ for every $e \in Ext(X)$ and all $x \in K_e$, and by $g(x) = x$ for all other $x \in X$. To prove that g is a *dp*-map, it suffices to show that it is continuous. Observe that, for every $Z \subseteq X$,

$$g^{-1}(Z) = (Z \setminus (\bigcup\{K_e \mid e \in Ext(X)\})) \cup h^{-1}(Z \cap Ext(X)).$$

If Z is closed then $g^{-1}(Z)$ is closed whenever $\bigcup\{K_e \mid e \in Ext(X)\}$ is open because h is continuous and $Ext(S) = Ext(X)$ is closed. Let Y be the nucleus isomorphic to the rudiment of any component of X, and let $f : X \longrightarrow Y$ be the surjective dp-map from 4.1. Then $f^{-1}(Ext(Y)) = \bigcup\{K_e \mid e \in Ext(X)\}$ and, because Y is finite and f is continuous, it follows that the set $\bigcup\{K_e \mid e \in Ext(X)\}$ is, in fact, clopen.

Thus $g : X \longrightarrow X$ is a dp-map. Since all such maps are invertible, it follows that $\mid K_e \mid = 1$ for every $e \in Ext(X)$. But $e \in K_e$ and the claim follows. $\square$

In particular, by 3.1, for any **V**-rigid or **V**-cyclic space X, the dual $h : X \longrightarrow S$ of the inclusion of the rudiment $Rud(D(X))$ into the algebra $D(X)$ satisfies $h^{-1}(h\{e\}) = \{e\}$ for all $e \in Ext(X)$.

Lemma 4.6. *Let $Y = P(F)$, where F is the nucleus associated with the dual $D(X)$ of a uniform **V**-rigid or **V**-cyclic dp-space X. Then $Mid(Y) = Y \setminus Ext(Y)$ has an order component with more than two elements.*

Proof. Suppose, for contradiction, that all components of $Mid(Y)$ are either singletons or two-element chains.

Let $h : X \longrightarrow Y$ be the dp-map from 4.1, and let C be an arbitrarily selected component of X. Then the restriction $k = h \restriction C$ maps C onto Y. Since k is a dp-map, $k(u) \leq k(v)$ is equivalent to $u \leq v$ whenever u or v is extremal. By 4.5, k is the identity on $Ext(C) = Ext(Y)$ and $k(Mid(C)) = Mid(Y)$ and, by 3.1, the order of $Y = k(C)$ is the induced quotient order. Since any order component of $Mid(Y)$ has at most two elements, $k(u) \leq k(v)$ in $Mid(Y)$ if and only if $u' \leq v'$ in $Mid(C)$ for some u' and v' satisfying $k(u') = k(u)$ and $k(v') = k(v)$. Hence there exists an order preserving mapping $f : Y \longrightarrow C$ for which $k \circ f = id_Y$. Since Y is finite and f is the identity on $Ext(Y) = Ext(C)$, it follows that f is a dp-map. The composite $f \circ h : X \longrightarrow X$ is invertible only when $h = k$ is one-to-one. But then X is isomorphic to the rudimentary space Y, so that X can be neither **V**-rigid nor, by 4.2, **V**-cyclic. $\square$

The proof of (P1) is now complete. The claim below provides a final step towards (P2).

Lemma 4.7. *Assume the hypothesis of 4.6. Then the identity is the only endomorphism of F whose dual fixes $Mid(Y)$ elementwise.*

Proof. Set $Y = P(F)$ as in 4.6, and suppose that $f : Y \longrightarrow Y$ is a dp-map such that $f(y) = y$ for every $y \in Mid(Y)$.

Let $h : X \longrightarrow Y$ be the dp-map from 4.1, and let C_Q be a component of X. Then h maps $Ext(C_Q)$ bijectively onto $Ext(Y)$ and $h(Mid(C_Q)) = Mid(Y)$, so that there is a unique mapping $g_Q : C_Q \longrightarrow C_Q$ such that g_Q is the identity on $Mid(C_Q)$ and $(f \circ h) \restriction C_Q = (h \circ g_Q) \restriction C_Q$. By 3.4, the *dp*-map f is a permutation of $Ext(Y)$ and, consequently, the mapping g_Q is a permutation of C_Q whose all nontrivial orbits, if any, are contained in $Ext(C_Q)$. In addition, $Ext(g_Q(x)) = Ext(x)$ for any non-extremal $x \in C_Q$. Since the action of g_Q copies that of f on $Ext(C_Q))$, it follows that g_Q preserves the order and $g_Q(Ext(x)) = Ext(g_Q(x))$ for all $x \in C_Q$. The continuity of g_Q follows from the fact that the (finitely many) extremal points it permutes are open in C_Q. Therefore each $g_Q : C_Q \longrightarrow C_Q$ is an invertible *dp*-map.

The mapping $g : X \longrightarrow X$ defined as the joint extension of all g_Q thus preserves order and satisfies $g(Ext(x)) = Ext(g(x))$ and $f \circ h = h \circ g$.

Next we show that g is continuous. Since X is a totally disconnected compact space it suffices to show that $g^{-1}(Z)$ is clopen for every clopen $Z \subseteq X$. The mapping h is continuous and $Y = h(X)$ is finite, so that we may assume that $Z \subseteq h^{-1}(y)$ for some $y \in Y$. If $y \in Mid(Y)$ then $g^{-1}(Z) = Z$ because $Z \subseteq Mid(X)$ and g is the identity on $Mid(X)$. Secondly, for $y \in Min(Y)$ we have $Z \subseteq Min(X)$ and, because $D(X)$ is of a finite range n, the union $Z^{n(*+)}$ of all components that intersect Z is clopen. Clearly, $g^{-1}(Z) = Z^{n(*+)} \cap (f \circ h)^{-1}\{y\}$. From the continuity of $f \circ h$ and the fact that the singleton $\{y\}$ is clopen in Y it follows that $g^{-1}(Z) \subseteq X$ is clopen as well. Since an analogous argument applies when $y \in Max(Y)$, the continuity of g follows.

Therefore $g : X \longrightarrow X$ is a *dp*-map. If X is $\mathbf{V}$-rigid then $g = id_X$ follows immediately. For a $\mathbf{V}$-cyclic X we apply 4.2 to obtain the same conclusion. Thus $f \circ h = h$, and $f = id_Y$ follows because h is surjective. ☐

Corollary 4.8. *If* $\mathbf{V}$ *is a finitely generated universal variety of distributive double p-algebras, then* $\mathbf{V}$ *contains a nucleus whose dp-space* X *satisfies (P1) and (P2).* ☐

This completes the proof of the implications (4) $\Rightarrow$ (7) and (6) $\Rightarrow$ (7) in Theorem 1.1.

5. Smaller nuclei

To prove the implication (7) $\Rightarrow$ (8) of Theorem 1.1, we shall assume the existence of a nucleus $F \in \mathbf{V}$ whose *dp*-space $X = P(F)$ satisfies (P1) and (P2), and construct a nucleus $G \in \mathbf{V}$ whose *dp*-space $Y = P(G)$ satisfies the following two conditions:

- (Y1) $Mid(Y)$ has exactly one three-element order component C and at most three other components, all of them singletons, and
- (Y2) no *dp*-map $g : Y \longrightarrow Y$ other than the identity fixes all members of the three-element component C.

Recall that, for any $m \in Mid(X)$, the induced subposet $E(m) = Ext(X) \cup \{m\}$ of X is the dual of a subdirectly irreducible algebra from $\mathbf{V}$.

Let $N \subseteq Mid(X)$, and let $\leq$ be any partial order on N contained in the restriction of the extremal order $\leq_E$ of X described in 3.2(b) to the set N; in other words,

$n_0 \leq n_1$ implies, but it is not necessarily equivalent to, $Min(n_0) \subseteq Min(n_1)$ and $Max(n_0) \supseteq Max(n_1)$. On the disjoint union $Ext(X) \cup N$ we now define an extension $E(N, \leq)$ of $(N, \leq)$ by the requirement that, for every $n \in N$, the subposet $Ext(X) \cup \{n\}$ of $E(N, \leq)$ coincide with $E(n)$. It is clear that the inclusion of $E(n)$ into $E(N, \leq)$ is a dp-map, and that $E(N, \leq)$ is the union of all $E(n)$ with $n \in N$. Therefore, for any subposet $(N, \leq)$ of $(Mid(X), \leq_E)$, the poset $E(N, \leq)$ is the dp-space of a nucleus in the variety $\mathbf{V}$.

While it is clear that any variety $\mathbf{V}$ containing a nucleus F satisfying (P1) contains also a nucleus G for which (Y1) holds, in general, however, any such G may fail to satisfy (Y2). On the other hand, the subalgebra of F generated by any order connected triple of members of $Mid(F)$ satisfies (Y1) and (Y2), but need not be rudimentary. These difficulties will be resolved through careful selection of a generating set of G within a suitable quotient algebra of F. An adequate supply of suitable generators is ensured by the following claim.

Lemma 5.1. *If X is the dp-space of a finite nucleus such that the identity is the only dp-map $f : X \longrightarrow X$ which fixes $Mid(X)$, then*

(a) *$d, e \in Min(X)$ and $[d) \setminus \{d\} = [e) \setminus \{e\}$ imply $d = e$, and*
(b) *$d, e \in Max(X)$ and $(d] \setminus \{d\} = (e] \setminus \{e\}$ imply $d = e$.*

Proof. For distinct $d, e \in Min(X)$ with $[d) \setminus \{d\} = [e) \setminus \{e\}$, define a mapping $f : X \longrightarrow X$ by $f(d) = e$, $f(e) = d$ and $f(x) = x$ for all $x \in X \setminus \{d, e\}$. Then f is a nontrivial dp-map such that $f \restriction Mid(X)$ is the identity. This proves (a), and a similar argument leads to (b). □

We say that $x \in Mid(X)$ is <u>min-defective</u> whenever $Max(v) = Max(x)$ for all $v \in Min(x)$, and <u>max-defective</u> if $Min(u) = Min(x)$ for all $u \in Max(x)$. A consequence of the conclusion of 5.1 is that for every min-defective $x \in Mid(X)$ and any two distinct elements of $Min(x)$ there exists some $y \in Mid(X)$ such that $Min(y)$ contains exactly one of them.

Let A be a finite distributive double p-algebra. For any join irreducible $a \in A$, let $\overline{a}$ be the largest element of A with $\overline{a} \not\geq a$; then $\overline{a}$ is the join of all join irreducibles $j \not\geq a$. If $x \in P(A)$ represents the prime filter $[a) \subset A$, then $a \in A$ is represented by $(x] \subseteq P(A)$, and $\overline{a}$ corresponds to $P(A) \setminus [x)$. When there is no danger of confusion, we shall also write $\overline{x} = P(A) \setminus [x)$.

Lemma 5.2. *Let A be a finite distributive double p-algebra. Then the subalgebra B of A generated by the set $T(A) = Mid(A) \cup \{\overline{a} \mid a \in Mid(A)\}$ satisfies $Mid(B) \cong Mid(A)$.*

Moreover, if A is a nucleus with $\mid Mid(A) \mid \geq 2$, then the algebra B is rudimentary whenever $X = P(A)$ is such that

(1) *for every min-defective a which is minimal in $Mid(X)$ there is some $y \in Mid(X)$ which <u>splits</u> $Min(a)$ in the sense that both $Min(a) \cap Min(y)$ and $Min(a) \setminus Min(y)$ are nonvoid and, dually,*

(2) *for every max-defective b which is maximal in $Mid(X)$ there is some $z \in Mid(X)$ such that both $Max(b) \cap Max(z)$ and $Max(b) \setminus Max(z)$ are nonvoid.*

Proof. Let $h : X \longrightarrow Y$ be the surjective dp-map dual to the inclusion $B \subseteq A$. Then $h(x_0) \leq h(x_1)$ in Y if and only if $x_1 \in b$ implies $x_0 \in b$ for every $b \subseteq X$ representing a member of B. Equivalently, $h(x_0) \not\leq h(x_1)$ exactly when there is a $b \in B$ such that $x_1 \in b$ and $x_0 \notin b$.

Let $x_0 \in Mid(X)$. Then $(x_0]$ and $\overline{x}_0$ represent members of B. If $x_1 \not\leq x_0$ then $x_0 \in (x_0] \not\ni x_1$ and the above observation implies that $h(x_1) \not\leq h(x_0)$. Similarly, for any $x_1 \not\geq x_0$ it follows that $x_1 \in \overline{x}_0 \not\ni x_0$ and hence $h(x_0) \not\leq h(x_1)$. In particular, h is an order isomorphism of $Mid(X)$ onto an order subspace of Y.

Should $x_0 \in Mid(X)$ and $h(x_0) \in Min(Y)$, then $x \not\geq x_0$ and $h(x) = h(x_0)$ for every $x \in Min(x_0)$, a contradiction. Dually, $h(x_0) \notin Max(Y)$. Hence $h(Mid(X)) \subseteq Mid(Y)$ and, since $h : X \longrightarrow Y$ is a surjective dp-map, $h(Mid(X)) = Mid(Y)$. Therefore h gives an isomorphism of $Mid(X)$ onto $Mid(Y)$.

Furthermore, if $x_0 \in Mid(X)$ and $h(x) \in Min(h(x_0))$, then $x \in Min(X)$ and $x \leq x_0$. Thus $h^{-1}(Min(h(x_0)) = Min(x_0)$ and, dually, $h^{-1}(Max(h(x_0)) = Max(x_0)$, for every $x_0 \in Mid(X)$.

Let A be a nucleus. Suppose that $Ext(w_0) = Ext(w_1)$ in Y, and let $w_i = h(x_i)$ for $i = 0, 1$.

Let $w_0 \in Mid(Y)$, so that $x_0 \in Mid(X)$. For any $x \in Min(x_1)$ we have $h(x) \in Min(w_1) = Min(h(x_0))$, and hence $x \in Min(x_0)$. Together with a dual observation, this shows that $Ext(x_1) \subseteq Ext(x_0)$. If also $w_1 \in Mid(Y)$, then $Ext(x_0) = Ext(x_1)$, and $w_0 = w_1$ follows because X is rudimentary.

Next assume that $w_0 \in Mid(Y)$ and $w_1 \in Min(Y)$. Then $Ext(h(x_0)) = \{w_1\} \cup Max(w_1)$ and, since $x_0 \in Mid(X)$, we have $Max(x_0) = h^{-1}(Max(w_1))$ and $Min(x_0) = h^{-1}\{w_1\}$. If $z \leq x_0$ and $u \in Max(z)$, then $h(u) \in Max(w_1)$, and hence $u \in Max(x_0)$. But $Max(x_0) \subseteq Max(z)$ for all $z \leq x_0$, and $Max(x_0) = Max(z)$ follows. Should $z \in Mid(X)$, then $Min(z) = f^{-1}\{w_1\}$, so that $Ext(x_0) = Ext(z)$, and $x_0 = z$ because X is rudimentary. Therefore x_0 is minimal in $Mid(X)$ and, because $Max(z) = Max(x_0)$ for every $z \in Min(x_0)$, the element x_0 is also min-defective. If $Min(x_0) \cap Min(y) \neq \emptyset$ for some $y \in Mid(X)$, then for every $z \in Min(x_0)$ we have $h(z) = w_1 \leq h(y)$ and, consequently, $z \in Min(y)$. Thus no $y \in Mid(X)$ splits $Min(x_0)$, in contradiction to (1). A dual argument uses (2) to show that $Ext(w_0) \neq Ext(w_1)$ for any $w_0 \in Mid(Y)$ and $w_1 \in Max(Y)$.

Since Y is connected, for extremal w_0 and w_1 we need only consider the case when $Min(Y) = \{w_0\}$ and $Max(Y) = \{w_1\}$. But X contains at least two distinct $x, x' \in Mid(X)$; as shown earlier, $Ext(h(x)) \neq Ext(h(x'))$ in Y, so that this case cannot occur. $\square$

Let X be the dp-space of a nucleus F satisfying 1.1(7), and let $\leq_E$ be the extremal order on $Mid(X)$. First we construct a three-element component C of $\leq_E$ as follows.

If $\leq_E$ contains a three-element chain, we select a $\leq_E$-chain $C = C_0 = \{a, c, b\}$ in which a is an $\leq_E$-minimal and b is an $\leq_E$-maximal member of $Mid(X)$.

Let C_1 denote the poset in which $a < c$ and $b < c$, and where a is incomparable to b; let C_2 denote the dual of C_1. If $\leq_E$ has no three-element chains in $Mid(X)$, then $(Mid(X), \leq_E)$ must contain a copy of C_1 or of C_2; we select $C = C_i$ accordingly and note that a is incomparable to b in the extremal order $\leq_E$. Furthermore, in all three cases, the extremal elements of C_i are also extremal in the poset $(Mid(X), \leq_E)$.

In each of the three cases, we shall select a subset K of $Mid(X)$ ordered so that C is a component of K while $K \setminus C$ is an antichain, and show that (Y1) and (Y2) hold for the subalgebra $G = D(Y)$ generated by the set $T(E) = K \cup \{\overline{n} \mid n \in K\}$ in the algebra dual to $E = E(K, \leq)$.

We begin with an observation that will be needed in all three cases.

Assume that a minimal element $a \in C$ is min-defective. Then $Min(a)$ has at least two elements because X is rudimentary; by 5.1,

$$M_0(a) = \{y \in Mid(X) \mid Min(a) \setminus Min(y) \neq \emptyset \neq Min(a) \cap Min(y)\} \neq \emptyset.$$

For any $y \in M_0(a)$ we have $Max(y) \subseteq Max(a)$, so that for each $u \in Max(y)$ it follows that $Min(u) \setminus Min(y) \neq \emptyset$; thus y is not max-defective. Either $Max(a) \setminus Max(y) \neq \emptyset$ and hence y is not min-defective, or else $Max(y) = Max(a)$ and then, since a is minimal in the extremal order of $Mid(X)$, we must have $Min(y) \setminus Min(a) \neq \emptyset$, and hence also $a \in M_0(y)$. This shows that adding any $y \in M_0(a)$ to an arbitrary poset containing $C \cup Ext(X)$ produces a space in which both a and y satisfy 5.2(1) and 5.2(2).

Dually, in the case of a max-defective maximal $b \in C$ the set $M_1(b)$ of all $z \in Mid(X)$ which split $Max(b)$ in the sense that $Max(b) \setminus Max(z) \neq \emptyset \neq Max(b) \cap Max(z)$ is nonvoid, no member of $M_1(b)$ is min-defective, and $b \in M_1(z)$ for any max-defective $z \in M_1(b)$. Thus b and z satisfy (1) and (2) of 5.2 as well.

Case 0. The component $C_0 = \{a, c, b\}$ with $a < c < b$.

If $M_0(a)$ intersects $M_1(b)$, select $y \in M_0(a) \cap M_1(b)$ arbitrarily and note that y is neither max-defective nor min-defective. Set $K = C_0 \cup \{y\}$, and order K so that C_0 and the singleton $\{y\}$ are the components of K. As indicated earlier, we set $Y = P(G)$, where G is generated within $E(K, \leq)$ by $K \cup \{\overline{k} \mid k \in K\}$. Then $Mid(Y) = K$ and Y is rudimentary by 5.2. Therefore Y satisfies (Y1). Any dp-map g of the rudimentary space Y into itself is invertible, so that it maps the three-element chain C_0 identically onto itself, and hence $g(y) = y$; thus $g(k) = k$ and, by the invertibility of g also $g(\overline{k}) = \overline{k}$ for all $k \in K$. Altogether, g is the identity since it fixes every generator of $G = D(Y)$. Hence G is, in fact, rigid, and (Y2) follows.

If $M_0(a)$ and $M_1(b)$ are nonvoid and disjoint, select $y \in M_0(a)$ and $z \in M_1(b)$ arbitrarily. This time set $K = C_0 \cup \{y, z\}$, again with the trivial extension of the order of C_0. If y is min-defective then $a \in M_0(y)$, and dually for z, so that 5.2 applies again to yield a rudimentary Y with $Mid(Y) = K$ which satisfies (Y1). To see that Y is rigid, note that, as before, any dp-map $g : Y \longrightarrow Y$ is invertible, fixes elements of C_0 and hence preserves the antichain $\{y, z\}$. Since $y \notin M_1(b)$, either $Max(b) \subseteq Max(y)$ or $Max(b) \cap Max(y) = \emptyset$, while z splits $Max(b)$; thus $g(y) = y$ and $g(z) = z$ as required by (Y2).

If a is the only defective element in C_0 then we set $K = C_0 \cup \{y\}$ with an arbitrarily selected $y \in M_0(a)$, and make a dual selection when b is the only defective member of C_0. The arguments for these two cases coincide with those already used. Finally, when C_0 has no defective elements, we set $K = C_0$.

In every possible instance we thus obtain a rigid nucleus $G_0 = G$ such that $Mid(G_0)$ is the union of a three-element chain $C_0 = \{a, c, b\}$ with at most two other order components, both of which are singletons.

Case 1. The component $C_1 = \{a, b, c\}$ with $a < c$ and $b < c$.

Recall that all three elements of C_1 are extremal in the extremal ordering on $Mid(X)$.

Assume first the existence of some $v \in Min(a) \cap Min(b)$. If a is min-defective, then $Max(b) \subseteq Max(v) = Max(a)$. Since $a \not\leq_E b$, we must have $Min(a) \backslash Min(b) \neq \emptyset$, and this shows that b splits $Min(a)$. Hence 5.2(1) holds true for a and, by symmetry, also for b, whenever $Min(a) \cap Min(b) \neq \emptyset$. If c is max-defective, we select $z \in M_1(c)$ arbitrarily and define K as the disjoint union of the component C_1 and the singleton $\{z\}$; otherwise we set $K = C_1$. Lemma 5.2 applies to either case, and the rudimentary space Y satisfies $Mid(Y) \cong K$. Thus (Y1) holds. To demonstrate (Y2), we again recall that any dp-map $g : Y \longrightarrow Y$ is invertible; since g fixes C_1 elementwise, it must also fix the complementary singleton component $\{z\}$ whenever there is one included in $Mid(Y) \cong K$.

Secondly, for $Min(a) \cap Min(b) = \emptyset$, no member of C_1 can split the extremal elements of another one, and we proceed as in the case of the chain component. To obtain K, we extend C_1 by a least size antichain Z intersecting each set $M_0(a)$, $M_0(b)$ and $M_1(c)$ which is nonvoid. Then $| Z | \leq 3$ and, because of the minimality requirement, every $z \in Z$ is uniquely determined by its inclusion in, and its exclusion from each of these three sets. As in all previous cases, 5.2 applies and produces a nucleus whose dp-space Y satisfies $Mid(Y) \cong C_1 \cup Z$, and hence also (Y1). Any dp-map $g : Y \longrightarrow Y$ fixing C_1 elementwise permutes Z and, since each $z \in Z$ is uniquely determined by the set of members of C_1 whose extremals it splits, the permutation $g \restriction Z$ must be the identity on Z. Thus Y satisfies (Y2) as well.

The remaining case of the component C_2 submits to arguments dual to those used for C_1. Altogether, 1.1(8) follows from 1.1(7), and gives the following consequence.

Corollary 5.3. *Any finitely generated universal variety* $\mathbf{V}$ *contains one of the nuclei* G_i *with* $i \in \{0, 1, 2\}$ *such that* $Mid(G_i)$ *is the union of a component isomorphic to* C_i *and an antichain of at most three elements. Furthermore, the identity is the only endomorphism of* G_i *which is the identity on* C_i. $\square$

This also concludes the proof of 1.2.

6. The representation

In this section we prove the remaining implication (8) $\Rightarrow$ (1) of Theorem 1.1 by constructing, for each $i \in \{0,1,2\}$, a full embedding Φ_i of a universal category $\mathbf{D}_i$ of suitably augmented Priestley spaces into the dual of the variety $Var(G_i)$ generated by the nucleus G_i of 5.3.

Each category $\mathbf{D}_i$ is formed by Priestley spaces with two distinct open points u and v and by all continuous order preserving mappings g for which $\{g(u), g(v)\} = \{u, v\}$. In any object of $\mathbf{D}_0$ one of the elements u, v is maximal and the other is minimal, while both u and v are maximal for all spaces of $\mathbf{D}_1$ and both are minimal in all spaces of $\mathbf{D}_2$.

The following result of Koubek [5] will be used.

Theorem 6.1. [5]. *Let* $\mathbf{H}$ *be the universal (cf.* [12]*) category of all undirected graphs and all their compatible mappings. Then, for each* $i \in \{0,1,2\}$, *there is a full contravariant embedding* $\Psi_i : \mathbf{H} \longrightarrow \mathbf{D}_i$ *such that any* $\mathbf{D}_i$*-morphism* $g : \Psi_i(H) \longrightarrow \Psi_i(H')$ *satisfies* $g(u) = u$ *and* $g(v) = v$. $\square$

For each $i \in \{0,1,2\}$ and any graph H, a connected *dp*-space $\Phi_i(H)$ dual to an algebra with the rudiment G_i will be constructed so that the Priestley space $\Psi_i(H)$ from 6.1 replaces the element c of $P(G_i)$ as follows.

Set $Y = P(G_i)$ and let $\Psi_i(H) = (X, \tau, \leq, u, v)$, where u and v are the two distinguished elements of $\Psi_i(H) \in \mathbf{D}_i$. Define $\Phi_i(H) = \Phi(H) = (W, \sigma, \leq)$ so that

$W = (Y \setminus \{c\}) \cup X$, where the union is disjoint,
σ is the union of τ and the discrete topology on the finite set $Y \setminus \{c\}$,
the partial order $\leq$ on W coincides with the respective orders of $Y \setminus \{c\}$ or of X on these subsets, and satisfies
$([x) \cup (x]) \cap Y = Ext(c)$ for all $x \in X \setminus \{u, v\}$,
$([u) \cup (u]) \cap Y = Ext(c) \cup \{a\}$ and $([v) \cup (v]) \cap Y = Ext(c) \cup \{b\}$
in such a way that, in the latter two clauses, $u \geq a$ when u is maximal in X while $u \leq a$ when u is minimal and, similarly, $v \geq b$ for a maximal v while $v \leq b$ when v is minimal in X.

It is routine to verify that, in each of the three cases, this defines a partial order on W satisfying

$$([a) \cup (a]) \cap X = \{u\} \text{ and } ([b) \cup (b]) \cap X = \{v\}.$$

Lemma 6.2. *The ordered space* $(W, \sigma, \leq)$ *is the dp-space of an algebra* A *from the variety* $Var(G_i)$ *generated by the nucleus* $G = G_i$, *and* $G = Rud(A)$.

Proof. Since (X, τ) is compact and $Y \setminus \{c\}$ is finite, the space (W, σ) is compact.

For any $w \in W \setminus X$, the set $(w] \cap X$ is $\emptyset$ or X, or one of the open singletons $\{u\}$, $\{v\}$, and the same is true for the set $[w) \cap X$. Since $(X, \tau, \leq)$ is totally order disconnected, so is its extension $(W, \sigma, \leq)$ by finitely many open points $y \in Y \setminus \{c\}$

such that $([y) \cup (y]) \cap X$ is τ-open. Furthermore, if $A \subseteq W$ is increasing or decreasing then $(A] = (Max(A)]$ or $[A) = [Min(A))$, respectively; since $Ext(A) \subseteq W \setminus X$, either of the latter two sets intersects X in one of the σ-open sets $\emptyset$, X, $\{u\}$, $\{v\}$, or $\{u, v\}$. Thus $(W, \sigma, \leq)$ is a *dp*-space.

Since $Ext(x) = Ext(c)$ for all $x \in X$ and because $X \subseteq W$ is clopen and convex in (W, σ), the finite space Y is the rudimentary quotient of W. Depending on whether or not $w \in X$, the finite subspace $E(w) = Ext(W) \cup \{w\} = Ext(Y) \cup \{w\}$ is either isomorphic, or equal to, a subspace of Y. Therefore $\Phi(H)$ is the *dp*-space of an algebra in the variety generated by its nucleus G. □

For any morphism $f : H \longrightarrow H'$ of $\mathbf{H}$ we now define $\Phi(f) : \Phi(H) \longrightarrow \Phi(H')$ by

$\Phi(f)(x) = \Psi_i(f)(x)$ for all $x \in X$, and
$\Phi(f)(y) = y$ for all $y \in Y \setminus \{c\}$.

The mapping $\Phi(f)$ is continuous and order preserving since $\Psi_i(f)$ is a morphism in $\mathbf{D}_i$ and because $\Phi(f)$ is the identity on $W \setminus X$. The latter fact also implies that $\Phi(f)$ maps $Ext(w)$ onto $Ext(\Phi(f)(w))$ for every $w \in \Phi(H)$. Therefore Φ is a contravariant functor from the category $\mathbf{H}$ of all graphs and all their compatible mappings into the category of all *dp*-spaces of algebras from the variety $\mathbf{V}$.

Once we prove that the functor Φ is full, from 6.1 it will follow that the category $\mathbf{H}$ has a full covariant embedding into the variety $\mathbf{V}$.

To do this, assume $g : \Phi(H) \longrightarrow \Phi(H')$ to be a *dp*-map, and write $\Phi(H) = W = ((Y \setminus \{c\}) \cup X, \sigma, \leq)$ and $\Phi(H') = W' = ((Y \setminus \{c\}) \cup X', \sigma', \leq)$.

By 3.3, the homomorphism $D(g)$ maps the rudiment of $D(W')$ into the rudiment of $D(W)$; since the nucleus G is the rudiment of either algebra, the restriction of $D(g)$ to G is an automorphism of $D(G)$, by 3.4. Thus the unique three-element component $C_i = \{a, b, c\}$ of $Mid(G)$ is preserved by $D(g)$. Since c is, respectively, the unique non-extremal, maximal, or minimal member of $Mid(G_i)$ for $i = 0, 1, 2$, it follows that $D(g)(c) = c$ and $\{D(g)(a), D(g)(b)\} = \{a, b\}$ in all three cases.

When interpreted through the duality, this implies that $g(X) \subseteq X'$, that g permutes $Y \setminus \{c\} = W \setminus X = W' \setminus X'$, and that $\{g(a), g(b)\} = \{a, b\}$. Since u is the only member of X or X' comparable to a and v is the only element of X or X' comparable to b in each of the three cases, it follows that $\{g(u), g(v)\} = \{u, v\}$. Hence the restriction of g to X is a morphism in the appropriate category $\mathbf{D}_i$. By 6.1, the restriction of g to X is the image $\Psi_i(f)$ of some morphism $f : H' \longrightarrow H$ of $\mathbf{H}$, and $g(u) = u$ and $g(v) = v$. Since a and b are uniquely determined non-extremal elements of $Y \setminus \{c\}$ comparable to u and v, respectively, the restriction of $D(g)$ to $C_i = \{a, b, c\}$ is the identity. But then $D(g)$ is the identity on the rudiment G and hence g maps $W \setminus X$ identically onto $W' \setminus X'$. Altogether, $g = \Phi(f)$, and the functor Φ is full, as required.

The proof of Theorem 1.1. is now complete.

7. Concluding remarks

There exists a universal variety $\mathbf{V}$ of range one generated by two finite subdirectly irreducibles with the same monolith quotient. To see this, let $\mathbf{V} = Var(F)$, where F is the nucleus whose poset $X = \{p, q, r, s, a, b, c, u\}$ of join irreducibles is given by the following requirements:

$Max(X) = \{u\}$ and $Min(X) = \{p, q, r, s\}$,
$Min(a) = \{p, q\}$, $Min(b) = \{q, r\}$ and $Min(c) = \{p, q, r\}$,
$a \leq c$ and $b \leq c$.

It is easily seen that $E(a) = Ext(X) \cup \{a\}$ is isomorphic to $E(b)$ but not to $E(c)$. These posets are dp-spaces of subdirectly irreducible members of $\mathbf{V}$ whose common monolith quotient is represented by the dp-space $Ext(X)$. The variety $\mathbf{V}$ is of range one: since $Max(X)$ is a singleton, we have $f^+ = 1$ for any $f \in F \setminus \{1\}$. The universality of $\mathbf{V}$ then easily follows when 1.1(8) is applied.

This example also shows why the lower limit in 1.2(2) cannot be higher than two.

We are tempted to call for a characterization of all universal varieties of distributive double p-algebras, even though such project may be a little too ambitious at this time. A more realistic approach might attempt a syntactic characterization of finitely generated universal varieties suitable for a description of minimal ones, or aim to describe minimal finitely generated universal varieties of a small finite range.

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