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SIMULTANEOUS REPRESENTATIONS IN UNIFORM SPACES

by M. HUŠEK and V. TRNKOVÁ

In memory of Jan REITERMAN

Résumé. Pour chaque monoïde S et sous-monoïde S', il existe un espace uniforme X tel que le monoïde formé de toutes les applications uniformément continues non-constantes X → X soit isomorphe à S' et que le monoïde formé de toutes les applications uniformément continues non-constantes MX → MX soit isomorphe à S (où M est une modification localement fine, coz-fine, précompacte, κ-précompacte...). Tous ces résultats et d'autres analogues résultent de considérations assez générales sur les représentations de foncteurs fidèles vers les espaces uniformes, par exemple: pour chaque cardinal κ, il existe un espace topologique admettant un nombre κ d'espaces uniformes avec la même compactification, formant de plus un ensemble rigide pour les applications uniformément continues.

1. Introduction

Representations of small faithful functors by modification functors in topological spaces were investigated in [Trnková, Hušek, 1988], where historical remarks can be found. For convenience of readers, we will repeat the main scheme. Most of the concepts used in this paper are defined in [Adámek, Herrlich, Strecker, 1990], [Engelking, 1989], and [Isbell, 1964].

A functor F from a category $\mathcal{K}$ into a construct $\mathcal{C}$ (like **Top** or **Unif**) is said to be an *almost full embedding* (see e.g. [Pultr, Trnková, 1980]) if it is faithful and F maps the set $\mathcal{K}(K, L)$ onto the set of nonconstant morphisms in $\mathcal{C}(FK, FL)$, for every pair of objects K, L of $\mathcal{K}$.

Let $\mathcal{C}_1, \mathcal{C}_2$ be constructs and M be a faithful functor from $\mathcal{C}_1$ into $\mathcal{C}_2$. We say that a faithful functor $G : \mathcal{K}_1 \to \mathcal{K}_2$ has a *simultaneous representation by M* if there exist almost full embeddings $F_i : \mathcal{K}_i \to \mathcal{C}_i$ such that $F_2 \circ G = M \circ F_1$, i.e., the following square commutes:

It was proved in [Trnková, 1986] that every faithful functor between small categories has a simultaneous representation by the functor $\Gamma : \mathcal{C}_1 \to \mathcal{C}_2$, where $\mathcal{C}_1 = \mathcal{G}$ is the construct of all directed connected graphs without loops: the objects are pairs (V, R) of a set V and a relation $R \subset V \times V$ disjoint with the diagonal and such that the only transitive relation on V containing $R \cup R^{-1}$ is $V \times V$; the morphisms $(V, R) \to (V', R')$ are graph homomorphisms, i.e., the mappings $f : V \to V'$ such that $(f \times f)(R) \subset R'$.

The construct $\mathcal{C}_2 = \mathcal{H}$ has for objects triples (V, R, S), where (V, R) is an object of $\mathcal{G}$ and $S \subset R$; morphisms $(V, R, S) \to (V', R', S')$ are those morphisms $f : (V, R) \to (V', R')$ of $\mathcal{G}$ with $(f \times f)(S) \subset S'$.

The functor Γ is the forgetful one that forgets S, i.e., $\Gamma(V, R, S) = (V, R)$.

An endofunctor M of a construct $\mathcal{C}$ is said to be a *modification* if it is idempotent (i.e., $M \circ M = M$) and preserves underlying sets.

A modification M in a construct $\mathcal{C}$ is called *comprehensive* if every faithful functor between small categories has a simultaneous representation by m.

To show that a modification M is comprehensive it suffices to prove that the above functor Γ has a simultaneous representation by M. That was the procedure in [Trnková, Hušek, 1988] and it will be our procedure in the present paper, too.

Our main concern will be to find comprehensive modifications in the category **Unif** of uniform spaces and uniformly continuous maps. Mostly we shall be interested in upper or lower modifications M, i.e., when X is always finer (or coarser, resp.) than MX. In other words, upper modifications are bireflections, and lower modifications are bicoreflections (i.e., coreflections with the exception of that one corresponding to the coreflective subcategory consisting of the empty space only). Upper modifications in **Unif** form a "large" complete lattice with the identity functor as the smallest element and the indiscrete functor as the greatest element. Similarly for lower modifications (discrete functor, identity functor).

We restrict our consideration to modifications preserving topology. (The general case will be considered in a forthcoming paper.) It follows that we must consider upper modifications smaller than the precompact (or totally bounded) modification, and lower modifications that are greater than the (topologically) fine modification.

Our method requires that for upper modifications M (preserving topology) there exists a separated uniform space X not containing a metrizable continuum and such that $MX \neq X$. We shall call such modifications (in agreement with [Trnková, Hušek, 1988]) *essentially nonidentical*. Which upper modifications are essentially nonidentical? If we denote by $\mathcal{C}$ the class of uniform spaces which are either precompact or do not contain a continuum, and by M_u the modification corresponding to the bireflective hull in **Unif** of $\mathcal{C}$, then it is easy to see that essentially nonidentical modifications are precisely those upper modifications that are finer than the precompact modification and not finer than M_u.

As to lower modifications M, we shall need more. Take an infinite set Z and a free ultrafilter $\mathcal{F}$ on Z; the space $Z \times \{0,1\}$ with the base of covers of the form $\{\{(z,i)\} : z \in Z, i = 0,1\} \cup \{\{(z,0),(z,1)\} : z \in F\}$ for $F \in \mathcal{F}$, is an atom (i.e., a space that is not uniformly discrete but the only strictly finer uniformity is uniformly discrete). Let $\mathcal{A}$ be the class of all atoms of this form. It is well-known that the coreflective hull in **Unif** of $\mathcal{A}$ is the whole **Unif**. We say that a lower modification is *decent* if it is coarser than the fine modification and there exists $X \in \mathcal{A}$ such that the restriction of $M(K(X))$ to the base of $K(X)$ is uniformly discrete (where K(X) and its base is defined in the next section).

We shall prove the following results:

Theorem 1 *Every essentially nonidentical upper modification in* **Unif** *is comprehensive.*

Theorem 2 *Every decent lower modification of* **Unif** *is comprehensive.*

Proofs will be given in Section 3, several applications and examples in Section 4. We do not know whether Theorem 2 holds for essentially nonidentical lower modifications.

2. Constructions

We shall now describe several constructions in **Unif** needed later:

Glueing. Let P be a uniform space with a uniformly discrete subspace A, and let X be a uniform space with $|X| = |A|$. Suppose a bijection $\phi : A \to X$ is given. Then $P_A X$ will denote the quotient of the sum (coproduct) of P and of X sewing together A and X along the bijection ϕ (i.e., along the equivalence on the disjoint sum $P + X$ equal to the union of the identity of $P + X$ and of $\phi \cup \phi^{-1}$).

Cones. Let H be the metrizable hedgehog with κ spines, i.e., as a set, H is the quotient of the sum of κ copies of the unit interval, sewing together the top (all the points 1). The metric ρ on H gives the usual metric when restricted to the spines, and $\rho(x,y) = |1-x|+|1-y|$ when x, y belong to different spines. Let A be the subspace of H formed by all the points 0; then A is uniformly discrete. For a uniform space X of cardinality κ let $K(X)$ be the space $H_A X$ from the preceding paragraph. We may regard the points of $K(X)$ as the top point 1 and the points (x,t) for $t \in [0,1[$; to simplify the procedure, we shall also consider points $(x,1)$ that are all equal to the top point 1.

It is easy to see that the map assigning $(x,0)$ to $x \in X$ is an embedding of X into $K(X)$, denote it by h; the image $h(X)$ is called a base of $K(X)$.

Mapping from $P_A X$ into $K(X)$. We shall now describe a canonical mapping of $P_A X$ into $K(X)$. Let d be a uniformly continuous pseudometric on P such that $d(a,a') > 4$ for different points $a, a' \in A$. For each $a \in A$ and $x \in P$ let

$$p_a(x) = \frac{d(x, B_{a,1})}{d(x, B_{a,1}) + d(x, P - B_{A,2})},$$

where $B_{S,r}$ denotes the d-ball around the set or point S having the radius r. Thus p_a is a uniformly continuous function on P into $[0,1]$ having value 0 at the points of $B_{a,1}$ and the value 1 at the points of $P - B_{A,2}$. Define the mapping p' from P into $K(X)$ in the following way:

$$p'(x) = 1 \text{ if } x \notin B_{A,2}, \quad p'(x) = (\phi(a), p_a(x)) \text{ if } x \in B_{a,2}$$

(observe that for $x \in B_{A,2}$ there is a unique $a \in A$ with $x \in B_{a,2}$). Now, the requested mapping $p : P_A X \to K(X)$ is generated by the mappings $p' : P \to K(X)$ and $h : X \to K(X)$, giving a mapping from $P + X \to K(X)$ which factors via the quotient $P + X \to P_A X$, hence giving a uniformly continuous mapping from $P_A X$ into $K(X)$.

Spaces F(V,R,S). Let X, Y be uniform spaces with the same underlying set, where we choose three points p_1, p_2, p_3; let V be a set, and $S \subset R \subset V \times V$. We define

$$F(V, R, S)_{X,Y} \text{ is a quotient of } (X \times S) \cup (Y \times (R \setminus S))$$

obtained by identifying

$$(p_1, (u, v)) \text{ with } (p_1, (u, v')),$$
$$(p_2, (u, v)) \text{ with } (p_2, (u', v)),$$
$$(p_1, (u, v)) \text{ with } (p_2, (u', u)),$$
$$(p_3, (u, v)) \text{ with } (p_3, (u', v')),$$

for every $(u, v), (u, v'), (u', v), (u', u), (u', v')$ from R.

We can endow the sets $F(V, R, S)$ with various uniformities. If $S, R \setminus S$ bear the uniformly discrete uniformities and we take the above quotient of the sum of two products, the resulting space will be denoted by $F_u(V, R, S)_{X,Y}$. Instead of the product uniformities on $X \times S$ and on $Y \times (R \setminus S)$ we may take the sum (coproduct) uniformities, i.e.. we can regard the set as the coproduct of S-copies of the space X and of $R \setminus S$-copies of the space Y. Then we take the quotient uniformity again and denote the result as $F_s(V, R, S)_{X,Y}$. We have thus obtained two uniformities on $F(V, R, S)$ and any other between them can be used in our construction.

Since the equivalence used to obtain the spaces $F(V, R, S)$ is not too complicated, it is not difficult to describe the uniformities on e.g. $F_u(V, R, S)_{X,Y}$. Take uniform covers $\mathcal{U}, \mathcal{V}$ of X, Y resp., take their product with the finest cover of $S, R \setminus S$, resp. (covers by singletons), and the quotient image $\mathcal{W}$ of the resulting cover on $(S \times X) \cup ((R \setminus S) \times Y)$. If a point p is an image of more than one point by the quotient map, we must add $\mathrm{Star}_{\mathcal{W}}(p)$ to $\mathcal{W}$. The resulting cover is said to be determined by the covers $\mathcal{U}, \mathcal{V}$ and such covers form a base of $F_u(V, R, S)_{X,Y}$. If we want to describe the uniformity on spaces of the type F_s, the only difference is that we must start with choosing uniform covers $\mathcal{U}_r$ for every $r \in R$.

3. Proofs of Theorems

In [Trnková, Hušek, 1988], for every infinite cardinal κ a complete metric space P was constructed that has a uniformly discrete subset $A \cup \{p_1, p_2, p_3\}$ of cardinality κ and that is convenient for the construction needed. The main property is that if one uses the spaces $P_A X$ or their modifications $mP_A X$ as X, Y in the construction of $F(V, R, S)_{X,Y}$, then nonconstant continuous mappings $F(V, R, S)_{X,Y} \to F(V', R', S')_{X,Y}$ must map identically every copy of X or Y onto the corresponding copy in the other space. This assertion was proved in [Trnková, Hušek, 1988, Lemma IV.13] for the topologies on F_u, but the proof for F_s or for other topologies between those two is the same.

From now on, a space P means just the above space for some cardinal κ, where κ is determined by the cardinality of the chosen space X.

Proof of Theorem 1. Let M be an essentially nonidentical upper modification in **Unif**. We are going to prove that Γ has a simultaneous representation by M. By our assumption, there is a uniform space X containing no continuum and such that MX is strictly coarser than X. Then also $MP_A X$ is strictly coarser than $P_A X$: since X embeds canonically into $P_A X$ there is a one-to-one mapping of MX into $MP_A X$; consequently, $MP_A X$ must be strictly coarser than $P_A X$. Clearly, the topologies of $P_A X$ and of $MP_A X$ coincide.

Define the functors $G_1 : \mathcal{H} \to \mathbf{Unif}, G_2 : \mathcal{G} \to \mathbf{Unif}$ as follows:

$$G_1(V, R, S) = F_s(V, R, S)_{MP_A X, P_A X}$$

$$G_2 = M \circ G, \text{ where } G(V, R) = F_s(V, R, \emptyset)_{MP_A X, MP_A X} .$$

Images of morphisms are defined in a canonical way – see [Trnková, Hušek, 1988, p.758] for details. We shall show that G_1, G_2 are almost full embeddings.

Our construction is such that $T \circ G_2 : \mathcal{G} \to \mathbf{Top}$ (by T we denote the canonical functor from **Unif** to **Top**) is an almost full embedding, as proved in [Trnková, Hušek, 1988] in the second Observation on p. 758. So if $(V, R), (V', R')$ are objects of $\mathcal{G}$ and $g : TG_2(V, R) \to TG_2(V', R')$ is a continuous map, then either g is constant or there exists $f : (V, R) \to (V', R')$ in $\mathcal{G}$ such that $g = TG_2(f)$. However in the last case, $G_2(f)$ is uniformly continuous. This implies that $G_2 : \mathcal{G} \to \mathbf{Unif}$ is an almost full embedding.

Now, we prove that $M \circ G_1 = G_2 \circ \Gamma$, which means that $M(G(V, R)) = M(G_1(V, R, S))$ for every object (V, R, S) of $\mathcal{H}$. To prove the last equality it suffices to show that $F_s(V, R, \emptyset)_{MP_A X, MP_A X}$ is finer than $M(F_s(V, R, S)_{MP_A X, P_A X})$, which follows from the following more general reasoning. Denote by $/\sim$ an equivalence, and by $\geq$ the relation to be coarser than. For every uniform space Z we have

$$M((\sum_I Z)/\sim) = M((\sum_I MZ))/\sim) \geq (\sum_I MZ)/\sim .$$

Now, using our special quotient equivalence from the description of F_s-spaces for $/\sim$, and P_AX for Z, we get that $F_s(V,R,\emptyset)_{MP_AX,MP_AX}$ is finer than the space $M(F_s(V,R,S)_{MP_AX,P_AX})$.

It remains to prove that $\mathcal{G}_1$ is an almost full embedding.Let (V,R,S), (V',R',S') be objects in $\mathcal{H}$, let $h : G_1(V,R,S) \rightarrow G_1(V',R',S')$ be a nonconstant uniformly continuous mapping. Then $Mh : G_2(V,R) \rightarrow G_2(V',R')$ is a nonconstant uniformly continuous map, hence there exists $f : (V,R) \rightarrow (V',R')$ in $\mathcal{G}$ such that $G_2(f) = Mh$. Since h is uniformly continuous and the uniformity of MP_AX (replacing arrows in S) is strictly coarser than the uniformity of P_AX (replacing arrows in $R\setminus S$), f necessarily sends arrows in S into arrows in S', thus $f = \Gamma\bar{f}$ for $\bar{f} : (V,R,S) \rightarrow (V',R',S')$ in $\mathcal{H}$, i.e. $h = G_1(\bar{f})$, which means that $G_1 : \mathcal{H} \rightarrow \mathbf{Unif}$ is an almost full embedding. □

The last but one paragraph of the above proof is easy because we used sums (and quotients) in the definitions of G_i (since we used spaces of the type F_s). That has one disadvantage, namely if we start with nice spaces P, X, the representing spaces $G_i(V)$ may lose those nice properties that are not preserved by sums and quotients. We must use quotients anyway; fortunately, our quotients are not "too wild" and preserve many properties. Unlike in **Top**, the sums in **Unif** may kill some nice properties, e.g., metrizability. For that reason it may be more convenient to use the spaces of the type F_u; but we are not able to use the same procedure. One needs that for a uniformly discrete space D and a space Z, the space $(D \times MZ)/\sim$ is finer than the space $M((D \times Z)/\sim)$, which need not be true in general. But whenever that holds, the spaces of the type F_u may be used for the above construction.

PROOF of Theorem 2. The idea of this proof is the same as that of the proof of Theorem 1. Let X be an atom witnessing the fact that our modification M is decent; thus X is a discrete space (but not uniformly discrete) and the modification $MK(X)$ is strictly finer than $K(X)$, MX is uniformly discrete. We shall use the metric space P as before but now endowed with its fine uniformity. The definitions of G_i are modified as follows:

$$G_1(V,R,S) = F_u(V,R,S)_{P_AX,MP_AX}$$

$$G_2 = M \circ G, \quad \text{where } G(V,R) = F_u(V,R,\emptyset)_{MP_AX,MP_AX} .$$

The proof that G_1, G_2 are almost full embeddings is the same as in the proof of Theorem 1. It remains to show that $M \circ G_1(V,R,S) = M \circ G(V,R)$ for every object (V,R,S) of $\mathcal{H}$. Since $G(V,R)$ is finer than $G_1(V,R,S)$, it suffices to show that $MG_1(V,R,S)$ is finer than $G(V,R)$.

First, notice that there is no uniformity strictly between P_AX and the fine modification of $P_AX = P$; this follows directly from the fact that atoms are determined by ultrafilters. Consequently, MP_AX coincides with the fine modification of $P_AX = P$. Notice also that if $\approx$ is an equivalence on P_AX identifying only two points, then $M(P_A X/\approx) = (M P_A X)/\approx$. If we denote by D a uniformly discrete space and by $\sim$ an equivalence of our type, then we must show that

$$M((D \times P_A X)/\sim) \text{ is finer than } (D \times M P_A X)/\sim .$$

Take a uniform cover $\bar{\mathcal{U}}$ of $(D \times M P_A X)/\sim$ determined by a uniform cover $\mathcal{U}$ of $M P_A X$ as described at the end of Section 2. We show that $\bar{\mathcal{U}}$ is a uniform cover of $M((D \times P_A X)/\sim)$. Take the uniform cover $\bar{\mathcal{W}}$ on $(D \times P_A X)/\sim$ determined by the cover $\mathcal{W}$ on $P_A X$ that is obtained from $\mathcal{U}$ by glueing the canonical two-point-set cover of X. Let $\approx$ be the equivalence on $P_A X$ sewing together the points p_1, p_2. Then there is a canonical mapping $q : (D \times P_A X)/\sim \longrightarrow P_A X/\approx$ (roughly described by $q(d, [y]) = [y]$), and we get that $q : M((D \times P_A X)/\sim) \to M(P_A X/\approx)$ is uniformly continuous. Since the last range coincides with $(M P_A X)/\approx$, we can take a uniform cover $\bar{\mathcal{V}}$ on $M((D \times P_A X)/\sim)$ to be $q^{-1}(\mathcal{V}')$, where $\mathcal{V}'$ is the uniform cover of $(M P_A X)/\approx$ determined by the cover $\mathcal{U}$. Now it is easy to see that our original cover $\bar{\mathcal{U}}$ is refined by the joint refinement of $\bar{\mathcal{V}}$ and of $\bar{\mathcal{W}}$ which are uniform covers of $M((D \times P_A X)/\sim)$. □

It is convenient to use atoms for our space X because of the proof becomes easier, but we are loosing some nice properties of the resulting spaces. If we start with countable small categories, and can use a metric spaces X then there is a chance that the representation by G_1 will use metrizable spaces only. To use a metric space X instead of an atom (and the metric uniformity on P) means to show that $M((D \times P_A X)/\sim)$ is finer than $(D \times M P_A X)/\sim$. This works for some specific modification M, e.g. for the fine modification.

4. Special modifications

Upper modifications. We have proved that every upper modification that is finer than the precompact modification and not finer than the modification M_u is comprehensive. Practically all used upper modifications in **Unif** preserving topology have the above property, i.e., they are not finer than M_u. For instance, all κ-precompact modifications are of this kind, where κ is any infinite cardinal. These modifications have a base of uniform covers of cardinality less than κ and the corresponding bireflective subcategories do not contain all uniformly discrete spaces, so that we may choose X to be a uniformly discrete space. In fact, every upper modification which is not fixed on uniformly discrete spaces and preserves topology is comprehensive. An example different from the above cardinal reflections is the modification c generated by the reals (i.e., the bireflective hull of the reals in **Unif**).

Among bireflective classes containing all precompact and all uniformly discrete spaces we have for instance point-κ-spaces or star-κ-spaces, where again κ is an infinite cardinal. The first class is composed of all uniform spaces having a base of uniform covers with the property that every point is contained in less than κ

members of the cover. The latter class consists of uniform spaces having a base of uniform covers with the property that every member of the cover meets less than κ members of the cover. The classes of the first type do not contain all the spaces $U_u(D, Q)$ of uniformly continuous mappings from uniformly discrete spaces D into rationals Q, endowed with the sup-norm metric. The classes of the second type do not contain all the hedgehogs made of rationals in [0,1].

A class C of uniform spaces is said to be rigid if the only nonconstant uniformly continuous maps between members of C are the identity-selfmaps.

Summary: (a) *For any infinite cardinal κ, the following upper modifications are comprehensive:*

κ-precompact, star-κ-modification, point-κ-modification, the modification c.

(b) *Every bireflective class in* Unif *that contains all precompact spaces and does not contain all uniformly discrete spaces is comprehensive.*

The second assertion of Abstract follows from the following statement (a) (recall that proximities and compactifications on a given topological space are in a one-to-one correspondence).

Applications: (a) *For any cardinal κ and any monoid S there exists a proximity space X for which the set of nonconstant proximally continuous selfmaps is isomorphic to S, and a rigid set of uniform spaces of cardinality κ inducing the proximity space X.*

(b) *For every pair of monoids $S_1 \subset S_2$ there exists a proximity space X for which the set of nonconstant proximally continuous selfmaps is isomorphic to S_2 and which is induced by a uniform space Y for which the set of nonconstant uniformly continuous selfmaps is isomorphic to S_1.*

Lower modifications. To find out whether a lower modification M preserving topology is decent, one must find an atom X such that the restriction of $M(K(X))$ to the base of $K(X)$ is uniformly discrete. It is easy to see that it suffices to find a uniform space X such that the restriction of $M(K(X))$ to the base of $K(X)$ is strictly finer than X.

Observe that if a lower modification M is decent then so are all finer lower modifications preserving topology.

The proximally fine modification M is comprehensive. If one takes for X an infinite discrete topological space endowed with the Čech uniformity (the finest precompact uniformity), then MX is the uniformly discrete space and $M(K(X))$ is also strictly finer than $K(X)$. Consequently, the coz-fine and the fine modifications are comprehensive (the coz-fine modification is positioned between the proximally fine and the fine modifications).

The modification corresponding to the coreflective class of uniform spaces Z such that the set $U(Z)$ of uniformly continuous real-valued functions is closed under taking $1/f$ for $f \neq 0$, is comprehensive. Indeed, take for X the rationals in the open interval]0,1[; it is not inversion closed and neither is $K(X)$.

Using Katětov extension theorem for uniform spaces other lower modifications can be described by bounded uniformly continuous real-valued mappings (as the inversion-closed modification). If the property is described by unbounded mappings, we need not get a decent modification. This is the case for the coreflective subcategory of **Unif** defined by the property that $U(Z)$ is algebra, i.e., that products of uniformly continuous real-valued mappings are uniformly continuous; the reason is that every uniformly continuous real-valued mapping on a cone is bounded and, hence, every cone belongs to the coreflective subcategory.

Since the following modifications are positioned between the above inversion-closed coreflection and the topologically fine coreflection, they are comprehensive: locally fine modification; the coreflection corresponding to the coreflective subcategory consisting of spaces having the property that inverse of every uniformly continuous function bounded from zero is uniformly continuous (call it bounded-inversion-closed modification).

Summary: *The following lower modifications of* **Unif** *are comprehensive:*

fine, locally fine, bounded-inversion-closed, inversion-closed, proximally fine, coz-fine.

Applications: (a) *For every pair of monoids $S_1 \subset S_2$ there exists a uniform space X for which the set of nonconstant uniformly continuous selfmaps is isomorphic to S_1 and the set of nonconstant coz-continuous selfmaps is isomorphic to S_2.*

Using the remark following the proof of Theorem 2 we can show:
(b) *For every pair of countable monoids $S_1 \subset S_2$ there exists a topological space X for which the set of nonconstant continuous selfmaps is isomorphic to S_2 and which is induced by a metrizable uniform space Y for which the set of nonconstant uniformly continuous selfmaps is isomorphic to S_1.*

We could formulate the same assertion for lower modifications as Application (a) in the previous part for upper modifications. Or, instead of taking a "discrete" small category we may take a poset and get a representation of the poset in the set of uniformities on some topological space (the order in the set of uniformities is "finer").

In the above assertions (a), (b) one may take more monoids than just two.

Other modifications. If M is a modification preserving topology that is neither lower or upper modification but can be written as $M = M_1 \circ M_2$ where M_2 is an upper or a lower modification satisfying the assumption of Theorem 1 or of Theorem 2, resp. then M is comprehensive. Indeed, our construction yields functors G_1, G into **Unif** such that $M_2 \circ G = M_2 \circ G_1$. Consequently, $M \circ G = M \circ G_1$ and we may define $G_2 = M \circ G$ to get what is needed.

An example is $M = M_1 \circ M_2$ where, say, M_1 is a lower modification and M_2 is an upper modification which commute. If M_2 is essentially nonidentical then M is comprehensive.

It remains to look at modifications that do not preserve topology. The situation is more complicated. The same example as in **Top** shows that the zero-dimensional modification is not comprehensive, and similarly for lower modifications corresponding to coreflective subcategories containing with any its connected member every coarser space.

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