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C. CASTELLINI
J. KOSLOWSKI
G. E. STRECKER

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AN APPROACH TO A DUAL OF REGULAR CLOSURE OPERATORS

by C. CASTELLINI*, J. KOSLOWSKI & G.E. STRECKER

Résumé. Nous introduisons des techniques nouvelles pour obtenir, dans un contexte catégorique, un analogue complètement symétrique de la fermeture régulière de Salbany. Pour cette étude nous utilisons des correspondances de Galois et des structures de factorisation criblée. Nous obtenons des factorisations pour les correspondances de Galois les plus importantes qui représentent des constructions de fermeture, et nous donnons des formules détaillées pour construire une fermeture idempotente canonique et une fermeture faiblement héréditaire à partir de classes arbitraires d'objets et de classes spéciales de morphismes.

Introduction

There are two standard (and in some sense dual) ways to construct the closure of a subset A of a topological space X. The first way is to form the intersection (or infimum in the lattice of subobjects of X) of the family of all closed sets that contain A. The second way is to add limit points to A, i.e., to form the union (or supremum in the lattice of subobjects of X) of the family of all sets that contain A as a dense subset. In general a given closure operator on a category $\mathcal{X}$ cannot be reconstructed using either of the above approaches, since these two processes usually yield different results. In the case of a weakly hereditary and idempotent closure operator, however, (as is the case with the usual closure in topology) both processes do recover the original closure operator. In fact, weakly hereditary and idempotent closure operators may be characterized by this fact; i.e., they can be reconstructed from the corresponding class of "closed" subobjects as well as from the corresponding class of "dense" subobjects.

Both constructions described above generalize to arbitrary classes of subobjects. For classes of subobjects induced by a class of objects in a specific way, closure operators that result from the first (or infimum) procedure are called "regular" or closure operators of "Salbany" type. In this paper we continue to investigate them and also concentrate our attention on closure operators that arise from object-induced classes of subobjects via the second (or supremum) approach. It turns out that in an $(\mathbf{E}, \mathcal{M})$-category for sinks the crucial new notion needed for this analogue is that of $\mathbf{E}$-sink stability.

In Section 1 we present preliminary definitions and results that are necessary for the remainder of the paper.

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Section 2 contains our main results. A canonical idempotent closure operator is obtained from any pullback-stable family of $\mathcal{M}$-subobjects. Also this construction is shown to give rise to a natural factorization of the global Galois connection the coadjoint part of which associates to each idempotent closure operator its class of closed $\mathcal{M}$-subobjects. Similarly, a canonical weakly hereditary closure operator is obtained from any $\mathbf{E}$-sink stable family of $\mathcal{M}$-subobjects. This construction is shown to be symmetric to the first construction in that it gives rise to a natural factorization of the global Galois connection the adjoint part of which associates to each weakly hereditary closure operator its class of dense $\mathcal{M}$-subobjects. Finally, in a slightly more special setting a new Galois connection is introduced that relies on squares of objects and diagonal morphisms. It turns out that this connection is the key link between objects in our category and the canonical weakly hereditary closure construction discussed above. Surprisingly, its dual turns out to be its own symmetric analogue, which fits as the link between objects in our category and the Salbany regular closure construction.

Section 3 contains applications of the theory.

Notice that some of the results presented in this paper can be obtained from the general theory developed in [6]. However, to keep this paper reasonably self-contained we have included the corresponding proofs.

We use the terminology of [1] throughout the paper [1].

1 Preliminaries

We begin by recalling the following

1.1 Definition A category $\mathcal{X}$ is called an $(\mathbf{E},\mathcal{M})$-*category for sinks*, if there exists a collection $\mathbf{E}$ of $\mathcal{X}$-sinks, and a class $\mathcal{M}$ of $\mathcal{X}$-morphisms such that:

(0) each of $\mathbf{E}$ and $\mathcal{M}$ is closed under compositions with isomorphisms;

(1) $\mathcal{X}$ has $(\mathbf{E},\mathcal{M})$-factorizations (of sinks); i.e., each sink s in $\mathcal{X}$ has a factorization $\mathrm{s} = m \circ \mathrm{e}$ with $\mathrm{e} \in \mathbf{E}$ and $m \in \mathcal{M}$, and

(2) $\mathcal{X}$ has the unique $(\mathbf{E},\mathcal{M})$-diagonalization property; i.e., if $B \xrightarrow{g} D$ and $C \xrightarrow{m} D$ are $\mathcal{X}$-morphisms with $m \in \mathcal{M}$, and $\mathrm{e} = (A_i \xrightarrow{e_i} B)_I$ and $\mathrm{s} = (A_i \xrightarrow{s_i} C)_I$ are sinks in $\mathcal{X}$ with $\mathrm{e} \in \mathbf{E}$, such that $m \circ \mathrm{s} = g \circ \mathrm{e}$, then there exists a unique diagonal $B \xrightarrow{d} C$ such that for every $i \in I$ the following diagrams commute:

$$\begin{tikzcd} A_i \arrow[r, "e_i"] \arrow[d, "s_i"'] & B \arrow[dl, "d"] \\ C & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & B \arrow[dl, "d"'] \arrow[d, "g"] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

[1] Paul Taylor's commutative diagrams macro package was used to typeset the diagrams.

That $\mathcal{X}$ is an $(\mathbf{E},\mathcal{M})$-category implies the following features of $\mathcal{M}$ and $\mathbf{E}$ (cf. [1] for the dual case):

1.2 Proposition (0) *Every isomorphism is in both $\mathcal{M}$ and $\mathbf{E}$ (as a singleton sink).*

(1) *Every m in $\mathcal{M}$ is a monomorphism.*

(2) *$\mathcal{M}$ is closed under $\mathcal{M}$-relative first factors, i.e., if $n \circ m \in \mathcal{M}$, and $n \in \mathcal{M}$, then $m \in \mathcal{M}$.*

(3) *$\mathcal{M}$ is closed under composition.*

(4) *Pullbacks of $\mathcal{X}$-morphisms in $\mathcal{M}$ exist and belong to $\mathcal{M}$.*

(5) *The $\mathcal{M}$-subobjects of every $\mathcal{X}$-object form a (possibly large) complete lattice; suprema are formed via $(\mathbf{E},\mathcal{M})$-factorizations and infima are formed via intersections.* □

Throughout the paper we assume that $\mathcal{X}$ has equalizers and is an $(\mathbf{E},\mathcal{M})$-category for sinks and that $\mathcal{M}$ contains all regular monomorphisms.

1.3 Definition A *closure operator* C on $\mathcal{X}$ (with respect to $\mathcal{M}$) is a family $\{(\ )_X^C\}_{X\in\mathcal{X}}$ of functions on the $\mathcal{M}$-subobject lattices of $\mathcal{X}$ with the following properties that hold for each $X \in \mathcal{X}$:

(a) [*growth*] $m \leq (m)_X^C$, for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$;

(b) [*order-preservation*] $m \leq n \Rightarrow (m)_X^C \leq (n)_X^C$ for every pair of $\mathcal{M}$-subobjects of X;

(c) [*morphism-consistency*] If p is the pullback of the $\mathcal{M}$-subobject $M \xrightarrow{m} Y$ along some $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ and q is the pullback of $(m)_Y^C$ along f, then $(p)_X^C \leq q$, i.e., the closure of the inverse image of m is less than or equal to the inverse image of the closure of m.

The growth condition (a) implies that for every closure operator C on $\mathcal{X}$, every $\mathcal{M}$-subobject $M \xrightarrow{m} X$ has a canonical factorization

$$\begin{tikzcd} M \arrow[r, "t"] \arrow[dr, "m"'] & (M)_X^C \arrow[d, "(m)_X^C"] \\ & X \end{tikzcd}$$

where $((M)_X^C, (m)_X^C)$ is called the *C-closure* of the subobject (M, m).

When no confusion is likely we will write m^C rather than $(m)_X^C$ and for notational symmetry we will denote the morphism t by m_C.

If $M \xrightarrow{m} X$ is an $\mathcal{M}$-subobject, $X \xrightarrow{f} Y$ is a morphism and $M \xrightarrow{e} f(M) \xrightarrow{f(m)} Y$ is the $(\mathbf{E},\mathcal{M})$-factorization of $f \circ m$ then $f(m)$ is called the *direct image* of m along f.

1.4 Remark (1) Notice that in the presence of (b) (order-preservation) the morphism-consistency condition (c) of the above definition is equivalent to the following statement concerning direct images: if $M \xrightarrow{m} X$ is an $\mathcal{M}$-subobject and $X \xrightarrow{f} Y$ is a morphism, then $f((m)^C_X) \leq (f(m))^C_Y$, i.e., the direct image of the closure of m is less than or equal to the closure of the direct image of m; (cf. [9]).

(2) Provided that (a) holds (growth), both order-preservation and morphism-consistency, i.e., conditions (b) and (c) together are equivalent to the following: given (M, m) and (N, n) $\mathcal{M}$-subobjects of X and Y, respectively, if f and g are morphisms such that $n \circ g = f \circ m$, then there exists a unique morphism d such that the following diagram commutes

$$\begin{tikzcd}
M \arrow[rr, "g"] \arrow[dd, "m"'] \arrow[dr, "m_C"] & & N \arrow[dd, "n"] \arrow[dr, "n_C"] & \\
& M^C \arrow[rr, "d"] \arrow[dl, "m^C"] & & N^C \arrow[dl, "n^C"] \\
X \arrow[rr, "f"'] & & Y &
\end{tikzcd}$$

(3) If we regard $\mathcal{M}$ as a full subcategory of the arrow category of $\mathcal{X}$, with the codomain functor from $\mathcal{M}$ to $\mathcal{X}$ denoted by U, then the above definition can also be stated in the following way: A *closure operator* on $\mathcal{X}$ (with respect to $\mathcal{M}$) is a pair $C = (\gamma, F)$, where F is an endofunctor on $\mathcal{M}$ that satisfies $UF = U$, and γ is a natural transformation from $id_{\mathcal{M}}$ to F that satisfies $(id_U)\gamma = id_U$ (cf. [9]).

1.5 Definition Given a closure operator C, we say that $m \in \mathcal{M}$ is *C-closed* if m_C is an isomorphism. An $\mathcal{X}$-morphism f is called *C-dense* if for every $(\mathbf{E}, \mathcal{M})$-factorization (e, m) of f we have that m^C is an isomorphism. We call C *idempotent* provided that m^C is C-closed for every $m \in \mathcal{M}$. C is called *weakly hereditary* if m_C is C-dense for every $m \in \mathcal{M}$.

Notice that morphism-consistency 1.3(c) implies that pullbacks of C-closed $\mathcal{M}$-subobjects are C-closed.

We denote the collection of all closure operators on $\mathcal{M}$ by $CL(\mathcal{X}, \mathcal{M})$ pre-ordered as follows: $C \sqsubseteq D$ if $m^C \leq m^D$ for each $m \in \mathcal{M}$ (where $\leq$ is the usual order on subobjects). Notice that arbitrary suprema and infima exist in $CL(\mathcal{X}, \mathcal{M})$, they are formed pointwise on the $\mathcal{M}$-subobject fibers. $iCL(\mathcal{X}, \mathcal{M})$ and $wCL(\mathcal{X}, \mathcal{M})$ will denote the collection of all idempotent and all weakly hereditary closure operators, respectively.

For more background on closure operators see, e.g., [4], [6], [9], [10], [14].

We now recall a few basic facts concerning Galois connections. For a pre-ordered class $\mathcal{X} = (X, \sqsubseteq)$ we denote the dually ordered class $(X, \sqsupseteq)$ by $\mathcal{X}^{\mathrm{op}}$.

1.6 Definition For pre-ordered classes $\mathcal{X} = (X, \sqsubseteq)$ and $\mathcal{Y} = (Y, \preceq)$, a *Galois connection* $F = \langle F_*, F^* \rangle$ from $\mathcal{X}$ to $\mathcal{Y}$ (denoted by $\mathcal{X} \xrightarrow{F}\!\!\bullet\; \mathcal{Y}$) consists of a pair of order preserving functions ($\mathcal{X} \xrightarrow{F^*} \mathcal{Y}, \mathcal{Y} \xrightarrow{F_*} \mathcal{X}$) that satisfy $F_* \dashv F^*$, i.e., $x \sqsubseteq F^*F_*(x)$ for every $x \in X$ and $F_*F^*(y) \preceq y$ for every $y \in Y$. (F^* is the *adjoint part* and F_* is the *coadjoint part*).

Notice that for every such Galois connection F from $\mathcal{X}$ to $\mathcal{Y}$ up to equivalence F_* and F^* uniquely determine each other, and we also have a *dual* Galois connection $F^{\mathrm{op}} = \langle F^*, F_* \rangle$ from $\mathcal{Y}^{\mathrm{op}}$ to $\mathcal{X}^{\mathrm{op}}$. Moreover, given a Galois connection $G = \langle G_*, G^* \rangle$ from $\mathcal{Y}$ to $\mathcal{Z}$, one can form the *composite* Galois connection $G \circ F = \langle G_* \circ F_*, F^* \circ G^* \rangle$ from $\mathcal{X}$ to $\mathcal{Z}$.

1.7 Definition For a class A we let $P(A)$ denote the collection of all subclasses of A, partially ordered by inclusion. Any relation $\mathcal{R}$ between classes A and B, i.e., $\mathcal{R} \subseteq A \times B$ induces a Galois connection $P(A) \xrightarrow{\phi}\!\!\bullet\; P(B)^{\mathrm{op}}$, called a *polarity*, whose adjoint and coadjoint parts are given by

$$\phi_*(U) = \{\, b \in B : \forall\, a \in U, (a,b) \in \mathcal{R} \,\} \quad \text{for} \quad U \subseteq A$$
$$\phi^*(V) = \{\, a \in A : \forall\, b \in V, (a,b) \in \mathcal{R} \,\} \quad \text{for} \quad V \subseteq B$$

Various properties and many examples of Galois connections can be found in [11].

2 General results

We recall from [6] the following commutative diagram of Galois connections

$$\begin{tikzcd}
wCL(\mathcal{X},\mathcal{M}) \arrow[r, "\ddot{\Delta}"] & CL(\mathcal{X},\mathcal{M}) \arrow[r, "\dot{\nabla}"] & iCL(\mathcal{X},\mathcal{M}) \arrow[d, "\ddot{\nabla}"] \\
P(\mathcal{M}) \arrow[u, "\dot{\Delta}"] \arrow[rr, "\nu"'] & & P(\mathcal{M})^{\mathrm{op}}
\end{tikzcd}$$

The above Galois connections are as follows.

(1) $\dot{\Delta}^*$ associates to each weakly hereditary closure operator its class of C-dense $\mathcal{M}$-subobjects and $\dot{\Delta}_*$ is its corresponding coadjoint. Notice that $\dot{\Delta}_* \circ \dot{\Delta}^* \simeq id_{wCL(\mathcal{X},\mathcal{M})}$ (cf. [6], Proposition 2.12(0)).

(2) $\ddot{\Delta}^*$ associates to each closure operator its *weakly hereditary core* (i.e., the supremum of all smaller weakly hereditary closure operators) and $\ddot{\Delta}_*$ is the inclusion.

(3) $\dot{\nabla}_*$ associates to each closure operator its *idempotent hull* (i.e., the infimum of all larger idempotent closure operators) and $\dot{\nabla}^*$ is the inclusion.

(4) $\ddot{\nabla}_*$ associates to each idempotent closure operator its class of C-closed $\mathcal{M}$-subobjects and $\ddot{\nabla}^*$ is its corresponding adjoint. Notice that $\ddot{\nabla}^* \circ \ddot{\nabla}_* \simeq id_{iCL(\mathcal{X},\mathcal{M})}$ (cf. [6], Proposition 2.12(0)).

(5) ν is the polarity induced by the relation $\perp \subseteq \mathcal{M} \times \mathcal{M}$ defined by: $m \perp n$ iff for every pair of morphisms f, g such that $f \circ m = n \circ g$, there exists a unique morphism d such that both triangles of the following diagram commute

$$\begin{tikzcd} \bullet \arrow[r, "g"] \arrow[d, "m"'] & \bullet \arrow[d, "n"] \\ \bullet \arrow[r, "f"'] \arrow[ur, "d"] & \bullet \end{tikzcd}$$

A direct consequence of the general theory of Galois connections (cf. [11]) and the above statements is the following proposition.

2.1 Proposition (1) *Let ∇ be the composite $CL(\mathcal{X},\mathcal{M}) \xrightarrow{\dot\nabla} iCL(\mathcal{X},\mathcal{M}) \xrightarrow{\ddot\nabla} P(\mathcal{M})^{\mathrm{op}}$. Then, the components $CL(\mathcal{X},\mathcal{M}) \xrightarrow{\nabla_*} P(\mathcal{M})^{\mathrm{op}}$ and $P(\mathcal{M})^{\mathrm{op}} \xrightarrow{\nabla^*} CL(\mathcal{X},\mathcal{M})$ are given by:*

$$\nabla_*(C) = \{\, m \in \mathcal{M} : m \text{ is } C\text{-closed} \,\}$$
$$\nabla^*(\mathcal{N}) = \sup\{\, C \in CL(\mathcal{X},\mathcal{M}) : \nabla_*(C) \supseteq \mathcal{N} \,\}$$

(2) *Let Δ be the composite $P(\mathcal{M}) \xrightarrow{\dot\Delta} wCL(\mathcal{X},\mathcal{M}) \xrightarrow{\ddot\Delta} CL(\mathcal{X},\mathcal{M})$. Then, the components $P(\mathcal{M}) \xrightarrow{\Delta_*} CL(\mathcal{X},\mathcal{M})$ and $CL(\mathcal{X},\mathcal{M}) \xrightarrow{\Delta^*} P(\mathcal{M})$ are given by:*

$$\Delta^*(C) = \{\, m \in \mathcal{M} : m \text{ is } C\text{-dense} \,\}$$
$$\Delta_*(\mathcal{N}) = \inf\{\, C \in CL(\mathcal{X},\mathcal{M}) : \Delta^*(C) \supseteq \mathcal{N} \,\}$$

□

A few remarks concerning the symmetry of the diagram above are in order. Let $\mathcal{M} \diamond \mathcal{M}$ be the collection of composable pairs of morphisms in $\mathcal{M}$, and let W be the composition functor from $\mathcal{M} \diamond \mathcal{M}$ to $\mathcal{M}$. In [6] this functor was shown to be a bifibration, i.e., the notions of W-inverse image and of W-direct image make sense. If one restricts W from $\mathcal{M} \diamond \mathcal{M}$ to $\mathcal{M}$ via the second projection (which was the vital ingredient in the definition of $\ddot\nabla$) the notion of stability under W-inverse images translates to the usual notion of stability under pullbacks. If, on the other hand, one restricts the notion of stability under W-direct images along the first projection from $\mathcal{M} \diamond \mathcal{M}$ to $\mathcal{M}$, the following notion of $\mathbf{E}$-sink stability is obtained:

2.2 Definition (1) A subclass $\mathcal{N}$ of $\mathcal{M}$ is called $\mathbf{E}$-*sink stable*, if for every commutative square

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "m"'] & N \arrow[d, "n"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

with $n \in \mathcal{M}$ and the 2-sink $(g, n) \in \mathbf{E}$ we have that $m \in \mathcal{N}$ implies $n \in \mathcal{N}$.

(2) $P_{es}(\mathcal{M})$ denotes the collection of all **E**-sink stable subclasses of $\mathcal{M}$, ordered by inclusion.

(3) $P_{pb}(\mathcal{M})$ denotes the collection of all pullback-stable subclasses of $\mathcal{M}$, ordered by inclusion.

In view of the construction of $\ddot{\Delta}$ and $\ddot{\nabla}$ as given in [6], one can expect these Galois connections to factor through $P_{es}(\mathcal{M})$ and through $P_{pb}(\mathcal{M})$, respectively. We now provide the details of these factorizations.

2.3 Theorem (1) *Let $\mathcal{N} \in P_{pb}(\mathcal{M})$. If for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$, we define:*

$$m^{S_{\mathcal{N}}} = \inf\{\, m' \in \mathcal{N} : M' \xrightarrow{m'} X \text{and } m \leq m' \,\}$$

then $S_{\mathcal{N}}$ is an idempotent closure operator with respect to $\mathcal{M}$.

(2) *Let $\mathcal{N} \in P_{es}(\mathcal{M})$. If for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$, we define:*

$$m^{C_{\mathcal{N}}} = \sup\{\, (N \xrightarrow{n} X) \in \mathcal{M} : \exists (M \xrightarrow{t} N) \in \mathcal{N} \text{with } n \circ t = m \,\}$$

then $C_{\mathcal{N}}$ is a weakly hereditary closure operator with respect to $\mathcal{M}$.

Proof (1) Clearly, for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$, we have that $m \leq m^{S_{\mathcal{N}}}$.

To prove order-preservation, we just observe that if $M \xrightarrow{m} X$ and $N \xrightarrow{n} X$ are $\mathcal{M}$-subobjects such that $m \leq n$, then any $\mathcal{M}$-subobject $N' \xrightarrow{n'} X$ that satisfies $n \leq n'$ also satisfies $m \leq n'$. Therefore, taking the intersection yields $m^{S_{\mathcal{N}}} \leq n^{S_{\mathcal{N}}}$.

To show morphism-consistency, let $X \xrightarrow{f} Y$ be an $\mathcal{X}$-morphism, let $N \xrightarrow{n} Y$ be an $\mathcal{M}$-subobject and let $(N_i \xrightarrow{n_i} Y)_{i \in I}$ be the family of all $\mathcal{M}$-subobjects in $\mathcal{N}$ such that $n \leq n_i$, for every $i \in I$. By taking the pullbacks $f^{-1}(n)$ and $f^{-1}(n_i)$ of n and n_i along f, respectively, we obtain the following commutative diagram

$$\begin{tikzcd}
f^{-1}(N) \arrow[rr] \arrow[dr] \arrow[dd, "f^{-1}(n)"'] & & N \arrow[dr] \arrow[dd, "n"' near start] & \\
& f^{-1}(N_i) \arrow[rr] \arrow[dl, "f^{-1}(n_i)"] & & N_i \arrow[dl, "n_i"] \\
X \arrow[rr, "f"'] & & Y &
\end{tikzcd}$$

Since $\mathcal{N}$ is pullback-stable, we have that $f^{-1}(n_i) \in \mathcal{N}$ for every $i \in I$. By the definition of pullback, we have that $f^{-1}(n) \leq f^{-1}(n_i)$, for every $i \in I$. Thus, because pullbacks and intersections commute we have that $(f^{-1}(n))^{S_{\mathcal{N}}} \leq \cap f^{-1}(n_i) \simeq f^{-1}(\cap n_i) \simeq f^{-1}(n^{S_{\mathcal{N}}})$.

Thus $S_{\mathcal{N}}$ is a closure operator; i.e., it satisfies Definition 1.3.

To show that $S_{\mathcal{N}}$ is idempotent let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject. We first observe that by definition we have that $m^{S_{\mathcal{N}}} \leq (m^{S_{\mathcal{N}}})^{S_{\mathcal{N}}}$. Now if $n \in \mathcal{N}$ satisfies $m \leq n$, then it also satisfies $m^{S_{\mathcal{N}}} \leq n$. This clearly implies that $(m^{S_{\mathcal{N}}})^{S_{\mathcal{N}}} \leq m^{S_{\mathcal{N}}}$. Therefore, $(m^{S_{\mathcal{N}}})^{S_{\mathcal{N}}} \simeq m^{S_{\mathcal{N}}}$.

(2) This follows by the symmetry mentioned above. However, for clarity and completeness we provide the following proof.

It is clear from the definition of $C_{\mathcal{N}}$ that for every monomorphism $M \xrightarrow{m} X$, $m \leq m^{C_{\mathcal{N}}}$.

To prove order-preservation, let us consider the following commutative diagram

$$\begin{tikzcd}
M \arrow[rr, "g"] \arrow[dd, "t"'] \arrow[dr, "m"'] & & N \arrow[dd, "e_1"] \arrow[dl, "n"] \\
& X & \\
M' \arrow[rr, "e_2"'] \arrow[ur, "m'"] & & P \arrow[ul, "p"']
\end{tikzcd}$$

where $m \leq n$ are two $\mathcal{M}$-subobjects, $t \in \mathcal{N}$ and $((e_1, e_2), p)$ is the $(\mathbf{E}, \mathcal{M})$-factorization of the 2-sink (n, m'). Notice that the closure of $\mathcal{M}$ under $\mathcal{M}$-relative first factors implies that $e_1 \in \mathcal{M}$. $\mathbf{E}$-sink stability of $\mathcal{N}$ yields that $e_1 \in \mathcal{N}$. This, together with the fact that suprema are formed via $(\mathbf{E}, \mathcal{M})$-factorizations immediately yields via the $(\mathbf{E}, \mathcal{M})$-diagonalization property a diagonal morphism d with $n^{C_{\mathcal{N}}} \circ d = m^{C_{\mathcal{N}}}$. Therefore we can conclude that $m^{C_{\mathcal{N}}} \leq n^{C_{\mathcal{N}}}$.

To show morphism-consistency, let $X \xrightarrow{f} Y$ be a morphism and let $m = m''_\iota \circ m'_\iota$ be a factorization of m with $m'_\iota \in \mathcal{N}$ and $m''_\iota \in \mathcal{M}$. By taking the direct images of m and m''_ι along f, we obtain the following commutative diagram where t_ι is induced by the $(\mathbf{E}, \mathcal{M})$-diagonalization property.

$$\begin{tikzcd}
M \arrow[rr, "e"] \arrow[dd, "m"'] \arrow[dr, "m'_\iota"] & & f(M) \arrow[dd, "f(m)"' near start] \arrow[dr, "t_\iota"] & \\
& M'_\iota \arrow[rrr, "e''_\iota"' near start, crossing over] \arrow[dl, "m''_\iota"] & & & f(M'_\iota) \arrow[dll, "f(m''_\iota)"] \\
X \arrow[rr, "f"'] & & Y & &
\end{tikzcd}$$

Let $(\tilde{e}, m^{C_{\mathcal{N}}})$ and $(\hat{e}, (f(m))^{C_{\mathcal{N}}})$ be the $(\mathbf{E}, \mathcal{M})$-factorizations that yield the $C_{\mathcal{N}}$-closures of m and $f(m)$ and suppose that $\tilde{e}$ and $\hat{e}$ are indexed by I and J, respectively. For every $i \in I$ notice that since e''_ι belongs to $\mathbf{E}$, so does the 2-sink (e''_ι, t_ι). Since $\mathcal{N}$ is $\mathbf{E}$-sink stable, we conclude that $t_i \in \mathcal{N}$. Therefore for every $i \in I$ there exists some $j(i) \in J$ such that the following diagram commutes

$$\begin{tikzcd}
M'_i \arrow[rrr, "e''_i"] \arrow[ddd, "m''_i"'] \arrow[dr, "\tilde{e}_i"] & & & f(M'_i) \arrow[ddd, "f(m''_i)"] \arrow[dl, "\hat{e}_{j(i)}"'] \\
& M^{C_{\mathcal{N}}} \arrow[ddl, "m^{C_{\mathcal{N}}}"] & (f(M))^{C_{\mathcal{N}}} \arrow[ddr, "(f(m))^{C_{\mathcal{N}}}"'] & \\
& & & \\
X \arrow[rrr, "f"'] & & & Y
\end{tikzcd}$$

Using the $(\mathbf{E}, \mathcal{M})$-diagonalization property we obtain a morphism $M^{C_{\mathcal{N}}} \xrightarrow{d} f(M)^{C_{\mathcal{N}}}$ such that $d \circ \tilde{e}_i = \hat{e}_{j(i)} \circ e''_i$ and $f \circ m^{C_{\mathcal{N}}} = (f(m))^{C_{\mathcal{N}}} \circ d$. Let $(e^{C_{\mathcal{N}}}, f(m^{C_{\mathcal{N}}}))$ be the $(\mathbf{E}, \mathcal{M})$-factorization of $f o m^{C_{\mathcal{N}}}$. Since we have that $f(m^{C_{\mathcal{N}}}) o e^{C_{\mathcal{N}}} = (f(m))^{C_{\mathcal{N}}} o d$, the diagonalization property implies the existence of a morphism $\tilde{d}$ such that $(f(m))^{C_{\mathcal{N}}} \circ \tilde{d} = f(m^{C_{\mathcal{N}}})$. Therefore we can conclude that $f(m^{C_{\mathcal{N}}}) \leq (f(m))^{C_{\mathcal{N}}}$, i.e., that condition (c) of Definition 1.3 is satisfied (cf. Remark 1.4(1)).

To show that $C_{\mathcal{N}}$ is weakly hereditary, let us consider the following commutative diagram

$$\begin{tikzcd}
\bullet \arrow[r, "(m_{C_{\mathcal{N}}})^{C_{\mathcal{N}}}"] & M^{C_{\mathcal{N}}} \arrow[d, "m^{C_{\mathcal{N}}}"] \\
M \arrow[u, "(m_{C_{\mathcal{N}}})_{C_{\mathcal{N}}}"] \arrow[ur, "m_{C_{\mathcal{N}}}"] \arrow[r, "m"'] & X
\end{tikzcd}$$

Clearly we have $(m_{C_{\mathcal{N}}})^{C_{\mathcal{N}}} \circ (m_{C_{\mathcal{N}}})_{C_{\mathcal{N}}} = m_{C_{\mathcal{N}}}$. On the other hand, any $M \xrightarrow{m'} M' \in \mathcal{N}$ used in the construction of $m^{C_{\mathcal{N}}}$ is also used in the construction of $(m_{C_{\mathcal{N}}})^{C_{\mathcal{N}}}$, which implies $m^{C_{\mathcal{N}}} \leq m^{C_{\mathcal{N}}} \circ (m_{C_{\mathcal{N}}})_{C_{\mathcal{N}}}$. Thus $(m_{C_{\mathcal{N}}})^{C_{\mathcal{N}}}$ is an isomorphism. This completes the proof. $\square$

2.4 Theorem (1) *Let* $iCL(\mathcal{X}, \mathcal{M}) \xrightarrow{R_*} P_{pb}(\mathcal{M})^{\mathrm{op}}$ *and* $P_{pb}(\mathcal{M})^{\mathrm{op}} \xrightarrow{R^*} iCL(\mathcal{X}, \mathcal{M})$ *be defined by:*

$$R_*(C) = \{\, m \in \mathcal{M} : m \text{ is } C\text{-closed}\}$$

$$R^*(\mathcal{N}) = S_{\mathcal{N}}$$

Then, $iCL(\mathcal{X}, \mathcal{M}) \xrightarrow{R} P_{pb}(\mathcal{M})^{\mathrm{op}}$ *is a Galois connection that is a coreflection; i.e.,* $R^* \circ R_* \simeq id_{iCL(\mathcal{X}, \mathcal{M})}$.

(2) *Let* $wCL(\mathcal{X}, \mathcal{M}) \xrightarrow{K^*} P_{es}(\mathcal{M})$ *and* $P_{es}(\mathcal{M}) \xrightarrow{K_*} wCL(\mathcal{X}, \mathcal{M})$ *be defined by:*

$$K^*(C) = \{\, m \in \mathcal{M} : m \text{ is } C\text{-dense}\}$$

$$K_*(\mathcal{N}) = C_{\mathcal{N}}$$

Then, $P_{es}(\mathcal{M}) \xrightarrow{K} wCL(\mathcal{X}, \mathcal{M})$ *is a Galois connection that is a reflection; i.e.,* $K_* \circ K^* \simeq id_{wCL(\mathcal{X}, \mathcal{M})}$.

Proof (1) By the preceding lemma and the definition of closure operator, it is clear that R^* and R_* have domains and codomains as indicated.

If $C \sqsubseteq C'$, then C'-closed implies C-closed, which implies $R_*(C') \subseteq R_*(C)$, i.e., $R_*(C) \leq R_*(C')$ in $P_{pb}(\mathcal{M})^{\text{op}}$. Hence R_* is order-preserving.

Clearly, $\mathcal{M}_1 \leq \mathcal{M}_2 \Rightarrow \mathcal{M}_2 \subseteq \mathcal{M}_1 \Rightarrow S_{\mathcal{M}_1} \sqsubseteq S_{\mathcal{M}_2}$. Thus R^* is order-preserving.

If $\mathcal{N} \in P_{pb}(\mathcal{M})^{\text{op}}$ and $m \in \mathcal{N}$, then m is $S_{\mathcal{N}}$-closed, which implies that $\mathcal{N}$ is a subclass of $(R_* \circ R^*)(\mathcal{N})$. Thus, $(R_* \circ R^*)(\mathcal{N}) \leq \mathcal{N}$.

Let C be a closure operator, let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject and let $(N_i \xrightarrow{n_i} X)_I$ be the family of all $\mathcal{M}$-subobjects with the property that $m \leq n_i$ and n_i is C-closed. Then, for each $i \in I$, $m^C \leq n_i^C \simeq n_i$, which implies that $m^C \leq \cap\{n_i : i \in I\} \simeq m^{(R^* \circ R_*)(C)}$. Thus $C \sqsubseteq (R^* \circ R_*)(C)$. Also, since C is idempotent, m^C is C-closed and so is one of the n_i's originally chosen. Therefore, $m^{R^* \circ R_*} \leq m^C$. Thus $R^* \circ R_* \simeq id_{CL(\mathcal{X},\mathcal{M})}$.

(2) This again follows by the symmetry mentioned above. However, for clarity and completeness we provide the following proof.

If $C \sqsubseteq C'$, then C-dense implies C'-dense, which implies $K^*(C) \subseteq K^*(C')$, i.e., $K^*(C) \leq K^*(C')$.

Clearly, if $\mathcal{M}_1 \leq \mathcal{M}_2$, i.e., $\mathcal{M}_1 \subseteq \mathcal{M}_2$, then $C_{\mathcal{M}_1} \leq C_{\mathcal{M}_2}$.

If $\mathcal{N} \in P_{es}(\mathcal{M})$ and $M \xrightarrow{t} N \in \mathcal{N}$, we can consider the factorization $t = id_N \circ t$. Clearly this implies that $(M)_N^{C_{\mathcal{N}}} \simeq N$, i.e., $t \in (K^* \circ K_*)(\mathcal{N})$. Thus, $\mathcal{N} \leq (K^* \circ K_*)(\mathcal{N})$.

Now first notice that if C is a closure operator then $K^*(C)$ is **E**-sink stable, so that $K^*(C) \in P_{es}(\mathcal{M})$. Let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject and let $m = n_i \circ t_i$ be a family of factorizations with t_i C-dense for every $i \in I$. Clearly, we have that $n_i \simeq (m)_N^C \leq (m)_X^C$. Therefore, $n_i \leq (m)_X^C$ implies that $(m)_X^{C_{\mathcal{N}}} \simeq \sup\ n_i \leq (m)_X^C$. Thus $(K_* \circ K^*)(C) \sqsubseteq C$. Also, since C is weakly hereditary, m_C is C-dense and so is one of the t_i's originally chosen. Therefore, $m^C \leq m^{(K_* \circ K^*)(C)}$. Thus $(K_* \circ K^*)(C) \simeq C$. □

2.5 Remark A direct consequence of the above results is that a closure operator $C \in CL(\mathcal{X},\mathcal{M})$ is weakly hereditary and idempotent if and only if $C \simeq \dot{\nabla}^* \circ R^* \circ R_* \circ \dot{\nabla}_*(C) \simeq \ddot{\Delta}_* \circ K_* \circ K^* \circ \ddot{\Delta}^*(C)$.

Unfortunately the notion of **E**-sink stability is less easy to verify than stability under pullbacks. Therefore next we give some criteria that are easier to check and are equivalent to it.

2.6 Proposition (Characterization of $\mathbf{E}$-sink stability) *A subclass $\mathcal{N}$ of $\mathcal{M}$ is $\mathbf{E}$-sink stable if and only if it satisfies the following two conditions:*

(a) *whenever $M \xrightarrow{m} X$ and $N \xrightarrow{n} X$ are $\mathcal{M}$-subobjects with $m \leq n$, then for every factorization $m = m'' \circ m'$ with $m' \in \mathcal{N}$ and $m'' \in \mathcal{M}$, there exists a factorization $n = n'' \circ n'$ of n with $n' \in \mathcal{N}$ and $n'' \in \mathcal{M}$ such that $m'' \leq n''$.*

(b) *$\mathcal{N}$ is closed under direct images along $\mathbf{E}$-morphisms.*

Proof Suppose that $\mathcal{N}$ is $\mathbf{E}$-sink stable. Let $M \xrightarrow{m} X$ be an $\mathcal{N}$-subobject and let $X \xrightarrow{g} Y$ be an $\mathbf{E}$-morphism. If $(e, g(m))$ is the $(\mathbf{E}, \mathcal{M})$-factorization of $g \circ m$, then the 2-sink $(g, g(m))$ belongs to $\mathbf{E}$ and $\mathbf{E}$-sink stability immediately implies that $g(m) \in \mathcal{N}$, i.e., (b) holds. To show that (a) also holds, let us consider the following commutative diagram

$$\begin{tikzcd}
M \arrow[rr, "t"] \arrow[dr, "m"] \arrow[dd, "m'"'] & & N \arrow[dl, "n"'] \arrow[dd, "e_1"] \\
& X & \\
M' \arrow[ur, "m''"] \arrow[rr, "e_2"'] & & P \arrow[ul, "p"']
\end{tikzcd}$$

where m, n and m'' are $\mathcal{M}$-subobjects, $m' \in \mathcal{N}$ and $((e_1, e_2), p)$ is the $(\mathbf{E}, \mathcal{M})$-factorization of the 2-sink (n, m''). Notice that the closure of $\mathcal{M}$ under $\mathcal{M}$-relative first factors implies that $e_1 \in \mathcal{M}$. $\mathbf{E}$-sink stability yields that $e_1 \in \mathcal{N}$. Thus (a) holds.

Now let us assume that $\mathcal{N}$ satisfies (a) and (b) and consider the following commutative diagram

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "m"'] & N \arrow[d, "n"] \\
X \arrow[r, "g"'] & Y
\end{tikzcd}$$

with $m \in \mathcal{N}$, $n \in \mathcal{M}$ and the 2-sink $(g, n) \in \mathbf{E}$. Now let (e, p) be the $(\mathbf{E}, \mathcal{M})$-factorization of g, and let $(e', e(m))$ be the $(\mathbf{E}, \mathcal{M})$-factorization of $e \circ m$. Due to the $(\mathbf{E}, \mathcal{M})$-diagonalization property, we obtain a morphism d such that the following diagram is commutative.

$$\begin{tikzcd}
M \arrow[rr, "f"] \arrow[dr, "e'"] \arrow[ddd, "m"'] & & N \arrow[ddd, "n"] \\
& e(M) \arrow[ur, "d"'] \arrow[d, "e(m)"] & \\
& P \arrow[dr, "p"] & \\
X \arrow[ur, "e"] \arrow[rr, "g"'] & & Y
\end{tikzcd}$$

Notice that condition (b) implies that $e(m) \in \mathcal{N}$. Now let us consider the $\mathcal{M}$-subobject $p \circ e(m)$ of Y. Clearly, $p \circ e(m) \leq n$ and from condition (a) there exists a factorization of n, $n = n'' \circ n'$ with $n' \in \mathcal{N}$ and $n'' \in \mathcal{M}$ such that $p \leq n''$. Let h be the morphism such that $p = n'' \circ h$. We therefore obtain the following commutative diagram

$$\begin{tikzcd}
 & N \arrow[dd, "n'"] \arrow[dr, "n"] & \\
X \arrow[rr, "g"] \arrow[dr, "h \circ e"'] & & Y \arrow[d, "id_Y"] \\
 & N' \arrow[r, "n''"'] & Y
\end{tikzcd}$$

Since the 2-sink $(g, n) \in \mathbf{E}$, the $(\mathbf{E}, \mathcal{M})$-diagonalization property implies the existence of a morphism $Y \xrightarrow{k} N'$ such that $n'' \circ k = id_Y$. Since n'' is a monomorphism and a retraction, we have that n'' is an isomorphism. Thus, $n'' \in \mathbf{E}$, so since $n = n'' \circ n'$, we have that n is a direct image of n' along an $\mathbf{E}$-morphism and so by (b) it belongs to $\mathcal{N}$. Hence $\mathcal{N}$ is $\mathbf{E}$-sink stable. □

The proof of the following proposition is rather easy so we omit it.

2.7 Proposition (1) *Let* $P_{pb}(\mathcal{M})^{\mathrm{op}} \xrightarrow{Q_*} P(\mathcal{M})^{\mathrm{op}}$ *and* $P(\mathcal{M})^{\mathrm{op}} \xrightarrow{Q^*} P_{pb}(\mathcal{M})^{\mathrm{op}}$ *be defined by:*

$$Q_*(\mathcal{M}') = \mathcal{M}'$$
$$Q^*(\mathcal{N}) = \{\, m \in \mathcal{M} : m \text{ is a pullback of some } n \in \mathcal{N} \,\}$$

Then, $P_{pb}(\mathcal{M})^{\mathrm{op}} \overset{Q}{\longrightarrow\!\!\bullet} P(\mathcal{M})^{\mathrm{op}}$ *is a Galois connection and* $\ddot{\nabla} = Q \circ R$.

(2) *Let* $P(\mathcal{M}) \xrightarrow{L_*} P_{es}(\mathcal{M})$ *and* $P_{es}(\mathcal{M}) \xrightarrow{L^*} P(\mathcal{M})$ *be defined by:*

$$L_*(\mathcal{M}') = \{\, n \in \mathcal{M} : n \circ f = g \circ m \text{ for some } m \in \mathcal{M}' \text{ and some } \mathcal{X}\text{-morphisms } f \text{ and } g \text{ with } (g, n) \in \mathbf{E} \,\}$$
$$L^*(\mathcal{N}) = \mathcal{N}$$

Then, $P(\mathcal{M}) \overset{L}{\longrightarrow\!\!\bullet} P_{es}(\mathcal{M})$ *is a Galois connection and* $\dot{\Delta} = K \circ L$. □

2.8 Definition If $\mathcal{A}$ is a class of $\mathcal{X}$-objects, we say that an $\mathcal{X}$-monomorphism m is $\mathcal{A}$-*regular* if it is the equalizer of some pair of morphisms with codomain in $\mathcal{A}$.

In the case that $\mathcal{M}$ contains all regular monomorphisms, an important special class of idempotent closure operators can be defined as follows. Given any class $\mathcal{A}$ of $\mathcal{X}$-objects and $M \xrightarrow{m} X$ in $\mathcal{M}$, the $\mathcal{A}$-closure of m is given by the intersection of all $\mathcal{A}$-regular subobjects

n of X that satisfy $m \leq n$. This generalizes the Salbany construction of closure operators induced by classes of topological spaces, cf. [17], and may also be viewed as a relativization of Isbell's notion of *dominion* (cf. [13]). Following other authors, we will call such a closure operator *the regular (or Salbany) closure operator induced by* $\mathcal{A}$. The fact that in the presence of squares equalizers can be expressed as pullbacks of suitable diagonal morphisms allows us to build a symmetric counterpart of the Salbany operator. Both constructions fit nicely in our diagram of Galois connections given at the beginning of this section.

From now on we assume that our category $\mathcal{X}$ has squares and that $\mathcal{M}$ contains all regular monomorphisms.

2.9 Definition We call a monomorphism m *essentially a diagonal for* an object X, if m is an equalizer of the two projections for some product $(X \times X, \pi_1, \pi_2)$ of X with itself.

The corresponding relation between objects in $\mathcal{X}$ and monomorphism in $\mathcal{M}$ immediately induces a Galois connection as follows:

2.10 Proposition *Let* $P(Ob\mathcal{X}) \xrightarrow{H_*} P(\mathcal{M})$ *and* $P(\mathcal{M}) \xrightarrow{H^*} P(Ob\mathcal{X})$ *be defined by:*

$$H_*(\mathcal{A}) = \{\, m \in \mathcal{M} : m \text{ is essentially a diagonal for some } X \in \mathcal{A}\,\}$$
$$H^*(\mathcal{N}) = \{\, X \in Ob\mathcal{X} : \text{ every essential diagonal } m \text{ for } X \text{ belongs to } \mathcal{N}\,\}$$

Then, $P(Ob\mathcal{X}) \xrightarrow{H} P(\mathcal{M})$ *is a Galois connection that is a coreflection, i.e.,* $H^* \circ H_* = id_{Ob\mathcal{X}}$. □

2.11 Proposition *For any class* $\mathcal{A}$ *of* $\mathcal{X}$*-objects,* $(R^* \circ Q^* \circ H^{\mathrm{op}*})(\mathcal{A})$ *is precisely the Salbany closure operator induced by* $\mathcal{A}$.

Proof Let $\mathcal{A} \in P(Ob\mathcal{X})^{\mathrm{op}}$. First notice that since for every $X \in \mathcal{A}$, the diagonal morphism δ_X is $\mathcal{A}$-regular and pullbacks of $\mathcal{A}$-regular morphisms are $\mathcal{A}$-regular, we obtain that $(Q^* \circ H^{\mathrm{op}*})(\mathcal{A})$ is a class of $\mathcal{A}$-regular morphisms. On the other hand, if $M \xrightarrow{m} X$ is $\mathcal{A}$-regular, i.e., $m = eq(f,g)$, where f,g are two morphisms with codomain $Y \in \mathcal{A}$, then m is isomorphic to the pullback of δ_Y along the morphism $X \xrightarrow{<f,g>} Y \times Y$. Therefore $(Q^* \circ H^{\mathrm{op}*})(\mathcal{A})$ consists of all $\mathcal{A}$-regular morphisms. Consequently, $(R^* \circ Q^* \circ H^{\mathrm{op}*})(\mathcal{A})$ is the Salbany closure induced by $\mathcal{A}$. □

The next proposition follows from the general theory of Galois connections (cf.[11]).

2.12 Proposition (1) *Let* $iCL(\mathcal{X},\mathcal{M}) \xrightarrow{S_*} P(Ob\mathcal{X})^{\mathrm{op}}$ *and* $P(Ob\mathcal{X})^{\mathrm{op}} \xrightarrow{S^*} iCL(\mathcal{X},\mathcal{M})$ *be defined by:*

$$S_*(C) = \{\, X \in Ob\mathcal{X} : \delta_X \text{ is } C\text{-closed}\}$$
$$S^*(\mathcal{A}) = \sup\{\, C \in iCL(\mathcal{X},\mathcal{M}) : S_*(C) \supseteq \mathcal{A}\}$$

Then, $iCL(\mathcal{X},\mathcal{M}) \xrightarrow{S} P(\mathcal{M})^{\mathrm{op}}$ *is a Galois connection.*

(2) *Let* $P(Ob\mathcal{X}) \xrightarrow{T_*} wCL(\mathcal{X},\mathcal{M})$ *and* $wCL(\mathcal{X},\mathcal{M}) \xrightarrow{T^*} P(Ob\mathcal{X})$ *be defined by:*

$$T^*(C) = \{\, X \in Ob\mathcal{X} : \delta_X \text{ is } C\text{-dense}\}$$
$$T_*(\mathcal{A}) = \inf\{\, C \in wCL(\mathcal{X},\mathcal{M}) : T^*(C) \supseteq \mathcal{A}\}$$

Then, $P(Ob\mathcal{X}) \xrightarrow{T} wCL(\mathcal{X},\mathcal{M})$ *is a Galois connection.* □

2.13 Theorem (1) $S \simeq H^{\mathrm{op}} \circ \ddot{\nabla} = H^{\mathrm{op}} \circ Q \circ R$

(2) $T \simeq \dot{\Delta} \circ H = K \circ L \circ H$.

Proof (1) To see that $S \simeq H^{\mathrm{op}} \circ \ddot{\nabla}$ it is enough to observe that $S_* = H_*^{\mathrm{op}} \circ \ddot{\nabla}_*$. Consequently by the essential uniqueness of the adjoint we obtain that $S^* \simeq \ddot{\nabla}^* \circ H^{\mathrm{op}*}$. The equality follows from Proposition 2.7(1).

(2) Similarly, it is enough to recall Proposition 2.7(2) and to observe that $T^* = H^* \circ \dot{\Delta}^*$. Consequently by the essential uniqueness of the coadjoint we obtain that $T_* \simeq \dot{\Delta}_* \circ H_*$. □

2.14 Corollary *For any class* $\mathcal{A}$ *of* $\mathcal{X}$*-objects,* $S^*(\mathcal{A})$ *is precisely the Salbany closure operator induced by* $\mathcal{A}$. □

The following two commutative diagrams of Galois connections help to visualize the previous results.

$$\begin{tikzcd} P(Ob\mathcal{X}) \arrow[r, "T"] \arrow[d, "H"'] & wCL(\mathcal{X},\mathcal{M}) \\ P(\mathcal{M}) \arrow[r, "L"'] \arrow[ur, "\dot{\Delta}"] & P_{es}(\mathcal{M}) \arrow[u, "K"'] \end{tikzcd} \qquad \begin{tikzcd} iCL(\mathcal{X},\mathcal{M}) \arrow[r, "S"] \arrow[d, "R"'] \arrow[dr, "\ddot{\nabla}"] & P(Ob\mathcal{X})^{\mathrm{op}} \\ P_{pb}(\mathcal{M})^{\mathrm{op}} \arrow[r, "Q"'] & P(\mathcal{M})^{\mathrm{op}} \arrow[u, "H^{\mathrm{op}}"'] \end{tikzcd}$$

These diagrams show that the Galois connection T may well be viewed as a "symmetric counterpart" of the Galois connection S induced by the Salbany construction. Notice that S actually can be defined without reference to squares (cf. [7], Theorem 2.5 and Corollary 2.10). For T, however, this does not seem to be possible.

3 Applications in concrete categories

In this section $\mathcal{X}$ always denotes a category with squares that satisfies all the hypotheses of Sections 1 and 2.

We start with a simple example that nicely supports our view of S and T as symmetric counterparts of each other.

3.1 Example Let $\mathcal{X} = (X, \leq)$ be a partially ordered set (considered as a category) with the property that for every $x \in X$ the lower segment $\downarrow x$ is a complete lattice. For $\mathcal{M}$ we take $Mor\mathcal{X}$, so $\mathbf{E}$ consists of all supremum sinks. Clearly, $(X, \leq)$ has squares, namely $x \cong x \wedge x$ for every $x \in X$; consequently all diagonal morphisms are isomorphisms. For every subset $A \subseteq X$, the closure operator $S^*(A)$ turns out to be indiscrete (i.e., the largest closure operator). To see this notice that if $x \leq y$, there is at most one subobject of y that is the pullback of some diagonal and dominates x, and when there is one it is y itself. Thus the formation of the intersection of all these subobjects always yields y. On the other hand, the closure operator $T_*(A)$ turns out to be discrete (i.e., the smallest closure operator). To see this notice that if $x \leq y$, there is exactly one factorization $x \leq z \leq y$ such that $x \leq z$ is induced by a diagonal via an $\mathbf{E}$-sink, and this is $x \leq x \leq y$. Hence the formation of the supremum of all the corresponding subobjects $z \leq y$ always yields the original $x \leq y$.

Next we consider two special relations on the class of objects of $\mathcal{X}$.

3.2 Definition Let $\mathcal{C} \subseteq Ob\mathcal{X} \times Ob\mathcal{X}$ be the relation defined by $(A, B) \in \mathcal{C}$ iff every morphism from A to B is a constant morphism (cf. [12], 8.2–8.8), and let $\mathcal{K} \subseteq Ob\mathcal{X} \times Ob\mathcal{X}$ be the relation induced by the Galois connection $H^{\mathrm{op}} \circ \nu \circ H$, i.e., $(A, B) \in \mathcal{K}$ iff all essential diagonals m of A and n of B satisfy $m \perp n$ (cf. point (5) at the beginning of Section 2).

3.3 Proposition (1) *$\mathcal{K} \subseteq \mathcal{C}$.*

(2) *$\mathcal{C} \subseteq \mathcal{K}$ iff $\mathcal{X}$ satisfies the following property (P): for every commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \delta_A}\downarrow & & \downarrow{\scriptstyle \delta_B} \\ A \times A & \xrightarrow[g]{} & B \times B \end{array} \tag{3-1}$$

with $(A, B) \in \mathcal{C}$ and δ_A and δ_B essential diagonals for A and B, the two projections π_i from $B \times B$ to B satisfy $\pi_1 \circ g = \pi_2 \circ g$.

Proof (1) Consider $(A,B) \in \mathcal{K}$ and a morphism $A \xrightarrow{f} B$. For any choice of essential diagonals $A \xrightarrow{\delta_A} A \times A$ and $B \xrightarrow{\delta_B} B \times B$ the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\delta_A"'] & B \arrow[d, "\delta_B"] \\ A \times A \arrow[r, "f \times f"'] & B \times B \end{tikzcd}$$

Orthogonality yields a morphism d such that $d \circ \delta_A = f$ and $\delta_B \circ d = f \times f$. For any two morphisms $X \xrightarrow{r} A$ and $X \xrightarrow{s} A$ we have

$$< f \circ r, f \circ s >= (f \times f) \circ < r, s >= \delta_B \circ d \circ < r, s >$$

where $X \xrightarrow{<r,s>} A \times A$ and $X \xrightarrow{<f\circ r, f\circ s>} B \times B$ are induced by the universal property of the respective products. Therefore $f \circ r = d\circ < r, s >= f \circ s$, i.e., f is a constant morphism.

(2) For $(A,B) \in \mathcal{C}$ consider a commutative diagram (3-1) with δ_A and δ_B essential diagonals. By hypothesis f is constant.

Suppose that $\mathcal{C} \subseteq \mathcal{K}$. Since $\delta_A \perp \delta_B$ there exists a unique morphism d such that $d \circ \delta_A = f$ and $\delta_B \circ d = g$. Now $\pi_1 \circ g = \pi_1 \circ \delta_B \circ d = d = \pi_2 \circ \delta_B \circ d = \pi_2 \circ g$. Hence $\mathcal{X}$ satisfies (P).

Conversely, suppose that $\mathcal{X}$ satisfies the property (P). Thus $\pi_1 \circ g = \pi_2 \circ g$. Since δ_B is an equalizer of π_1 and π_2, there exists a unique d with $\delta_B \circ d = g$. Since δ_B is a monomorphism, it also follows that $d \circ \delta_A = f$, which establishes $\delta_A \perp \delta_B$. Thus $(A,B) \in \mathcal{K}$. □

3.4 Proposition (1) *In the categories* **Grp** *of groups and* **Ab** *of abelian groups the relations* $\mathcal{K}$ *and* $\mathcal{C}$ *coincide.*

(2) *In the category* **Top** *of topological spaces the relations* $\mathcal{K}$ *and* $\mathcal{C}$ *coincide.*

Proof By part (1) of Proposition 3.3 we need only show that for these categories $\mathcal{C} \subseteq \mathcal{K}$; and so by part (2) it suffices to show that they satisfy property (P). Suppose every morphism from A to B is constant, and consider arbitrary morphisms $A \xrightarrow{f} B$ and $A \times A \xrightarrow{g} B \times B$ for which the diagram (3-1) commutes.

(1) Let i_1 and i_2 denote the two injections of A into $A \times A$ and let π_1 and π_2 be the two projections of $B \times B$ into B. By hypothesis, $\pi_1 \circ g \circ i_1$ and $\pi_2 \circ g \circ i_1$ are both constant and therefore so is $g \circ i_1$. Similarly we obtain that $g \circ i_2$ is also constant. Since $g \circ i_1$ and $g \circ i_2$ both equal the constant e-valued homomorphism from A to $B \times B$, it follows that g itself is the constant e-valued homomorphism from $A \times A$ to $B \times B$. In particular, $\pi_1 \circ g = \pi_2 \circ g$.

(2) Clearly, both $\pi_1 \circ g$ and $\pi_2 \circ g$ are constant on the subspace $A \times \{ a \}$ for each $a \in A$. Since each of these subspaces intersects the diagonal, on which g is constant by hypothesis, it follows immediately that $\pi_1 \circ g$ and $\pi_2 \circ g$ agree on the union of $A \times \{ a \}$ with the diagonal of A, and consequently agree on all of $A \times A$. □

Notice that the relations $\mathcal{K}$ and $\mathcal{C}$ coincide in any construct $\mathcal{X}$ that has the property that for each $\mathcal{X}$-object A there is an epi-sink $(A_i \xrightarrow{e_i} A \times A)_I$ with the property that for each $i \in I$ both $A_i = A$ and the images of A under e_i and δ_A have a nonempty intersection. In Proposition 3.4(1) the two injections i_1 and i_2 from A into $A \times A$ constitute such an epi-sink. In 3.4(2) the sink is comprised of "horizontal sections" of the space $A \times A$.

Next we illustrate the situation in the two cases in which $\mathcal{X}$ is the category **Ab** of abelian groups or the category **Grp** of all groups. In these categories we assume that $\mathcal{M}$ is the class of all injective homomorphisms and consequently **E** consists of all epi-sinks. We use the notation $X \leq Y$ to mean that X is isomorphic to a subgroup of Y and we use X/M for the quotient group of X *mod* M.

3.5 Example Let $\mathcal{X}$ be the category **Ab**, and let **Sing** denote the class of all singleton groups. By Proposition 3.4 $(\mathbf{Sing}, Ob\mathbf{Ab})$ is a pair of corresponding fixed points of the Galois connection $H^{\mathrm{op}} \circ \nu \circ H$. Since every monomorphism in **Ab** is regular, it is immediate to see that $S^*(Ob\mathbf{Ab}) = T_*(\mathbf{Sing})$ is the discrete (i.e., smallest) closure operator. Furthermore, $(Ob\mathbf{Ab}, \mathbf{Sing})$ is also a pair of corresponding fixed points. $S^*(\mathbf{Sing})$ is the indiscrete (i.e., largest) closure operator. To see that $T_*(Ob\mathbf{Ab})$ is also the indiscrete operator requires more effort. We would like to show that the class $(L_* \circ H_*)(Ob\mathbf{Ab})$ consists of all monomorphisms. First notice that the inclusion $\{0\} \xrightarrow{0_Y} Y$ belongs to $(L_* \circ H_*)(Ob\mathbf{Ab})$ for every abelian group Y. This is true since $\{0\} \xrightarrow{0_Y} Y$ is the direct image of the diagonal δ_Y along the epimorphism $Y \times Y \xrightarrow{\pi_1 - \pi_2} Y$. Now notice that for every monomorphism $M \xrightarrow{n} Y$ we have that $id_Y \circ 0_Y = n \circ 0_N$, where $\{0\} \xrightarrow{0_N} N$ is the inclusion. Therefore, **E**-sink stability implies that n belongs to $(L_* \circ H_*)(Ob\mathbf{Ab})$.

3.6 Example Let $\mathcal{F}$ denote the class of all torsion-free abelian groups. It is well known that the regular closure operator $S^*(\mathcal{F})$ induced by $\mathcal{F}$ in **Ab**, is weakly hereditary and idempotent and has the morphisms in $(Q^* \circ H^{\mathrm{op}*})(\mathcal{F})$ corresponding to closed subgroups. Clearly, $(\ddot{\Delta}^* \circ \dot{\nabla}^*)(S^*(\mathcal{F})) \simeq S^*(\mathcal{F})$ and such a closure operator has as dense subgroups all subgroups $M \xrightarrow{m} X$ such that $X/M \in \mathcal{T}$, where $\mathcal{T}$ is the class of all torsion abelian groups. Clearly from Proposition 3.4 we have that $\mathcal{T} = (H^* \circ \Delta^* \circ \nabla^* \circ H^{\mathrm{op}*})(\mathcal{F}) = (H^* \circ \nu^* \circ H^{\mathrm{op}*})(\mathcal{F})$ and $\mathcal{F} = (H_*^{\mathrm{op}} \circ \nu_* \circ H_*)(\mathcal{T})$.

3.7 Example Let $\mathcal{R}$ denote the class of reduced abelian groups; i.e., those abelian groups which have no nontrivial divisible subgroup. It is well known that the regular closure operator $S^*(\mathcal{R})$ in **Ab** is weakly hereditary and idempotent and has the morphisms in $(Q^* \circ H^{\mathrm{op}*})(\mathcal{R})$ corresponding to closed subgroups. Clearly, $(\ddot{\Delta}^* \circ \dot{\nabla}^*)(S^*(\mathcal{R})) \simeq S^*(\mathcal{R})$ and such a closure operator has as dense subgroups all subgroups $M \xrightarrow{m} X$ such that $X/M \in \mathcal{D}$, where $\mathcal{D}$ is the class of all divisible abelian groups. Again from Proposition 3.4 we have that $\mathcal{D} = (H^* \circ \Delta^* \circ \nabla^* \circ H^{\mathrm{op}*})(\mathcal{R}) = (H^* \circ \nu^* \circ H^{\mathrm{op}*})(\mathcal{R})$ and $\mathcal{R} = (H_*^{\mathrm{op}} \circ \nu_* \circ H_*)(\mathcal{D})$.

3.8 Example Let $\mathcal{X}$ be the category **Grp** and let $\mathcal{B}$ be the class of all abelian groups. The $S^*(\mathcal{B})$-closure of a subgroup $M \xrightarrow{m} X$ is given by the intersection of all normal subgroups N of X such that X/M is abelian. Since the normal subgroup relation is not transitive, $S^*(\mathcal{B})$ is not weakly hereditary. From Proposition 3.4 we have that $\mathcal{A} = (H^* \circ \nu^* \circ H^{\text{op}*})(\mathcal{B})$ consists of all perfect groups, i.e., all groups X such that $X = X'$, where X' denotes the subgroup generated by the commutators. Moreover, $(H_*^{\text{op}} \circ \nu_* \circ H_*)(\mathcal{A})$ is the class of all groups which do not have any non-trivial perfect subgroup (cf. [5]).

For the last three examples we consider the special case where $\mathcal{X}$ is the category **Top** of topological spaces (and continuous functions), $\mathcal{M}$ is the class of all embeddings and **E** consists of all epi-sinks.

3.9 Example Let $\mathcal{A} = \mathbf{Top_0}$ be the class of all T_0 spaces and let $\mathcal{B} = \mathbf{Ind}$ be the class of all indiscrete spaces. Then $(\mathcal{B}, \mathcal{A})$ is a pair of corresponding fixed points of the connectedness-disconnectedness Galois connection induced by $\mathcal{C}$ in **Top** (cf. [2]). It is well known that the regular closure operator $S^*(\mathbf{Top_0})$ is the b-closure. We recall that if M is a subset of a topological space X, a point x belongs to the b-closure of M if for every neighborhood U of x, $U \cap M \cap Cl(x) \neq \emptyset$, where $Cl(x)$ denotes the topological closure of the subset $\{x\}$ (cf. [3], [15]). Since the b-closure is weakly hereditary and idempotent, we have that $(\ddot{\Delta}^* \circ \dot{\nabla}^* \circ S^*)(\mathbf{Top_0}) \simeq S^*(\mathbf{Top_0})$. From Proposition 3.4 we have that $\mathbf{Ind} = (H^* \circ \Delta^* \circ \nabla^* \circ H^{\text{op}*})(\mathbf{Top_0}) = (H^* \circ \nu^* \circ H^{\text{op}*})(\mathbf{Top_0})$ and $(H_*^{\text{op}} \circ \nu_* \circ H_*)(\mathbf{Ind}) = \mathbf{Top_0}$.

3.10 Example Let $\mathcal{A} = \mathbf{Top_1}$ be the class of all T_1 topological spaces. It is well known that $S^*(\mathbf{Top_1})$ is weakly hereditary and idempotent (cf.[8]), therefore $(\ddot{\Delta}^* \circ \dot{\nabla}^* \circ S^*)(\mathbf{Top_1}) \simeq S^*(\mathbf{Top_1})$. Consequently, $(H^* \circ \nu^* \circ H^{\text{op}*})(\mathbf{Top_1})$ consists of all topological spaces whose diagonal mapping is $S^*(\mathbf{Top_1})$-dense. From Proposition 3.4 we have that $(H^* \circ \nu^* \circ H^{\text{op}*})(\mathbf{Top_1})$ equals the class **Aconn** of all absolutely connected topological spaces. We recall that a topological space X is called *absolutely connected* if it cannot be decomposed into a disjoint family $\mathcal{L}$ of non-empty closed subsets such that $|\mathcal{L}| > 1$ (cf. [16]). Since **Aconn** and $\mathbf{Top_1}$ are corresponding fixed points of the connectedness-disconnectedness Galois connection induced by $\mathcal{C}$ in **Top**, we also have that $\mathbf{Top_1} = (H_*^{\text{op}} \circ \nu_* \circ H_*)(\mathbf{Aconn})$.

3.11 Example Let $\mathcal{A} = \mathbf{Dis}$ be the class of all discrete topological spaces. From Proposition 3.4 we have that $(H^* \circ \nu^* \circ H^{\text{op}*})(\mathbf{Dis})$ equals the class **Conn** of all connected topological spaces. Again from Proposition 3.4 we have that $(H_*^{\text{op}} \circ \nu_* \circ H_*)(\mathbf{Conn})$ is the class **Tdisc** of all totally disconnected topological spaces (cf. [2]).

Further examples that illustrate the relationship between closure operators and their classes of "closed" and "dense" subobjects can be found in [6].

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Gabriele Castellini
Department of Mathematics
University of Puerto Rico, Mayagüez campus
P.O. Box 5000
Mayagüez, PR 00681-5000, U.S.A.

Jürgen Koslowski
Voltastraße 31
D-30165 Hannover, Germany

George E. Strecker
Department of Mathematics
Kansas State University
Manhattan, KS 66506-2602, U.S.A.