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REMARKS ON THE MAL'CEV COMPLETION OF TORSION-FREE LOCALLY NILPOTENT GROUPS

by Temple H. FAY *

This paper is dedicated to the memory of my colleague and friend Professor Dr. H. S. P. Grässer.

Abstract

It is shown that the category of locally nilpotent groups, which contains the classes of all abelian, nilpotent, hypercentral, Fitting, and Baer groups, enjoys a factorization structure for homomorphisms which gives rise to a class of categorically compact groups: a locally nilpotent group G is categorically compact provided $G/\tau G$ is complete (has all n^{th}-roots). Here τG denotes the maximal torsion subgroup of G. We use the well known result of Mal'cev that a torsion-free locally nilpotent group can be embedded in a torsion-free complete locally nilpotent group to show that the class of torsion-free complete locally nilpotent groups (*i.e.*, the torsion-free categorically compact groups) is strongly-dense reflective in the full category of locally nilpotent groups. In this manner it is shown that the Mal'cev completion behaves not like a compactification, but rather like a topological completion.

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1 Introduction.

Completions and compactifications are, generally, familiar examples of epireflections in topology. However, completions exhibit a well behaved phenomenon which distinguishes them from compactifications (called firmness, see [4] and [3]). The best known examples of completions arise from topology (metric spaces and uniform spaces), but a few are from algebra, primarily of the sort related to a hereditary torsion theory (torsion-free divisible abelian groups form a firm reflective subcategory of the category of torsion-free abelian groups, the reflection of G being $G \rightarrow G^*$ where G^* denotes the divisible hull of G). The purpose of this work is twofold. On the one hand we introduce a new factorization structure for homomorphisms between locally nilpotent groups, called the (strongly-dense, isolated-mono) factorization structure, which in turn gives rise to the notion of (isolated-mono) categorically compact group. We then show that these categorically compact groups are the same as those determined by a closure operator studied in [8], and [9]. These groups, when torsion-free, are precisely the complete groups (a group is complete if it contains all n^{th}-roots). On the other hand, we use the well known theorem of Mal'cev giving an embedding of a torsion-free locally nilpotent group into a torsion-free complete locally nilpotent group to show that the class of torsion-free complete locally nilpotent groups produces a firm reflection in the category of torsion-free locally nilpotent groups (an extensive class of groups). This gives an algebraic example, not occurring in an abelian category, of an epireflection which behaves more like the completion of a metric space rather than the Stone-Cech compactification of a completely regular space. This will be of interest to categorical topologists and those interested in studying reflections.

2 Preliminaries.

The class of locally nilpotent groups is a well known class of groups containing the class of all abelian, nilpotent, hypercentral, Fitting, and Baer groups. Robinson [22] remarks "... yet relatively little is known about the structure, or even the normal structure, of locally nilpotent groups". It will be indicated herein, however, that the class is extremely well behaved.

Recall that a group G is *locally nilpotent* provided every finitely generated subgroup is nilpotent. It is immediate that the class of locally nilpotent groups is closed under formation of subgroups, factor groups, and finite products. Each of the subclasses mentioned in the previous paragraph is also closed under these formations. The elements of finite order form a characteristic subgroup which we denote by τG. Finally, a torsion-free locally nilpotent group has unique extraction of roots in the sense that $x^n = y^n$, for some $n > 0$, imply that $x = y$ (such groups are called

R-groups, see [2], [15], [20], and [21]).

A fundamental construction is that of the isolator of a subgroup. A subgroup H of a group G is called *isolated* if $x \in G$ and $x^n \in H$ for some $n > 0$, implies $x \in H$. It follows that the intersection of isolated subgroups is isolated, hence if H is a subgroup of G, then the intersection of all isolated subgroups containing H is denoted $I(H)$ and is called *the isolator of H* (in G).

Proposition 1 *[20]. The following hold for a group G.*

(1) If G is a locally nilpotent group with subgroups H and K having H normal in K, then $I(H)$ is normal in $I(K)$.

(2) If G is a locally nilpotent torsion-free group, then for every pair of subgroups H and K, $I(H \cap K) = I(H) \cap I(K)$.

(3) If G is a locally nilpotent torsion-free group and H is a subgroup with an ascending central series, then $I(H)$ has an ascending central series and the classes of H and of $I(H)$ are the same.

(4) If G is a torsion-free group having unique extraction of roots, and H is a locally nilpotent subgroup, then $I(H)$ is locally nilpotent.

(5) If G is a locally nilpotent torsion-free group, and H is a subgroup, then $I(H) = \{x \in G : x^n \in H \text{ for some } n > 0\}$.

Kurosh [15] has a nice exposition on complete locally nilpotent torsion-free groups, from which we summarize. A completion of the torsion-free locally nilpotent group G is defined to be an arbitrary minimal complete torsion-free locally nilpotent group G^* containing G. Consequently, it follows that a complete torsion-free locally nilpotent group G^* is the completion of its subgroup G if and only if some positive power of each element of G^* belongs to G. A more modern treatment of this completion as a localization can be found in [24] Chapter VI. This localization has applications in algebraic topology (see [26], Chapter 8 and the references there) and in homology (see [24]).

Theorem 1 *(Mal'cev [18], see also [17], [24], [25]). The Mal'cev completion is given by a localization functor $G \longrightarrow G^*$ which embeds every torsion-free locally nilpotent in a complete torsion-free locally nilpotent group G^* in such a way that if G is complete, then $G = G^*$.*

3 Factorization Structures for Homomorphisms.

Call a map $e : X \to Y$ *strongly-dense* provided for each $y \in Y$, there is an $n > 0$, so that $y^n \in e(X)$. This is equivalent to saying $I(e(X)) = Y$. Call a monomorphism $m : A \to B$ an *isolated-mono* provided for $b \in B$ and $b^n \in A$ for some $n > 0$, imply $b \in A$; that is, isolated-monos are precisely the inclusions of isolated subgroups. If $f : A \to B$ is an arbitrary map, then by taking the isolator of $f(A)$ in B, we obtain a (strongly-dense, isolated-mono) factorization of f . It is virtually trivial to see that the diagonalization property above holds for these classes, and so the category of locally nilpotent groups enjoys the (strongly-dense, isolated-mono) factorization structure.

With each $(\mathcal{E}, \mathcal{M})$-factorization structure, there is a class of objects called the $\mathcal{M}$-separated objects (see [13]) which play the role of Hausdorff spaces in the category of topological spaces when using the (dense, closed embedding) factorization structure there. Call a group G *$\mathcal{M}$-separated* provided $\Delta : G \to G \times G$, sending g to the ordered pair (g, g), belongs to $\mathcal{M}$. If $\mathcal{M}$ is the class of isolated-monos, then an $\mathcal{M}$-separated group will be called *i-separated.*

Call a homomorphism $e : X \to Y$ *τ-dense* provided it has a torsion cokernel. Call a momomorphism $m : X \to Y$ a *τ-closed embedding* provided X is normal in Y and the quotient Y/X is torsion-free. In[8] and [9] it was erroneously claimed that the category of (locally) nilpotent groups enjoys the (τ-dense, τ-closed embedding) factorization structure; property (ii) above holds but not every homomorphism has a factorization. Fortunately, this does not invalidate any of the group-theoretic results in [8] and [9], see [11]. If G is a locally nilpotent group and H is a subgroup, let $I<H>$ denote the isolator of the normal closure of H, and observe that the inclusion of $I<H>$ into G is a τ-closed embedding. It is easy to check that $I<->$ is an idempotent closure operator of the type studied in [6]; that is, if K is a subgroup of H, then $I<K> \subset I<H>$ and $I<I<H>> = I<H>$. Separatedness with respect to a closure operator is defined in a similar way as for factorization structures: call a group G *τ-separated* provided $\Delta : G \to G \times G$ is a τ-closed embedding.

Proposition 2 . *A locally nilpotent group G is τ-separated if and only if G is torsion-free abelian.*

Proof. Observe that the map $\Delta : G \to G \times G$ is the inclusion of a normal subgroup if and only if G is abelian, in which case, Δ is a τ-closed embedding if and only if its cokernel, G, is torsion-free.

Proposition 3 . *A locally nilpotent group G is i-separated if and only if G is torsion-free.*

Proof. If $\Delta : G \to G \times G$ is an isolated-mono, and $x \in G$ with $x^n = 1$ for some $n > 0$, then $(x, 1)^n$ belongs to the diagonal subgroup. Isolatedness implies $(x, 1)$ belongs to the diagonal as well, so $x = 1$, and G is torsion-free.

Conversely, if G is torsion-free, and $(x_1,\ x_2)^n$ belongs to the diagonal subgroup, then ${x_1}^n = {x_2}^n$. But within G extraction of roots is unique, hence $x_1 = x_2$.

4 Categorical Compactness.

Following [13], given an $(\mathcal{E}, \mathcal{M})$ factorization structure, a group G is called *$\mathcal{M}$-compact* provided for each group H, the second projection map $\pi_2 : G \times H \to H$ sends $\mathcal{M}$-

subgroups of $G \times H$ onto $\mathcal{M}$-subgroups of H. This is a categorical interpretation of Kuratowski's Theorem that a topological space X is compact if and only if for each space Y, $\pi_2 : X \times Y \to Y$ is a closed map. If $\mathcal{M}$ is the class of isolated monos, the we call an $\mathcal{M}$-compact group G *i-compact.* See [13] for many interesting topological examples. The class of $\mathcal{M}$-compact groups is closed under formation of $\mathcal{M}$-subgroups, factor groups, and finite products.

Categorical compactness for a closure operator is defined in exactly the same way as for factorization structures. A group G is called *τ-compact* provided for each group H, and each τ-closed subgroup A of $G \times H$, $\pi_2(A)$ is a τ-closed subgroup of H. Categorical compactness with respect to closure operators has been studied by Castellini [5]. Objects compact with respect to a closure operator are closed under homomorphic images and finite products. However in this case, a closed subobject of a compact object need not be compact. This property does hold for τ-compact groups, but is not a consequence of categorical theorems, but rather follows from the characterization of τ-compactness. Thus this property cannot be used as a proof technique. However, certain τ-closed subgroups of τ-compact groups are easily seen to be τ-compact.

Theorem 2 . *If G is a τ-compact locally nilpotent group and K is a τ-closed subgroup contained in the center of G, then K is τ-compact.*

Proof. Let H be an arbitrary locally nilpotent group and A be a τ-closed subgroup of $K \times H$. Then it is clear that A is also a τ-closed subgroup of $G \times H$ since K is contained in the center (normality is the only thing to be checked). Hence $\pi_2(A)$ is a τ-closed subgroup of H, and K is τ-compact.

We have two notions of categorical compactness, one relative to τ-closed embeddings, and the other relative to isolated-monos. The main purpose of this section is to show that these two notions coincide.

Theorem 3 . *A locally nilpotent group G is τ-compact if and only if $G/\tau G$ is complete.*

Proof. A *series of general order type* in a group G is a set of subgroups $\mathcal{S}$ which is totally ordered by inclusion and which satisfies:

(i) if $1 \neq x \in G$, then $\cup\{A \in \mathcal{S} : x \notin A\} = V_x \in \mathcal{S}$;

(ii) if $1 \neq x \in G$, then $\cap\{A \in \mathcal{S} : x \in A\} = \Lambda_x \in \mathcal{S}$;

(iii) V_x is a normal subgroup of Λ_x;

(iv) every member of $\mathcal{S}$ is of the form V_x or Λ_x for some $x \in G$.

The subgroups V_x and Λ_x are called the *terms* and Λ_x/V_x the *factors* of the series $\mathcal{S}$. The series $\mathcal{S}$ is called *invariant* if each of V_x and Λ_x are normal in G, and an invariant series is called *central* if the factors Λ_x/V_x are contained in the center of G/V_x for every $x \in G$. In the general theory of locally nilpotent torsion-free groups, Glushkov's Theorem [12] is fundamental: *Every locally nilpotent torsion-free group has a central system of isolated subgroups.*

Suppose G is τ-compact, then, as homomorphic images of G are also τ-compact, $G/\tau G$ is τ-compact. Hence there is no loss of generality to assume G is torsion-free. Let $G \to G^*$ be the Mal'cev completion of G. By Glushkov's Theorem, G^* has a central system of isolated (normal) subgroups (V_x^*, Λ_x^*). Since each of V_x^* and Λ_x^* are closed subgroups of G^*, it follows that they are complete as well. Hence Λ_x^*/V_x^* is torsion-free divisible abelian. Setting $V_x = V_x^* \cap G$ and $\Lambda_x = \Lambda_x^* \cap G$, it is easy to see that (V_x, Λ_x) is a central system of isolated (normal) subgroups for G. Now we have $\Lambda_x/V_x = \Lambda_x^* \cap G/V_x^* \cap G \cong (G \cap \Lambda_x^*) \cdot V_x^*/V_x^* \subset \Lambda_x^*/V_x^*$. Since G is τ-compact, so is G/V_x, and since Λ_x/V_x is a τ-closed subgroup of the center of G/V_x, it too is τ-compact and hence divisible abelian (τ-compact torsion-free abelian groups are divisible [7], [8]). This implies that Λ_x/V_x splits as a subgroup of Λ_x^*/V_x^*(*i.e.*, is a direct summand). Now if $x \in G^*$, there exists an $n > 0$ so that $x^n \in G$. This means $\bar{x}^n \in \Lambda_x/V_x$. But Λ_x/V_x a summand implies $\bar{x} \in \Lambda_x/V_x$ and so $x \in G$. This shows $G = G^*$.

Conversely, if G is complete, every homomorphic image is complete as well, and this implies that every torsion-free homomorphic image has a divisible center. By Theorem 4.2 of [9], G is τ-compact.

Next we consider the notion of i-compactness. Since every closed embedding is an isolated-mono and the surjective image of a normal subgroup is normal, i-compactness implies τ-compactness.

Proposition 4 . *If G is a complete locally nilpotent group, then G is i-compact.*

Proof. Let H be locally nilpotent and A be an isolated subgroup of $G \times H$. Let $A \twoheadrightarrow B \to H$ be the (surjective, injective) factorization of $A \to G \times H \twoheadrightarrow H$. If $h \in H$ with $h^n \in B$, then there exists a $g \in G$ so that $(G, h^n) \in A$. But G being complete implies there exists an $x \in G$ so that $x^n = g$, and thus $(x^n, h^n) \in A$; A isolated implies $(x, h) \in A$, so $h \in B$. This shows that B is isolated in H and that G is i-compact.

Corollary 1 . *If G is torsion-free locally nilpotent, then G is τ-compact if and only if it is i-compact.*

Theorem 4 . *If G is locally nilpotent, then G is τ-compact if and only if it is i-compact.*

Proof. It suffices to show that if $G/\tau G$ is complete, then G is i-compact. To that end, let H be an arbitrary locally nilpotent group and A be an isolated subgroup of $G \times H$. Observe that if $g \in G$ is of finite order, say $g^n = 1$, then $(g, 1)^n \in A$, so $(g, 1) \in A$. Thus $\tau G \times \{1\} \subset A$. Let $A \twoheadrightarrow B \to H$ be the (surjective, injective) factorization of $A \to G \times H \twoheadrightarrow H$; let $A \twoheadrightarrow A/\tau G \times \{1\} \xrightarrow{\sigma} G/\tau G \times H$ be the (surjective, injective) factorization of $A \to G \times H \to G/\tau G \times H$; and finally let $A/\tau G \times \{1\} \twoheadrightarrow C \to H$ be the (surjective, injective) factorization of $A/\tau G \times$

$\{1\} \to G/\tau G \times H \to\to H$. If $(\bar{g}, h)^n \in A/\tau G \times \{1\}$, then there exists an element $(a, b) \in G \times H$ so that $(a, b) \in A$ and $(\bar{a}, \bar{b}) = (\bar{g}^n, h^n) \in A/\tau G \times \{1\}$. Hence $b = \bar{b} = h^n$ and a is mapped to $\bar{a} = \bar{g}^n$. Thus $a = g_n \cdot t$ where $t \in \tau G$. But then $(t^{-1}, 1) \in A$ and so $(g^n, h^n) \in A$; A isolated implies $(g, h) \in A$ and hence $(\bar{g}, h) \in A/\tau G \times \{1\}$. This shows that σ is an isolated-mono. By the uniqueness of the (surjective, injective) factorization, it is clear that B and C are isomorphic as subobjects of H. Since $G/\tau G$ is complete, it is i-compact, and so σ being an isolated mono implies that $C \to H$ is an isolated-mono. Thus B is an isolated subgroup of H and G is i-compact.

The class of i-compact locally nilpotent groups is also mono-coreflective in the class of all locally nilpotent groups. This follows from a theorem of Glushkov [12] (see also [15] p258) that the product of all complete subgroups of a torsion-free locally nilpotent group G is a complete (normal) subgroup, denoted dG. It is clear that $d(-)$ is a subfunctor of the identity on the class of locally nilpotent torsion-free groups. For an arbitrary locally nilpotent group G, let cG denote the inverse image of $d(G/\tau G)$ along the natural map $G \to G/\tau G$. Then it is clear that cG is i-compact and that $c(-)$ is a subfunctor of the identity on the class of all locally nilpotent groups.

Theorem 5 . *The functors $c(-)$ and $d(-)$ are idempotent radicals.*

Thus if G is a torsion-free locally nilpotent group, $d^2G = dG$, and $d(G/dG) = 0$. If G is an arbitrary locally nilpotent group, then $c^2G = cG$, and $c(G/cG) = 0$.

5 Completeness as a Completion.

In this section we point out that the class of torsion-free complete locally nilpotent groups is strongly-dense-mono *firmly* reflective in the category of torsion-free locally nilpotent groups. Theorem 1.2 implies that any two strongly-dense-mono embeddings of a group G into complete groups are canonically isomorphic, which is precisely the definition of *firmness* used in [3] and in [4]. In this manner, the Mal'cev localization behaves more like the completion of a metric space than like a compactification.

Theorem 6 . *(1) The class of torsion-free complete locally nilpotent groups is strongly-dense-mono firmly reflective in the class of torsion-free locally nilpotent groups.*

(2) The class of torsion-free complete locally nilpotent groups is strongly-dense reflective in the class of all locally nilpotent groups.

(3) If G belongs to the class of Baer (respectively, Fitting, hypercentral of class α, nilpotent of class k, abelian) groups, then so does its completion.

Proof. We prove only (3), (1) and (2) are immediate from Theorem 1.2. Let G be a locally nilpotent group. Since each class is closed under images, it suffices to assume that G is torsion-free. If G is hypercentral of class α (nilpotent of class k, or abelian), then G has an upper central series $1 \leq G_1 \leq \cdots \leq G_\alpha = G$, and so the completion G^* of G has the upper central series $1 \leq I(G_1) \leq \cdots \leq I(G_\alpha) = G^*$.

If G is Fitting, and $\{x_1, \cdots, x_n\} \subset G^*$ then there exist integers $k_1, \cdots, k_n$ so that $\{x_1^{k_1}, \cdots, x_n^{k_n}\} \subset N \subset G$ where N is a normal nilpotent subgroup. Then we have $\{x_1, \cdots, x_n\} \subset I(N) \subset G^*$ and $I(N)$ is normal and nilpotent. Thus G^* is Fitting. Similarly one shows that if G is Baer, then so is G^*.

If $f : G \to H$ is a strongly-dense monomorphism between locally nilpotent groups, then the induced map $G/\tau G \to H/\tau H$ is also a strongly-dense monomorphism, and so the induced map from the Mal'cev completion of $G/\tau G$ to the Mal'cev completion of $H/\tau H$ is an isomorphism. From these observations we have the following result.

Corollary 2 . *A torsion-free complete locally nilpotent group is injective with respect to strongly-dense-monomorphisms in the class of locally nilpotent groups.*

Indeed, it is easy to see either directly or by appealing to Theorem 2.5 of [4] that the class of torsion-free complete locally nilpotent groups is precisely the class of strongly-dense mono injectives in the category of torsion-free locally nilpotent groups. Finally, the topological result that a continuous bijection from a compact space to a Hausdorff space is necessarily a homeomorphism has a group theoretic interpretation.

Corollary 3 . *If $f : G \to H$ is a strongly dense monic from a τ-compact locally nilpotent group G to a torsion-free locally nilpotent group H, then f is an isomorphism.*

Proof. The strongly dense inclusion map $f(G) \to H$ splits since $f(G)$ is τ-compact torsion-free. Hence $H/f(G)$ is simultaneously torsion and torsion-free, thus trivial.

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