CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 34, nº 4 (1993), p. 321-346

<http://www.numdam.org/item?id=CTGDC_1993__34_4_321_0>

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VOL. XXXIV-4 (1993)

SIMPLICIAL HOMOLOGY OF PRECONVEXITY SPACES

by Robert J. MacG. DAWSON

RESUME. Une préconvexité sur un ensemble consiste en une famille de sous-ensembles (ensembles convexes) fermée par intersections arbitraires. Les morphismes des espaces à proximité sont les fonctions de Darboux, celles qui préservent les ensembles convexes. Ici, on définit un foncteur homologique simplicial sur la catégorie des espaces à préconvexité, et on examine ses propriétés.

1. Introduction and Definitions.

A *preconvexity* **K** on a set X is a family of subsets of X such that the intersection of any nonempty subfamily of **K** is a member of **K** . Such a pair (X,**K**) is called a *preconvexity space,* and the elements of **K** are called *convex sets.* Note that we have specifically not required that X itself be convex; if it is, we will call (X,**K**) a *convex* preconvexity space. If **K** is also closed under directed unions, we will call (X, **K**) a *convexity space.*

The canonical example of a convexity space is $\mathbb{R}^n$ with **K** taken to be the family of convex sets in the usual sense. The compact convex sets in $\mathbb{R}^n$ form a preconvexity, but not a convexity. Other examples are given in [2,4,8]. Convexity spaces and preconvexity spaces generalise the linear structure of Euclidean space in much the same way that topological spaces generalise its metric structure. In this paper, it will be shown that homological properties of some familiar spaces, conventionally thought of as arising from the topology of the spaces, can also be derived from their linear structure.

Convex preconvexity spaces may also be thought of as spaces equipped with closure operators, taking every subset to its "convex hull", the intersection of all the convex sets that contain it. If (X,**K**) is not convex, a subset may not be contained in any convex set, and thus may fail to have any convex hull. These spaces may be thought of as sets equipped with suitable partial closure operators [2].

A simple example of nonconvex convexity space is obtained by removing the unit ball from $\mathbb{R}^n$, and letting $\mathbf{K}$ consist of all convex sets in the remainder (Figure 1).

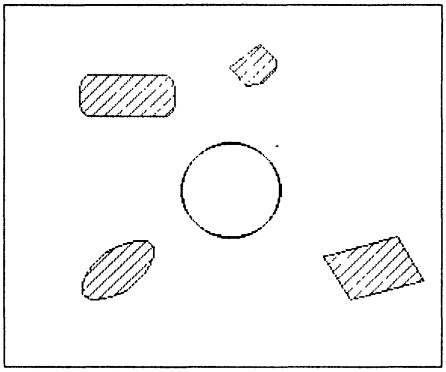


FIGURE 1

Two different types of morphisms between preconvexity spaces have been studied. Historically, the convexity-reflecting, or *monotone functions* (those such that the inverse image of any convex set is convex) have seniority [4,8]. However, as is shown in the next section, monotone functions do not generate a well-behaved homology functor, and we will obtain more interesting results using *Darboux functions* (those for which every direct image of a convex set is convex).

It is a categorical tradition to use different names for the same objects being acted upon by different maps, as many common mathematical constructions such as products depend not only on the internal structure of the objects but on the maps between them. Thus [2,4] we will henceforth reserve the names *(pre)convexity* and *(pre)convexity space* for objects and structures in the category whose morphisms are Darboux maps while the objects of the category with the "same" objects and monotone maps will be called *(pre)aligned spaces*, consisting of sets with *(pre)alignments.* The term alignment was introduced by Jamison-Waldner [8], who pioneered the categorical study of abstract convexity using (in the terminology of this paper) the category of convex aligned spaces.

We shall construct a homology functor on the category *Precxy* of preconvexity spaces and Darboux maps. In the third section of this paper, and thereafter, we will see that the restriction of this homology functor to *Cxy*, the category of convexity spaces, is particularly tractable, and essentially defines the functor over all of *Precxy*. Another important subcategory of *Precxy* is *DcPrecxy* , the category of *downclosed* preconvexity spaces. A preconvexity space is downclosed if every subset of a convex set is itself convex. These subcategories, and various functors associated with them which will be used later in this paper, are discussed in [4]; the reader is referred to that paper for those definitions and results.

Various separation axioms are useful in convexity theory. These correspond roughly to the separation axioms T_i of topology; the axioms used here are taken from [8], with the exception of the first, which is introduced here for the first time.

S_{-1}) *Every nonempty convex set contains a convex point.*

S_0) *Given two distinct points in* X, *at least one is contained in a convex set that does not contain the other.*

S_1) *Given two distinct points in* X, *each is contained in a convex set that does not contain the other.*

Other separation axioms are found in the literature, but will not be used here. It is easily seen that every point of an S_1 space is convex; thus S_1 implies both S_0 and S_{-1}. However, S_0 does not generally imply S_{-1}, although we have the following partial result:

PROPOSITION 1.1. *Any finite* S_0 *preconvexity space is* S_{-1}.

PROOF. Let A be a convex set in $(X,\mathbf{K})$; we will show that it has a convex singleton subset. Suppose not; then there exists a finite convex $B \subset A$ with minimal cardinality, containing two distinct points, x and y. By the S_0 axiom, one of these points is contained in a convex set C that does not contain the other; but then $B \cap C$ is a convex subset of B with smaller cardinality, a contradiction. ∎

This does not generalise to infinite preconvexity spaces, as shown by the example of the natural numbers, with the convexity consisting of all sets of the form $\uparrow j$. Also, S_{-1} does not imply S_0; a simple example is provided by the "trivial" or "dust" convexity on a set X with two or more points, in which no sets at all are convex. The resulting space, X_T, is trivially S_{-1}, but not S_0.

The "trivial" convexity construction, and the "power set" convexity construction in which all subsets of a set X are declared to be convex, are functorial. The functors $(-)_T$ and $(-)_P$ are, like the indiscrete and discrete topology functors into *Top*, left and right adjoint respectively to the forgetful functor to

Set. This implies the following proposition:

PROPOSITION 1.2. *The forgetful functor* U: *Precxy* $\to$ *Set preserves all limits and colimits that exist in Precxy.*

For the details of this proof, and other results on limits, colimits, and functorial constructions on *Precxy*, the reader is referred to [4].

2. Simplicial complexes and homology.

In this section, we consider three constructions for simplicial complexes in preconvexity spaces. Two of them resemble the Čech and Vietoris constructions in algebraic topology, while the third is a modification of the Čech construction; all three give rise to equivalent homology functors.

DEFINITION 2.1. A *Vietoris n-simplex* in a preconvexity space $(X,\mathbf{K})$ is an ordered $(n+1)$-tuple $(x_0, x_1, \dots, x_n)$ of points of X, contained in a common convex set. A *Čech n-simplex* in a preconvexity space $(X,\mathbf{K})$ is an ordered $(n+1)$-tuple $(A_0, A_1, \dots, A_n)$ of elements of $\mathbf{K}$, containing a common point. An *order n-simplex* in a preconvexity space $(X,\mathbf{K})$ is a Čech n-simplex in which $A_0 \subset A_1 \subset \dots \subset A_n$. These form, respectively, the *Vietoris simplicial compex* L^V, the *Čech simplicial complex* $\check{L}$, and the *order simplicial complex* $L^<$.

THEOREM 2.2. *The Vietoris, Čech, and order constructions for simplicial complexes in preconvexity spaces give rise to equivalent absolute (and relative) homology functors from the category of (pairs of) preconvexity spaces to the category of graded abelian groups.*

PROOF. The equivalence of the Čech and Vietoris homology constructions is most easily proved via Dowker's well-known construction in [6], in which generally defined Čech and Vietoris homologies on pairs of sets linked by a relation (in this case, X and $\mathbf{K}$, linked by the membership relation) are shown to be equivalent. The equivalence of the Čech and order constructions in homology is easily shown by constructing a chain map $\psi: C^\vee(X,\mathbf{K}) \to C^<(X,\mathbf{K})$ which, composed left or right with the inclusion $C^<(X,\mathbf{K}) \hookrightarrow C^\vee(X,\mathbf{K})$ is chain-homotopic to the identity. Let

$$\psi:\ (A_i: 0 \le i \le n) \mapsto \sum_{S_{n+1}} \Big(\bigcap_{j=i}^{n} A_{\sigma(j)}: 0 \le i \le n \Big)(-1)^\sigma$$

where the summation runs over the symmetric group on $(0,1,\dots,n)$ with the usual parity convention. This is a chain map, for:

$$\partial\psi(A_i\colon 0\le i\le n) = \sum_{k=0}^{n}\sum_{S_{n+1}}\Big(\bigcap_{j=i}^{n} A_{\sigma(j)}\colon 0\le i\le n,\ i\ne k\Big)(-1)^{\sigma}(-1)^{k}.$$

For any $k < n$,

$$\Big(\bigcap_{j=i}^{n} A_{\sigma(j)}\colon 0\le i\le n,\ i\ne k\Big) = \Big(\bigcap_{j=i}^{n} A_{((k\leftrightarrow k+1)\circ\sigma)(j)}\colon 0\le i\le n;\ i\ne k\Big)$$

as the only term in the two ordered sets that differs has been deleted. But, for any element σ of S_{n+1}, σ and $(k\leftrightarrow k+1)\circ\sigma$ have different parities; so every term in the outer summation with $k < n$ vanishes. Thus

$$\begin{aligned}\partial\psi(A_i\colon 0\le i\le n) &= \sum_{S_{n+1}}\Big(\bigcap_{j=i}^{n} A_{\sigma(j)}\colon 0\le i\le n-1\Big)(-1)^{n}(-1)^{\sigma}\\ &= \sum_{S_n}\sum_{k=0}^{n}\Big(\bigcap_{j=i}^{n-1} A_{\sigma\circ\gamma_k(j)}\colon 0\le i\le n-1\Big)(-1)^{k}(-1)^{\sigma}\end{aligned}$$

where

$$\gamma_k(j) = \begin{cases} j & : j < k\\ j+1 & : k\le j < n\\ k & : j = n\end{cases}$$

(Note that γ_k is the product of $(n-k)$ exchanges, and thus has the same parity as $(n-k)$.) But

$$\gamma_k(0,1,\dots,n-1) = (0,1,\dots,k-1,\dots,n);$$

so

$$\partial\psi(A_i\colon 0\le i\le n) = \psi\partial(A_i\colon 0\le i\le n).$$

It remains to show that ψ is chain-homotopic to the identity on $C^{<}(X,\mathbf{K})$ and on $C^{\vee}(X,\mathbf{K})$. We will construct an acyclic geometric carrier function that is defined on $C^{\vee}$ (and hence on $C^{<}$), and maps into $C^{<}$ (and hence into $C^{\vee}$). As ψ maps 0-simplexes to 0-simplexes, it is augmentable, so constructing this function will suffice.

Given any subset Y of $\mathbf{K}$ with nonempty intersection, let N(Y) be the family of all nested subsets of Y. Any element $A\in Y$ contains the intersection of Y, so N(Y) is the set of chains of a poset with a bottom element, and is thence acyclic. Therefore

$$P\colon (A_i\colon 0\le i\le n)\mapsto N\Big(\bigcap_{i\in I}(A_i\colon \varnothing\ne I\subset(0,1,\dots,n))\Big)$$

is an acyclic geometric carrier function $L^{\vee}\to L^{<}$; and it is easy to verify that it carries both ψ and the identity of $L^{\vee}$ or $L^{<}$. ∎

Finally, the homology construction is functorial on *Precxy*, as the image of a Vietoris simplex. There is not a generally-defined homology functor of this type on *Prealn*. There are several reasons for this. At the most obvious level, neither the image nor the inverse image of a simplex under a monotone map need be a simplex. More fundamentally, the standard constructions on *Prealn* behave in a rather ungeometric way. For instance, if X and Y are prealigned spaces without "dust" points, their coproduct is always connected. Thus, we would not expect to find any analogue to the Excision Theorem or to the Mayer-Vietoris sequence.

The following facts follow immediately from the homology construction of the previous theorem:

PROPOSITION 2.3. *If* $(X,\mathbf{K})$ *is convex,* $H_0(X,\mathbf{K}) = \mathbb{Z}$, *while all other homology groups are* 0.

PROPOSITION 2.4 *(Exactness). Given preconvexity spaces* $(X,\mathbf{K})$, $(Y,\mathbf{L})$ *and* $(Z,\mathbf{M})$, *with* $Z \subset Y \subset X$, $\mathbf{M} \subset \mathbf{L} \subset \mathbf{K}$, *there exists a long exact sequence in homology:*

$$\longrightarrow H_n((Y,\mathbf{L}),(Z,\mathbf{M})) \xrightarrow{i_*} H_n((X,\mathbf{K}),(Z,\mathbf{M})) \xrightarrow{j_*}$$
$$H_n((X,\mathbf{K}),(Y,\mathbf{L})) \xrightarrow{\partial} H_{n-1}((Y,\mathbf{L}),(Z,\mathbf{M})) \xrightarrow{i_*} \cdots$$

in which i_* *and* j_* *are induced by the inclusions* $Y \subset X$ *and* $Z \subset Y$.

PROPOSITION 2.5. $H_n(X_T) = 0$ *for all* n.

To illustrate the nature of the homology construction, we will now examine a few examples. The first example shows the similarities between the homology theory that we have just defined and the traditional one.

EXAMPLE 2.6. Let X be an open subset of the plane, with the usual convexity $\mathbf{E}$ (in which a set is convex if, with every two of its points, it contains the line segment joining them). Then $H(X,\mathbf{E}) = H(X)$, the singular topological homology group.

This is a special case of

EXAMPLE 2.7. Let X be an open subset of a Riemannian manifold M, with convexity $\mathbf{K}$ consisting of all subsets that contain, with every two of their points, the unique minimal geodesic joining them. Then $H(X,\mathbf{K})$ is isomorphic to the singular homology group of X considered as a topological subspace of M. To show this, we will construct a pair of chain maps between the Vieris convexity chain group and the singular topological chain group that are inverses up to chain homotopy.

THEOREM 2.7.1. *The convexity and topological homology groups of an open subset* X *of a Riemannian manifold* M *are isomorphic.*

PROOF. Let

$$S_n = \{\underline{s}\colon \underline{s} \in \mathbb{R}^{n+1},\ \Sigma s_i = 1,\ s_i \geq 0\}.$$

Then the singular simplexes in X are the continuous maps $\lambda\colon S_n \to X$. For each point $x \in \lambda S_n$, $d(x, M\setminus X) > 0$, so there exists $\varepsilon_x > 0$ such that $B(x,\varepsilon_x) \subset X$. Furthermore [1], we may choose ε_x small enough that $B(x,\varepsilon_x) \in \mathbf{K}$. $\lambda(S_n)$ is compact; hence a finite subset $\{B(x_i,\varepsilon_{x_i})\}$ of these open balls cover $\lambda(S_n)$. Thus, there exists a barycentric subdivision β of S_n fine enough that $\lambda\beta_n$ consists entirely of singular simplexes contained in these balls $\{B(x_i,\varepsilon_{x_i})\}$. This induces a chain map $\beta\colon C^s(X) \to C^s(X)$ which is chain-homotopic to the identity. But, for any simplex $\mu(S_n)$ of the subdivided simplicial complex, the vertices $(\mu s_i\colon 0 \leq i \leq n)$ are a Vietoris simplex of $(X,\mathbf{K})$. Call the induced map $\vartheta\colon C^s(X) \to C^V(X,\mathbf{K})$: it is invertible up to chain homotopy by the following map which "fills in" a singular simplex between any suitable set of vertices.

Let $\varphi(x_0)\colon s_0 \to x_0$; this is a singular 0-simplex. Assume that φ is defined in dimension n-1, and proceed inductively. For a Vietoris n-simplex $\underline{x} = (x_i\colon 0 \leq i \leq n)$, let $\varphi(x)$ map a point $\underline{s} = (s_j\colon 0 \leq j \leq n) \in S_n$ to the point of the arc from x_n to $\varphi(x_i\colon 0 \leq i \leq n\text{-}1)((s_j/1-s_n)\colon 0 \leq j \leq n\text{-}1)$ at a distance

$$(1-s_n)\, d(x_n, \varphi(x_i\colon 0 \leq i \leq n-1)((s_j/1-s_n)\colon 0 \leq j \leq n-1))$$

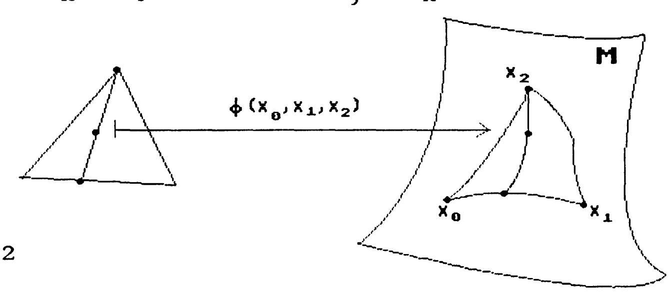


FIGURE 2

from x_n (see Figure 2). As $(x_i\colon 0 \leq i \leq n)$ has a convex hull in the unique minimal geodesic convexity, this point exists, is uniquely defined, and induces a continuous function from S_n to X. (Continuity follows from the continuity on such a region of geodesic polar coordinates; see, for instance, [1], §4.7). Furthermore, $\vartheta\varphi$ is the identity on $C^V(X,\mathbf{K})$, while $\varphi\vartheta$ is chain-homotopic to the identity on $\beta C^s(X)$. ■

Note that this result is not generally true on other subsets of Riemannian manifolds, or even of $\mathbb{R}^n$. A simple example is an arc of a circle in $\mathbb{R}^2$, which has trivial homology as a topological subspace of $\mathbb{R}^2$, while as a convexity subspace of $\mathbb{R}^2$ every point is a component and H_0 is thus uncountably generated. A more sphisticated example is the subset of the plane shown in Figure 3, which is the closure of its interior but is topologically connected, whereas the point 0 is a separate component in convexity.

FIGURE 3

A family of convexities on the circle, the ϑ-convexities, is obtained by defining arcs of length ϑ, their intersections, and the directed unions of those intersections to be convex. For $\vartheta < \pi$, $H(S^1,\mathbf{K}_\vartheta)$ takes the usual values of $\mathbb{Z}$ in dimensions 0 and 1, and 0 in all other dimensions. However, for larger values of ϑ, it is not difficult to show that certain higher homology groups become non-zero. For instance, it was shown in [2] that $H_1(S^1,\mathbf{K}_\pi) = 0$, while $H_2(S^1,\mathbf{K}_\pi)$ is uncountably generated. The calculation, which will not be reproduced here, is a lengthy but routine application of the Mayer-Vietoris sequence (see Section 5 of this paper). It is conjectured that for

$$2n\pi/(n+2) < \vartheta < (2n+2)\pi/(n+3),$$

$(S^1,\mathbf{K}_\vartheta)$ has the homology of the n-sphere, $\mathbb{Z}$ in dimensions 0 and n, and 0 in every other dimension. So far, however, I have been unable to compute this, and can only prove the following result:

PROPOSITION 2.8. $H_n(S^1,\mathbf{K}_\vartheta) \neq 0$ *for*

$$2n\pi/(n+2) < \vartheta < (2n+2)\pi/(n+3).$$

PROOF. Let Z^n be the $(n+2)$-point space with all proper subsets convex. It is easily verified that the n-simplexes of Z^n form a cycle, and this cannot be a boundary as Z^n contains no $(n+1)$-simplex. Thus $H_n(Z^n) \neq 0$.

Let the points of Z^n be $(0,1,\ldots,n+1)$, and let $f: Z^n \to S^1$ take j to $2j\pi/(n+2)$. Let $g: S^1 \to Z^n$ take the interval

$$[(2j\pi-1)/(n+2), (2j\pi+1)/(n+2)]$$

to j; then gf is the identity on Z^n, g is Darboux, and f is Darboux into the downclosure of $\mathbf{K}_\vartheta$ (see Section 3), so $H(f)$ is a monomorphism from a nontrivial group into $H_n(S^1,\mathbf{K}_\vartheta)$. ■

3. Downclosure and convexification.

Recall that a preconvexity space is *downclosed* if every subset of a convex set is convex. While this may seem to be a rather non-geometric property, it is possessed, for instance, by simplicial complexes with the usual convexity. As a result of this, it will be seen that the reflective subcategory *DcCxy* of *Precxy*, containing the downclosed convexity spaces, is the real arena in which the homology that we have defined takes place.

PROPOSITION 3.1. $H \circ Dc = H \circ Cx = H$.

PROOF. $(x_0, x_1, \ldots, x_n)$ is a Vietoris simplex in $Dc(X,\mathbf{K})$ or in $Cx(X,\mathbf{K})$ iff it is a Vietoris simplex in $(X,\mathbf{K})$. ■

DEFINITION 3.2. Given functors $F: C \to D$ and $G: C \to C$ such that $FG = F$, we say that *F absorbs G.* If F absorbs G and G absorbs every functor that F absorbs, we say that *F absorbs G universally.*

LEMMA 3.2.1. *If F absorbs G (universally) and G absorbs H (universally), then F absorbs H (universally).*

PROOF. If F absorbs G and G absorbs H, $FH = FGH = FG = F$. If F absorbs G universally, and G absorbs H universally, then in addition

$$FJ = F \Rightarrow GJ = G \Rightarrow HJ = H;$$

so F absorbs H universally. ■

THEOREM 3.3. *Let U: Precxy $\to$ Set be the forgetful functor; then (U,H): Precxy$\to$Set$\times$GrAb absorbs DcCx universally.*

PROOF. Neither Dc nor Cx change the underlying set of a preconvexity space; and, by Proposition 3.1, neither functor changes the homology. Thus, by Lemma 3.2.1, it follows that (U,H) absorbs $Dc \circ Cx$. It remains to show that this is universal.

To show this, note that a set A is convex in $DcCx(X,\mathbf{K})$ iff every finite subset of A is in $L^V(X,\mathbf{K})$. Hence, it suffices to show that if F is absorbed by U and by H, F is absorbed by

$L^V(-)$; in other words, that F does not create or destroy any Vietoris simplex.

We will first show, by induction on n, that F cannot destroy any n-simplex. For any n-simplex $(a_i : 0 \leq i \leq n)$ in $(X,\mathbf{K})$, let $(A,\mathbf{L})$ be the hull of $(a_i : 0 \leq i \leq n)$ with convexity

$$\mathbf{L} = \{\mathrm{hull}(V) : V \subset (a_i : 0 \leq i \leq n)\};$$

and let $j:(A,\mathbf{L}) \to (X,\mathbf{K})$ be the inclusion map. Note that j is Darboux. In the case $n = 0$, consider any 0-simplex $\{a_0\}$. $(A,\mathbf{L})$ is the hull in $(X,\mathbf{K})$ of $\{a_0\}$, with the indiscrete convexity. This is convex, and hence (Proposition 2.3) has $H_0 = \mathbb{Z}$. In order to preserve H_0, some subset B of $F(A,\mathbf{L})$ must be convex (otherwise $L^V = \varnothing$). But any epimorphism $f:(A,\mathbf{L}) \twoheadrightarrow (A,\mathbf{L})$ is Darboux; in particular, there exists such a function such that $a_0 \in f\mathrm{B}$. (As U absorbs F, $\mathrm{B} \subset \mathrm{A}$.) $jf:(A,\mathbf{L}) \to (X,\mathbf{K})$ is Darboux; so $F(jf)$: $F(A,\mathbf{L}) \to F(X,\mathbf{K})$ is also Darboux. As U absorbs F, $a_0 \in jf(\mathrm{B}) = F(jf(\mathrm{B}))$, which last is convex in $F(X,\mathbf{K})$. Thus, F preserves Vietoris 0-simplexes (Figure 4).

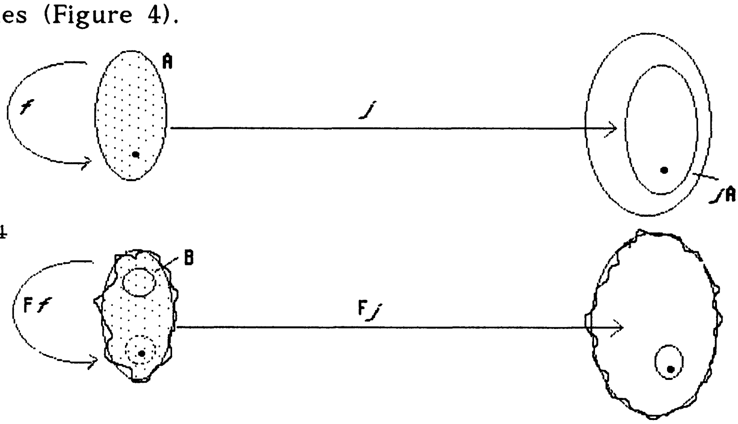


FIGURE 4

To continue the induction, let F preserve any $(n-1)$-simplex. Let $(a_i : 0 \leq i \leq n)$ be a Vietoris n-simplex in $(X,\mathbf{K})$, and let $(A,\mathbf{L})$ be as defined above. By hypothesis, the faces of the n-simplex $(a_i : 0 \leq i \leq n)$ have convex hulls in $F(A,\mathbf{L})$. In order that $H_{n-1}(F(A,\mathbf{L}))$ may equal $H_{n-1}(A,\mathbf{L})$, $(a_i : 0 \leq i \leq n)$ must also have a convex hull in $F(A,\mathbf{L})$; and, as the inclusion map Fj is Darboux, this maps to a convex set in $F(X,\mathbf{K})$ containing $(a_i : 0 \leq i \leq n)$ (Figure 5).

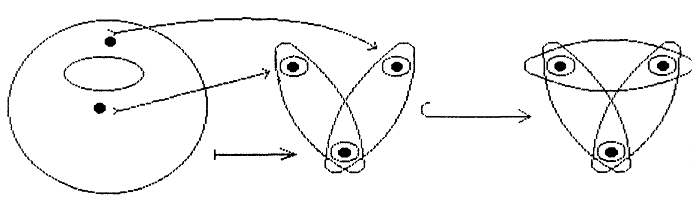
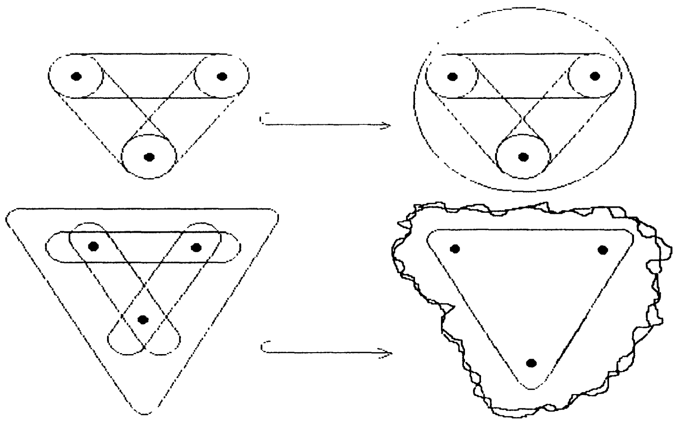


FIGURE 5

It remains to show that F does not create new Vietoris simplexes. Suppose $(a_i: 0 \leq i \leq n)$ is not a simplex in $(X,\mathbf{K})$. Let $A = (0,1,\dots,n+1)$ with the power set convexity, $A^* = (0,1,\dots,n+1)$ with all proper subsets convex, and $A^{**} = (0,1,\dots,n+1)$ with all proper subsets except for $(0,1,\dots,n)$ convex. Let $f: (X,\mathbf{K}) \to A^{**}$ take a_i to i, and every other point of X to $(n+1)$. As every subset of X except for X itself and $(a_i: 0 \leq i \leq n)$ maps to a convex subset of A^{**}, f is Darboux. Let $j: A^{**} \to A^*$ be the identity on the underlying set; this is clearly also Darboux. Thus Ff and Fj must be Darboux. F does not destroy Vietoris simplexes, so $F(A^*) = A^*$ or A. The latter is ruled out, as $H_n(A) \neq H_n(A^*)$. Fj is Darboux, so $F(A^{**})$ is not convex; thus $F(A^{**}) = A^{**}$ or A^*. Again, the latter is ruled out, as $H_n(A^*) \neq H_n(A^{**})$. But $(0,1,\dots,n)$ is not convex in A^{**}, and Ff is Darboux; so $(a_n: 0 \leq i \leq n)$ has no convex hull in $F(X,\mathbf{K})$ (Figure 6).

FIGURE 6



This theorem states, in effect, that we can simplify the study of homology in *Precxy* to that of homology in *DcCxy* without changing either the underlying set or the homology of any space; but that there does not exist any further simplification. As the homology of a space may be considered as the homology of its downclosure, another equivalent definition of homology suggests itself:

DEFINITION 3.4. A *singular n-simplex* in $(X,\mathbf{K})$ is a Darboux map $\lambda: (0,1,\dots,n)_P \to Dc(X,\mathbf{K})$. The boundary of λ is

$$\sum_{i=0}^{n} \lambda \circ d_i (-1)^i, \text{ where } d_i(j) = \begin{cases} j & \text{for } j < i \\ j+1 & \text{for } j \geq i \end{cases}$$

PROPOSITION 3.4.1. *The singular homology functor derived from* *the simplicial complex whose simplexes are defined in Definition* 3.4 *is naturally equivalent to the Vietoris homology functor.*

PROOF. If $(x_0, x_1, \dots, x_n)$ has a convex hull in $(X,\mathbf{K})$, then it and every subset of it are convex in $Dc(X,\mathbf{K})$, and hence λ: $i \to x_i$ is Darboux. Conversely, a map from a set with the power set convexity to $Dc(X,\mathbf{K})$ can only be Darboux if its image has a convex hull. ∎

As the singular homology funçtor defined above is not defined for any space for which the Čech and Vietoris homology functors are not defined, and as it agrees with them in every case, it will not be particularly important as an additional tool. However, it is worth defining, if only for its simplicity, and categorically nice presentation.

4. Contiguity and homotopy.

A fundamental result in algebraic topology is the Homotopy Theorem, which relates the topological and combinatorial structures. This was used by Eilenberg and Steenrod [7] as one of their axioms for homology, although "homotopy-free" axiomatisations of homology have been given by Kelly [9] and, for simplicial homology, by the present author [5]. The corresponding concept for discrete structures is "contiguity", defined below; we shall see that this extends naturally to an equivalence relation on maps that is very similar to homotopy in form.

DEFINITION 4.1. Two maps $f_0, f_1: (X,\mathbf{K}) \to (Y,\mathbf{L})$ are *contiguous* if, for every convex set $A \subset (X,\mathbf{K})$, $f_0(A) \cup f_1(A)$ has a convex hull in $(Y,\mathbf{L})$. If there exists a chain of maps $f_0, f_1, \dots, f_n: (X,\mathbf{K}) \to (Y,\mathbf{L})$ such that for each i, f_i and f_{i+1} are contiguous, we will call f_0 and f_n *homotopic.*

PROPOSITION 4.2. *Let* P_n *be the convexity space whose points are* $(0,1,\dots,n)$ *and whose convex sets are all points and all adjacent pairs of points. Then two Darboux maps* $f,g: (X,\mathbf{K}) \to (Y,\mathbf{L})$ *are homotopic iff for some* n *there exists a Darboux map*

$h: P_n \odot Dc(X,\mathbf{K}) \to Dc(Y,\mathbf{L})$ *such that* $h(0,-) = f$ *and* $h(n,-) = g$.

PROOF. If f and g are homotopic, with a chain of contiguous maps $f = f_0, f_1, \dots, f_n = g$, then let $h(i,-) = f_n(-)$; this is Darboux, as the convex sets of $P_n \odot Dc(X,\mathbf{K})$ are precisely the subsets of sets of the form $\{i, i+1\} \odot A$, where A is convex in $(X,\mathbf{K})$.

Conversely, given such an h, it follows that the maps $h(i,-)$ are Darboux and pairwise contiguous. ∎

PROPOSITION 4.3. *If* $f, g: (\mathrm{X},\mathbf{K}) \to (\mathrm{Y},\mathbf{L})$ *are homotopic, then* $f_*, g_*: \mathrm{H}(\mathrm{X},\mathbf{K}) \to \mathrm{H}(\mathrm{Y},\mathbf{L})$ *are equal.*

PROOF. It is sufficient to show that *contiguous* maps induce the same map in homotopy. By abuse of notation, let f and g also represent the maps that they induce from $\mathrm{C}^{\mathrm{V}}(\mathrm{X},\mathbf{K})$ to $\mathrm{C}^{\mathrm{V}}(\mathrm{Y},\mathbf{L})$. Let $(x_i: 0 \leq i \leq n)$ be a Vietoris simplex of $Dc(\mathrm{X},\mathbf{K})$; by hypothesis

$$\{fx_i: 0 \leq i \leq n\} \cup \{gx_i: 0 \leq i \leq n\}$$

is convex in $Dc(\mathrm{Y},\mathbf{L})$. Therefore,

$$\mathrm{P}:\ (x_i\ : 0 \leq i \leq n) \mapsto \mathrm{L}^{\mathrm{V}}(\{fx_i: 0 \leq i \leq n\} \cup \{gx_i: 0 \leq i \leq n\})_{\mathrm{P}}$$

is an acyclic carrier function which carries both f and g. ∎

DEFINITION 4.4. Two spaces $(\mathrm{X},\mathbf{K})$ and $(\mathrm{Y},\mathbf{L})$ are *homotopy equivalent* if there are Darboux maps

$$f: (\mathrm{X},\mathbf{K}) \to (\mathrm{Y},\mathbf{L}) \text{ and } g: (\mathrm{Y},\mathbf{L}) \to (\mathrm{X},\mathbf{K})$$

such that fg and gf are each homotopic to the identity on the appropriate space.

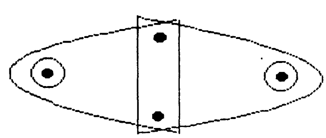
DEFINITION 4.5. A space which is homotopy equivalent to a point is called *contractible*. A space which is homotopy equivalent to a convex set is called *almost contractible*. A space which is the union of convex sets containing a common point x_0 is said to be *starshaped about* x_0.

LEMMA 4.5.1. *All starshaped* S_1 *spaces are contractible.*

PROOF. Let $(\mathrm{X},\mathbf{K})$ be starshaped about x_0. The maps $f: \{x_0\} \subset \mathrm{X}$ and $g: \mathrm{X} \to \{x_0\}$ form a homotopy equivalence, as fg is contiguous to the identity on X, while gf is the identity on x_0. ∎

Note, however, that the separation axiom S_1 (or at least some condition guaranteeing the convexity of the point x_0 about which $(\mathrm{X},\mathbf{K})$ is starshaped) is needed; Figure 7 shows a starshaped convexity space that is not even almost contractible.

FIGURE 7



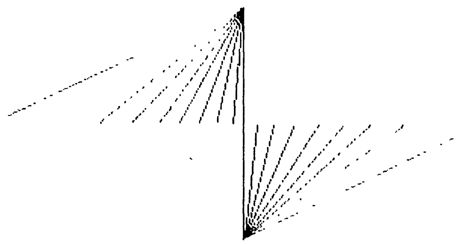
PROPOSITION 4.6. *Homotopy equivalent preconvexity spaces have isomorphic homology groups.*

PROOF. If (f,g) is a homotopy equivalence between a pair of spaces, then by Proposition 4,3, (f_*,g_*) is an isomorphism between their homology groups. ∎

COROLLARY 4.6.1. *A preconvexity space which is starshaped, contractible, or almost contractible has homology groups isomorphic to those of a point.*

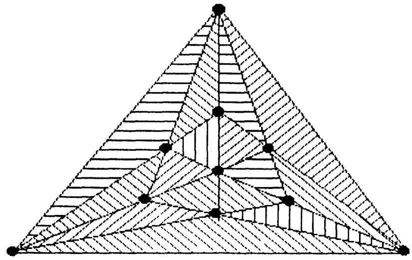
PROOF. The only part of the proof which requires comment is the proof for starshaped spaces; if $(X,\mathbf{K})$ is starshaped, $Dc(X,\mathbf{K})$ is starshaped and S_1, and has isomorphic homology groups. By Lemma 4.5.1, $Dc(X,\mathbf{K})$ is contractible, and thus has homology groups isomorphic to those of a point. ∎

FIGURE 8



As in topology, there are homologically trivial spaces which are not contractible. However, these pathologies occur for different reasons. A typical topological example is the "double broom" of Figure 8; a simple example of a convexity space which is not contractible is the "infinite path", consisting of $\mathbb{Z}$ with single points and adjacent pairs $(i,i+1)$ convex. The space in Figure 7, although starshaped, fails to be contractible because it is not S_1 and there do not exist any map taking convex points to the "middle". A third possibility is shown in Figure 9 (in which the triangles, edges, and vertices are convex); it is down-closed and homologically trivial, but fails to be contractible because there is no other 3-cycle contiguous to the outer one. This example demonstrates most specifically the difference between topological and convexity homotopy; the space cannot contract because of the "rigidity" of the outer boundary.

FIGURE 9



On the other hand, we can easily prove:

PROPOSITION 4.7. *A preconvexity space* $(X,\mathbf{K})$ *is connected iff* $H_0(X,\mathbf{K}) = \mathbb{Z}$. (See [4] for the definition of connectedness.)

PROOF. A 0-boundary in $C^V(X,\mathbf{K})$ consists of a formal sum $\Sigma a_i(x_i)$ where $\Sigma a_i = 0$ and the 0-simplexes (x_i) can be paired off into opposite ends of finite sequences $(x_0 = a_0, a_1, a_2, \dots, a_n = x_1)$, where all (a_i, a_{i+1}) have a convex hull. If any 0-cycle differs from an element of the free abelian group on a given 0-simplex (x_0) by a 0-boundary, this implies that every other vertex is in the same component of $(X,\mathbf{K})$ as x_0. Conversely, if every vertex is in the same component, $\Sigma a_i(x_i)$ differs from $(\Sigma a_i)(x_0)$ only by a 0-boundary. ∎

5. Excision.

If two topological spaces have a common subspace, we can construct their union, as a special case of the pushout construction in *Top*. In this section, we will use pushouts and pullbacks to define unions and intersections in *Precxy*. It may appear that we are keeping more generality than is really needed, but the extra generality will be important in the next chapter, when the analogues to the topological excision property that we develop below will be used as axioms for the homology of preconvexity spaces.

DEFINITION 5.1. In a bicartesian square of monomorphisms

$$\begin{tikzcd}
(X_0,\mathbf{K}_0) \arrow[r, tail, "f_1"] \arrow[d, tail, "f_2"'] & (X_1,\mathbf{K}_1) \arrow[d, tail, "g_1"] \\
(X_2,\mathbf{K}_2) \arrow[r, tail, "g_2"] & (X,\mathbf{K})
\end{tikzcd}$$

$(X_0,\mathbf{K}_0)$ is the *intersection of* $((X_1,\mathbf{K}_1),(X_2,\mathbf{K}_2),g_1,g_2,(X,\mathbf{K}))$, or, by abuse of language, the *intersection of* $(X_1,\mathbf{K}_1)$ *and* $(X_2,\mathbf{K}_2)$: while $(X,\mathbf{K})$ is the *union of* $((X_0,\mathbf{K}_0),f_1,f_2,(X_1,\mathbf{K}_1),(X_2,\mathbf{K}_2))$ (or *of* $(X_1,\mathbf{K}_1)$ *and* $(X_2,\mathbf{K}_2)$).

PROPOSITION 5.2. *A diagram of the form*

$$\begin{tikzcd}
(X_0,\mathbf{K}_0) \arrow[r, tail, "f_1"] \arrow[d, tail, "f_2"'] & (X_1,\mathbf{K}_1) \arrow[d, tail, "g_1"] \\
(X_2,\mathbf{K}_2) \arrow[r, tail, "g_2"] & (X,\mathbf{K})
\end{tikzcd}$$

is a bicartesian square in Precxy iff the squares

$$
\begin{tikzcd}
X_0 \arrow[r, "f_1"] \arrow[d, tail, "f_2"'] & X_1 \arrow[d, tail, "g_1"] \\
X_2 \arrow[r, tail, "g_2"] & X
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbf{K}_0 \arrow[r, "f_1"] \arrow[d, tail, "f_2"'] & \mathbf{K}_1 \arrow[d, tail, "g_1"] \\
\mathbf{K}_2 \arrow[r, tail, "g_2"] & \mathbf{K}
\end{tikzcd}
$$

are bicartesian in Set. (By a slight abuse of notation, f_1, f_2, g_1, g_2 also represent the mappings that they induce on the preconvexity.)

PROOF. Let $h_i: (Y,\mathbf{L}) \to (X_i,\mathbf{K}_i)$ $(i=1,2)$ be such that $g_1h_1 = g_2h_2$. Then, by hypothesis, there exists a unique $h: Y \to X_0$ such that $hf_i = h_i$, and this induces $h: \mathbf{L} \to \mathbf{K}_0$. Thus the square is a pullback square; the proof that it is a pushout square is similar. It remains to show that all bicartesian squares with monomorphic maps are of this form. The existence of the bicartesian square on the underlying sets follows from Proposition 1.2. The preconvexity on X_0 must contain all convex sets common to $\mathbf{K}_1$ and $\mathbf{K}_2$ so that the inclusions $h_i: (X_0, \mathbf{K}_1 \cap \mathbf{K}_2) \to (X_i, \mathbf{K}_i)$ may factor through it; and it cannot contain any others, otherwise the pullback injections $f_i: (X_0,\mathbf{K}_0) \to (X_i,\mathbf{K}_i)$ would not be Darboux. Thus the square in *Set* on the preconvexities is a pullback; the proof that it is a pushout is similar. ■

PROPOSITION 5.3. *A diagram*

$$
\begin{tikzcd}
(X_0,\mathbf{K}_0) \arrow[r, tail, "f_1"] \arrow[d, tail, "f_2"'] & (X_1,\mathbf{K}_1) \\
(X_2,\mathbf{K}_2) &
\end{tikzcd}
$$

extends to a bicartesian square if one of the following conditions is satisfied:

i) $(X_0,\mathbf{K}_0)$ *is convex and* f_1, f_2 *are subspace inclusions;*

ii) $(X_1,\mathbf{K}_1)$ *and* $(X_2,\mathbf{K}_2)$ *is downclosed and* f_1, f_2 *are subspace inclusions.*

PROOF. If f_1 and f_2 are subspace inclusions, and $g_i: X_i \to X_1 \cup X_2$ are the pushout injections,

$$g_1f_1\mathbf{K}_0 = g_2f_2\mathbf{K}_0 = g_1\mathbf{K}_1 \cup g_2\mathbf{K}_2.$$

It remains to show that $g_1\mathbf{K}_1 \cup g_2\mathbf{K}_2$ is a preconvexity on $X_1 \cup X_2$ – that is, that for $A_1 \in \mathbf{K}_1$, $A_2 \in \mathbf{K}_2$, $g_1A_1 \cap g_2A_2 \in g_1\mathbf{K}_1 \cup g_2\mathbf{K}_2$. In case (i),

$$g_1A_1 \cap g_2A_2 \subset g_1X_1 \cap g_2X_2 = g_1f_1X_0 = g_2f_2X_0.$$

Therefore,

$$g_1A_1 \cap g_2A_2 = g_1(A_1 \cap f_1X_0) \cap g_2(A_2 \cap f_2X_0).$$

As f_1 and f_2 are subspace inclusions, $A_1 \cap f_1X_0$, which is convex, has a convex preimage $C_1 \subset X_0$ under f_1; and similarly $A_2 \cap f_2X_0$ has a convex preimage C_2 under f_2. As $g_1f_1 = g_2f_2$,

$$g_1A_1 \cap g_2A_2 = g_1f_1(C_1 \cap C_2) \in g_1\mathbf{K}_1.$$

In case (ii), let, without loss of generality, $(X_1,\mathbf{K}_1)$ be downclosed. Then

$$g_1A_1 \cap g_2A_2 \subset g_1A_1 \in g_1\mathbf{K}_1$$

and so $g_1A_1 \cap g_2A_2 \in g_1\mathbf{K}_1$. Thus, extending the diagram on the underlying sets to a bicartesian square induces a bicartesian square on the preconvexities, and by the previous proposition we have a bicartesian square in *Precxy*. ∎

Our objective in this chapter is to obtain an excision theorem for bicartesian squares in *Precxy*. This cannot be done for all bicartesian squares of monomorphisms; as in *Top*, we will require additional conditions, although the conditions that we will need differ from the closure conditions encountered in algebraic topology. Consider the following example:

EXAMPLE 5.4. Let $X_0 = X_1 = X_2 = X$ be the two-point set, with preconvexities

$$\mathbf{K}_0 = \varnothing,\ \mathbf{K}_1 = \{\{1,2\}\},\ \mathbf{K}_2 = \{\{1\},\{2\}\},\ \mathbf{K} = \{\{1\},\{2\},\{1,2\}\};$$

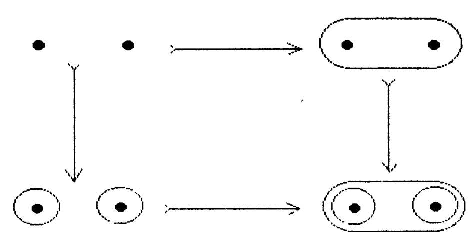
and let f_1, f_2, g_1, g_2 be the identity on the underlying set. Then

$$\begin{tikzcd}
(X_0,\mathbf{K}_0) \arrow[r, tail, "f_1"] \arrow[d, tail, "f_2"'] & (X_1,\mathbf{K}_1) \arrow[d, tail, "g_1"] \\
(X_2,\mathbf{K}_2) \arrow[r, tail, "g_2"] & (X,\mathbf{K})
\end{tikzcd}$$

is bicartesian; but (Figure 10)

$$H_0((X_1,\mathbf{K}_1),(X_0,\mathbf{K}_0)) = 0, \text{ while } H_0((X,\mathbf{K}),(X_2,\mathbf{K}_2)) = \mathbb{Z}^2.$$

FIGURE 10



This example shows that we can expect problems when we have simplexes (such as the 0-simplexes above) which are covered in different spaces by completely unrelated convex sets. This motivates the conditions in the next theorem.

THEOREM 5.5. *Let*

$$\begin{array}{ccc} (X_0,\mathbf{K}_0) & \xrightarrow{f_1} & (X_1,\mathbf{K}_1) \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ (X_2,\mathbf{K}_2) & \xrightarrow{g_2} & (X,\mathbf{K}) \end{array}$$

be bicartesian, with all edges mono, and suppose that one of the following is satisfied:

i) $(X_0,\mathbf{K}_0)$ *is convex;*

ii) $(X_1,\mathbf{K}_1)$ *and* $(X_2,\mathbf{K}_2)$ *are downclosed.*

Then $H((X_1,\mathbf{K}_1),(X_0,\mathbf{K}_0)) = H((X,\mathbf{K}),(X_2,\mathbf{K}_2))$.

PROOF. We must show that $L^V(X,\mathbf{K}) = L^V(X_1,\mathbf{K}_1) \cup L^V(X_2,\mathbf{K}_2)$ and that $L^V(X_0,\mathbf{K}_0) = L^V(X_1,\mathbf{K}_1) \cap L^V(X_2,\mathbf{K}_2)$, from which the theorem follows by a standard argument. A simplex of $L^V(X,\mathbf{K})$ is a finite subset of a convex set in $(X,\mathbf{K})$, which is in turn the image under one of the injections g_i of a finite subset of a convex set in $(X_i,\mathbf{K}_i)$. Thus,

$$L^V(X,\mathbf{K}) = L^V(X_1,\mathbf{K}_1) \cup L^V(X_2,\mathbf{K}_2).$$

If $(X_0,\mathbf{K}_0)$ is convex, $L^V(X_0,\mathbf{K}_0)$ contains all finite subsets of X_0. Furthermore, as f_1 and f_2 are Darboux, all finite subsets of X_1 that lie in the range of f_1 are simplexes in $L^V(X_1,\mathbf{K}_1)$: and similarly for X_2. Thus, in case (i),

$$L^V(X_0,\mathbf{K}_0) = L^V(X_1,\mathbf{K}_1) \cap L^V(X_2,\mathbf{K}_2).$$

In case (ii), suppose that the images f_1S and f_2S of a finite subset S of X_0 are both simplexes in the relevant spaces. Then, by downclosure, f_1S and f_2S are both convex; and by Proposition 5.2, S is convex and hence a simplex of $L^V(X_0,\mathbf{K}_0)$. ∎

COROLLARY 5.5.1 *(Mayer-Vietoris Sequence). Let* $(X_0,\mathbf{K}_0)$, $(X_1,\mathbf{K}_1)$, $(X_2,\mathbf{K}_2)$ *and* $(X,\mathbf{K})$ *be as in Theorem 5.5; then the Mayer-Vietoris sequence shown below (in which* $d(a) = (a,-a)$ *and* $s(b,c) = b+c$*) is exact.*

$$\xrightarrow{s} H_{n+1}(X,\mathbf{K}) \xrightarrow{\partial} H_n(X_0,\mathbf{K}_0) \xrightarrow{d} H_n(X_1,\mathbf{K}_1) \oplus H_n(X_2,\mathbf{K}_2) \xrightarrow{s} \cdots$$

We now have a powerful method for computing the homology of more complicated spaces from that of simpler ones, especially when used in conjunction with Propositions 3.1 and

4.3. However, there are limitations on the utility of this method, compared with that of the corresponding method on topology. Chief among these is that it is generally much more difficult to construct preconvexity spaces as unions of simpler spaces than it is to construct topological spaces in such a way; in particular, it is rarely possible to construct a preconvexity space as the union of two arbitrarily selected convex subspaces, while it is easy to build topological spaces as unions of open subspaces.

6. Axioms for the homology of preconvexity spaces.

Eilenberg and Steenrod [7] characterised the homology functor and boundary natural transformation on the category of pairs of compact topological spaces axiomatically. In this section, we will find a similar characterisation of homology for the category of pairs of preconvexity spaces, and show that on various subcategories of this category, weaker sets of axioms suffice. The axioms which we will use are:

CH1) (Dimension) *If* $(X,\mathbf{K})$ *is convex,* $H_0(X,\mathbf{K}) = \mathbb{Z}$ *and every other homology group of* $(X,\mathbf{K})$ *is* 0.

CH2) (Exactness) *The triangle*

$$\begin{tikzcd} & H((X,\mathbf{K}),(Y,\mathbf{L})) \arrow[dr, "\partial"] & \\ H(X,\mathbf{K}) \arrow[ur, "j_*"] & & H(Y,\mathbf{L}) \arrow[ll, "i_*"] \end{tikzcd}$$

is exact, where $i: Y \to X$ *and* j: $(X,\varnothing) \to (X,Y)$ *are the inclusion maps.*

CH3) (Excision) *The inclusion map*

$$i: ((X,\mathbf{K}),(X,\mathbf{K})\cap(Y,\mathbf{L})) \hookrightarrow ((X,\mathbf{K})\cup(Y,\mathbf{L}),(Y,\mathbf{L}))$$

induces an isomorphism in homology whenever $(X,\mathbf{K})\cap(Y,\mathbf{L})$ *is convex or* $(X,\mathbf{K})$ *and* $(Y,\mathbf{L})$ *are downclosed.*

CH4) (Triviality) $H_n(X_T) = 0$ *for all* n.

CH5) (Inductivity) H *preserves filtered colimits.*

CH6) (Comparability) *Any map between convex spaces induces an isomorphism in homology.*

The first three of these axioms correspond to axioms of Eilenberg and Steenrod, and suffice on their own to characterise the homology of finite S_1 downclosed preconvexity spaces. Adding in the triviality axiom allows us to extend this to non-S_1 finite downclosed preconvexity spaces, which differ only by "dust" points not contained in any convex set. The inductivity axiom extends the characterisation further, to all downclosed finitary preconvexity spaces; and finally, using the comparability axiom, we can extend it to all preconvexity spaces.

We must first show that the homology functor defined in §2 satisfies these axioms. (CH1,2,4) were proved in §2, (CH3) was proved in §5, and (CH6) follows from the fact that any map $f: X \to Y$ between convex spaces maps a generator of $H_0(X)$ to a generator of $H_0(Y)$. It remains to prove (CH5); this is not trivial, as the homology functor does *not* preserve *all* colimits.

PROPOSITION 6.1. *The Vietoris homology functor H preserves filtered colimits.*

PROOF. Let $\{q_j : (X_j, \mathbf{K}_j) \to (X, \mathbf{K})\}$ be a colimiting cocone under a filtered diagram in *Precxy* indexed by J. Then

$$\begin{aligned}(x_i : 0 \leq i \leq n) \in L^V(X,\mathbf{K}) &\Leftrightarrow (x_i : 0 \leq i \leq n) \in L^V(X,\mathbf{K}) \in Dc(\mathbf{K}) \\ &\Leftrightarrow \exists\, j \in ob(J),\ (x_i : 0 \leq i \leq n) \in Dc(\mathbf{K}_j) \text{ such that } x_i \in q_j(x_i) \\ &\Leftrightarrow (x_i : 0 \leq i \leq n) \in \operatorname{colim}_J L^V(X_j, \mathbf{K}_j).\end{aligned}$$

Suppose $a \in C^V(X,\mathbf{K})$. a is a formal sum $\sum_{i \leq m} a_i \lambda_i$, where λ_i is in $L^V(X,\mathbf{K})$. As we have just shown, any such simplex λ_i may be represented as $q_{j(i)}(\lambda'_i)$ for some $\lambda'_i \in L^V(X_{j(i)}, \mathbf{K}_{j(i)})$. But the diagram is filtered, so we may factor every $q_{j(i)}$ through some q_j and write

$$\lambda_i = q_i(\lambda''_i) \text{ for } \lambda''_i \in L^V(X_j, \mathbf{K}_j):$$

and thus $a = \sum_{i \leq m} a_i \lambda''_i$. Therefore, $C^V(X,\mathbf{K}) = \operatorname{colim}_J C^V(X_j, \mathbf{K}_j)$.

Let $z \in C^V(X,\mathbf{K})$. $z \in Z^V(X,\mathbf{K})$ iff $\partial z = 0$, $z = q_j z'$ for some $j \in ob(J)$, $z \in C^V(X_j, \mathbf{K}_j)$, so there must exist some Darboux morphism

$$f : (X_j, \mathbf{K}_j) \to (X_{j'}, \mathbf{K}_{j'}) \text{ such that } C^V(f)(\partial z') = 0.$$

Thus $\partial(C^V(f)(z')) = 0$: so there exists $z'' \in Z^V(X_{j'}, \mathbf{K}_{j'})$ such that $q_{j'}(z'') = z$. Thus, Z^V: *Precxy* $\to$ *GdAb* preserves filtered colimits.

Two elements $z_1, z_2 \in Z^V(X,\mathbf{K})$ are equivalent in homology iff they differ by a boundary. Thus if $z_1 = q_j z'_1$ and $z_2 = q_{j'} z'_2$, they are equivalent in homology iff there exist

$$\begin{gathered} k \in ob(J),\ f_1 : (X_j, \mathbf{K}_j) \to (X_k, \mathbf{K}_k),\ f_2 : (X_{j'}, \mathbf{K}_{j'}) \to (X_k, \mathbf{K}_k) \\ \text{and } a \in C^V(X_k, \mathbf{K}_k) \text{ such that } f_1 z'_1 - f_2 z'_2 = \partial a. \end{gathered}$$

But then $[f_1 z'_1] = [f_2 z'_2]$ in $H^V(X_k, \mathbf{K}_k)$, and consequently

$[z_1] = [z_2]$ in $\mathrm{colim}_J(X_j, \mathbf{K}_j)$. Thus, H preserves filtered colimits. ∎

DEFINITION 6.2. An *H-pair* on a category C is a pair (H,∂), where H is a functor from the category of pairs $(X \supset A)$ of objects of C to the category of graded abelian groups, and ∂ is a natural transformation of degree -1.

LEMMA 6.3. *The exactness of the Mayer-Vietoris sequence of any pair of spaces* $(X,\mathbf{K})$ *and* (Y,L) *whose intersection is convex or which are both downclosed is implied by* (CH2,3).

PROOF. See Eilenberg and Steenrod [7], I.15.3; note that no space or construction is used there which is not available under the hypotheses of this lemma. ∎

PROPOSITION 6.4. *There is a unique H-pair on $DcPrecxy_{1f}$, the category of downclosed finite S_1 preconvexity spaces, satisfying the axioms* (CH1-3).

PROOF. If, in the definition of a finite downclosed S_1 preconvexity space, we replace "point", "convex set", and "Darboux map" by "vertex", "simplex", and "simplicial map" respectively, we see that $DcPrecxy_{1f}$ is isomorphic to SC_f, the category of finite simplicial complexes, for which this result was proved in [3]. ∎

PROPOSITION 6.5. *There is a unique H-pair on $DcPrecxy_f$ satisfying the axioms* (CH1-4).

PROOF. Any downclosed preconvexity space $(X,\mathbf{K})$ is the disjoint union of an S_1 space $(X_1,\mathbf{K})$ and a set X^* of "dust" points contained in no convex set (and hence with the trivial convexity). By axiom (CH4), $H(X^*_T) = 0$; therefore, by Lemma 5.3 ,

$$H(X,\mathbf{K}) = H((X_1,\mathbf{K}) \cup X^*_T) = H(X_1,\mathbf{K}).$$

But, by the previous proposition, this is uniquely determined by (CH1-3). ∎

PROPOSITION 6.6. *There is a unique H-pair on FinDcPrecxy, the category of downclosed preconvexity spaces with finite convex sets, satisfying the axioms* (CH1-5).

PROOF. As shown in [4], every element of *FinDcPrecxy* is a filtered colimit of elements of $DcPrecxy_f$. Thus, by the axiom of inductivity, the homology of *FinDcPrecxy* is uniquely determined. ∎

In fact, (CH1-5) characterise the homology of a larger category, the category *Precxy*$_{-1}$ of S_{-1} preconvexity spaces. This can be proved directly; however, it is a simple corollary of the next theorem.

THEOREM 6.7. *There is a unique H-pair on Precxy satisfying axioms* (CH1-6).

PROOF. We are going to show that any homology theory on *Precxy* obeying (CH1-6) must absorb *Dc* and *Cx*, and thus relate the homology of *FinDcPrecxy* to that of *Cxy*.

LEMMA 6.7.1. *Let* (H,∂) *satisfy* (CH1-6) *on Precxy. Then*

$$H \circ Dc = H.$$

PROOF. Let $A \subset X$ be convex. Then $A_P \cup (X,\mathbf{K})$ is a preconvexity space with underlying set X, which may be thought of as a local downclosure of $(X,\mathbf{K})$. A_P and $A_P \cap (X,\mathbf{K})$ are both convex. We can thus construct a Mayer-Vietoris sequence for $(X,\mathbf{K}) \cup A_P$: we shall see that this generates an isomorphism between $H_n(X,\mathbf{K})$ and $H_n(A_P \cup (X,\mathbf{K}))$. For $n > 1$, this is trivial; the relevant section of the exact sequence is:

$$H_n(A_P \cap (X,\mathbf{K})) \to H_n(X,\mathbf{K}) \oplus H_n(A_P) \to H_n(A_P \cup (X,\mathbf{K})) \to H_{n-1}(A_P \cap (X,\mathbf{K})),$$

and as the first and last terms, and the second summand of the second term, are all 0 by the dimension axiom, the result follows.

To prove the isomorphism in dimensions 0 and 1, it is necessary to examine the sequence a bit more closely. In particular, consider the inclusion $j : (A_P \cap (X,\mathbf{K})) \hookrightarrow A_P$; by the axiom of comparability, j_* is an isomorphism. Thus,

$$i_* \oplus j_* : H_0(A_P \cap (X,\mathbf{K})) \hookrightarrow H_0(X,\mathbf{K}) \oplus H_0(A_P)$$

is a monomorphism. Therefore, we have in dimension 1:

$$0 \to H_1(X,\mathbf{K}) \oplus 0 \to H_1(A_P \cup (X,\mathbf{K})) \to H_0(A_P \cap (X,\mathbf{K})) \xrightarrow{i_* \oplus j_*} \cdots$$

and the map into $H_0(A_P \cap (X,\mathbf{K}))$ is the zero map, forcing isomorphism in dimension 1.

Furthermore, $i_* \oplus j_*$ is split mono, with left inverse $j_*^{-1}\pi_2$ (where π_2 is the projection onto the second summand). Therefore

$$0 \to H_0(A_P \cap (X,\mathbf{K})) \xrightarrow{i_* \oplus j_*} H_0(X,\mathbf{K}) \oplus H_0(A_P) \to H_0(A_P \cup (X,\mathbf{K})) \to 0$$

is split exact. As $H_0(A_P \cap (X,\mathbf{K})) = H_0(A_P)$, the other summand and the third term of the sequence must be isomorphic. Thus, we have isomorphism in dimension 0.

It is therefore possible to make all subsets of a convex set A convex without changing the homology. By repeating this process a few times, we can make all subsets of elements of any finite subset of **K** convex. The resulting partially downclosed spaces

$$\{(\mathrm{X},\mathbf{K}) \cup \bigcup_{\mathrm{A}\in\mathrm{F}} \mathrm{P}(\mathrm{A})) : \mathrm{F} \in \mathbf{F}(\mathbf{K})\}$$

are filtered by **F(K)**, and thus

$$H(Dc(\mathrm{X},\mathbf{K})) = H(\bigcup_{\mathrm{F}\in\mathbf{F}(\mathbf{K})} \{(\mathrm{X},\mathbf{K} \cup \bigcup_{\mathrm{A}\in\mathrm{F}} \mathrm{P}(\mathrm{A})) : \mathrm{F} \in \mathbf{F}(\mathbf{K})\}$$

$$= \operatorname{colim}_{\mathbf{F}(\mathbf{K})} H(\mathrm{X},\mathbf{K} \cup \bigcup_{\mathrm{A}\in\mathrm{F}} \mathrm{P}(\mathrm{A})) = H(\mathrm{X},\mathbf{K}). \quad \blacksquare$$

LEMMA 6.7.2. *Let* (H,∂) *satisfy* (CH1-6) *on Precxy. Then*

$$H \circ Cx = H.$$

PROOF. This proof proceeds much like the last one, locally convexifying over finite collections of directed families of convex sets. To avoid problems with unions that might not exist, we will prove the result for *DcPrecxy*; by the previous lemma, this is equivalent.

If $\{\mathrm{A}_\gamma : \gamma \in \Gamma\}$ is a directed family of convex sets in $(\mathrm{X},\mathbf{K})$ with union A, we can locally convexify $(\mathrm{X},\mathbf{K})$ to $(\mathrm{X},\mathbf{K}) \cup (\mathrm{A},\{\mathrm{A}_\gamma\} \cup \{\mathrm{A}\})$. $(\mathrm{A},\{\mathrm{A}_\gamma\} \cup \{\mathrm{A}\})$ is convex, and while its intersection with $(\mathrm{X},\mathbf{K})$ is not necessarily convex, it is the directed union of convex sets and thus, by the dimension and inductivity axioms, it has the same homology as a convex set. The sequence chasing and joining up the patches are done as in the previous lemma. ■

By Proposition 6.6, H is uniquely specified by axioms (CH1-5) on *FinDcPrecxy*. But, on *Precxy*,

$H \circ Fin \circ Dc$	$= H \circ Cx \circ Fin \circ Dc$	(Lemma 6.7.2)
	$= H \circ Cx \circ Dc$	([4], Theorem 3.10)
	$= H \circ Dc = H$	(Lemmas 6.7.2 and 6.7.1)

Thus, H is uniquely determined on *Precxy*. Finally, a unique extension of ∂ to *Precxy* is defined by the requirement that

$$\begin{tikzcd} \mathrm{H}(\mathrm{X}_j,\mathrm{A}_j) \arrow[r, "\partial_{\mathrm{X}_j,\mathrm{A}_j}"] \arrow[d, "(q_j)_*"] & \mathrm{H}(\mathrm{A}_j) \arrow[d, "(q_{j\mathrm{A}})_*"] \\ \mathrm{H}(\mathrm{X},\mathrm{A}) \arrow[r] & \mathrm{H}(\mathrm{A}) \end{tikzcd}$$

commutes for every map q_j in every colimiting cocone. ■

COROLLARY 6.7.3. *There is a unique H-pair on* $Precxy_{-1}$, *the category of* S_{-1} *preconvexity spaces, satisfying axioms* (CH1-5).

PROOF. The proof of Theorem 6.7 for any given space only uses axiom (CH5) as applied to subspaces of that space. It thus suffices, in light of Proposition 1.1, to show that for S_{-1} spaces, (CH1) $\Rightarrow$ (CH6).

For any convex S_{-1} space $(X,\mathbf{K})$, let p be a convex point in X, with inclusion $i: (\{p\},\{\{p\}\}) \hookrightarrow (X,\mathbf{K})$, and terminal map t: $(X,\mathbf{K}) \to (\{p\}, \{\{p\}\})$. Then ti is the identity, and so i_* is a split monomorphism from $\mathbb{Z}$ to $\mathbb{Z}$. Any such map must be an isomorphism. Similarly, given $f: (X,\mathbf{K}) \to (Y,\mathbf{L})$, with $(X,\mathbf{K})$ and $(Y,\mathbf{L})$ convex, $(fi)_*$ is an isomorphism; thus f_* is also an isomorphism. ∎

Having given a set of axioms characterising the homology functor on *Precxy*, it is natural to ask whether these axioms are all essential. The following examples justify the inclusion of various axioms in the set.

EXAMPLE 6.8. Let S_1 be the functor that makes every point convex; then $H \circ S_1$ obeys every axiom except for the axiom of triviality.

We can in fact, replace the dimension axiom in the form given here by the apparently weaker requirement that the homology group of a convex one-point space be $\mathbb{Z}$ in dimension 0 and 0 in all other dimensions; but this would be a rather trivial simplification, and would necessitate the use of axiom (CH6) in the proof of Propositions 6.4-6.6. Much more interesting is the question of whether we can eliminate axiom (CH6) in all cases. I claimed, incorrectly, in [2], that (CH1-5) were sufficient to characterise the homology of *Precxy*; this is in fact not the case, as the next example shows.

EXAMPLE 6.9. For any preconvexity space $(X,\mathbf{K})$, let $C^W(X,\mathbf{K})$ be the chain group generated by the Vietoris simplexes of $(X,\mathbf{K})$, with boundary map

$$\partial s = \sum_{i=0}^{n} \frac{w(s)}{w(\partial_i(s))} \partial_i(s)(-1)^i,$$

where $w(s) = 1$ if the simplex s contains a nonconvex vertex, and 2 otherwise, while ∂_i are the usual face maps. It is shown in [5] that this construction obeys the dimension, exactness, and excision axioms: the triviality axiom is obviously satisfied, and the inductivity axiom follows as in Proposition 6.1.

It is also shown in [5] that H^W is not the same as the homology functor H defined in this paper. For instance, the space shown in Figure 7 has $H^W_0 = \mathbb{Z} + \mathbb{Z}_2$ and $H_0 = \mathbb{Z}$; both of these can easily be confirmed by direct calculation.

This construction does not correspond to anything that can be done in the category *Top* of topological spaces and continuous maps. In *Top*, for any two points a and b in (possibly identical) spaces A and B, there exists a map $f\colon A \to B$ that takes a to b; in *Precxy*, this is not so. It is proved in [5] that a concrete category has a nontrivial functorial weighting on its points iff there is a pair of points (a,b) such that no morphism carries a to b. If such a weighting exists, it may give rise to a "homology theory" defined as in Example 6.9, in which boundaries appear with abnormally high coefficients, giving rise to unusual torsion elements in homology. For instance, Figure 11 shows the weighted simplicial complex corresponding to the space in Figure 7. The boundaries of its 1-simplexes are $2a-b$, $2a-c$, $b-c$, $b-2d$, $c-2d$; thus H^W_0 of this space has the presentation

$$(a,b,c,d\colon\ 2a = b = c = 2d),$$

which simplifies to $(a,c\colon\ 2a = 2c)$ and thence to $(a,e\colon 2e = 0)$, where $e = a-d$.

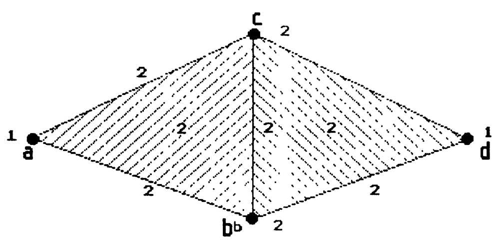


FIGURE 11

ACKNOWLEDGEMENTS. This research was supported by the Royal Commission for the Exhibition of 1851, and by the Killam Foundation.

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