CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

M. A. BATANIN

Coherent categories with respect to monads and coherent prohomotopy theory

Cahiers de topologie et géométrie différentielle catégoriques, tome 34, n° 4 (1993), p. 279-304

<http://www.numdam.org/item?id=CTGDC_1993__34_4_279_0>

CAHIERS DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE CATÉGORIQUES

VOL. XXXIV-4 (1993)

COHERENT CATEGORIES WITH RESPECT TO MONADS AND COHERENT PROHOMOTOPY THEORY

by M. A. BATANIN

RESUME. Le but de cet article est de développer une approche générale pour construire des catégories homotopiquement cohérentes. On présente une telle catégorie comme une catégorie de Kleisli pour une monade particulière. La méthode développée permet d'obtenir une équivalence entre la théorie de la forme forte de Lisica-Mardešič [15] et celle de Cathey-Segal [7].

1. Introduction.

There are different approaches for the strong shape theory of all topological spaces [2,7,15]. According to Lisica-Mardešič a strong shape category is a full subcategory of some special constructed category, CPH-Top, called the coherent prohomotopy category of topological spaces. The goal of our work is to give a general construction of the coherent homotopy categories. This leads us to the proof of the equivalence of the categories CPH-Top and ho(pro-Top) of Edwards-Hastings [12]. This implies that the strong shape category of Lisica-Mardešič is equivalent to that of Cathey-Segal [7].

There exists an immediate link between our work and the theory developed by J.-M. Cordier and T. Porter in a series of papers [8,9,10,11,25]. Some results of [10] may be obtained by directly dualizing our construction and applying them to the comonad Lan on F(Λ,K) (see §§2, 4 for notation), but in [8,10,11] this is considered within the general theory of homotopy coherence at a purely categorical level. This allows the authors to obtain deep results about the connection between different approaches to homotopy coherence [4,12,15,24,27], and give applications to strong shape theory [9,25].

Remark, finally, that the dual construction may be useful also in stong coshape theory [19] and, for instance, in the theory of iterated loop spaces (thus we obtain a natural structure of a comonad on a May bar construction [22,23]). It is possible also to find other obvious generalizations, for example, one can consider enriched Kan extensions [14] or the Bousfield-Kan R-completion on a category of diagrams of spaces, but in this paper we are not going to consider all possible applications.

2. The basic notion and definitions.

We shall need some definitions from enriched category theory [14]. In this section we introduce some notation and prove necessary auxiliary results. Let $\boldsymbol{A}$ be a monoidal category and K be an $\boldsymbol{A}$-category. We denote by $\mathrm{hom}^{\boldsymbol{A}}(X,Y)$ the set of morphisms from X to Y in $\boldsymbol{A}$. If $\boldsymbol{A}$ is closed [14], then $\mathrm{HOM}^{\boldsymbol{A}}(X,Y)$ is an internal HOM functor in $\boldsymbol{A}$ that is a representing object for $\mathrm{hom}^{\boldsymbol{A}}(-\otimes X,Y)$. Let HOM^{K} denote an $\boldsymbol{A}$-enrichment functor for K. For K, one can define an underlying category K_0. It has the same objects as K, and

$$\mathrm{hom}^{K_0}(X,Y) = \mathrm{hom}^{\boldsymbol{A}}(I,\mathrm{HOM}(X,Y)),$$

where I is the unit object in $\boldsymbol{A}$ [14].

We shall say that an $\boldsymbol{A}$-category K is *complete* (*cocomplete*) if the underlying category K_0 has all limits (colimits). Let $\boldsymbol{A}'$ be a subcategory of $\boldsymbol{A}$. We shall say that an $\boldsymbol{A}$-category K has *$\boldsymbol{A}'$-products* (*$\boldsymbol{A}'$-degrees*) provided there exist a functor

$$E\times X\colon \boldsymbol{A}'\times K_0 \to K_0 \qquad (X^E\colon (\boldsymbol{A})^{op}\times K_0 \to K_0)$$

and a natural isomorphism

$$\mathrm{HOM}^{K}(E\times X,Y) \approx \mathrm{HOM}^{\boldsymbol{A}}(E,\mathrm{HOM}^{K}(X,Y))$$
$$(\mathrm{HOM}^{K}(X,Y^E) \approx \mathrm{HOM}^{\boldsymbol{A}}(E,\mathrm{HOM}^{K}(X,Y)\ ).$$

For basic examples of symmetric monoidal categories, we consider the category of simplicial sets S, the category of Kan simplicial sets Kan, and the category of compactly generated spaces $\boldsymbol{K}$. We shall consider also the category Top of topological spaces with the natural enrichment

$$\mathrm{HOM}_n^{Top}(X,Y) = \mathrm{hom}^{Top}(X\times|\Delta(n)|,Y),$$

where

$$|\Delta(n)| = \{(u_1,\dots,u_n)\in\mathbb{R}^n \mid 0\le u_1\le \dots \le u_n\le 1\}$$

is the geometric realization of the standard n-simplex $\Delta(n)$ [5].

For a small category Λ and an $\boldsymbol{A}$-category K, we define $F(\Lambda,K)$ to be the category of functors from Λ to K_0. We denote its objects as $\{X_\lambda\}$, and consider it with a natural $\boldsymbol{A}$-enrichment $\mathrm{HOM}^{F(\Lambda,K)}(X,Y)$ defined to be the kernel of a pair of morphisms

$$\prod_{\lambda\in\Lambda}\mathrm{HOM}^{K}(X_\lambda,Y_\lambda) \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \prod_{\lambda\to\lambda'}\mathrm{HOM}^{K}(X_\lambda,Y_{\lambda'}),\ \varphi = \prod\varphi_\sigma,\ \psi = \prod\psi_\sigma,$$

where φ_σ with index $\sigma\colon\lambda\to\lambda'$ is defined as the composition

$$\prod_{\lambda\in\Lambda}\mathrm{HOM}^{K}(X_\lambda,Y_\lambda) \longrightarrow \mathrm{HOM}^{K}(X_\lambda,Y_\lambda) \xrightarrow{\mathrm{HOM}(1,\sigma)} \mathrm{HOM}^{K}(X_\lambda,Y_{\lambda'}),$$

and ψ_σ as

$$\prod_{\lambda\in\Lambda} \mathrm{HOM}^{\mathrm{K}}(\mathrm{X}_\lambda,\mathrm{Y}_\lambda) \longrightarrow \mathrm{HOM}^{\mathrm{K}}(\mathrm{X}_{\lambda'},\mathrm{Y}_{\lambda'}) \xrightarrow{\mathrm{HOM}(\sigma,1)} \mathrm{HOM}^{\mathrm{K}}(\mathrm{X}_\lambda,\mathrm{Y}_{\lambda'}).$$

If K has $\boldsymbol{A}'$-products or $\boldsymbol{A}'$-degrees, then $\mathrm{F}(\Lambda,\mathrm{K})$ has them also:

$$\mathrm{E}\times\{\mathrm{X}_\lambda\} = \{\mathrm{E}\times\mathrm{X}_\lambda\},\ \{\mathrm{X}_\lambda\}^{\mathrm{E}} = \{\mathrm{X}_\lambda^{\mathrm{E}}\}.$$

Suppose K has $\boldsymbol{A}'$-degrees, let $\mathrm{M}\colon \Lambda\to\boldsymbol{A}'$ be a functor and $\{\mathrm{X}_\lambda\}$ be an object of $\mathrm{F}(\Lambda,\mathrm{K})$. Then we can consider a kernel of the pair

$$\prod_{\lambda\in\Lambda} \mathrm{X}_\lambda^{\mathrm{M}_\lambda} \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \prod_{\lambda\to\lambda'} \mathrm{X}_{\lambda'}^{\mathrm{M}_\lambda}.$$

We call it a *realization of* $\{\mathrm{X}_\lambda\}$ with respect to M and denote it by $\{\mathrm{X}\}^{\mathrm{M}}$.

Let Δ be the category whose objects are finite ordered sets, $[n] = \{0,1,\dots,n\}$, and whose morphisms are the monotonic maps. For the categories $\mathrm{F}(\Delta,\mathrm{K})$ and $\mathrm{F}(\Delta\times\Delta,\mathrm{K})$, the categories of cosimplicial and of bicosimplicial objects of a category K, we use the standard notations cK and $\mathrm{c}^2\mathrm{K}$ [5] correspondingly. In cK one can consider a notion of cosimplicial homotopy [22,23]. A family of morphisms in K,

$$\mathrm{H}^i\colon \mathrm{X}^{q+1}\to \mathrm{Y}^q,\ 0\le i\le q,\ q=0,1,\dots,$$

is called a *cosimplicial homotopy* between cosimplicial morphisms $f,g\colon \mathrm{X}\to\mathrm{Y}$ provided

$$\mathrm{H}^0 d^0 = f^q, \qquad \mathrm{H}^q d^{q+1} = g^q$$

$$\mathrm{H}^i d^j = \begin{cases} d^j\mathrm{H}^{i-1} & \text{when } i > j \\ \mathrm{H}^i d^{j-1} & \text{when } i = j > 0 \\ d^{j-1}\mathrm{H}^i & \text{when } i > j-1 \end{cases}.$$

For an S-category K, there is a more "geometrical" notion of homotopy. We say that $f,g \in \mathrm{HOM}^{\mathrm{K}}_1(\mathrm{X},\mathrm{Y})$ are *homotopic* if there is a 1-simplex

$$h \in \mathrm{HOM}^{\mathrm{K}}_1(\mathrm{X},\mathrm{Y}) \text{ such that } d_0 h = f,\ d_1 h = g.$$

This relation generates an equivalence relation on $\mathrm{hom}^{\mathrm{K}_0}(\mathrm{X},\mathrm{Y})$. We shall denote the corresponding factor category by $\pi\mathrm{K}$, and $\mathrm{hom}^{\pi\mathrm{K}}(\mathrm{X},\mathrm{Y})$ by $[\mathrm{X},\mathrm{Y}]$. Thus, for an S-category K, we have in cK two types of homotopy. We need a lemma to connect these notions.

Let $\boldsymbol{\Delta}$ be the cosimplicial simplicial set which in codimension n consists of the standard n-simplex $\Delta(n)$ and for which the cofaces and codegeneracies are the standard maps between them. Let $\Delta[s]$ be the cosimplicial simplicial S-skeleton of $\mathbf{\Delta}$. Then in these notations, for a cosimplicial simplicial set X^* we get that

$$(X^*)^{\Delta[s]} \approx \mathrm{Tot}_s X^*$$

which is the Bousfield-Kan total space. Recall also that in cS there is a structure of a cosimplicial closed model category [5], and one can talk about fibrant and cofibrant objects in cS.

LEMMA 2.1. *Let* X^*, Y^* *be cosimplicial objects in an S-category* K, *and* f^*, $g^*: X^* \to Y^*$ *be cosimplicial morphisms. Assume there exist realizations* $(X^*)^\Delta$, $(Y^*)^\Delta$ *in* K, *and*

$$X^{*\wedge} = \mathrm{HOM}^{K}((X^*)^\Delta, X^*), \quad Y^{*\wedge} = \mathrm{HOM}^{K}((X^*)^\Delta, Y^*)$$

are fibrant cosimplicial simplicial sets. If f^* *and* g^* *are cosimplicial homotopic, then* $(f^*)^\Delta$ *and* $(g^*)^\Delta$ *are homotopic morphisms in* K.

PROOF. It suffices to show that f^* and g^* induce homotopic morphisms of simplicial sets

$$\begin{aligned} f,g\colon \mathrm{HOM}((X^*)^\Delta,(X^*)^\Delta) &= \mathrm{Tot}(\mathrm{HOM}((X^*)^\Delta,X^*) \\ &\to \mathrm{Tot}(\mathrm{HOM}((X^*)^\Delta,Y^*) = \mathrm{HOM}((X^*)^\Delta,(Y^*)^\Delta). \end{aligned}$$

Indeed, this means that there is

$$h\colon \mathrm{HOM}_0((X^*)^\Delta,(X^*)^\Delta) \to \mathrm{HOM}_0((X^*)^\Delta, ((Y^*)^\Delta)^{\Delta(1)})$$

such that $d^0 h = f$, $d^1 h = g$. Then $h(1_{(X^*)^\Delta})$ is the required homotopy. Further, since $X^{*\wedge}$, $Y^{*\wedge}$ are fibrant, we have homotopy equivalences

$$\mathrm{Tot}\, X^{*\wedge} \approx S(|X^*|^{|\Delta|}), \quad \mathrm{Tot}\, Y^{*\wedge} \approx S(|Y^*|^{|\Delta|}).$$

Therefore, we reduce the problem to the following statement. Let X^* and Y^* be cosimplicial simplicial topological spaces, $f^*, g^*: X^* \to Y^*$ be cosimplicial maps and h^* be a cosimplicial homotopy between them, then $(f^*)^{|\Delta|}$ and $(g^*)^{|\Delta|}$ are homotopic.

Let us construct a homotopy $H\colon (X^*)^{|\Delta|} \to ((Y^*)^{|\Delta|})^{|\Delta(1)|}$. Consider a subdivision of

$$|\Delta(1) \times |\Delta(n)| = \{(t,u_1,\dots,u_n) \mid 0 \le t \le 1,\ 0 \le u_1 \le \dots \le u_n\}$$

into subspaces

$$\mathbb{R}_n^p = \{(t,u_1,\dots,u_n) \mid 0 \le u_1 \le \dots \le u_p \le t \le u_{p+1} \le \dots \le u_n \le 1\}.$$

There is an obvious homeomorphism $\gamma_n^p\colon \mathbb{R}_n^p \to |\Delta(n+1)|$. Let $\varphi^*: |\Delta| \to X^*$ be a cosimplicial map. Then we put

$$H\varphi(u)_t = h^p \circ \varphi^{n+1} \circ \gamma_n{}^p(t,u), \text{ for } (t,u) \in \mathbb{R}_n{}^p.$$

The definition of cosimplicial homotopy implies that $H\varphi_t$ is defined correctly, and it is evident that

$$H\varphi(u)_0 = f^*\varphi, \; H\varphi(u)_1 = g^*\varphi.$$

Lastly we recall the notions of a monad on a category and of the corresponding Kleisli category [20].

Let $\boldsymbol{A}$ be a monoidal category, and K be an $\boldsymbol{A}$-category.

DEFINITION 2.1. Let R be an $\boldsymbol{A}$-endofunctor on an $\boldsymbol{A}$-category K, $\mu: R^2 \to R$ and $\varepsilon: I \to R$ be $\boldsymbol{A}$-natural transformations, where I is the identity $\boldsymbol{A}$-functor. We say that a triple (R, μ, ε) is an ***A-monad*** on K, with unit ε and multiplication μ, provided the following diagrams are commutative:

$$\begin{tikzcd} R^3 \arrow[r, "R\mu"] \arrow[d, "\mu \circ R"'] & R^2 \arrow[d, "\mu"] \\ R^2 \arrow[r, "\mu"] & R \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\varepsilon \circ R"] \arrow[dr, "1"'] & R^2 \arrow[d, "\mu"] & R \arrow[l, "R\varepsilon"'] \arrow[dl, "1"] \\ & R & \end{tikzcd}$$

One can associate, with any monad (R, μ, ε) on a category K, a category whose objects are those of K and whose set of morphisms from X to Y is $\hom^K(X, RY)$. The identity morphism is defined by $\varepsilon: X \to RX$ and the composition of $f: X \to RY$ and $g: Y \to RZ$ by

$$X \xrightarrow{f} RY \xrightarrow{Rg} R^2Z \xrightarrow{\mu} RZ.$$

DEFINITION 2.2. The above category is called the *Kleisli category of the monad* (R, ε, μ), and denote it by Kl_R-K.

We need finally a lemma from [5], that we call the Bousfield-Kan Lemma.

BOUSFIELD-KAN LEMMA. *Let* $R: K \to K$ *be a functor,* $\varepsilon: I \to R$ *be a natural transformation. Let there exist a natural associative pairing*

$$\mu_{X,Y,Z} : \hom(X, RY) \times \hom(Y, RZ) \longrightarrow \hom(X, RZ)$$

and let

$$\mu_{X,X,Y}(\varepsilon, f) = f = \mu_{X,Y,Y}(f, \varepsilon).$$

Then there is a natural transformation $\mu: R^2 \to R$ *such that* (R, μ, ε) *is a monad on* K.

PROOF. We define μ_X as the morphism $\mu_{R^2X, RX, X}(1_{R^2X}, 1_{RX})$.

3. Tensor product of cosimplicial objects and cosimplicial H-comonoids.

Let $\boldsymbol{A}$ be a category with finite colimits, and let $X = \{X^{p,q}\}$ be a bicosimplicial object in $\boldsymbol{A}$. Let us construct a sequence $B^n(X)$, $n \geq 0$ as follows:

$B^0(X) = X^{0,0}$,

$B^{n+1}(X)$ is the colimit of the diagram

$$\begin{tikzcd}
X^{0,n+1} & & & & \\
X^{0,n} \arrow[u, "X(1\times d^0) = d_2^0"] \arrow[r, "X(d^1\times 1) = d_1^1"'] & \cdots \arrow[r] & X^{p,q+1} & & \\
& & X^{p,q} \arrow[u, "d_2^0"] \arrow[r, "d_1^{p+1}"] & X^{p+1,q} & \\
& p+q=n & & \cdots \arrow[u] & \\
& & & X^{n,0} \arrow[u, "d_2^0"] \arrow[r, "d_1^{n+1}"] & X^{n+1,0}
\end{tikzcd}$$

For $0 \leq k \leq n+2$, consider a system of morphisms

$$(3.1)\qquad \begin{cases} d_2^{k-p} : X^{p,q} \to X^{p,q+1}, & p \leq k-1,\ p+q = n \\ d_1^{k} : X^{p,q} \to X^{p+1,q}, & p \geq k,\quad p+q = n \end{cases}$$

and for $0 \leq k \leq n+1$, a system

$$(3.2)\qquad \begin{cases} s_2^{k} : X^{p,q+1} \to X^{p,q}, & p \leq k,\quad p+q = n \\ s_1^{k} : X^{p+1,q} \to X^{p,q}, & p \geq k+1,\ p+q = n \end{cases}$$

PROPOSITION 3.1. *The systems* (3.1), (3.2) *define morphisms*

$$d^k : B^n(X) \to B^{n+1}(X) \text{ and } s^k : B^{n+1}(X) \to B^n(X)$$

correspondingly. The sequence $B^n(X)$ *with the morphisms* s^k *and* d^k *is a cosimplicial object in* $\boldsymbol{A}$.

PROOF. Immediate from the definition of $B^*(X)$.

Let now $\boldsymbol{A}$ be a closed symmetric monoidal category with tensor product $\otimes_{\boldsymbol{A}}$, unit object I and finite colimits. For two cosimplicial objects X and Y in $\boldsymbol{A}$, one can construct a bicosimplicial object

$$X\hat{\otimes}Y : (X\hat{\otimes}Y)^{p,q} = X^p \otimes_{\boldsymbol{A}} Y^q,\ d_1^i = d^i \otimes_{\boldsymbol{A}} 1,\ d_2^i = 1 \otimes_{\boldsymbol{A}} d^i,$$

$$s_1^i = s^i \otimes_{\boldsymbol{A}} 1, \quad s_2^i = 1 \otimes_{\boldsymbol{A}} s^i.$$

We shall denote the constant cosimplicial object $\{I^n\}$ by I. Then we define a tensor product of cosimplicial objects X and Y by the formula

$$X^* \otimes Y^* = B(X^* \hat{\otimes} Y^*).$$

PROPOSITION 3.2. *There exist natural isomorphisms*

$$I \otimes X \approx X \otimes I \approx X, \quad X \otimes (Y \otimes Z) \approx (X \otimes Y) \otimes Z,$$

which make $c\boldsymbol{A}$ *a monoidal category.*

Let now $\boldsymbol{A}$ be as above and let in addition $\boldsymbol{A}$ be a complete S-category with enrichment $HOM^{\boldsymbol{A}}: \boldsymbol{A}^{op} \times \boldsymbol{A} \to S$, with finite S-products. We shall say then that $\boldsymbol{A}$ is a monoidal S-category provided there is a natural isomorphism $k \times (X \otimes_{\boldsymbol{A}} Y) \approx (k \times X) \otimes_{\boldsymbol{A}} Y$ for each finite simplicial set k.

PROPOSITION 3.3. *Let* $\boldsymbol{A}$ *be a monoidal S-category. Then* $c\boldsymbol{A}$ *and* $c^2\boldsymbol{A}$ *are S-categorties with finite S-products and* $B: c^2\boldsymbol{A} \to c\boldsymbol{A}$ *is a simplicial functor. Furthermore, there are natural isomorphisms*

$$k \times (X \otimes Y) \approx (k \times X) \otimes Y \approx X \otimes (k \times Y)$$

for each finite simplicial set k *and* $X, Y \in \boldsymbol{A}$. *Finally we have a natural transformation*

$$D: HOM^{c\boldsymbol{A}}(E,F) \times HOM^{c\boldsymbol{A}}(X,Y) \to HOM^{c\boldsymbol{A}}(E \otimes X, F \otimes Y)$$

and a map $d: \Delta(0) \to HOM^{c\boldsymbol{A}}(I,I)$ *which make* $HOM^{c\boldsymbol{A}}: (c\boldsymbol{A})^{op} \times \boldsymbol{A} \to S$ *a monoidal functor.*

PROOF. We prove the last statement only, because the others are obvious. In $c\boldsymbol{A}$ we have a natural transformation

$$D_0: hom^{c\boldsymbol{A}}(E,F) \times hom^{c\boldsymbol{A}}(X,Y) \to hom^{c\boldsymbol{A}}(E \otimes X, F \otimes Y).$$

But

$$HOM_n^{c\boldsymbol{A}}(X,Y) = hom^{c\boldsymbol{A}}(\Delta(n) \times X, Y).$$

Define D_n by the composition

$$\begin{aligned} HOM_n^{c\boldsymbol{A}}(E,F) \times HOM_n^{c\boldsymbol{A}}(X,Y) &\to hom^{c\boldsymbol{A}}((\Delta(n) \times E) \otimes (\Delta(n) \times X), F \otimes Y) \\ &\longrightarrow HOM_n^{c\boldsymbol{A}}(E \otimes X, F \otimes Y). \end{aligned}$$

The last map is induced by

$$\delta \times 1: \Delta(n) \times (E \times F) \to \Delta(n) \times \Delta(n) \times (E \otimes F) \approx (\Delta(n) \times E) \otimes (\Delta(n) \times F),$$

where $\delta: \Delta(n) \to \Delta(n) \times \Delta(n)$ is the diagonal map.

As basic monoidal S-categories, we shall consider the categories S and $\boldsymbol{K}$. Note that the functor of fibrewise realization

$|-|: cS \to C\boldsymbol{K}$ is simplicial and strong monoidal.

DEFINITION 3.1. Let $\boldsymbol{A}$ be a monoidal simplicial S-category, and N a cosimplicial object in $\boldsymbol{A}$. Let $\rho: N \to N \otimes N$, $\eta: N \to I$ be two cosimplicial morphisms. We shall call a triple (N, ρ, η) a *cosimplicial* H-*comonoid* provided the following diagrams are commutative up to homotopy:

$$\begin{tikzcd} N\otimes N\otimes N & N\otimes N \arrow[l, "1\otimes\rho"'] \\ N\otimes N \arrow[u, "\rho\otimes 1"] & N \arrow[l, "\rho"] \arrow[u, "\rho"'] \end{tikzcd} \qquad \begin{tikzcd} N & N\otimes N \arrow[l, "1\otimes\eta"'] \arrow[r, "\eta\otimes 1"] & N \\ & N \arrow[ul, "1"] \arrow[u, "\rho"] \arrow[ur, "1"'] & \end{tikzcd}$$

THEOREM 3.1. *For each $s \geq -1$ there is a morphism*

$$\rho(s): |\Delta[s]| \overset{\cdot}{\to} |\Delta[s]| \otimes |\Delta[s]|$$

in $c\boldsymbol{K}$ *which, with* $\eta(s): |\Delta[s]| \to |\Delta[-1]|$ *makes* $|\Delta[s]|$ *into a cosimplicial* H-*comonoid.*

PROOF. The idea of the proof of this theorem is based on the Lisica-Mardesič construction of the composition of coherent maps [15]. Consider the following subdivisions of $|\Delta(n)|$ [15]. For $0 \leq \tau \leq 2$, $p, q \geq 0$, $p+q = n$, let

$P^{p,q}[\tau] = \{(u_1, \dots, u_n) \in \mathbb{R}^n \mid 0 \leq u_1 \leq \dots \leq u_p \leq \tau/2 \leq u_{p+1} \leq \dots \leq u_n \leq 1\}$.

For $0 \leq \tau \leq 1$, $r, s, t \geq 0$, $r+s+t = n$, let

$Q^{r,s,t}[\tau] = \{(u_1, \dots, u_n) \in \mathbb{R}^n \mid$

$0 \leq u_1 \leq \dots \leq u_r \leq (1+\tau)/4 \leq u_{r+1} \leq \dots \leq u_{r+s} \leq (2+\tau)/4 \leq u_{r+s+1} \leq \dots \leq 1\}$.

For $0 < \tau \leq 2$, let

$$\alpha^{p,q}[\tau](u_1, \dots, u_n) = (2/\tau)(u_1, \dots, u_n), \quad \alpha^{p,q}[\tau]: P^{p,q}[\tau] \to |\Delta(p)|.$$

For $0 \leq \tau < 2$, let

$$\beta^{p,q}[\tau](u_1, \dots, u_n) = (1/2-\tau)(2u_{p+1}-\tau, \dots, 2u_{p+q}-\tau),$$

$\beta^{p,q}[\tau]: P^{p,q}[\tau] \to |\Delta(p)|$. And finally let $\varphi^{r,s,t}[\tau]$, $\psi^{r,s,t}[\tau]$, $\pi^{r,s,t}[\tau]$ be the maps from $Q^{r,s,t}[\tau]$ to $|\Delta(r)|$, $|\Delta(s)|$, $|\Delta(t)|$ defined as follows:

$$\varphi^{r,s,t}[\tau](u_1, \dots, u_n) = (4/(1+\tau))(u_1, \dots, u_n),$$

$$\psi^{r,s,t}[\tau](u_1, \dots, u_n) = (4u_{r+1}-(1+\tau), \dots, 4u_{r+s}-(1+\tau)),$$

$$\pi^{r,s,t}[\tau](u_1, \dots, u_n) = (1/(2-\tau))(4u_{r+s}-(2+\tau), \dots, 4u_{r+s+t}-(2+\tau)),$$

correspondingly. Now we define a map $\rho: |\Delta(n)| \to B^n(|\Delta| \otimes |\Delta|)$ with the help of the formula

$$\rho^n(u) = b(\alpha^{p,q}[1](u), \beta^{p,q}[1](u)), \quad u \in P^{p,q}[1], \ p+q = n$$

where b is the canonical map to the colimit. The sequence of maps ρ^n defines a cosimplicial morphism $\rho: |\Delta| \to |\Delta| \otimes |\Delta|$. Consider a cosimplicial morphism $h: |\Delta(1)| \times |\Delta| \to |\Delta| \otimes |\Delta|$ defined by the formula

$$h^n(\tau, u) = b(\alpha^{p,q}[1+\tau](u), *), \quad u \in P^{p,q}[\tau].$$

Then $h^n(\tau, u) \in B^n(|\Delta| \otimes |\Delta[-1]|)$, and we have: for $u \in P^{p,q}[1]$,

$$h^n(0, u) = b(\alpha^{p,q}[1](u), *) = (1 \otimes \eta) \circ \rho(u),$$

for $u \in P^{p,q}[2]$, $h(1, u) = b(\alpha^{p,q}[2])$. But

$$|\Delta(n)| = P^{n,0}[2] \supset P^{n-1,n}[2] \supset \ldots \supset P^{0,n}[2] = *,$$

$\alpha^{n,0}[2] = 1: |\Delta(n)| \to |\Delta(n)|$. Then h is a homotopy between 1 and $(1 \otimes \eta) \circ \rho$. Similarly we have a homotopy between 1 and $(\eta \otimes 1) \circ \rho$. Finally the map

$$H(\tau, u) = b(\varphi^{r,s,t}[\tau](u), \psi^{r,s,t}[\tau](u), \pi^{r,s,t}[\tau](u)), \quad u \in Q^{r,s,t}[\tau],$$

$0 \leq \tau \leq 1$ gives us a homotopy between $(\rho \otimes 1) \circ \rho$ and $(1 \otimes \rho) \circ \rho$. Remark now that ρ, and the homotopies constructed above, map the s-skeleton of $|\Delta|$ to itself. Thus $\rho[s]: |\Delta[s]| \to |\Delta[s]| \otimes |\Delta[s]|$ is defined.

4. Coherent homotopy categories.

Let $\boldsymbol{A}$ be a monoidal S-category, and K an $\boldsymbol{A}$-category. We can define for K an "underlying" S-category K_s by the formula

$$HOM^{K_s}(X,Y) = HOM^{\boldsymbol{A}}(I, HOM^K(X,Y)).$$

Then we can define a category πK as πK_s. An $\boldsymbol{A}$-functor between $\boldsymbol{A}$-categories induces an S-functor between the "underlying" categories, the same holds for $\boldsymbol{A}$-natural transformations.

Let now (K, μ, ε) be an $\boldsymbol{A}$-monad on the category K. Then we have also an S-monad on K_s and a usual monad on πK. Let us associate with each object Y of K a cosimplicial object R^*Y. By definition

$$(R^*Y) = R^{n+1}Y, \; d^i = R^{n-i+1} \circ \varepsilon \circ R^i, \; s^i = R^{n-i} \circ \mu \circ R^i.$$

There is a natural augmentation morphism $\varepsilon: Y \to R^*Y$. We obtain also a cosimplicial object in the category $\boldsymbol{A}$ by applying $HOM^K(X,-)$ to R^*Y fibrewise. We shall denote it by $HOM^K(X, R^*Y)$. For objects X, Y and Z in K, consider the family of morphisms

$$m^{p,q}_{X,Y,Z}: HOM^K(X, R^{p+1}Y) \otimes_{\boldsymbol{A}} HOM^K(Y, R^{q+1}Z) \xrightarrow{1 \otimes R^{p+1}}$$
$$HOM^K(X, R^{p+1}Y) \otimes_{\boldsymbol{A}} HOM^K(R^{p+1}Y, R^{p+1} \circ R^{q+1}Z) \longrightarrow$$

$$HOM^K(X,R^{p+1}\circ R^{q+1}Z) \xrightarrow{HOM^K(1,R^p\circ\mu\circ R^q)} HOM^K(X,R^{p+q+1}Z).$$

This family gives a morphism

$$M_{X,Y,Z}: HOM^K(X,R^*Y)\otimes HOM^K(Y,R^*Z) \longrightarrow HOM^K(X,R^*Z).$$

Define also $E_X: I\to HOM^K(X,R^*X)$ by the composition

$$E^n_X \longrightarrow HOM^K(X,X) \xrightarrow{HOM^K(1,R^n\varepsilon)} HOM^K(X,R^{n+1}X)$$

for $n \geq 0$. It is easy to check that we thus obtain an enrichment of K in c$\boldsymbol{A}$.

Let now (N,ρ,η) be a cosimplicial H-comonoid in $\boldsymbol{A}$.

DEFINITION 4.1. We shall call the *coherent homotopy category of the monad* (R,μ,ε) with respect to (N,ρ,η) the category, denoted by CHR_N-K, defined as follows:

CHR_N-K has the same objects as K,

$CHR_N\text{-}K(X,Y) = \pi_0(HOM^{c\boldsymbol{A}}(N,HOM^K(X,R^*Y)),$

an identity morphism is the image of the point $*$ under the composition

$$* \longrightarrow \pi_0(HOM^{c\boldsymbol{A}}(I,I)) \xrightarrow{\pi_0 HOM^{c\boldsymbol{A}}(\eta,\varepsilon)} \pi_0(HOM^{c\boldsymbol{A}}(N,HOM^K(X,R^*X)),$$

a composition $\mu_{X,Y,Z}$ is defined by the map

$$CHR_N\text{-}K(X,Y)\times CHR_N\text{-}K(Y,Z) \xrightarrow{\pi_0(D)}$$
$$\pi_0(HOM^{c\boldsymbol{A}}(N\otimes N,HOM^K(X,R^*Y))\otimes HOM^K(Y,R^*Z))$$
$$\xrightarrow{\pi_0 HOM^K(\rho,M)} \pi_0(HOM^{c\boldsymbol{A}}(N,HOM^K(X,R^*Z)).$$

The category CHR_N-K admits another description.

PROPOSITION 4.1. *If in the category* K_s *there exists* $(R^*X)^N$ *for each object* X, *then we have a monad* $(R^N,\mu^N,\varepsilon^N)$ *on* πK, *such that* CHR_N-K *is isomorphic to* $Kl_{R^N}\text{-}\pi K$.

PROOF. Define $R^N(X) = (R^*X)^N$. The augmentation $\varepsilon: X\to R^*X$ induces a natural transformation $\varepsilon^N: I\to R^N$. We shall use the Bousfield-Kan Lemma for the definition of the multiplication in R^N. By construction $(R^*)^N$ is the kernel of the pair of morphisms

$$(4.1) \qquad \prod_{n\geq 0} (R^nY)^{N^n} \overset{\varphi}{\underset{\psi}{\rightrightarrows}} \prod_{[n]\to[n']} (R^{n'}Y)^{N^n}$$

Applying $\pi_0(\mathrm{HOM}^K(X,-))$ to (4.1), we obtain

$$\hom^{\pi K}(X,R^N Y) = \pi_0 \mathrm{HOM}^{K_s}(X,R^N Y)$$
$$= \pi_0 \mathrm{HOM}^{\boldsymbol{A}}(N,\mathrm{HOM}^K(X,R^*Y)) = \mathrm{CHR}_N\text{-}K(X,Y).$$

It is evident that for the identity functor 1 and the cosimplicial H-comonoid I in $c\boldsymbol{A}$, there is an isomorphism $\mathrm{CHI}_I\text{-}K \approx \pi K$. Then $\varepsilon: I \to R$ and $\eta: N \to I$ induce a canonical functor $P^N: \pi K \to \mathrm{CHR}_N\text{-}K$. Assume that the condition of Proposition 4.1 holds, then we obtain immediately that P^N has a right adjoint $Q^N: \mathrm{CHR}_N\text{-}K \to \pi K$, and $Q^N \circ P^N = R^N$.

The case $\boldsymbol{A} = S$ will play an important role in our work. Unfortunately the cosimplicial simplicial set $\boldsymbol{\Delta}$ is not a cosimplicial H-comonoid. To rectify this kind of difficulty we introduce some additional notions.

DEFINITION 4.2. We say that the S-monad (R,μ,ε) on an S-category K is *fibrant* provided $\mathrm{HOM}^K(X,R^*Y)$ is a fibrant object in cS for every objects X and Y in K.

DEFINITION 4.3. Let (N,ρ,η) be a cosimplicial object in S. We say that N is a *cofibrant cosimplicial* H-*comonoid* provided that N is a cofibrant object in cS and $|N|$ is a cosimplicial H-comonoid in $\boldsymbol{K}$.

Finally remark that for N cofibrant and X fibrant objects in cS, there is a natural homotopy equivalence

$$\mathrm{HOM}^{cS}(N,X) \approx \pi_0(\mathrm{HOM}^{c\boldsymbol{K}}(|N|,|X|).$$

Therefore we can now give a definition of the category $\mathrm{CHR}_N\text{-}K$ putting

$$\mathrm{CHR}_N\text{-}K(X,Y) = \pi_0(\mathrm{HOM}^{cS}(N,\mathrm{HOM}^K(X,R^*Y))),$$

where R is a fibrant monad and N is a cofibrant cosimplicial H-comonoid.

EXAMPLES. 1. As shown in [5], the cosimplicial simplicial sets $\Delta[s]$, $0 \leq s \leq \infty$ are cofibrant cosimplicial simplicial objects in cS.

2. Let R be a commutative ring with unit. For a simplicial set X, let $R\otimes X$ denote the simplicial R-module freely generated by the simplices of X. Then $RX \subset R\otimes X$ is the subset consisting of the simplices $\Sigma r_i x_i$ with $\Sigma r_i = 1$. There are maps $\varphi: I \to R$ and $\psi: R^2 \to R$, which make R into a fibrant S-monad on S [5]. We shall call it a *Bousfield-Kan monad.*

3. Let Λ be a small category, and K an S-category. As was

remarked above, $F(\Lambda,K)$ has a natural S-enrichment. If in addition K has products, then on $F(\Lambda,K)$ the following S-monad $(\mathrm{Ran},\mu,\varepsilon)$ can be defined:

Let Λ_0 be a maximum discrete subcategory of Λ. The inclusion $i:\Lambda_0\to\Lambda$ induces a functor $i^*: F(\Lambda,K)\to F(\Lambda_0,K)$. Then i^* has a right adjoint Ran_i (a right Kan extension along i) [20]. The pair (i^*,Ran_i) induces a monad $\mathrm{Ran}=\mathrm{Ran}_i\circ i^*$ on $F(\Lambda,K)$. More explicitly, $(\mathrm{Ran},\mu,\varepsilon)$ may be described by the formulas:

$$(\mathrm{Ran}\,X)_\lambda = \prod_{\lambda_0\leftarrow\lambda} X_{\lambda_0},\quad \varepsilon_\lambda : X_\lambda \to \prod_{\lambda_0\leftarrow\lambda} X_{\lambda_0} \quad \mu_\lambda: \prod_{\lambda_0\leftarrow\lambda_1\leftarrow\lambda} X_{\lambda_0} \to \prod_{\lambda_0\leftarrow\lambda} X_{\lambda_0}$$

where $\varepsilon_\lambda = \prod_\sigma \varepsilon_{\lambda,\sigma}$, $\varepsilon_{\lambda,\sigma}$ with index $\sigma:\lambda\to\lambda_0$ is the morphism $X(\sigma): X_\lambda\to X_{\lambda_0}$ and μ_λ is the projection on the factor with index $\lambda_1\leftarrow\lambda_0\leftarrow\lambda$.

DEFINITION 4.3. An S-category K is called a *locally Kan category* provided for any object X and Y in K, $\mathrm{HOM}^K(X,Y)$ is a Kan simplicial set.

PROPOSITION 4.2. *Let* Λ *be a small category, and* K *a locally Kan category. Then the monad* $(\mathrm{Ran},\mu,\varepsilon)$ *on* $F(\Lambda,K)$ *is fibrant.*

PROOF. Let $\tilde{\Lambda}_*$ be the nerve of Λ. Let $\tilde{\lambda}_p\in\tilde{\Lambda}_p$, $\tilde{\lambda}_p=(\lambda_0\leftarrow\ldots\leftarrow\lambda_p)$. Then we can write

$$\mathrm{HOM}^{F(\Lambda,K)}(X,\mathrm{Ran}^{p+1}Y) \approx \prod_{\tilde{\lambda}_p\in\tilde{\Lambda}_p} \mathrm{HOM}^K(X_{\lambda_p},Y_{\lambda_0}).$$

The codegeneracy s^i for the factor with index

$$\tilde{\lambda}_{p+1} = (\lambda_0\leftarrow\ldots\leftarrow\lambda_{p+1})$$

is defined as the composition

$$\prod_{\tilde{\lambda}_{p+2}\in\tilde{\Lambda}_p} \mathrm{HOM}^K_{\tilde{\lambda}_{p+2}}(Y_{\lambda_p},Y_{\lambda_0}) \to \mathrm{HOM}^K_{s_i(\tilde{\lambda}_{p+1})}(Y_{\lambda_{p+1}},Y_{\lambda_0})$$
$$\to \mathrm{HOM}^K_{\tilde{\lambda}_{p+1}}(X_{\lambda_{p+1}},X_{\lambda_0}).$$

We recall now the condition for a cosimplicial simplicial set X^* to be fibrant [5].

For $n\geq 0$ let M^n denote the limit of the following diagram

$$\begin{tikzcd} X^n_0 \ \cdots\cdots & X^n_i \ \cdots\cdots & & X^n_j \ \cdots\cdots\ X^n_n \\ & & X^{n-1}_{ij} \arrow[ul, "s^{j-1}"'] \arrow[ur, "s^i"'] & \end{tikzcd} \qquad 0\leq i<j\leq n.$$

For $n=-1$, one puts $M^{-1}(X)=\Delta(0)$. The maps $s^i: X^{n+1}\to X^n$ induce

$r^n: X^{n+1} \to M^n(X)$. Then, according to [5], X^* is fibrant iff all r^n, $n \geq -1$ are Kan fibrations. We prove the following lemma for application to $\mathrm{HOM}^{F(\Lambda,K)}(X, \mathrm{Ran}^* Y)$.

LEMMA 4.1. *Let $\tilde{\Lambda}_*$ be a simplicial set, $\{F_\lambda\}$ a family of sets with index from $\Lambda = \bigcup_{n \geq 0} \tilde{\Lambda}_n$. Let C be a category whose objects are subsets of Λ and whose morphisms are their inclusions. Consider the following cofunctor F from C to the category of sets: $F(U) = \prod_{\lambda \in U} F_\lambda$, and for $U \subset V$, the map $F(V) \to F(U)$ is the obvious projection.* Since in $\tilde{\Lambda}_*$ all degeneracies are inclusions, we can consider the following diagram

$$\begin{tikzcd}[column sep=small]
F(\tilde{\Lambda}_n^0) \arrow[r, dotted, no head] & F(\tilde{\Lambda}_n^i) \arrow[r, dotted, no head] & & F(\tilde{\Lambda}_n^j) \arrow[r, dotted, no head] & F(\tilde{\Lambda}_n^n) \\
& & F(\tilde{\Lambda}_{n-1}^{ij}) \arrow[ul, "F(s_{j-1})"] \arrow[ur, "F(s^i)"'] & &
\end{tikzcd} \tag{4.2}$$

The degeneracies $s_i: \tilde{\Lambda}_n^i \to \tilde{\Lambda}_{n+1}^i$, $0 \leq i \leq n$ induce a map from $F(\tilde{\Lambda}_{n+1})$ to the limit of the diagram (4.2). Then this map is a projection on a factor.

PROOF. Let $D_n(\Lambda)$ be a colimit of the diagram of sets

$$\begin{tikzcd}[column sep=small]
\tilde{\Lambda}_n^0 \arrow[r, dotted, no head] & \tilde{\Lambda}_n^i \arrow[r, dotted, no head] & & \tilde{\Lambda}_n^j \arrow[r, dotted, no head] & \tilde{\Lambda}_n^n \\
& & \tilde{\Lambda}_n^{ij} \arrow[ul, "s_{j-1}"] \arrow[ur, "s^i"'] & &
\end{tikzcd}$$

Then we have a map $\psi: D_n(\Lambda) \to \tilde{\Lambda}_{n+1}$ induced by the degeneracy maps $s_i: \tilde{\Lambda}_n^i \to \tilde{\Lambda}_{n+1}^i$, $0 \leq i \leq n$. Let us prove that ψ is an inclusion. Indeed, if $\tilde{\lambda}_i, \tilde{\lambda}_j \in \tilde{\Lambda}_n$ are such that $s_i(\tilde{\lambda}_i) = s_j(\tilde{\lambda}_j)$, then

$$\tilde{\lambda}_i = d_i s_j(\tilde{\lambda}_j) = s_{j-1}(d_i \tilde{\lambda}_j).$$

Thus we obtain

$$d_{j-1} \tilde{\lambda}_i = d_{j-1} s_{j-1}(d_i \tilde{\lambda}_j) = d_i \tilde{\lambda}_j.$$

If $j = i+1$, then $\tilde{\lambda}_j = d_j s_i(\tilde{\lambda}_i) = \tilde{\lambda}_i$, and if $j > i+1$ then

$$\tilde{\lambda}_j = d_j s_i(\tilde{\lambda}_i) = s_i(d_{j-1}(\tilde{\lambda}_i)) = s_i(d_i \tilde{\lambda}_j).$$

The subsequent part of the proof is evident.

Now Lemma 4.1 implies the conclusion of Proposition 4.2 because K is a locally Kan category.

Finally we remark that the above theory may be easy to develop for the pointed S_*-categories. Instead of $\Delta[s]$ we may use in this case the pointed cofibrant cosimplicial H-comonoids $\Delta_+[s] = \Delta[s] \cup \Delta(0)$.

5. Coherent homotopy categories and localization.

Here we consider S-categories and fibrant S-monads. As a cosimplicial H-comonoid, we shall use one of the cosimplicial simplicial sets $\Delta[s]$. For the category $\mathrm{CHR}_{\Delta[s]}$-K, we shall write simply CHR_s-K, the monad $\mathrm{R}^{\Delta[s]}$ will be denoted by R_s, the canonical functor from πK to CHR_s-K will be denoted P_s, and the adjoint to P_s by Q_s. Then we have a commutative diagram

(5.1)
$$\begin{tikzcd} & & \pi K \arrow[dll, "P_\infty"'] \arrow[d, "P_s"] \arrow[drr, "P_0"] & & \\ CHR_\infty\text{-}K \arrow[r] & \dots \arrow[r] & CHR_s\text{-}K \arrow[r] & \dots \arrow[r] & CHR_0\text{-}K = Kl_R\text{-}\pi K \end{tikzcd}$$

Let Σ_s be the class of morphisms f in πK such that $\mathrm{P}_s f$ is an invertible morphism, and let Σ_R be the class of morphisms f in πK such that $\mathrm{R}f\colon \mathrm{RX}\to\mathrm{RY}$ is invertible in πK.

PROPOSITION 5.1. $\Sigma_\infty = \dots = \Sigma_s = \dots = \Sigma_0 = \Sigma_\mathrm{R}$.

PROOF. Make one useful remark. Let $(\mathrm{R},\mu,\varepsilon)$ be a monad on K and let $\mathrm{EM_R}$-K be the category of R-algebras of this monad [20]. If $\mathrm{G}\colon \mathrm{K}\to\mathrm{M}$ is a functor with a right adjoint D, then it yields a monad $(\mathrm{D}\circ\mathrm{G},\rho,\varphi)$ on K. Assume $(\mathrm{R},\mu,\varepsilon)=(\mathrm{D}\circ\mathrm{G},\rho,\varphi)$. Then the following diagram is commutative [20]

$$\begin{tikzcd} & K \arrow[dl, "k"'] \arrow[d, "G"] \arrow[dr, "e"] & \\ Kl_R\text{-}K \arrow[r] & M \arrow[r] & EM_R\text{-}K \end{tikzcd}$$

where k and e are canonical functors. Let Σ_k, Σ_G, Σ_e be the classes of morphisms f in K such that the morphisms $k(f)$, $\mathrm{G}(f)$, $e(f)$ are invertible correspondingly. From the above diagram, we see that $\Sigma_k\subset\Sigma_\mathrm{G}\subset\Sigma_e$. Let $f\colon \mathrm{X}\to\mathrm{Y}$ be a morphism with $f\in\Sigma_e$. Then there exists $g\colon \mathrm{RY}\to\mathrm{RX}$ such that the diagram

$$\begin{tikzcd} R^2Y \arrow[r, "Rg"] \arrow[d, "\mu"'] & R^2X \arrow[d, "\mu"] \\ RY \arrow[r, "g"] & RX \end{tikzcd}$$

commutes and $\mathrm{R}f\circ g = 1_{\mathrm{RY}}$, $g\circ\mathrm{R}f = 1_{\mathrm{RX}}$. Consider the morphism g in $\mathrm{Kl_R}$-K, defined by the composition

$$Y \xrightarrow{\varepsilon} RY \xrightarrow{g} RX.$$

It is easy to see that g is the inverse to $k(f)$ in Kl_R-K. Consequently $\Sigma_k = \Sigma_G = \Sigma_e$.

Let us return to the proof of the proposition. From the remark above and the diagram (5.1), we see that

$$\Sigma_\infty \subset \ldots \subset \Sigma_s \subset \ldots \subset \Sigma_0 \subset \Sigma_R .$$

Let $f\colon X \to Y$ with $f \in \Sigma_R$. Then for each object Z in K, and $p \geq 0$, f induces a homotopy equivalence of simplicial sets:

$$HOM(1,R^{p+1}f) : HOM^K(Z, R^{p+1}X) \longrightarrow HOM^K(Z, R^{p+1}Y).$$

This means that we have a weak equivalence of fibrant cosimplicial simplicial sets:

$$Hom(1,R^* f)\colon HOM^K(Z,R^*X) \longrightarrow HOM^K(Z,R^*Y).$$

Therefore for each Z, the morphism f induces a bijection

$$CHR_\infty\text{-}K(Z,X) \longrightarrow CHR_\infty\text{-}K(Z,Y).$$

EXAMPLES. 1. Let (R,μ,ε) be the Bousfield-Kan monad. Then the class Σ_R consists of such $f\colon X \to Y$ that $Rf\colon RX \to RY$ is a homotopy equivalence. The last condition is equivalent to

$$H_* f\colon H_*(X,R) \longrightarrow H_*(Y,R)$$

being an isomorphism.

2. Let Λ be a small category and $\Lambda_0 \subset \Lambda$ be its maximum discrete subcategory. Since $Ran = Ran_i \circ [i^*]$, we have $\Sigma_{Ran} = \Sigma_{[i^*]}$, so it is a class of levelwise homotopy equivalences.

If $G\colon K \to M$ is a functor, with a right adjoint, then the class Σ_G satisfies the Gabriel-Zisman axioms for a calculus of left fractions and one can define a category $K[\Sigma_G^{-1}]$ [13]. By Proposition 5.1, we have the following commutative diagram of categories and functors

$$\begin{tikzcd}
\pi K \arrow[rr, "P_{\Sigma_R}"] \arrow[d, "P_\infty"'] & & \pi K[\Sigma_R^{-1}] \arrow[dll, "L_\infty"'] \arrow[d, "L_s"] \arrow[drr, "L_0"] & & \\
CHR_\infty\text{-}K \arrow[r] & \ldots \arrow[r] & CHR_s\text{-}K \arrow[r] & \ldots \arrow[r] & CHR_0\text{-}K
\end{tikzcd}$$

where P_{Σ_R}, L_s, $0 \leq s \leq \infty$ are the canonical functors [13].

THEOREM 5.1. *Let* (R,μ,ε) *be a fibrant S-monad on a S-category* K, *and suppose for some s there exists a monad* $(R_s,\mu_s,\varepsilon_s)$.

Then the functor L_s: $\pi K[\Sigma_R^{-1}] \to CHR_s\text{-}K$ *is an equivalence of categories iff for each object* X *in* K, *the morphism* $\varepsilon_s: X \to R_s K$ *belongs to* Σ_R.

PROOF. Assume L_s is an equivalence of categories. Then Q_s is fully faithful, and the morphism of adjunction, $\Phi: P_s Q_s \to I$, is an isomorphism [13], but the composition

$$P_s X \xrightarrow{P_s \varepsilon_s} P_s Q_s P_s X \xrightarrow{\Phi P_s} P_s X$$

is an identity morphism, hence $P_s \varepsilon_s$ is invertible in CHR_s-K and by Proposition 5.1, $\varepsilon_s \in \Sigma_R$.

Let now $\varepsilon_s \in \Sigma_R$. We shall show that Q_s is fully faithful. This will be sufficient for the proof of Theorem 5.1. Let $f: X \to R_s Y$ be a morphism in πK, that is a morphism from X to Y in $Kl_{Q_s P_s}\text{-}\pi K$. The functor Q_s maps it to the composition

$$R_s X \xrightarrow{R_s f} R_s^2 Y \xrightarrow{\mu_s} R_s Y.$$

The map $q: \pi K(X, R_s Y) \to \pi K(R_s X, R_s Y)$ induced by Q_s has a right inverse p, which is defined as follows: for $g: R_s X \to R_s Y$, p associates the composition

$$X \xrightarrow{\varepsilon_s} R_s X \xrightarrow{g} R_s Y.$$

It is evident that $p \circ q = 1$. To show that $q \circ p = 1$, it is sufficient to show that the following diagram is commutative in πK:

(5.2)
$$\begin{tikzcd} R_s^2 X \arrow[r, "R_s g"] & R_s^2 Y \arrow[d, "\mu_s"] \\ R_s X \arrow[u, "R_s \varepsilon_s"] \arrow[r, "f"] & R_s Y \end{tikzcd}$$

By Proposition 5.1, we have $R_s \varepsilon: R_s X \to R_s^2 X$ is an invertible morphism, therefore $\varepsilon_s R_s = R_s \varepsilon_s = \mu_s^{-1}$. Now the commutative diagram

$$\begin{tikzcd} R_s^2 X \arrow[r, "R_s f"] & R_s^2 Y \\ R_s X \arrow[u, "\varepsilon_s R_s"] \arrow[r, "f"] & R_s Y \arrow[u, "\varepsilon_s R_s"'] \end{tikzcd}$$

yields the commutativity of (5.2).

EXAMPLE. As was shown in [5] for the Bousfield-Kan monad on S_*, the unit $\varepsilon_\infty: X \to R_\infty X$ does not necessarily induce an isomorphism in R-homology. However on the category of nilpotent spaces we have an isomorphism [5]

$$(\varepsilon_\infty)_*: H_*(X,R) \longrightarrow H_*(R_\infty X,R).$$

THEOREM 5.2. *Let* (R,μ,ε) *be a fibrant* S*-monad on the* S*-category* K, *and suppose* R *preserves limits and* S*-degrees. If there exists a monad* $(R_\infty,\mu_\infty,\varepsilon_\infty)$, *then the functor* L_∞: $CHR_\infty\text{-}K \to \pi K[\Sigma_R^{-1}]$ *is an equivalence of categories.*

PROOF. From the conditions of the theorem, we have a homotopy equivalence of $RR_\infty(X)$ and $(R(R^*X))^\Delta$, where $R(R^*X)$ is a cosimplicial object obtained by fibrewise application of R to R^*X. Let $(RX)^*$ be the constant cosimplicial object with $(RX)^n = RX$. Then ε induces a cosimplicial morphism $R\varepsilon$: $(RX)^* \to R(R^*X)$. Thus there exists a cosimplicial morphism $\rho: R(R^*X) \to (RX)^*$ such that $\rho \circ R\varepsilon = 1$ and there exists a cosimplicial homotopy between $R\varepsilon\circ\rho$ and 1 [22]. Then for any object Z in K, we have a bijection

$$[1,R\varepsilon_\infty]: [Z,RX] = \pi_0(\mathrm{Tot}(\mathrm{HOM}^K(Z,(RX)^*))) \longrightarrow$$
$$\pi_0(\mathrm{Tot}(\mathrm{HOM}^K(Z,R(R^*X))) = [Z,RR_\infty X],$$

and hence $\varepsilon_\infty \in \Sigma_R$.

COROLLARY 1. *Let* K *be a locally Kan* S*-category, and* Λ *a small category. If there exists a monad* $(\mathrm{Ran}_\infty,\mu_\infty,\varepsilon_\infty)$ *on* $F(\Lambda,K)$, *then* $CH\mathrm{Ran}_\infty\text{-}F(\Lambda,K)$ *is the localization of* $\pi F(\Lambda,K)$ *with respect to levelwise homotopy equivalences.*

COROLLARY 2. *Let* K *be a simplicial closed model category,* K_f *be its full subcategory of fibrant objects, and suppose each object of* K *is cofibrant. If there exists* Ran_∞ *on* $F(\Lambda,K_f)$, *then the categories* $CH\mathrm{Ran}_\infty\text{-}F(\Lambda,K_f)$ *and* $\mathrm{ho}(K^\Lambda)$ *of Edwards-Hastings* [12] *are equivalent.*

COROLLARY 3. $CH\mathrm{Ran}_\infty\text{-}F(\Lambda,\mathrm{Top})$ *is equivalent to the category* $\varkappa\Lambda$ *of Vogt* [27].

Consider now the category naturally associated with a coherent category. This category has the same objects as K. As set of morphisms, we take the kernel of the pair of morphisms

$$[X,RY] \underset{[1,d^1]}{\overset{[1,d^0]}{\rightrightarrows}} [X,R^2Y].$$

We shall denote this category by W_R-K. By construction we have a natural functor $W_0: W_R\text{-}K \to CHR_0\text{-}K$. We can also construct functors $W_s: CHR_s\text{-}K \to W_R\text{-}K$ for each $s \leq \infty$. Indeed, if X^* is a cosimplicial simplicial set, then we have a map p:

$\pi_0 \mathrm{Tot}_1 X^* \to \pi_0 \mathrm{Tot}_0 X^*$. From the definition of Tot_s we obtain $d^0 p = d^1 p$, where $d^0, d^1 : \pi_0 X^0 \to \pi_0 X^1$ are induced by the coface maps. The remark above applied to HOM(X,R*Y) yields a commutative diagram of categories and functors

(5.3)
$$\begin{tikzcd} \pi K \arrow[rr, "W"] \arrow[d, "P_s"'] & & W_R\text{-}K \arrow[d, "W_0"] \\ CHR_s\text{-}K \arrow[r] \arrow[urr, "W_s"] & \dots \arrow[r] & CHR_0\text{-}K \end{tikzcd}$$

Let Σ_R^W denote the class of morphisms f in πK such that $W(f)$ is invertible. From (5,3) we see that

$$\Sigma_\infty = \dots = \Sigma_s = \dots = \Sigma_0 = \Sigma_R = \Sigma_R^W.$$

PROPOSITION 5.2. *A morphism* $f: X \to Y$ *in* CHR_s-K *is invertible iff* $W_s f$ *is invertible in* W_R-K.

PROOF. As above, let Q_s be adjoint to P_s. Then $P_s Q_s f = f$, but

$$W(Q_s f) = W_s(P_s Q_s f) = W_s(f)$$

is invertible by assumption. This means that $Q_s f \in \Sigma_R^W = \Sigma_s$, and consequently $P_s Q_s f = f$ is invertible.

EXAMPLES. 1. For the Bousfield-Kan monad on S_*, we have that W_R-$S_*(X,Y)$ is $E_2^{0,0}$ term of the unstable Adams spectral sequence [6].

2. W_{Ran}-$F(\Lambda,K) \approx F(\Lambda,\pi K)$.

Finally remark that in any S_*-category, we have a spectral sequence of Bousfield-Kan type

$$\pi^p(\Sigma^q X, R^* Y) \Rightarrow CHR_\infty\text{-}K(\Sigma^{q-p} X, Y).$$

Here Σ is the suspension functor.

6. Coherent prohomotpy categories.

Now we shall expand the above constructions on a category pro-K. Let us recall some definitions. As usually we can consider an ordered set Λ as a small category. Then an inverse system over Λ in a category K is defined as a functor on Λ to K. We shall denote an inverse system by $X = \{X_\lambda\}_{\lambda \in \Lambda}$ or simply $\{X_\lambda\}$, and the morphism corresponding to $\lambda \leq \lambda'$ by $P_{\lambda\lambda'}: X_{\lambda'} \to X_\lambda$. A directed set Λ is called cofinite if each element of Λ has only finitely many predecessors.

Let Λ, M be directed cofinite sets and let $\varphi: M \to \Lambda$ be an increasing function. Then we have a functor $\varphi^*: F(\Lambda,K) \to F(M,K)$. We define then a morphism from the inverse system $\{X_\lambda\}_{\lambda\in\Lambda}$ to $\{Y_\mu\}_{\mu\in M}$ as a pair (φ,f), where f is a morphism $\varphi^*\{X_\lambda\} \to \{Y_\mu\}$ in $F(M,K)$. A morphism (φ,f) is called a *level morphism* provided $\Lambda = M$ and $\varphi = 1_\Lambda: \Lambda \to \Lambda$. We have thus a category inv-K, whose objects are all inverse systems in K and whose morphisms are morphisms of inverse systems.

Let (φ,f), $(\psi,g): \{X_\lambda\} \to \{Y_\mu\}$ be two morphisms of inverse systems. The morphism (ψ,g) is said to be *congruent to* (φ,f) provided $\psi \geq \varphi$ and for for each $\mu \in M$ the following diagram commutes

$$\begin{tikzcd}
X_{\psi(\mu)} \arrow[d, "P_{\varphi(\mu)\psi(\mu)}"'] \arrow[dr, "g_\mu"] & \\
X_{\varphi(\mu)} \arrow[r, "f_\mu"] & Y_\mu
\end{tikzcd}$$

The category pro-K will be the following category. The objects of pro-K are all inverse systems in K over cofinite directed sets. A morphism $f: X \to Y$ is an equivalence class of morphisms of inverse systems with respect to the equivalence relation generated by the relation of congruence above. As is proved in [21], this definition of pro-K is equivalent to the usual definition of

$$\text{pro-K}(\{X_\lambda\},\{Y_\mu\}) = \lim_\mu \operatorname{colim}_\lambda \hom^K(X_\lambda, Y_\mu).$$

A last formula prompts the definition of S-enrichment for pro-K, provided K is an S-category [12] :

$$\mathrm{HOM}^{\text{pro-K}}(\{X_\lambda\},\{Y_\mu\}) = \lim_\mu \operatorname{colim}_\lambda \mathrm{HOM}^K(X_\lambda, Y_\mu).$$

If K has S-degrees, then pro-K also has S-degrees:

$$\{X_\lambda\}^E = \{X_\lambda^E\} \text{ and } \mathrm{HOM}_n^{\text{pro-K}}(\{X_\lambda\},\{Y_\mu\}) = \text{pro-K}(\{X_\lambda\},\{Y_\mu^{\Delta(n)}\}).$$

We are going to construct now a monad $(\mathrm{Ran}_\infty, \mu_\infty, \varepsilon_\infty)$ on $\pi(\text{pro-K})$. Notice that we have a monad $(\mathrm{Ran},\mu,\varepsilon)$ on inv-K. As above, for $X = \{X_\lambda\}$, let $(\mathrm{Ran}X)_\lambda = \prod_{\lambda_0 \leq \lambda} X_{\lambda_0}$. For a morphism $(\varphi,f): \{X_\lambda\} \to \{Y_\mu\}$, as $\mathrm{Ran}(\varphi,f)$ we take $(\varphi, \tilde{\mathrm{Ran}} f)$, where $\tilde{\mathrm{Ran}} f$ is the composition of $\varphi^*: \mathrm{Ran}X \to \mathrm{Ran}\varphi^*X$ and $\mathrm{Ran}f: \mathrm{Ran}\varphi^*X \to \mathrm{Ran}Y$. A multiplication μ, and a unit ε are induced by the multiplication and the unit of the monad $(\mathrm{Ran},\mu,\varepsilon)$ on $F(\Lambda,K)$. It is easy to see that we cannot consider $(\mathrm{Ran},\mu,\varepsilon)$ as a monad on pro-K, because Ran does not preserve the congruence relation on morphisms. Nevertheless we can define a monad $(\mathrm{Ran}_\infty, \mu_\infty, \varepsilon_\infty)$ on $\pi(\text{pro-K})$.

Let $\tilde{\Lambda}_*$ be the nerve of a directed set Λ. We shall denote by the symbol $\tilde{\lambda}_n$ a m-simplex of $\tilde{\Lambda}_*$, that is a chain $\lambda_0 \leq \ldots \leq \lambda_n$. If $\varphi: M \to \Lambda$ is an increasing function, then $\varphi(\tilde{\mu}_n)$ will be the n-simplex $\varphi(\mu_0) \leq \ldots \leq \varphi(\mu_n)$. The notation $\tilde{\lambda}_n \leq \lambda$ will mean that $\lambda_0 \leq \ldots \leq \lambda_n \leq \lambda$, and $X_{\tilde{\lambda}_n}$ for $\tilde{\lambda}_n = (\lambda_0 \leq \ldots \leq \lambda_n)$ will mean the object X_{λ_0}. With these notations, we can write $\mathrm{Ran}^{n+1}\{X_\lambda\}$ as an inverse system over Λ of the type

$$(\mathrm{Ran}^{n+1}X)_\lambda = \prod_{\tilde{\lambda}_n \leq \lambda} X_{\tilde{\lambda}_n}.$$

Now assume that, for each directed set Λ, there exists the monad Ran_∞ on $F(\Lambda, K)$. Then we define a functor Ran_∞ on inv-K by the formula

$$\mathrm{Ran}_\infty\{X_\lambda\} = (\mathrm{Ran}^*\{X_\lambda\})^\Delta.$$

PROPOSITION 6.1. *Let* $(\varphi, f), (\psi, g): \{X_\lambda\} \to \{Y_\mu\}$ *be two morphisms in* inv-K, *and let* (ψ, g) *be congruent to* (φ, f). *Then there is a morphism* F *in* inv-K *such that*:

1. F *is congruent to* $\mathrm{Ran}_\infty f$.
2. F *is homotopic to* $\mathrm{Ran}_\infty g$.

PROOF. We define F by the composition

$$\psi^*\mathrm{Ran}_\infty X \longrightarrow \varphi^*\mathrm{Ran}_\infty X \xrightarrow{\mathrm{Ran}_\infty f} \mathrm{Ran}_\infty Y.$$

Then F is congruent to $\mathrm{Ran}_\infty f$ by construction. On the other hand, F is a realization of the following morphism of cosimplicial inverse systems:

$$F^n: \psi^*(\mathrm{Ran}^{n+1}X) \longrightarrow \mathrm{Ran}^{n+1}Y,$$

for $\tilde{\mu}_n \leq \mu$, defined by the composition

$$\prod_{\tilde{\lambda}_n \leq \psi(\mu)} X_{\tilde{\lambda}_n} \xrightarrow{\pi_{\varphi(\tilde{\mu}_n)}} X_{\varphi(\tilde{\mu}_n)} \xrightarrow{f_{\mu_0}} Y_{\tilde{\mu}_n}$$

where $\pi_{\varphi(\tilde{\mu}_n)}$ is a projection to the corresponding factor, but $\mathrm{Ran}^{n+1}g$, for $\tilde{\mu}_n \leq \mu$, is defined by the composition

$$\prod_{\tilde{\lambda}_n \leq \psi(\mu)} X_{\tilde{\lambda}_n} \xrightarrow{\pi_{\psi(\tilde{\mu}_n)}} X_{\psi(\tilde{\mu}_n)} \xrightarrow{g_{\mu_0}} Y_{\tilde{\mu}_n}.$$

Then we can construct a family of morphisms

$$H^q: \psi^*\mathrm{Ran}^{n+2}X \longrightarrow \mathrm{Ran}^{n+1}Y, \text{ for } 0 \leq q \leq n,$$

with the help of the composites

$$(\mathrm{Ran}^{n+2}X)_{\psi(\mu)} \xrightarrow{\pi_{\eta^q(\tilde{\mu}_n)}} X_{\eta^q(\tilde{\mu}_n)} \xrightarrow{f_{\mu_0}} Y_{\tilde{\mu}_n},$$

where

$$\eta^q(\mu_0 \leq \ldots \leq \mu_n) = (\varphi(\mu_0) \leq \ldots \leq \varphi(\mu_q) \leq \psi(\mu_q) \leq \ldots \leq \psi(\mu)).$$

It is easy to show that the family H^q is a cosimplicial homotopy between F^* and $\mathrm{Ran}^* g$. Then the realizations are homotopic in inv-K.

According to Proposition 6.1, we have a functor

$$\mathrm{Ran}_\infty : \pi(\text{pro-K}) \longrightarrow \pi(\text{pro-K}).$$

The unit and the multiplication of the monad $(\mathrm{Ran}_\infty, \mu_\infty, \varepsilon_\infty)$ on $\pi F(\Lambda, K)$ induce natural transformations on π(pro-K), namely $\mu_\infty : \mathrm{Ran}_\infty^2 \to \mathrm{Ran}_\infty$ and $\varepsilon_\infty : I \to \mathrm{Ran}_\infty$. Thus we have obtained a monad $(\mathrm{Ran}_\infty, \mu_\infty, \varepsilon_\infty)$ on π(pro-K).

THEOREM 6.1. *The category* $\mathrm{Kl}_{\mathrm{Ran}_\infty}$-$\pi$(pro-Top) *and the coherent prohomotopy category* (CPH-Top) *of Lisica-Mardešić are equivalent.*

PROOF. Let $f : \{X_\lambda\} \to \{Y_\mu\}$ be a morphism in $\mathrm{Kl}_{\mathrm{Ran}_\infty}$-$\pi$(pro-Top), defined by $\varphi : M \to \Lambda$ and $f_\mu : X_{\varphi(\mu)} \to (\mathrm{Ran}_\infty Y)_\mu$. From the definition, $\mathrm{Ran}_\infty Y$ is a subspace of the space $\prod_{n\geq 0} \prod_{\tilde\mu_n \leq \mu} Y_{\tilde\mu_n}^{|\Delta(n)|}$. Thus, for each $\tilde\mu_n \leq \mu$, we have a morphism

$$f_{\tilde\mu_n \leq \mu} : X_{\varphi(\mu)} \longrightarrow Y_{\mu_0}^{|\Delta(n)|}.$$

The exponential law gives a map

$$\hat f_{\tilde\mu_n \leq \mu} : X_{\varphi(\mu)} \times |\Delta(n)| \longrightarrow Y_{\mu_0}.$$

For $\tilde\mu_n = (\mu_0 \leq \ldots \leq \mu_n)$, we define

$$(Gf)_{\tilde\mu_n} = \hat f_{\tilde\mu_n \leq \mu} : X_{\varphi(\mu)} \times |\Delta(n)| \longrightarrow Y_{\mu_0}.$$

It is easy to verify that the function $G\varphi(\tilde\mu_n) = \varphi(\tilde\mu_n)$ and the family $(Gf)_{\tilde\mu_n}$, $\tilde\mu_n \in \tilde M_n$, produce a special coherent morphism in the sense of [15]. It is clear in addition that G may be expanded to the map

$$G : \pi\,\text{pro-Top}(\{X_\lambda\}, \mathrm{Ran}_\infty\{Y_\mu\}) \longrightarrow \text{CPH-Top}(\{X_\lambda\}, \{Y_\mu\}).$$

Now we shall construct a map H inverse to G. Let $f : \{X_\lambda\} \to \{Y_\mu\}$ be a special coherent morphism defined by

$$f_{\tilde\mu_n} : X_{\varphi(\mu)} \times |\Delta(n)| \longrightarrow Y_{\mu_0}, \text{ for } \tilde\mu_n = (\mu_0 \leq \ldots \leq \mu_n).$$

For $\tilde\mu_n \leq \mu$, we can consider the composition

$$(Hf)_{\tilde\mu_n \leq \mu} : X_{\varphi(\mu)} \xrightarrow{P_{\varphi(\mu_n)\varphi(\mu)}} X_{\varphi(\mu_n)} \xrightarrow{\bar f_{\tilde\mu_n}} Y_{\mu_0}^{|\Delta(n)|},$$

where $\bar f_{\tilde\mu}$ corresponds to $f_{\tilde\mu}$ by the exponential law. Thus we have a map

$$(Hf)_\mu ; X_{\varphi(\mu)} \longrightarrow \prod_{n\geq 0} \prod_{\tilde\mu_n \leq \mu} Y_{\tilde\mu_0}^{|\Delta(n)|}.$$

The coherent conditions [15] show that it is a morphism with codomain $(\mathrm{Ran}_\infty Y)_\mu$. It is easy to check that we have $H \circ G = 1$ and $G \circ H = 1$. Thus the conditions of the Bousfield-Kan Lemma hold and we obtain a monad $(\mathrm{Ran}'_\infty, \mu'_\infty, \varepsilon'_\infty)$ on π(pro-Top) such that CPH-Top is equivalent to $\mathrm{Kl}_{\mathrm{Ran}'_\infty}$-$\pi$(pro-Top). In addition $(\mathrm{Ran}'_\infty, \mu'_\infty, \varepsilon'_\infty)$ can differ from $(\mathrm{Ran}_\infty, \mu_\infty, \varepsilon_\infty)$ by multiplication only, but the multiplication in Ran'_∞ is defined by a composition of level morphisms in CPH-Top. Therefore it suffices to prove that the composition of level morphisms in CPH-Top and that in CHRan_∞-$F(\Lambda,\mathrm{Top})$ coincide. By definition a morphism in CHRan_∞-$F(\Lambda,\mathrm{Top})$ is determined by the cosimplicial map f: $\Delta \to \mathrm{Hom}(\{X_\lambda\}, \mathrm{Ran}^*\{Y_\lambda\})$. Hence, for each $\bar{\lambda} = (\lambda_0 \le \ldots \le \lambda_p)$, we have a map

$$f_{\bar{\lambda}_p}\colon X_{\lambda_p} \times |\Delta(p)| \longrightarrow Y_{\lambda_0},$$

and for the family $\{f_{\bar{\lambda}_p}\}$, the coherent conditions hold. If now a morphism $g\colon \Delta \to \mathrm{HOM}(\{Y_\lambda\}, \mathrm{Ran}^*\{Z_\lambda\})$ is defined by the family

$$g_{(\lambda_0 \le \ldots \le \lambda_q)}\colon Y_{\lambda_q} \times |\Delta(q)| \longrightarrow Z_{\lambda_0},$$

then $g \circ f$ is defined by the composition:

$$|\Delta| \xrightarrow{\rho} |\Delta \otimes \Delta| \xrightarrow{|f \otimes g|} |\mathrm{HOM}(\{X_\lambda\}, \mathrm{Ran}^*\{Y_\lambda\}) \otimes \mathrm{HOM}(\{Y_\lambda\}, \mathrm{Ran}^*\{Z_\lambda\})|$$
$$\xrightarrow{M} |\mathrm{HOM}(\{X_\lambda\}, \mathrm{Ran}^*\{Z_\lambda\})|,$$

or

$$(g \circ f)_{(\lambda_0 \le \ldots \le \lambda_{p+q})}(x, t) = g_{(\lambda_0 \le \ldots \le \lambda_q)}(f_{(\lambda_q \le \ldots \le \lambda_{p+q})}(x, \alpha^{p,q}[1](t), \beta^{p,q}[1](t)),$$

for $t \in P^{p,q}[1]$, but this is the formula for the composition of coherent level morphisms [15].

This theorem justifies the following definition.

DEFINITION 6.1. Let K be a locally Kan S-category, and suppose that for each directed ordered set Λ and for each inverse system $\{X_\lambda\}$ in K, there exists $\mathrm{Ran}_\infty\{X_\lambda\}$. We now define the *coherent prohomotopy category* for the category K as the category $\mathrm{Kl}_{\mathrm{Ran}_\infty}$-$\pi$(pro-K). We shall denote it by CPH-K. If K' is a full S-subcategory of K, then we can consider the full subcategory of CPH-K generated by the objects of K'. We shall denote it by CPH-K'.

THEOREM 6.2. *The category* CPH-K *is the localization of the category* π(pro-K) *with respect to levelwise homotopy equivalences.*

PROOF. It is clear that $P_\infty\colon \pi(\text{pro-K}) \to$ CPH-K inverts the levelwise homotopy equivalences. In addition $\varepsilon_\infty : X \to \mathrm{Ran}_\infty X$ is a levelwise homotopy equivalence too. Let $F: \pi(\text{pro-K}) \to L$ be a functor inverting the levelwise homotopy equivalences. We define $F': \text{CPH-K} \to L$ as follows: on objects, $F'(X) = F(X)$, and for $f: X \to \mathrm{Ran}_\infty Y$, the morphism $F'(f)$ is $F(\varepsilon_\infty)^{-1} \circ F(f): F(X) \to F(Y)$.

As in Theorem 5.1, one can prove that the functor Q_∞ adjoint to P_∞ is fully faithful, and therefore the counit of the adjunction $\Phi: P_\infty Q_\infty \to I$ is an isomorphism [13]. Now for the functor G: $\text{CPH-K} \to L$ such that $G \circ P_\infty = F$, and for $f: X \to Y$ in CPH-K we have a commutative diagram

$$\begin{tikzcd}
G(X) & G(P_\infty Q_\infty X) = F(Q_\infty X) \arrow[l, "\Phi(X)"'] \arrow[d, "G(P_\infty Q_\infty f)"', "F(Q_\infty f)"] \\
G(Y) \arrow[from=u, "G(f)"'] & G(P_\infty Q_\infty Y) = F(Q_\infty Y) \arrow[l, "\Phi(Y)"']
\end{tikzcd}$$

If for another functor G': $\text{CPH-K} \to L$, we have $G' \circ P_\infty = F$, then $G'(\Phi) \circ G^{-1}(\Phi): G \to G'$ is an isomorphism of functors.

COROLLARY 1. *Let* K *be a simplicial closed model category,* K_f *be its full subcategory of fibrant objects, let each object of* K *be cofibrant and let the Edwards-Hastings conditions for the existence of* ho(pro-K) *hold* [12]. *Then* CPH-K_f *and* ho(pro-K) *are equivalent categories.*

COROLLARY 2. *The strong shape category of all topological spaces of Lisica-Mardešič* [15] *and that of Cathey-Segal* [7] *are equivalent.*

Finally we make some remarks about homology theories on the strong shape category of pointed topological spaces. Let E be a cofibrant simplicial spectrum in the sense of Thomason [26]. For an inverse system $\{X_\lambda\}$ of pointed topological spaces, we define the homology with coefficients in E by the formula

$$E_*(\{X_\lambda\}) = \pi_* \mathrm{holim}\{Q(SX_\lambda \wedge E)\},$$

where S is the singular complex functor, for each spectrum E, Q gives an equivalent fibrant spectrum (it may be defined by the formula $Q^n = \mathrm{colim}_k \Omega^k \mathrm{E}\times^\infty X_{n+k}$ [26]). Now we can define the E-homology of a topological space X as the E-homology of its ANR-resolution. The resulting theory on the strong shape category has the following properties:

1. If X is a paracompact Hausdorff space, and A is a closed subspace, then there is an exact sequence

$$\to E_{n+1}(X/A) \to E_n(A) \to E_n(X) \to E_n(X/A) \to E_{n-1}(A) \to E_{n-1}(X) \to$$

2. If $p: X \to \{X_\lambda\}$ is the Mardešič resolution such that all X_λ are normal topological spaces, then there is the spectral sequence of Thomason [26]

$$\lim{}^{(p)} E_q\{X_\lambda\} \Rightarrow E_{q-p}(X).$$

In particular on the category of compact Hausdorff spaces, we have a homology theory for which all Steenrod-Sitnikov axioms hold [3,12]. The details are in the author's preprint [1]. Remark that the relations between our homology and that of Lisica-Mardešič [16,17,18] are not clear.

References.

1. BATANIN M.A., About the coherent prohomotopy category of Lisica-Mardešič and generalized Steenrod homology theories (Russian), Preprint 1986.
2. BAUER F.W., A shape theory with singular homology, *Pacific J. Math.* 64 (1976), 25-76.
3. BAUER F.W., Under what condition are shape homology $\bar{E}_*$ and Steenrod homology ${}^sE_*$ isomorphic? *Lecture Notes in Math.* 870, Springer (1981), 186-214.
4. BOARDMANN J.M. & VOGT R.M., Homotopy invariant algebraic structures in topological spaces, *Lecture Notes in Math.* 347, Springer (1973).
5. BOUSFIELD A.K. & KAN D.M., Homotopy limits, completions and localizations, *Lecture Notes in Math.* 304, Springer (1972).
6. BOUSFIELD A.K. & KAN D.M., The homotopy spectral sequence of a space with coefficients in a ring, *Topology* 11 (1972), 79-106.
7. CATHEY F.W. & SEGAL J., Strong shape theory and resolutions, *Topology and its Appl.* 15 (1983), 119-130.
8. CORDIER J.-M., Extensions de Kan simplicialement cohérentes, Prépublications Amiens 1985.
9. CORDIER J.-M., Homologie de Steenrod-Sitnikov et limites homotopiques algébriques, *Manuscripta Math.* 59 (1987), 35-52.
10. CORDIER J.-M., Comparaison de deux catégories d'homotopie de morphismes cohérents, *Cahiers Top. & Geom. Diff. Cat.* XXX (1989), 257-275.
11. CORDIER J.-M. & PORTER T., Vogt's Theorem on categories of homotopy coherent diagrams, *Math. Proc. Cambridge Phil. Soc.* 100 (1986), 65-90.
12. EDWARDS P.A. & HASTINGS H.M., Čech and Steenrod homotopy theories with applications to Geometric Topology, *Lecture Notes in Math.* 542, Springer (1976).
13. GABRIEL P. & ZISMAN M., *Calculus of fractions and homotopy theory,* Springer 1967.
14. KELLY G.M., Basic concepts of enriched category theory, *Lecture Notes in Math.* 764, Springer (1982).
15. LISICA J.T. & MARDESIČ S., Coherent prohomotopy and strong shape theory, *Glasnic Matematiči* s.111, v. 19 (1984), 335-400.
16. LISICA J.T. & MARDESIČ S., Strong homology of inverse systems I, *Top. & its Appl.* 19 (1985), 29-43.
17. LISICA J.T. & MARDESIČ S., Strong homology of inverse systems II, *Top. & its Appl.* 19 (1985), 45-64.

18. LISICA J.T. & MARDEŠIČ S., Strong homology of inverse systems III, *Top. & its Appl.* 20 (1985), 29-37.
19. LISICA J.T., On strong theory of coshapes (Russian), Sibirski Mat. Z. 24 (1983), 81-99.
20. MAC LANE S., *Categories for the working mathematician,* Springer 1971.
21. MARDEŠIČ S. & SEGAL J., *Shape Theory,* North-Holland, 1982.
22. MAY J.P., The geometry of iterated loop spaces, *Lecture Notes in Math.* 271, Springer (1972).
23. MEYER J.P., Bar and cobar constructions I, *J. Pure & Appl. Algebra* 33 (1984), 163-207.
24. PORTER T., Coherent prohomotopy theory, *Cahiers Top. & Geom. Diff. Cat.* XIX (1978), 3-46.
25. PORTER T., On the two definitions of Ho(pro-C), *Top. & its Appl.* 28 (1988), 289-293.
26. THOMASON R.W., Algebraic K-theory and etale cohomology, *Ann. Scient. Ecole Normale Sup.* XVIII-3 (1985), 437-552.
27. VOGT R.M., Homotopy limits and colimits, *Math. Z.* 134 (1973), 11-53.

INSTITUTE OF MATHEMATICS
UNIVERSITETSKIY PROSPECT 4
630090 NOVOSIBIRSK 90.
RUSSIA