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UNIVERSAL CONCRETE CATEGORIES AND FUNCTORS

by Věra TRNKOVÁ

Dedicated to the memory of Jan Reiterman

Résumé. La catégorie $\mathcal{S}$ des semigroupes topologiques admet un foncteur $F : \mathcal{S} \to \mathcal{S}$ et une congruence $\sim$ préservée par F tel que le foncteur

$$F/\sim : \mathcal{S}/\sim \to \mathcal{S}/\sim$$

est universel dans le sens suivant: pour chaque foncteur $G : \mathcal{K}_1 \to \mathcal{K}_2$, $\mathcal{K}_1$ et $\mathcal{K}_2$ des catégories arbitraires, il existe des foncteurs pleins injectifs $\Phi_i : \mathcal{K}_i \to \mathcal{S}/\sim$, $i = 1, 2$, tels que $(F/\sim) \circ \Phi_1 = \Phi_2 \circ G$. Ce résultat découle de notre théoréme principal. Le probléme d'un prolongement d'un foncteur défini sur une sous-catégorie pleine joue un rôle essentiel dans cet article.

I. Introduction and the Main Theorem

By [14], there exists a *universal category* $\mathcal{U}$, i.e. a category containing an isomorphic copy of every category as its full subcategory (we live in a set theory with sets and classes, see e.g. [1]). This result was strengthened in [15]: if $\mathcal{C}$ is a *C-universal category* (i.e. a concretizable category such that every concretizable category admits a full one-to-one functor into $\mathcal{C}$), then there exists a congruence $\sim$ on $\mathcal{C}$ such that $\mathcal{C}/\sim$ is a universal category. The proof of this result is based on the Kučera theorem that every category is a factorcategory of a concretizable one (see [11]). Hence a universal category $\mathcal{U}$ is a factorcategory of a concretizable one, say $\mathcal{K}$, so that $\mathcal{U}$ is isomorphic to $\mathcal{K}/\simeq$ for a congruence $\simeq$ on $\mathcal{K}$. If $\mathcal{C}$ is a C–universal category, $\mathcal{K}$ is (isomorphic to) a full subcategory of $\mathcal{C}$. If $\sim$ is a congruence on $\mathcal{C}$ extending $\simeq$, then $\mathcal{C}/\sim$ contains an isomorphic copy of $\mathcal{U}$ as a full subcategory hence it is universal.

Since the category $\mathcal{T}$ of topological spaces and open continuous maps and the category $\mathcal{M}$ of metric spaces and open uniformly continuous maps are known to be C–universal (see [13]), we get that their suitable factorcategories are universal categories. Under a set-theoretical statement

(M) there is only a set of measurable cardinals,

many varieties of universal algebras, categories of presheaves, and other categories are known to be C–universal (see [13]) hence their suitable factorcategories are universal, too.

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In the present paper, we also show that the category $\mathcal{S}$ of topological semigroups is C–universal (without any special set-theoretical assumption).

In the present paper, we investigate a "simultaneous version" of the above field of problems. We investigate functors $F : \mathcal{U} \to \mathcal{U}$ ***universal*** in the following sense: for every functor $G : \mathcal{K}_1 \to \mathcal{K}_2$, where $\mathcal{K}_1$ and $\mathcal{K}_2$ are arbitrary categories, there exist full one-to-one functors $\Phi_i : \mathcal{K}_i \to \mathcal{U}$, $i = 1, 2$, such that the square

$$\begin{array}{ccc} \mathcal{K}_1 & \xrightarrow{G} & \mathcal{K}_2 \\ {\scriptstyle \Phi_1}\downarrow & & \downarrow{\scriptstyle \Phi_2} \\ \mathcal{U} & \xrightarrow{F} & \mathcal{U} \end{array}$$

commutes. Such a functor does exist, it is constructed in [15]. Now, we would like to proceed as above and express $\mathcal{U}$ as a full subcategory of $\mathcal{C}/\sim$. However, how to handle the functor F? How to extend it on $\mathcal{C}/\sim$? Let us state explicitly that we do not know whether for *every* C–universal category $\mathcal{C}$ there exists an endofunctor F of $\mathcal{C}$ and a congruence $\sim$ on $\mathcal{C}$ preserved by F such that $F/\sim : \mathcal{C}/\sim \to \mathcal{C}/\sim$ is universal. Our Main Theorem is a little weaker: C–universality is replaced by a stronger notion of COE–universality, introduced below. Under this restriction we are able to formulate and prove a satisfactory Extension Lemma and, using it, to construct F and $\sim$ preserved by F such that $F/\sim$ is universal. Fortunately, the categories $\mathcal{T}$, $\mathcal{M}$, $\mathcal{S}$, ... are COE–universal.

Our constructions require to distinguish between *concretizable categories*, i.e. categories admitting faithful functors into Set, and *concrete categories* $(\mathcal{K}, U)$, where a faithful functor $U : \mathcal{K} \to$ Set is already specified. The following sorts of full one-to-one functors of concrete categories $\Psi : (\mathcal{K}_1, U_1) \to (\mathcal{K}_0, U_0)$ have been investigated in literature or play a rôle in our constructions:

realization: $U_1 = U_0 \circ \Psi$;

extension: there is a monotransformation of U_1 into $U_0 \circ \Psi$;

strong extension: U_1 is a summand of $U_0 \circ \Psi$ i.e. there exists $H : \mathcal{K}_1 \to$ Set such that $U_0 \circ \Psi = U_1 \amalg H$;

strong embedding: there exists a faithful functor $H :$ Set $\to$ Set such that $U_0 \circ \Psi = H \circ U_1$.

In this paper, the word "functor" always means a covariant functor unless the contravariance is explicitely stated. This is just now: we investigate also ***strong co-embedding*** (see [13]), i.e. full one-to-one contravariant functor $\Psi : (\mathcal{K}_1, U_1) \to (\mathcal{K}_0, U_0)$ such that there exists a faithful contravariant functor $H :$ Set $\to$ Set such that $U_0 \circ \Psi = H \circ U_1$.

We say that a full one-to-one functor $\Psi : (\mathcal{K}_1, U_1) \to (\mathcal{K}_0, U_0)$ is an ***ordinary embedding*** if it can be expressed as a finite composition of strong extensions, strong embeddings and strong co-embeddings. We say that a concrete category $(\mathcal{C}, U)$ is *COE-universal* if every Concrete category $(\mathcal{K}_1, U_1)$ admits an Ordinary Embedding in it.

Main Theorem. *Let $\mathcal{C}$ admit a faithful functor $U : \mathcal{C} \to$ Set such that $(\mathcal{C}, U)$ is COE-universal. Then there exists an endofunctor $F : \mathcal{C} \to \mathcal{C}$ and a congruence $\sim$ on $\mathcal{C}$ preserved by F such that $F/\sim : \mathcal{C}/\sim \to \mathcal{C}/\sim$ is a universal functor.*

Problem: Does every C–universal category $\mathcal{C}$ admit a functor $U : \mathcal{C} \to$ Set such that $(\mathcal{C}, U)$ is COE–universal?

Let us describe briefly the contents of the paper. In the part II, we present and prove the Extension Lemma. This lemma is the core of the proof of the Main Theorem but it could be of interest in itself. In the part III, we present a lemma which shows that, in the Extension Lemma, realization can be in a way replaced by an ordinary embedding. In the part IV, we prove the Main Theorem. In V, we indicate the parts of [13], which have to be inspected to see that $\mathcal{T}$ and $\mathcal{M}$ are COE–universal and that, under (M), Graph, Alg(1, 1) and others are COE–universal. Moreover, we prove that the category $\mathcal{S}$ of topological semigroups (with its natural forgetful functor), is COE–universal (its C–universality is also a new result). Hence, applying the Main Theorem on $\mathcal{S}$, we get the result stated in the Abstract.

II. The Extension Lemma

II.1. Let $\mathcal{K}_1$ be a full subcategory of $\mathcal{K}_0$, let $E : \mathcal{K}_1 \to \mathcal{K}_0$ be the inclusion. Let a functor

$$F_1 : \mathcal{K}_1 \to \mathcal{H}$$

be given. If we want to extend F_1 on $\mathcal{K}_0$ and we permit an enlarging of the range category, a push-out of E and F_1

$$\text{(s)} \qquad \begin{array}{ccc} \mathcal{K}_1 & \xrightarrow{F_1} & \mathcal{H} \\ {\scriptstyle E}\downarrow & & \downarrow{\scriptstyle J} \\ \mathcal{K}_0 & \xrightarrow{F_0} & \mathcal{C} \end{array}$$

seems to offer a solution (clearly, J would be full and one-to-one). However, there are troubles with push-out: for some $a, b \in \operatorname{obj}\mathcal{C}$, $\mathcal{C}(a, b)$ could be a proper class; if $\mathcal{C}(a, b)$ is always a set, $\mathcal{C}$ could be non-concretizable though $\mathcal{K}_0$, $\mathcal{K}_1$, $\mathcal{H}$ are concretizable. Simple examples are shown below. The Extension Lemma gives a solution satisfactory for our purposes.

II.2. Examples: a) Let $\mathcal{K}_1$ be a large discrete category (i.e. $\operatorname{obj}\mathcal{K}_1$ is a proper class and $\mathcal{K}_1$ has no other morphisms than identities), $\operatorname{obj}\mathcal{K}_0 = \{i\} \cup \operatorname{obj}\mathcal{K}_1$, i is an initial object of $\mathcal{K}_0$, $\operatorname{obj}\mathcal{H} = \operatorname{obj}\mathcal{K}_1 \cup \{t\}$, t is a terminal object of $\mathcal{H}$, both

E and F_1 are full inclusions. If the square (s) is a push-out, then $\mathcal{C}(i,t)$ is a proper class.

b) Let k be a category with precisely three objects, say a, b, c, and precisely seven morphisms, namely 1_a, 1_b, 1_c, $\alpha \in k(a,b)$, $\beta_1, \beta_2 \in k(b,c)$ and $\beta_1 \circ \alpha = \beta_2 \circ \alpha \in k(a,c)$. Let I be a proper class and let, for every $i \in I$, k_i be a copy of k (we denote its objects by a_i, b_i, c_i). Let $\mathcal{K}_0$ be a coproduct of all k_i, $i \in I$, $\mathcal{K}_1$ its full subcategory generated by $\{a_i, b_i \mid i \in I\}$ and let $\mathcal{H}$ be the category obtained from $\mathcal{K}_1$ by the merging all b_i, $i \in I$, into a unique object b and F_1 is the corresponding functor. If (s) is a push-out, then $\mathcal{C}$, though really a category, is not concretizable because b does not fulfil the condition in [7] (however $\mathcal{K}_0$, $\mathcal{K}_1$, $\mathcal{H}$ are concretizable, evidently).

II.3. We recall that, given functors $G_1, G_2 : \mathcal{K}_1 \to \mathcal{K}_2$, G_2 is *a retract* of G_1 if there exist a monotransformation $\mu : G_2 \to G_1$ and an epitransformation $\pi : G_1 \to G_2$ such that $\pi \circ \mu$ is the identity. (The notion "realization" is recalled in the part I.)

Extension Lemma: *Let $(\mathcal{K}_0, U_0)$, $(\mathcal{K}_1, U_1)$, $(\mathcal{H}, V)$ be concrete categories. Let $E : (\mathcal{K}_1, U_1) \to (\mathcal{K}_0, U_0)$ be a realization and $F_1 : \mathcal{K}_1 \to \mathcal{H}$ be a functor such that $V \circ F_1$ is a retract of U_1. Then there exist a concrete category $(\mathcal{C}, W)$, a realization $J : (\mathcal{H}, V) \to (\mathcal{C}, W)$ and a functor $F_0 : \mathcal{K}_0 \to \mathcal{C}$ such that the square* (s) *commutes.*

II.4. Remarks. a) The rest of part II. is devoted to the proof of the Extension Lemma. However in IV. where we use the lemma, we need also the following properties (i), (ii), (iii) of the constructed category $\mathcal{C}$ (for shortness, we suppose that E and J are inclusions):

(i) F_0 is one-to-one on $\operatorname{obj}\mathcal{K}_0 \setminus \operatorname{obj}\mathcal{K}_1$ and $F_0(a) \neq F_0(b)$ whenever $a \in \operatorname{obj}\mathcal{K}_1$, $b \in \operatorname{obj}\mathcal{K}_0 \setminus \operatorname{obj}\mathcal{K}_1$;

(ii) if $a \in \operatorname{obj}\mathcal{K}_0 \setminus \operatorname{obj}\mathcal{K}_1$ satisfies

$$(\emptyset) \qquad \mathcal{K}_0(a,c) = \mathcal{K}_0(c,a) = \emptyset \quad \text{for all } c \in \operatorname{obj}\mathcal{K}_1$$

then $\mathcal{C}(F_0(a), d) = \mathcal{C}(d, F_0(a)) = \emptyset$ for all $d \in \operatorname{obj}\mathcal{H}$;

(iii) if $a, b \in \operatorname{obj}\mathcal{K}_0 \setminus \operatorname{obj}\mathcal{K}_1$ both satisfy $(\emptyset)$ in (ii) and if $\mathcal{K}_0(a,b) = \mathcal{K}_0(b,a) = \emptyset$, then $\mathcal{C}(F_0(a), F_0(b)) = \mathcal{C}(F_0(b), F_0(a)) = \emptyset$.

The statements (i), (ii), (iii) are mentioned explicitely in II.10 below.

b) Inspecting the proof, one can see that the Extension Lemma is fulfilled also if the category Set is replaced by an arbitrary base category $\mathcal{B}$ in which every class of pairs of parallel morphisms with the same codomain has a joint coequalizer. But we do not use this level of generality: we investigate only categories concrete over Set.

II.5. Convention. We shall suppose, for shortness, that the realization $E : (\mathcal{K}_1, U_1) \to (\mathcal{K}_0, U_0)$ is an inclusion, i.e. $\mathcal{K}_1$ *is* a full subcategory of $\mathcal{K}_0$ and U_1 is the domain-restriction of U_0. We will construct $(\mathcal{C}, W)$ and $J : (\mathcal{H}, V) \to (\mathcal{C}, W)$ such that J will be an inclusion as well.

Since $(\mathcal{K}_0, U_0)$, $(\mathcal{H}, V)$ are concrete categories, we use the following convenient usual notation: every $\alpha \in \mathcal{K}_0(a,b)$ (or $\alpha \in \mathcal{H}(a,b)$) is a triple $\alpha = (b, \overline{\alpha}, a)$, where $\overline{\alpha}$ is a map $U_0(a) \longrightarrow U_0(b)$ (or $V(a) \longrightarrow V(b)$, respectively). Clearly, $\overline{\alpha} = U_0(\alpha)$ (or $\overline{\alpha} = V(\alpha)$) and $\overline{\beta} \circ \overline{\alpha} = \overline{\beta \circ \alpha}$ whenever $\beta \circ \alpha$ is defined.

II.6. We start the proof of the Extension Lemma.

Denote by $\mu : V \circ F_1 \longrightarrow U_1$ and $\pi : U_1 \longrightarrow V \circ F_1$ natural transformations such that $\pi \circ \mu$ is the identity. Hence, for every $c, d \in \operatorname{obj} \mathcal{K}_1$, $\alpha \in \mathcal{K}_1(c,d)$,

$$(*) \qquad \overline{F_1(\alpha)} = \pi_d \circ \overline{\alpha} \circ \mu_c \,.$$

In fact, $\pi_d \circ \overline{\alpha} \circ \mu_c = \overline{F_1(\alpha)} \circ \pi_c \circ \mu_c = \overline{F_1(\alpha)}$.

Now, we define an auxiliary concrete category $(\mathcal{C}_0, W_0)$ as follows:

$$\operatorname{obj} \mathcal{C}_0 = A \amalg B \text{ where } A = \operatorname{obj} \mathcal{H} \text{ and } B = \operatorname{obj} \mathcal{K}_0 \setminus \mathcal{K}_1 \,.$$

We define $W_0 : \operatorname{obj} \mathcal{C}_0 \longrightarrow \mathrm{Set}$ by

$W_0(a) = V(a)$ for all $a \in A$,
$W_0(b) = U_0(b)$ for $b \in B$.

For $a, b \in \operatorname{obj} \mathcal{C}_0$, we define a set $\operatorname{gen}(a,b)$ as follows:

α) if $a, b \in A$, then $\operatorname{gen}(a,b) = \mathcal{H}(a,b) = \{(b, \overline{\nu}, a) \mid \nu \in \mathcal{H}(a,b)\}$;
β) if $a, b \in B$, then $\operatorname{gen}(a,b) = \mathcal{K}_0(a,b) = \{(b, \overline{\nu}, a) \mid \nu \in \mathcal{K}_0(a,b)\}$;
γ) if $a \in A$, $b \in B$, then

$$\operatorname{gen}(a,b) = \{(b, \overline{\varrho} \circ \mu_c, a) \mid c \in \operatorname{obj} \mathcal{K}_1 \& F_1(c) = a, \varrho \in \mathcal{K}_0(c,b)\},$$
$$\operatorname{gen}(b,a) = \{(a, \pi_c \circ \overline{\sigma}, b) \mid c \in \operatorname{obj} \mathcal{K}_1 \& F_1(c) = a, \sigma \in \mathcal{K}_0(b,c)\}.$$

The definition in γ) is correct because the maps have the correct domains and codomains:

$$W_0(a) \xrightarrow{\ \mu_c\ } U_0(c) \xrightarrow{\ \overline{\varrho}\ } W_0(b)$$

and

$$W_0(b) \xrightarrow{\ \overline{\sigma}\ } U_0(c) \xrightarrow{\ \pi_c\ } W_0(a).$$

Moreover, though all the $c \in \operatorname{obj} \mathcal{K}_1$ with $F_1(c) = a$ could form a proper class, $\operatorname{gen}(a,b)$ and $\operatorname{gen}(b,a)$ are sets because there are only sets of maps $W_0(a) \longrightarrow W_0(b)$ and $W_0(b) \longrightarrow W_0(a)$.

The category $\mathcal{C}_0$ is defined by the closing of the above sets gen with respect to the composition, where, of course, $(c, \overline{\beta}, b) \circ (b, \overline{\alpha}, a) = (c, \overline{\beta} \circ \overline{\alpha}, a)$. The forgetful functor $W_0 : \mathcal{C}_0 \longrightarrow \mathrm{Set}$ is defined on objects; we put $W_0(\alpha) = \overline{\alpha}$ for morphisms.

II.7. Now, we describe the morphisms in $\mathcal{C}_0$, i.e. we describe all the triples $\alpha = (b, \overline{\alpha}, a)$ which can be obtained by the composition of elements of the sets gen:

(α) if $a, b \in A$, then $\mathcal{C}_0(a,b) = \operatorname{gen}(a,b) = \mathcal{H}(a,b)$;

(β) if $a,b \in B$, then $\mathcal{C}_0(a,b) = \mathcal{K}_0(a,b) \cup P(a,b)$ where $P(a,b)$ is the set of all triples $(b, \overline{\sigma} \circ \mu_d \circ \overline{\delta} \circ \pi_c \circ \overline{\varrho}, a)$,

$$W_0(a) \xrightarrow{\overline{\varrho}} U_0(c) \xrightarrow{\pi_c} W_0(c') \xrightarrow{\overline{\delta}}$$
$$\xrightarrow{\overline{\delta}} W_0(d') \xrightarrow{\mu_d} U_0(d) \xrightarrow{\overline{\sigma}} W_0(b)$$

where $c,d \in \operatorname{obj} \mathcal{K}_1$, $F_1(c) = c'$, $F_1(d) = d'$, $\varrho \in \mathcal{K}_0(a,c)$, $\delta \in \mathcal{H}(c',d')$ and $\sigma \in \mathcal{K}_0(d,b)$.

(γ) if $a \in A$ and $b \in B$, then every $\alpha \in \mathcal{C}_0(a,b)$ has the form $(b, \overline{\sigma} \circ \mu_d \circ \overline{\gamma}, a)$,

$$W_0(a) \xrightarrow{\overline{\gamma}} W_0(d') \xrightarrow{\mu_d} U_0(d) \xrightarrow{\overline{\sigma}} W_0(b)$$

where $d \in \operatorname{obj} \mathcal{K}_1$, $F_1(d) = d'$, $\gamma \in \mathcal{H}(a,d')$ and $\sigma \in \mathcal{K}_0(d,b)$ and every $\beta \in \mathcal{C}_0(b,a)$ has the form $(a, \overline{\gamma} \circ \pi_c \circ \overline{\varrho}, b)$,

$$W_0(b) \xrightarrow{\overline{\varrho}} U_0(c) \xrightarrow{\pi_c} W_0(c') \xrightarrow{\overline{\gamma}} W_0(a)$$

where $c \in \operatorname{obj} \mathcal{K}_1$, $F_1(c) = c'$, $\varrho \in \mathcal{K}_0(b,c)$ and $\gamma \in \mathcal{H}(c',a)$.

The proof of these statements (α), (β), (γ) requires to take arbitrary $a,b,c \in A \amalg B$ and to show that if $\alpha = (b, \overline{\alpha}, a)$ and $\beta = (c, \overline{\beta}, b)$ have the form described above for $\mathcal{C}_0(a,b)$ and $\mathcal{C}_0(b,c)$, then $\beta \circ \alpha$ can be expressed in the form of $\mathcal{C}_0(a,c)$. We show it for the case $a \in A$, $b \in B$, $c \in A$ and omit the other $2^3 - 1$ cases because they are analogous or easier. Thus, we may suppose that $\alpha = (b, \overline{\sigma} \circ \mu_d \circ \overline{\gamma}, a)$, where $d \in \operatorname{obj} \mathcal{K}_1$, $F_1(d) = d'$, $\gamma \in \mathcal{H}(a,d')$ and $\sigma \in \mathcal{K}_0(d,b)$ and $\beta = (c, \overline{\varepsilon} \circ \pi_e \circ \overline{\varrho}, b)$, where $e \in \operatorname{obj} \mathcal{K}_1$, $F_1(e) = e'$, $\varrho \in \mathcal{K}_0(b,e)$, $\varepsilon \in \mathcal{H}(e',c)$. We use $(*)$ and compute $\overline{\beta} \circ \overline{\alpha} = \overline{\varepsilon} \circ \pi_e \circ \overline{\varrho} \circ \overline{\sigma} \circ \mu_d \circ \overline{\gamma} = \overline{\varepsilon} \circ \pi_e \circ \overline{\varrho \circ \sigma} \circ \mu_d \circ \overline{\gamma} = \overline{\varepsilon} \circ \overline{F_1(\varrho \circ \sigma)} \circ \overline{\gamma} = \overline{\varepsilon \circ F_1(\varrho \circ \sigma) \circ \gamma}$, i.e. $\beta \circ \alpha = \varepsilon \circ F_1(\varrho \circ \sigma) \circ \gamma \in \mathcal{H}(a,c)$.

II.8. Now, we define a map $G : \mathcal{K}_0 \to \mathcal{C}_0$ as follows:

$G(a) = F_1(a)$ for all $a \in \operatorname{obj} \mathcal{K}_1$,
$G(b) = b$ for all $b \in \operatorname{obj} \mathcal{K}_0 \setminus \operatorname{obj} \mathcal{K}_1$;

α) if $a,b \in \operatorname{obj} \mathcal{K}_1$, $\alpha \in \mathcal{K}_0(a,b)$, we put $G(\alpha) = F_1(\alpha)$;
β) if $a,b \in B$, $\alpha \in \mathcal{K}_0(a,b)$, we put $G(\alpha) = \alpha$;
γ) if $a \in \operatorname{obj} \mathcal{K}_1$, $b \in B$, $\alpha \in \mathcal{K}_0(a,b)$, $\beta \in \mathcal{K}_0(b,a)$, we put

$$G(\alpha) = (b, \overline{\alpha} \circ \mu_a, F_1(a)) \text{ and } G(\beta) = (F_1(a), \pi_a \circ \overline{\beta}, b).$$

Then G maps $\operatorname{obj}\mathcal{K}_0$ into $\operatorname{obj}\mathcal{C}_0$ and, for every $a, b \in \operatorname{obj}\mathcal{K}_0$ and $\alpha \in \mathcal{K}_0(a,b)$, it maps α into $\mathcal{C}_0(G(a), G(b))$. Moreover, G preserves the units. But it is *not* a functor because it does not preserve the composition. We perform a factorization $P : \mathcal{C}_0 \to \mathcal{C}$ such that $F_0 = P \circ G : \mathcal{K}_0 \to \mathcal{C}$ is already a functor. ($\mathcal{C}$ and F_0 will already satisfy our Extension Lemma.)

For every $x \in \operatorname{obj}\mathcal{C}_0$, let us denote by T_x the class of all triples $(\varrho, \sigma, \alpha)$ where $\varrho \in \mathcal{K}_0(r,s)$, $\sigma \in \mathcal{K}_0(s,t)$, $\alpha \in \mathcal{C}_0(G(t), x)$.

$$\begin{array}{ccccc} r & \xrightarrow[\varrho]{} & s & \xrightarrow[\sigma]{} & t \\ & & & & G(t) \xrightarrow[\alpha]{} x \end{array}$$

Let us denote by $\varepsilon_x : W_0(x) \to \overline{\overline{x}}$ a joint coequalizer (in the category Set!) of all pairs of maps

$$\overline{\alpha} \circ \overline{G(\sigma)} \circ \overline{G(\varrho)} \text{ and } \overline{\alpha} \circ \overline{G(\sigma \circ \varrho)}$$

for all $(\varrho, \sigma, \alpha) \in T_x$.

If $x, y \in \operatorname{obj}\mathcal{C}_0$, $\beta \in \mathcal{C}_0(x,y)$, $\beta = (y, \overline{\beta}, x)$, then there exists precisely one map $\overline{\overline{\beta}} : \overline{\overline{x}} \to \overline{\overline{y}}$ such that $\overline{\overline{\beta}} \circ \varepsilon_x = \varepsilon_y \circ \overline{\beta}$ because, for every $(\varrho, \sigma, \alpha) \in T_x$, $(\varrho, \sigma, \beta \circ \alpha)$ is in T_y.

We define the category $\mathcal{C}$ as follows: $\operatorname{obj}\mathcal{C} = \operatorname{obj}\mathcal{C}_0$, $\mathcal{C}(x,y) = \{(y, \overline{\overline{\beta}}, x) | \beta = (y, \overline{\beta}, x) \in \mathcal{C}_0(x,y)\}$; the forgetful functor $W : \mathcal{C} \to \mathrm{Set}$ is defined by $W(x) = \overline{\overline{x}}$, $W(\beta) = \overline{\overline{\beta}}$. The factorization $P : \mathcal{C}_0 \to \mathcal{C}$ is given by $P(x) = x$, $P(y, \overline{\beta}, x) = (y, \overline{\overline{\beta}}, x)$. Then $F_0 = P \circ G$ already preserves the composition because, for $\varrho \in \mathcal{K}_0(r,s)$, $\sigma \in \mathcal{K}_0(s,t)$, $\alpha = 1 \in \mathcal{C}_0(G(t), G(t))$, we have $\varepsilon \circ \overline{G(\sigma)} \circ \overline{G(\varrho)} = \varepsilon \circ \overline{G(\sigma \circ \varrho)}$ hence $\overline{\overline{G(\sigma)}} \circ \overline{\overline{G(\varrho)}} \circ \varepsilon = \overline{\overline{G(\sigma \circ \varrho)}} \circ \varepsilon$ and ε is epi.

II.9. We show that $\mathcal{H}$ is a full subcategory of $\mathcal{C}$ and that F_0 is an extension of F_1. To show this, it is sufficient to prove that for every $a \in A = \operatorname{obj}\mathcal{H}$, the joint coequalizer $\varepsilon_a : W_0(a) \to \overline{\overline{a}}$ in II.8 is a bijection. Then we can choose $\varepsilon_a = 1_{W_0(a)}$ and, for $a, b \in A$ and $\alpha \in \mathcal{H}(a,b)$, $\alpha = (b, \overline{\alpha}, a)$ we have $\overline{\overline{\alpha}} = \overline{\alpha}$, hence $P(\alpha) = \alpha$. Consequently, for $c, d \in \operatorname{obj}\mathcal{K}_1$, $F_1(c) = a$, $F_1(d) = b$, $\varrho \in \mathcal{K}_1(c,d)$, we have $F_0(\varrho) = G(\varrho) = F_1(\varrho)$.

Thus, let $a \in A$. It is sufficient to show that for every $(\varrho, \sigma, \alpha) \in T_a$,

$$(**) \qquad \overline{\alpha} \circ \overline{G(\sigma)} \circ \overline{G(\varrho)} = \overline{\alpha} \circ \overline{G(\sigma \circ \varrho)}.$$

Let us suppose that

$$\begin{array}{ccccc} r & \xrightarrow[\varrho]{} & s & \xrightarrow[\sigma]{} & t \\ & & & & G(t) \xrightarrow[\alpha]{} a. \end{array}$$

We have to investigate all the possibilities $r, s, t \in \operatorname{obj}\mathcal{K}_1 \cup (\operatorname{obj}\mathcal{K}_0 \setminus \operatorname{obj}\mathcal{K}_1)$, i.e. 2^3 cases again. We show "the worst" case that $r \in \operatorname{obj}\mathcal{K}_0 \setminus \operatorname{obj}\mathcal{K}_1$, $s \in \operatorname{obj}\mathcal{K}_1$

and $t \in \operatorname{obj}\mathcal{K}_0 \setminus \operatorname{obj}\mathcal{K}_1$, the other cases are either analogous or easier. In our case, $G(\varrho) = (F_1(s), \pi_s \circ \overline{\varrho}, r)$, $G(\sigma) = (t, \overline{\sigma} \circ \mu_s, F_1(s))$, $G(\sigma \circ \varrho) = \sigma \circ \varrho$; then $\overline{G(\sigma \circ \varrho)} = \overline{\sigma \circ \varrho}$ and $\overline{G(\sigma)} \circ \overline{G(\varrho)} = \overline{\sigma} \circ \mu_s \circ \pi_s \circ \overline{\varrho}$; but, by II.7, $\alpha \in \mathcal{C}_0(G(t), a)$ can be expressed as $\alpha = (a, \overline{\gamma} \circ \pi_d \circ \overline{\delta}, t)$ where $d \in \operatorname{obj}\mathcal{K}_1$, $F_1(d) = c$, $\delta \in \mathcal{K}_0(t, d)$ and $\gamma \in \mathcal{H}(c, a)$, so that

$$\overline{\alpha} \circ \overline{G(\sigma)} \circ \overline{G(\varrho)} = \overline{\gamma} \circ \pi_d \circ \overline{\delta} \circ \overline{\sigma} \circ \mu_s \circ \pi_s \circ \overline{\varrho} = \overline{\gamma} \circ \overline{F_1(\delta \circ \sigma)} \circ$$
$$\circ\, \pi_s \circ \overline{\varrho} = \overline{\gamma} \circ \pi_d \circ \overline{\delta \circ \sigma} \circ \overline{\varrho} = \overline{\gamma} \circ \pi_d \circ \overline{\delta} \circ \overline{\sigma \circ \varrho} = \overline{\alpha} \circ \overline{G(\sigma \circ \varrho)},$$

hence $(**)$ is satisfied.

II.10. We conclude that the inclusion $J : (\mathcal{H}, V) \longrightarrow (\mathcal{C}, W)$ is a realization and $F_0 \circ E = j \circ F_1$. The statements (i), (ii), (iii) in II.4 follow from the fact that $\mathcal{C}$ is a mere factorcategory of $\mathcal{C}_0$, so that $\mathcal{C}$ satisfies them whenever $\mathcal{C}_0$ satisfies the corresponding statements. However this follows immediately from II.7.

III. Ordinary embeddings

III.1. Ordinary embeddings were introduced in the part I. as compositions of finitely many strong extensions, strong embeddings and strong co-embeddings. The notion of ordinary embedding is sitting between two requirements:

a) *to be weak enough*: so weak that every concrete category admits an embedding of this kind into the category of topological semigroups (as shown in part V of this paper) and into other comprehensive categories;

b) *to be strong enough*: so strong that the Extension Lemma can be, in a way, generalized to it. This is just stated in the Proposition below.

III.2. *A distinguished point* of a concrete category $(\mathcal{K}, U)$ is a collection $d = \{d_a \mid a \in \operatorname{obj}\mathcal{K}\}$ such that $d_a \in U(a)$ and $[U(\alpha)](d_a) = d_b$ for all $a, b \in \operatorname{obj}\mathcal{K}$, $\alpha \in \mathcal{K}(a, b)$.

Proposition. *Let $(\mathcal{K}_0, U_0)$, $(\mathcal{K}_1, U_1)$, $(\mathcal{H}, Y)$ be concrete categories, let $(\mathcal{H}, Y)$ have a distinguished point. Let $E : (\mathcal{K}_1, U_1) \longrightarrow (\mathcal{K}_0, U_0)$ be an ordinary embedding. Let $F_1 : \mathcal{K}_1 \longrightarrow \mathcal{H}$ be a functor such that $Y \circ F_1$ is a retract of U_1. Then there exists a faithful functor $V : \mathcal{H} \longrightarrow$ Set such that $V \circ F_1$ is a retract of $U_0 \circ E$.*

The rest of part III. is devoted to the proof of this Proposition.

III.3. In III.4–III.6, we suppose the following situation (we shall not repeat these presumptions): $(\mathcal{K}_1, U_1)$ and $(\mathcal{H}, Y)$ are concrete categories, $(\mathcal{H}, Y)$ has a distinguished point, say d; moreover, a functor $F_1 : \mathcal{K}_1 \longrightarrow \mathcal{H}$ and natural transformations $\pi : U_1 \longrightarrow Y \circ F$ and $\mu : Y \circ F \longrightarrow U_1$ are given such that $\pi \circ \mu$ is the identity.

III.4. **Lemma.** *Let $E : (\mathcal{K}_1, U_1) \longrightarrow (\mathcal{K}_0, U_0)$ be either a strong extension or a strong embedding. Then there exists a faithful functor $V : \mathcal{H} \longrightarrow$ Set such that*

a) *$(\mathcal{H}, V)$ has a distinguished point;*

b) *there exist natural transformations*

$$\tilde{\pi} : U_0 \circ E \to V \circ F_1, \quad \tilde{\mu} : V \circ F_1 \to U_0 \circ E$$

such that $\tilde{\pi} \circ \tilde{\mu}$ *is the identity.*

Proof. 1) Let E be a strong extension. Hence there exists $G : \mathcal{K}_1 \to \mathrm{Set}$ such that $U_0 \circ E = U_1 \amalg G$. We put $V = Y$, $\tilde{\pi}$ sends U_1 onto V as π and G on the distinguished point d (i.e. $\tilde{\pi}_a(x) = d_a$ for all $x \in G(a)$); $\tilde{\mu}$ is a composition of $\mu : V \circ F_1 \to U_1$ and the coproduct injection $U_1 \to U_1 \amalg G$. Since $V = Y$, $(\mathcal{H}, V)$ has a distinguished point.

2) Let E be a strong embedding, i.e. there exists a faithful functor G : Set $\to$ Set such that $U_0 \circ E = G_0 \circ U_1$. We put $V = G \circ Y$. Since G is faithful, there exists a monotransformation of the identical functor Id : Set $\to$ Set (see e.g. [13]), so that the concrete category $(\mathcal{H}, V)$ has a distinguished point again. We put $\tilde{\pi} = G \circ \pi$, $\tilde{\mu} = G \circ \mu$, hence V, $\tilde{\pi}$, $\tilde{\mu}$ have all the required properties.

III.5. We need an analogous result also for strong co-embedding, hence we have to work with opposite categories. Let $\mathcal{K}_1^{op}$, $\mathcal{H}^{op}$ be the categories opposite to $\mathcal{K}_1$, $\mathcal{H}$ and let us denote by $F_1^{op} : \mathcal{K}_1^{op} \to \mathcal{H}^{op}$ the functor opposite to $F_1 : \mathcal{K}_1 \to \mathcal{H}$. Given a strong co-embedding $E : (\mathcal{K}_1, U_1) \to (\mathcal{K}_0, U_0)$, let us denote by $\tilde{E} : \mathcal{K}_1^{op} \to \mathcal{K}_0$ the corresponding covariant (full one-to-one) functor.

Lemma. *Let* $E : (\mathcal{K}_1, U_1) \to (\mathcal{K}_0, U_0)$ *be a strong co-embedding. Then there exists a faithful functor* $V : \mathcal{H}^{op} \to \mathrm{Set}$ *such that*

a) $(\mathcal{H}^{op}, V)$ *has a distinguished point;*

b) *there exist natural transformations*

$$\tilde{\pi} : U_0 \circ \tilde{E} \to V \circ F_1^{op}, \quad \tilde{\mu} : V \circ F_1^{op} \to U_0 \circ \tilde{E}$$

such that $\tilde{\pi} \circ \tilde{\mu}$ *is the identity.*

Proof. Since E is a strong co-embedding, there exists a contravariant faithful functor G : Set $\to$ Set such that $U_0 \circ E = G \circ U_1$. Then $G \circ Y : \mathcal{H} \to \mathrm{Set}$ is a contravariant faithful functor, let $V : \mathcal{H}^{op} \to \mathrm{Set}$ be the corresponding covariant one. Since G is contravariant and faithful, the contravariant power-set functor P^- (i.e. $P^-X = \exp X$, $[P^-(f)](Z) = f^{-1}(Z)$) is a retract of G (see e.g. [13, pp. 85–86]); since P^- has a distinguished point (namely $\emptyset \in P^-X$), $(\mathcal{H}^{op}, V)$ also has a distinguished point. Now, we put $\tilde{\pi} = G \circ \mu$, $\tilde{\mu} = G \circ \pi$. Then V, $\tilde{\pi}$, $\tilde{\mu}$ have all the required properties.

III.6. Proof of the Proposition.

By the definition, $E : (\mathcal{K}_1, U_1) \to (\mathcal{K}_0, U_0)$ can be expressed as $E = E_n \circ \cdots \circ E_1$ where every $E_i : (\mathcal{K}_i, U_i) \to (\mathcal{K}_{i+1}, U_{i+1})$ is either a strong extension or a strong embedding or a strong co-embedding (and $(\mathcal{K}_{n+1}, U_{n+1})$ is our given

$(\mathcal{K}_0, U_0))$. Moreover, we may suppose that all the functors E_i, except possibly the last E_n, map the class obj $\mathcal{K}_i$ onto the class obj $\mathcal{K}_{i+1}$. Hence every E_i, $i = 1, \dots, n-1$, is either an isomorphism of $\mathcal{K}_i$ onto $\mathcal{K}_{i+1}$ (and then we may suppose that $\mathcal{K}_i = \mathcal{K}_{i+1}$ and E_i is identical, only U_i, U_{i+1} are possibly different) or a contravariant isomorphism (and then we may suppose that $\mathcal{K}_{i+1} = \mathcal{K}_i^{op}$ and E_i is the duality). If every E_i, $i = 1, \dots, n$ is either a strong extension or a strong embedding, we use Lemma III.4 finitely many times and we get the Proposition. If some of the E_i's is a strong co-embedding, we use also Lemma III.5: let j be the smallest element of $\{1, \dots, n\}$ such that E_j is a strong co-embedding. Since E is covariant, some E_{j+h} with $h \geq 1$ must be also a strong co-embedding; let h be the smallest natural number with this property. We apply Lemma III.4 on $E_1, \dots, E_{j-1}$ (we may suppose $\mathcal{K}_1 = \dots = \mathcal{K}_j$); then we use Lemma III.5 for E_j. For $E_{j+1}, \dots, E_{j+h-1}$, we use Lemma III.4 again, but applied on $\mathcal{K}_1^{op}$ and $F_1^{op} : \mathcal{K}_1^{op} \to \mathcal{H}^{op}$. Then, using Lemma III.5 for E_{j+h}, we change $\mathcal{K}_1^{op}$, $\mathcal{H}^{op}$ and F_1^{op} back to $\mathcal{K}_1$, $\mathcal{H}$ and F_1. We proceed similarly with $E_{j+h+1}, \dots, E_n$.

IV. The proof of the Main Theorem

IV.1. In this part, we present the proof of the Main Theorem, using the Extension Lemma of II. and the Proposition in III. By [15], a universal functor

$$\mathcal{U} \xrightarrow{\ H\ } \mathcal{U}$$

does exist. By the Kučera theorem [11], a concretizable category $\mathcal{H}$ and a congruence $\sim_0$ exist such that $\mathcal{H}/\sim_0$ is isomorphic to $\mathcal{U}$. Let us denote by $P : \mathcal{H} \to \mathcal{U}$ the corresponding surjective functor, one-to-one on obj $\mathcal{H}$ (i.e. $P(\alpha) = P(\beta)$ iff $\alpha \sim_0 \beta$). Let us form a pullback of $H \circ P$ and P,

$$\begin{array}{ccccc} \mathcal{H} & \xrightarrow{P} & \mathcal{U} & \xrightarrow{H} & \mathcal{U} \\ {\scriptstyle F_2}\uparrow & & & & \uparrow{\scriptstyle P} \\ \mathcal{K}_1 & = & \mathcal{K}_1 & \xrightarrow{F_1} & \mathcal{H} \end{array}$$

i.e. obj $\mathcal{K}_1$ is precisely the class of all pairs $(a_2, a_1) \in \text{obj}\,\mathcal{H} \times \text{obj}\,\mathcal{H}$ such that $H(P(a_2)) = P(a_1)$, $F_1(a_2, a_1) = a_1$, $F_2(a_2, a_1) = a_2$ and analogously for morphisms. One can verify that F_2 is surjective and that it is one-to-one on obj $\mathcal{K}_1$ (because the functor P, opposit to F_2 in the pullback, has also these properties), so that $P \circ F_2$ is also surjective and one-to-one on obj $\mathcal{K}_1$. Let $\sim_1$ be the congruence on $\mathcal{K}_1$ defined by

$$\alpha \sim_1 \beta \text{ iff } P(F_2(\alpha)) = P(F_2(\beta)).$$

Hence $\mathcal{K}_1/\sim_1$ is isomorphic to $\mathcal{U}$. Moreover, F_1 sends $\sim_1$-congruent morphisms to $\sim_0$-congruent morphisms so that the factorfunctor $\tilde{F}_1 : \mathcal{K}_1/\sim_1 \to \mathcal{H}/\sim_0$ of F_1 is a universal functor.

IV.2. Since $\mathcal{H}$ is concretizable, there is a faithful functor $X : \mathcal{H} \to \mathrm{Set}$. Let $C_1 : \mathcal{H} \to \mathrm{Set}$ be a functor sending each $a \in \mathrm{obj}\,\mathcal{H}$ on a one point set $\{d_a\}$ and let Y be a coproduct of X and C_1 so that $(\mathcal{H}, Y)$ is a concrete category with a distinguished point $d = \{d_a \mid a \in \mathrm{obj}\,\mathcal{H}\}$ (see III.2). Let us define

$$U_1 : \mathcal{K}_1 \to \mathrm{Set}$$

by $U_1(a_2, a_1) = Y(a_2) \times Y(a_1)$, $U_1(\alpha_2, \alpha_1) = Y(\alpha_2) \times Y(\alpha_1)$. Then $(\mathcal{K}_1, U_1)$ is a concrete category because U_1 is faithful. Moreover, $Y \circ F_1$ is a retract of U_1. In fact, if $a = (a_2, a_1) \in \mathrm{obj}\,\mathcal{K}_1$, then the maps $\pi_a : Y(a_2) \times Y(a_1) \to Y(a_1)$ sending (x_2, x_1) to x_1 determine an epitransformation $\pi : U_1 \to Y \circ F_1$ and the maps $\mu_a : Y(a_1) \to Y(a_2) \times Y(a_1)$ sending $x \in Y(a_1)$ to (d_{a_2}, x) determine a monotransformation $\mu : Y \circ F_1 \to U_1$ such that $\pi \circ \mu$ is the identity.

IV.3. Let a COE–universal category $(\mathcal{K}_0, U_0)$ be given. Then there exists an ordinary embedding $E : (\mathcal{K}_1, U_1) \to (\mathcal{K}_0, U_0)$. By Proposition III.2, there exists a faithful functor $V : \mathcal{H} \to \mathrm{Set}$ such that $V \circ F_1$ is a retract of $U_0 \circ E$. Hence $E : (\mathcal{K}_1, U_0 \circ E) \to (\mathcal{H}, V)$ is a realization, so that we can use the Extension Lemma: there exist a concrete category $(\mathcal{C}, W)$, a realization $J : (\mathcal{H}, V) \to (\mathcal{C}, W)$ and a functor $F_0 : \mathcal{K}_0 \to \mathcal{C}$ such that $F_0 \circ E = J \circ F_1$. Let $\Phi : \mathcal{C} \to \mathcal{K}_0$ be a full one-to-one functor (such functor does exists because $\mathcal{C}$ is concretizable and $\mathcal{K}_0$ is COE–universal). Then we have the commutative square

$$\begin{array}{ccc}
\mathcal{K}_1 & \xrightarrow{F_1} & \mathcal{H} \\
{\scriptstyle E}\downarrow & & \downarrow{\scriptstyle J} \\
\mathcal{K}_0 & \xrightarrow{F_0} & \mathcal{C} \\
\| & & \downarrow{\scriptstyle \Phi} \\
\mathcal{K}_0 & \xrightarrow{\Phi \circ F_0} & \mathcal{K}_0
\end{array}$$

and both E and $\Phi \circ J$ are full one-to-one functors. Moreover, we have congruences $\sim_1$ on $\mathcal{K}_1$ and $\sim_0$ on $\mathcal{H}$ such that the functor $\tilde{F}_1 : \mathcal{K}_1/{\sim_1} \to \mathcal{H}/{\sim_0}$ is universal.

IV.4. To finish the proof of the Main Theorem, it is sufficient to find a congruence $\sim$ on $\mathcal{K}_0$ such that the constructed functor $F = \Phi \circ F_0$ preserves it and $\sim$ agrees with $\sim_1$ on $E(\mathcal{K}_1)$ and $\sim$ agrees with $\sim_0$ on $\Phi(J(\mathcal{H}))$. Then, clearly,

$$F/{\sim} : \mathcal{K}_0/{\sim} \to \mathcal{K}_0/{\sim}$$

is universal. For this reason, we have to choose the functors $E : (\mathcal{K}_1, U_1) \to (\mathcal{K}_0, U_0)$ and $\Phi : \mathcal{C} \to \mathcal{K}_0$ in a special way. This is done in IV.6 below.

IV.5. Before the special construction of E and Φ, let us mention explicitely how a congruence $\simeq$ on a full subcategory $\mathcal{A}$ of a category $\mathcal{B}$ is extended on $\mathcal{B}$: For every $a, b \in \operatorname{obj} \mathcal{B}$, $\alpha, \beta \in \mathcal{B}(a, b)$, we define

$$\alpha R \beta \quad \text{iff} \quad \text{either } \alpha = \beta \text{ or } \alpha = \mu \circ \varrho \circ \nu, \quad \beta = \mu \circ \sigma \circ \nu \text{ and } \varrho \simeq \sigma.$$

Then the transitive envelope of R is a congruence $\cong$ on $\mathcal{B}$ and, since $\mathcal{A}$ is a full subcategory,

$$\alpha \cong \beta \text{ iff } \alpha \simeq \beta \text{ whenever } \alpha, \beta \in \mathcal{A}(a, b).$$

IV.6. We investigate two copies $(\mathcal{K}_0^1, U_0^1)$ and $(\mathcal{K}_0^2, U_0^2)$ of the given COE-universal category $(\mathcal{K}_0, U_0)$. The coproduct is a concrete category as well, hence there exists an ordinary embedding

$$\Lambda : (\mathcal{K}_0^1, U_0^1) \amalg (\mathcal{K}_0^2, U_0^2) \to (\mathcal{K}_0, U_0).$$

Let us denote by $K_i : (\mathcal{K}_0^i, U_0^i) \to (\mathcal{K}_0^1, U_0^1) \amalg (\mathcal{K}_0^2, U_0^2)$ the coproduct injections (they are realizations of the concrete categories). Let $E_1 : (\mathcal{K}_1, U_1) \to (\mathcal{K}_0^1, U_0^1)$ be an ordinary embedding and $\Phi_2 : \mathcal{C} \to \mathcal{K}_0^2$ be a full one-to-one functor. We choose $E : (\mathcal{K}_1, U_1) \to (\mathcal{K}_0, U_0)$ and $\Phi : \mathcal{C} \to \mathcal{K}_0$ in IV.3 of the form

$$E = \Lambda \circ K_1 \circ E_1 \quad \text{and} \quad \Phi = \Lambda \circ K_2 \circ \Phi_2.$$

Then E is an ordinary embedding and Φ is a full one-to-one functor, as required in IV.3.

IV.7. For shortness, let us suppose that $\mathcal{K}_0^1$, $\mathcal{K}_0^2$ are full subcategories of $\mathcal{K}_0$, $\mathcal{K}_1$ is a full subcategory of $\mathcal{K}_0^1$, $\mathcal{C}$ is a full subcategory of $\mathcal{K}_0^2$, $\mathcal{H}$ is a full subcategory of $\mathcal{C}$ and all the functors E_1, K_1, K_2, Λ, Φ_2, Φ, J are inclusions. Notice that $F : \mathcal{K}_0 \to \mathcal{K}_0$ sends the whole $\mathcal{K}_0$ into $\mathcal{C}$ and $\mathcal{K}_1$ into $\mathcal{H}$.

We investigate the following classes of objects of $\mathcal{K}_0$:

$$M_0 = \operatorname{obj} \mathcal{K}_1, \quad M_{j+1} = F(M_j).$$

We claim that

(a) if $i \neq j$, then $M_i \cap M_j = \emptyset$; in fact, $M_0 \cap M_j = \emptyset$ for all $j \geq 1$ because $M_0 \subseteq \operatorname{obj} \mathcal{K}_0^1$ and $M_j \subseteq \operatorname{obj} \mathcal{K}_0^2$ for all $j \geq 1$. If $M_0, \dots, M_n$ are supposed pairwise disjoint, then $M_0, M_1 = F(M_0), \dots, M_{n+1} = F(M_n)$ are pairwise disjoint, by II.4 (i);

(b) if $i \neq j$, $a \in M_i$, $b \in M_j$, then $\mathcal{K}_0(a, b) = \mathcal{K}_0(b, a) = \emptyset$; this can be proved analogously to (a), using (ii) and (iii) in II.4.

IV.8. Let us denote by $\mathfrak{M}_i$ the full subcategory of $\mathcal{K}_0$ generated by M_i and by $\mathfrak{M}$ the full subcategory of $\mathcal{K}_0$ generated by $\bigcup_{j=0}^{\infty} M_j$. The claims (a), (b) in IV.7 imply that a congruence $\simeq$ on $\mathfrak{M}$ is determined by congruences $\simeq_j$ on $\mathfrak{M}_j$, $j = 0, 1, \dots$. We define the congruences $\simeq_j$ as follows:

$$\simeq_0 = \sim_1, \quad \simeq_1 = \sim_0$$

and, for $j \geq 2$, we put $\alpha \simeq_j \beta$ whenever α and β in $\mathfrak{M}_j$ have the same domain and codomain.

Let $\sim$ be the congruence on $\mathcal{K}_0$ extending $\simeq$ as mentioned in IV.5. Then $F : \mathcal{K}_0 \to \mathcal{K}_0$ preserves $\sim$ and the square

$$\begin{array}{ccc} \mathcal{K}_1/\sim_1 & \xrightarrow{\bar{F}_1} & \mathcal{H}/\sim_0 \\ \downarrow & & \downarrow \\ \mathcal{K}_0/\sim & \xrightarrow{F/\sim} & \mathcal{K}_0/\sim \end{array}$$

commutes, where the vertical arrows are isofunctors onto full subcategories. We conclude that $F/\sim$ is universal.

IV.9. Remark. An easy modification of the proof of the Main Theorem gives the following variant of it: Let $(\mathcal{C}_i, U_i)$, $i = 1, 2$, be COE–universal categories. Then there exist congruences $\sim_i$ on $\mathcal{C}_i$, $i = 1, 2$, and a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ sending $\sim_1$-congruent morphisms to $\sim_2$-congruent ones such that the corresponding factor functor $\mathcal{C}_1/\sim_1 \to \mathcal{C}_2/\sim_2$ is universal.

V. COE–universal categories

V.1. A considerable part of full embeddings in [13] are in fact ordinary embeddings. However, this is not explicitly stated in [13]. In V.2–V.5 below, we indicate the parts of [13] which have to be inspected to see that the categories $\mathcal{T}$ and $\mathcal{M}$ are COE–universal and that, under (M), the categories Graph, Alg(1, 1), Alg(2) and others are COE–universal (all endowed with their natural forgetful functors). In V.6, we show that the category $\mathcal{S}$ of all topological semigroups and all continuous homomorphisms is COE–universal.

V.2. The categories $S(P^+)$, $S(P_0^+)$, $S(P^-)$ are often used in the embedding constructions in [13]. Let us recall that P^+, P^- are the covariant and the contravariant power-set functors Set $\to$ Set, $P^+(X) = P^-(X) = \{Z \mid Z \subseteq X\}$ and, for $f : X \to Y$, $[P^+(f)](Z) = f(Z)$ for every $Z \subseteq X$ and $[P^-(f)](Z) = f^{-1}(Z)$ for every $Z \subseteq Y$. The functor P_0^+ is a subfunctor of P^+, $P_0^+(Y) = \{Z \mid Z \subseteq X, Z \neq \emptyset\}$.

Given a functor F : Set $\to$ Set, covariant or contravariant, the category $S(F)$ is defined as follows: objects are all pairs (X, r), where X is a set and $r \subseteq F(X)$; $f : (X, r) \to (X', r')$ is a morphism of $S(F)$, if it is a map $X \to X'$ such that $[F(f)](r) \subseteq r'$ (or $[F(f)](r') \subseteq r$ if F is contravariant). The forgetful functor $S(F) \to$ Set sends (X, r) to X of course. $S(F_1, \ldots, F_n)$ are defined analogously (here, objects are all $(X, r_1, \ldots, r_n)$ with $r_j \subseteq F_j(X)$).

V.3. The categories $S(P^+)$, $S(P_0^+)$, $S(P^-)$ are COE–universal. In fact, given an arbitrary concrete category $(\mathfrak{A}, U)$, there exists a preordered class $(T, \leq)$ and a *realization* of $(\mathfrak{A}, U)$ into the category $\mathcal{S}(P^+ \circ Q, (T, \leq))$. This is precisely Theorem 3.3 of [13, pp. 95–96], the category $\mathcal{S}(P^+ \circ Q, (T, \leq))$ being defined on [13, p. 94].

(Let us recall that Q : Set $\rightarrow$ Set denotes the square functor $Q(X) = X \times X$, $Q(f) = f \times f$.) A full one-to-one functor $F : S(P^+ \circ Q, (T, \leq)) \rightarrow S(P^+, \ldots, P^+)$ is constructed in the proof of Theorem 4.3, in [13, pp. 97–99]. Inspecting the formula for F on p. 98, one can see that F can be expressed as a composition of a strong embedding, (carried by the set functor $P^+ \circ Q$) followed by a strong extension (adding the summand $\bigcup_{M \subseteq X \times X}(\{M\} \times Y(\alpha(M))))$. Finally, $S(P^+, \ldots, P^+)$ admits a strong embedding into $S(P_0^+)$, see [13, pp. 88–89]. We conclude that $S(P_0^+)$, hence $S(P^+)$, are COE–universal. A strong embedding of $S(P_0^+)$ into $S(P^-)$ is constructed in [13, pp.81–85], hence $S(P^-)$ is also COE–universal.

V.4. A strong embedding of $S(P_0^+)$ into $\mathcal{M}$ is constructed in [13, pp. 230–233]. A full embedding $\mathcal{U}_1$ of $S(P_0^+)$ into $\mathcal{T}$ is constructed in [13, pp. 244–246]; inspecting the construction, one can see that $\mathcal{U}_1$ can be expressed as a composition of a strong embedding followed by a strong extension. Thus, both $\mathcal{M}$ and $\mathcal{T}$ are COE–universal.

V.5. Under the set-theoretical statement (M), a strong embedding of $S(P^-)$ into Graph, the category of all directed graphs and all compatible maps, is constructed in [13, p. 80]. In [13, pp. 59–60], full embeddings of Graph into Alg(1, 1) and into Alg(1, 1, 0), the categories of all universal algebras with two unary resp. two unary and one nullary operations and all their homomorphisms, are constructed. Inspecting the construction (the construction is also mentioned in V.6 below), one can see that these full embeddings are strong extensions. Strong embeddings of Alg(1, 1) into Alg(2) and into Alg(2, 0), the categories of all algebras with one binary or one binary and one nullary operations, are constructed in [13, pp. 60–61]. Hence, under (M), all the categories Graph, Alg(1, 1), Alg(1, 1, 0), Alg(2), Alg(2, 0) are COE–universal. Moreover, a strong embedding of Alg(2) into the category Smg of all semigroups is constructed in [13, pp. 148–154] and a strong embedding of Alg(1, 1) into Id_0, the category of all integral domains of characteristic zero and all their ring homomorphisms is outlined in [13, pp. 158–160] (the full version see [4]). Thus, under (M), Smg and Id_0 are also COE–universal. Many varieties of unary algebras are shown to be COE–universal under (M) in [13, pp.173–185]. (Let us mention explicitly that every variety of universal algebras is investigated as a category, morphisms are formed by all homomorphisms.) In fact, if Alg(1, 1) can be fully embedded in a variety $\mathcal{V}$ of *unary* algebras, then it can be strongly embedded in it, by [12]. Hence, under (M), every C–universal variety of unary algebras is COE–universal. In [16], all the categories of presheaves in Set, which admit a full embedding of Graph into them, are characterized; the results are presented in [13, pp.201–202]. All the full embeddings constructed in [16] are strong extensions whenever the category of presheaves in question is endowed by the usual forgetsul functor sending any $\varphi : (P, \leq) \rightarrow$ Set to the coproduct of all sets $\varphi(p)$, $p \in P$. Hence again, under (M), if a category of presheaves in Set is C–universal, then it is COE–universal. A lot of full embeddings of Graph into varieties of universal algebras are constructed in the literature, mostly in *very* sophisticated way, see e.g. [2],

[5], [9], [10]. However these full embeddings imply the C–universality (and possibly COE–universality) of the varieties in question only under (M) — and better result cannot be obtained: no category of algebras is C–universal under non (M) (see [13, pp. 348–349]).

In the next part V.6, the statement (M) is *not* supposed.

V.6. Let us denote by Top Graph the category of all topological graphs, i.e. objects are all pairs (X, R) where X is a topological space and R is a *closed* subset of the space $X \times X$, morphisms are all continuous compatible maps; denote by Par Graph the full subcategory consisting of all (X, R) with X paracompact.

Let $\mathcal{V}$ be a variety of universal algebras, say of a type Δ, determined by a set of equations E. Let us denote by Top $\mathcal{V}$ (or Par $\mathcal{V}$) the corresponding category of topological (or paracompact) algebras, i.e. objects are all topological (or paracompact) spaces endowed by *continuous* operations of the type Δ satisfying the set of equations E, morphisms are all continuous homomorphisms.

Proposition: *The categories* Par Graph, Par Alg(1, 1), Par Alg(1, 1, 0), Par Alg(2), Par Alg(2, 0), Par Smg *(hence $\mathcal{S}$ =* Top Smg*), all investigated with their natural forgetful functors, are COE–universal.*

Proof: a) In [13], *almost full embeddings* into the category Top of all topological spaces and all continuous maps are investigated. Let us recall that a one-to-one functor $\Phi : \mathcal{K} \to$ Top is called an almost full embedding if, for every $a, b \in \operatorname{obj} \mathcal{K}$, Φ maps the set $\mathcal{K}(a, b)$ *onto* the set of *all* nonconstant continuous maps of $\Phi(a)$ into $\Phi(b)$. Let us denote by $\Phi(\mathcal{K})$ the subcategory of Top consisting of all $\Phi(a)$, $a \in \operatorname{obj} \mathcal{K}$, and all $\Phi(\alpha)$, $\alpha \in \operatorname{morph} \mathcal{K}$. Then $\mathcal{K}$ and $\Phi(\mathcal{K})$ are isomorphic, $\Phi(\mathcal{K})$ is a category of suitable topological spaces and *all* their nonconstant continuous maps (and these maps are closed with respect to the composition!).

b) In [13, §15, pp. 249–252], as variations of [8], almost full embeddings of $S(P^-)^{op}$ into Par, the category of all paracompact spaces and all continuous maps, are presented. Particularly, in 15.10 [13, p. 252], such almost full embedding $Q : S(P^-)^{op} \to$ Par is presented (let us investigate it as a contravariant functor $S(P^-) \to$ Par) that there exists a contravariant functor $F :$ Set $\to$ Set with $U \circ Q = F \circ V$, where $U :$ Par $\to$ Set and $V : S(P^-) \to$ Set are the natural forgetful functors. Hence, denoting by $\mathcal{D}$ the subcategory $Q(S(P^-))$ of Par, we get that the range-restriction $\bar{Q} : S(P^-) \to \mathcal{D}$ of Q is a strong co-embedding. This implies easily that $\mathcal{D}$ is COE–universal. In fact, given a concrete category $(\mathcal{C}, W)$, we denote by $D : (\mathcal{C}, W) \to (\mathcal{C}^{op}, P^- \circ W)$ the strong co-embedding which is just the duality on $\mathcal{C}$, carried by $P^- :$ Set $\to$ Set. Since $(S(P^-), V)$ is COE–universal, there is an ordinary embedding $E : (\mathcal{C}^{op}, P^- \circ W) \to (S(P^-), V)$, so that $\bar{Q} \circ E \circ D$ is an ordinary embedding of $(\mathcal{C}, W)$ into $(\mathcal{D}, U)$ (where $U : \mathcal{D} \to$ Set is the domain restriction of $U :$ Par $\to$ Set).

c) If we inspect §15 in [13, pp. 249–252] again, we see that $(\mathcal{D}, U)$ has two distinct distinguished points, say $d^1 = \{d_a^1 \mid a \in \operatorname{obj} \mathcal{D}\}$ and $d^2 \doteq \{d_a^2 \mid a \in \operatorname{obj} \mathcal{D}\}$ (in fact, $(\mathcal{D}, U)$ has $2^{\aleph_0}$ distinct distinguished points, see [13, pp. 148–152], and we choose arbitrary pair of them). This implies immediately that there is a realization $\Psi : (\mathcal{D}, U) \longrightarrow$ Par Graph namely, for every space $a = (U(a), t_a)$, we put $\Psi(a) = (a, R_a)$, where $R_a = \{(d_a^1, d_a^2)\}$. Clearly, if $f : a \longrightarrow a'$ is a morphism of $\mathcal{D}$ then it is a morphism of Par Graph and vice versa. We conclude that Par Graph is COE–universal.

d) Let us recall that in [13, p. 59], the following functor F : Graph $\longrightarrow$ Alg(1, 1) (or Graph $\longrightarrow$ Alg(1, 1, 0)) is defined: $F(X, R) = (Y, \alpha, \beta)$ (or $F(X, R) = (Y, \alpha, \beta, \omega)$) where

$$\begin{aligned}
&Y = X \amalg R \amalg \{p, q\} \text{ with}\\
&\alpha(x) = p, \alpha(x, y) = x, \alpha(p) = \alpha(q) = q,\\
&\beta(x) = q, \beta(x, y) = y, \beta(p) = \alpha(q) = p,\\
&\text{for } x \in X, (x, y) \in R, (\omega = p);
\end{aligned}$$

(the definition of F on morphisms is omitted).

In [13, pp. 59–60], the functor F is proved to be full. It is a strong extension, evidently. Moreover, if $a = (X, R) \in$ obj Top Graph, and Y is endowed by the topology t_a which coincides with the topology of a on X, X is closed-and-open in t_a and t_a is discrete on $R \amalg \{p, q\}$, then α, β are continuous and F determines a strong extension of Top Graph into Top Alg(1, 1) and into Top Alg(1, 1, 0). Moreover, if $a = (X, R) \in$ obj Par Graph, then the underlying space of $F(a)$ is paracompact as well. We conclude that Par Alg(1, 1) and Par Alg(1, 1, 0) are COE–universal.

e) In [13, p. 61] a strong embedding F of Alg(1, 1) into Alg(2) and into Alg(2, 0) is defined as follows: its sends an algebra (X, α, β) to an algebra $(G(X), \bullet)$ (or $(G(X), \bullet, \omega)$) where $G(X) = X \amalg \{p, q\}$ and

$$\begin{aligned}
&x \bullet y = p, p \bullet x = x \bullet p = \alpha(x). q \bullet x = x \bullet q = \beta(x) \text{ for all } x, y \in X,\\
&p \bullet p = q, q \bullet q = p, p \bullet q = q \bullet p = p, (\omega = p)
\end{aligned}$$

(the definition of F on morphisms is not necessary, F is carried by $G = \text{Ident} \amalg C_{\{p,q\}}$).

The proof that F is really a strong embedding (especially its fullness) is proved in [13, pp. 61–62]. We see immediately that if X is a topological space and α, β are continuous, then the binary operation $\bullet$ is continuous as well whenever we investigate $G(X)$ topologized such that it is a topological sum of the space X and a two-point discrete space $\{p, q\}$. Moreover, if X is paracompact, $G(X)$ is paracompact as well. We conclude that F determines strong embeddings Top Alg(1, 1) — Top Alg(2), Top Alg(1, 1) $\longrightarrow$ Top Alg(2, 0), Par Alg(1, 1) $\longrightarrow$ Par Alg(2), Par Alg(1, 1) — Par Alg(2, 0), so that Par Alg(2) and Par Alg(2, 0) are COE–universal.

f) In [13, pp. 152–154], a strong embedding Φ of Alg(2) into the category Smg of all semigroups is constructed as follows: a rigid semigroup D of [3], on two generators a, b and the defining relation $ab^2 = baba$, is decomposed into two sets, namely

$$M = \{a, ab, ba, aba, bab, baba\} \text{ and } N = D \setminus M.$$

The functor Φ is carried by $G = (\mathrm{Ident} \times C_N) \amalg C_M$, i.e. $G(X) = X \times M \amalg N$; given $(X, \odot) \in \mathrm{obj\,Alg}(2)$, $\Phi(X, \odot)$ is a semigroup $(G(X), \bullet)$ where its operation is defined by means of $\odot$ and also by means of the composition of the rigid semigroup D, see the rules (α), ... , (γ) on p. 152 in [13]. Inspecting these rules, one can see that if X is a topological space such that the binary operation $\odot$ is continuous, then $\bullet$ is continuous on $G(X)$ whenever $G(X)$ is investigated as the space $X \times M \amalg N$, M and N endowed with a discrete topology. This also implies that, for every continuous $f : (X, \odot) \to (X', \odot)$, the map $\Phi(f) : (GX, \bullet) \to (GX', \bullet)$, carried by $G(f)$, is also continuous. Conversely, if $h : (GX, \bullet) \to (GX', \bullet)$ is a continuous homomorphism, then necessarily $h = \Phi(f)$ where $f : (X, \odot) \to (X', \odot)$ is a homomorphism, by [13, pp. 148–154]; since Φ is carried by G and $G(f)$ is continuous, f must be continuous. We conclude that Φ determines strong embeddings Top Alg(2) $\to$ Top Smg $= \mathcal{S}$ and Par Alg(2) $\to$ Par Smg, so that $\mathcal{S}$ and Par Smg are COE–universal.

Remark. The C–universality and COE–universality of categories Top $\mathcal{V}$, where $\mathcal{V}$ is an algebraically universal variety (i.e. admitting a full one-to-one functor Alg(1, 1) $\to \mathcal{V}$) will be investigated in a forthcoming paper.

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