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AN ABSTRACT CHARACTERIZATION OF THE JET SPACES

by Ivan KOLAR

Résumé. Nous déduisons une caractéristique abstraite des jets d'applications lisses introduits par C. Ehresmann. Les jets d'ordre fini sont les seuls images homomorphes de dimension finie des germes d'applications lisses satisfaisants deux conditions naturelles.

1. Introduction.

Our research was inspired by two results in the foundations of differential geometry. On one hand, Palais-Terng, [8], and Epstein-Thurston, [3], deduced that every natural bundle over m-dimensional smooth manifolds has finite order. On the other hand, Kainz-Michor, [4], Eck, [1], and Luciano, [7], proved that every product preserving bundle functor on the whole category of smooth manifolds is a functor of the "infinitely near points" by A. Weil, [9]. A general idea in both results is that the fact we study finite-dimensional manifolds has far-reaching consequences on the structure of the geometric objects in question.

In the present paper we discuss the finite-dimensional homomorphic images of the germs of smooth maps. We deduce that the so-called product property formulated in section 2 and certain simple assumptions on the smoothness of the derived objects imply that the classical jets are the only possibility.

2. The result.

The classical construction of the r-th order jets transforms every pair M, N of manifolds into a fibered manifold $J^r(M,N)$ over $M \times N$ and every germ of a smooth map $f : M \to N$ at $x \in M$ into its r-jet $j^r(germ_x f) = j^r_x f \in J^r(M,N)$. The composition $B_2 \circ B_1$ of germs induces the composition of r-jets (denoted by the same symbol)

$$j^r(B_2) \circ j^r(B_1) = j^r(B_2 \circ B_1).$$

Hence the r-jets are finite-dimensional homomorphic images of the germs of smooth maps. We are going to point out some general properties of the pairs (J^r, j^r) for all $r \in \mathbb{N}$.

Denote by $G(M,N)$ the set of all germs of smooth maps from M into N. Consider a rule F transforming every pair M, N of manifolds into a fibered manifold $F(M,N)$ over $M \times N$ and a system φ of maps $\varphi_{M,N} : G(M,N) \to F(M,N)$ commuting with the projections $G(M,N) \to M \times N$ and $F(M,N) \to M \times N$ for all M, N. Then we can formulate the following requirements I-IV.

I. (Surjectiveness) Every $\varphi_{M,N} : G(M,N) \to F(M,N)$ is surjective.

II. (Composability) If $\varphi(B_1) = \varphi(\bar{B}_1)$ and $\varphi(B_2) = \varphi(\bar{B}_2)$, then $\varphi(B_2 \circ B_1) = \varphi(\bar{B}_2 \circ \bar{B}_1)$.

By I and II we have a well defined composition (denoted by the same symbol as the composition of germs and maps)

$$X_2 \circ X_1 = \varphi(B_2 \circ B_1)$$

for every $X_1 = \varphi(B_1) \in F_x(M,N)_y$ and $X_2 = \varphi(B_2) \in F_y(N,P)_z$. Let us write $\varphi_x f$ for $\varphi(germ_x f)$. Every local diffeomorphism $f : M \to \bar{M}$ and every smooth map $g : N \to \bar{N}$ induces a map $F(f,g) : F(M,N) \to F(\bar{M},\bar{N})$ defined by

$$F(f,g)(X) = (\varphi_y g) \circ X \circ \varphi_{f(x)}(f^{-1}), \qquad X \in F_x(M,N)_y,$$

where f^{-1} is constructed locally.

III. (Regularity) Each map $F(f,g)$ is smooth.

Consider the product $N_1 \xleftarrow{q_1} N_1 \times N_2 \xrightarrow{q_2} N_2$ of two manifolds. Then we have the induced maps $F(id_M, q_1) : F(M, N_1 \times N_2) \to F(M, N_1)$ and $F(id_M, q_2) : F(M, N_1 \times N_2) \to F(M, N_2)$. Both $F(M,N_1)$ and $F(M,N_2)$ are fibered manifolds over M.

IV. (Product property) $F(M, N_1 \times N_2)$ coincides with the fibered product $F(M,N_1) \times_M F(M,N_2)$ over M and $F(id_M,q_1)$, $F(id_M,q_2)$ are the induced projections.

The geometric role of the Axiom IV will be discussed in section 5. (At the level of germs, we have a canonical identification $G(M, N_1 \times N_2) = G(M,N_1) \times_M G(M,N_2)$.) The main result of the present paper is the following asertion.

Theorem 1. *For every pair (F,φ) satisfying I-IV there exists an integer $r \geq 0$ such that $(F,\varphi) = (J^r, j^r)$.*

3. The induced functor in dimension k.

For every $k \in \mathbb{N}$ we define an induced functor F_k on the category $\mathcal{M}f$ of all smooth manifolds and all smooth maps by

$$F_k M = F_0(\mathbb{R}^k, M)$$

where the subscript 0 indicates the fiber over $0 \in \mathbb{R}^k$, and by

$$F_k f = F_0(id_{\mathbb{R}^k}, f) : F_k M \to F_k N$$

for every smooth map $f : M \to N$. By III each map $F_k f$ is smooth and IV implies that *each functor F_k preserves products.*

Since F_k preserves products, it coincides with a Weil functor, see [1], [4], [7]. Let A_k be the corresponding Weil algebra and N_k be its nilpotent ideal. Let $E(k)$ be the ring of all germs of smooth functions on $\mathbb{R}^k$ at 0 and $\mathfrak{m}(k) = G_0(\mathbb{R}^k, \mathbb{R})_0$ be its maximal ideal. By [5] or [6], A_k is a factor algebra $E(k)/\mathcal{A}_k$, where $\mathcal{A}_k$ is an ideal of finite codimension, and the intersection $\mathcal{N}_k = \mathcal{A}_k \cap \mathfrak{m}(k)$ is the set of all germs $B \in G_0(\mathbb{R}^m, \mathbb{R})_0$ satisfying

$$\varphi(B) = \varphi(\hat{0}) \tag{1}$$

where $\hat{0}$ is the germ of the constant function 0. This yields the following "*substitution property*" of $\mathcal{N}_k$

$$B \in \mathcal{N}_k \text{ and } h \in G_0(\mathbb{R}^k, \mathbb{R}^k)_0 \text{ implies } B \circ h \in \mathcal{N}_k. \tag{2}$$

Since $\mathcal{N}_k$ is an ideal of finite codimension, there exists an integer r_k such that

$$\mathfrak{m}(k)^{r_k+1} \subset \mathcal{N}_k. \tag{3}$$

The minimum of such integers is called the *order* of A_k. Assume r_k is minimal. Then beside (3) it also holds

$$\mathfrak{m}(k)^{r_k} \not\subset \mathcal{N}_k. \tag{4}$$

We recall that T_k^r denotes the functor of k-dimensional velocities of order r by Ehresmann, [2].

Lemma 2. *The functor F_k is the velocities functor $T_k^{r_k}$.*

Proof. We have to prove

$$\mathfrak{m}(k)^{r_k+1} = \mathcal{N}_k. \tag{5}$$

We deduce the opposite inclusion to (3) by contradiction. We show that if there exists and element $B \notin \mathfrak{m}(k)^{r_k+1}$ satisfying $B \in \mathcal{N}_k$, then the substitution property (2) implies $\mathfrak{m}(k)^{r_k} \subset \mathcal{N}_k$. Indeed, by (3) we may assume that B is the germ of a non-zero polynomial Q. Since $\mathcal{N}_k$ is an ideal, we may further assume Q is a homogenous polynomial of degree r_k. Let $b \in \mathbb{R}^k$ be a point such that $Q(b) \neq 0$. Consider the map $h : \mathbb{R}^k \to \mathbb{R}^k$, $h(x) = x_1 b$, $x = (x_1, \dots, x_k) \in \mathbb{R}^k$. Then $Q \circ h = Q(b)(x_1)^{r_k}$. By the substitution property $Q(b)(x_1)^{r_k} \in \mathcal{N}_k$ and $Q(b) \neq 0$ implies $(x_1)^{r_k} \in \mathcal{N}_k$. Consider further the map $\bar{h} : \mathbb{R}^k \to \mathbb{R}^k$, $\bar{h}(x_1, \dots, x_k) = (c_1 x_1 + \dots + c_k x_k, 0, \dots, 0)$ with arbitrary $c_1, \dots, c_k \in \mathbb{R}$. Then $(c_1 x_1 + \dots + c_k x_k)^{r_k} \in \mathcal{N}_k$ for all $c_1, \dots, c_k$. If we evaluate this expression and interpret $c_1, \dots, c_k$ as undetermined, we obtain that each monomial of degree r_k belongs to $\mathcal{N}_k$. Hence $\mathfrak{m}(k)^{r_k} \subset \mathcal{N}_k$, QED.

4. The end of the proof.

In the next step we use the following property of r-jets, which is equivalent to their standard definition and can be verified by direct evaluation. *If two germs $germ_x f$, $germ_x g \in G_x(M, N)_y$ satisfy $j_0^r(f \circ \delta) = j_0^r(g \circ \delta)$ for all $\delta : \mathbb{R} \to M$ with $\delta(0) = x$, then $j_x^r f = j_x^r g$.* This is easily extended to the following assertion.

Lemma 3. *If $j_0^r(f\circ\varepsilon) = j_0^r(g\circ\varepsilon)$ for all $\varepsilon : \mathbb{R}^k \to M$ with $\varepsilon(0) = x$, then $j_x^r f = j_x^r g$.*

Proof. For every map $\delta : \mathbb{R} \to M$ we construct $\varepsilon = \delta \circ p_1$ where $p_1 : \mathbb{R}^k \to \mathbb{R}$ is the product projection to the first factor. Then $j_0^r(f \circ \varepsilon) = j_0^r(g \circ \varepsilon)$ has the same coordinate meaning as $j_0^r(f \circ \delta) = j_0^r(g \circ \delta)$, QED.

Lemma 4. *It holds $r_k = r_l$ for all $k, l \in \mathbb{N}$.*

Proof. By Lemma 2 we know that for every f, g: $\mathbb{R}^k \to M$ the condition $\varphi_0 f = \varphi_0 g$ means $j_0^{r_k} f = j_0^{r_k} g$ with maximal r_k. For every $h \in G_0(\mathbb{R}^l, \mathbb{R}^k)_0$ we have $\varphi_0(f\circ h) = \varphi_0(g \circ h)$. By definition of r_l this implies $j_0^{r_l}(f \circ h) = j_0^{r_l}(g \circ h)$. Hence $j_0^{r_l} f = j_0^{r_l} g$ by Lemma 3. Since r_k is maximal, we have $r_k \geq r_l$. Replacing k and l, we obtain the converse relation $r_l \geq r_k$, QED.

Denote by r the common value of all r_k. Then it suffices to deduce that each $F(M, N)$ is the associated fiber bundle $F(M, N) = P^r M(T_m^r N)$ to the rth order frame bundle $P^r M(M, G_m^r)$ with respect to the jet action of its structure group G_m^r on $T_m^r N$, $m = \dim M$. For every $v \in P_x^r M$, $v = \varphi(V)$ we define $\tilde{v} : F_x(M, N) \to T_m^r N$ by $\tilde{v}(\varphi(B)) = \varphi(B \circ V)$. It holds $\varphi(B \circ V) = \varphi(B) \circ \varphi(V)$ and $\varphi(V) = j^r V$, $\varphi(B \circ V) = j^r(B \circ V)$. For $W = V \circ H$, $j^r H \in G_m^r$, we have $\tilde{w}(\varphi(B)) = \varphi(B \circ V) \circ \varphi(H) = j^r(B \circ V) \circ j^r(H)$. Hence we have the standard situation of the smooth associated bundles. This completes the proof of Theorem 1.

5. Final remark.

We conclude with a clarification of the geometric meaning of the requirement IV. There is a general procedure how to construct the homomorphic images of the germs of smooth maps which uses an arbitrary bundle functor E on the category $\mathcal{M}f$ of all manifolds. Let $\mathcal{FM}$ denote the category of smooth fibered manifolds and their morphisms and let $B : \mathcal{FM} \to \mathcal{M}f$ be the base functor. By a bundle functor on $\mathcal{M}f$ we mean a functor $E : \mathcal{M}f \to \mathcal{FM}$ satisfying $B \circ E = id_{\mathcal{M}f}$ and the localization condition:

> for every manifold M and every open subset $i : U \hookrightarrow M$, EU is the restriction of the fibered manifold $EM \to M$ over U and Ei is the inclusion $EU \hookrightarrow EM$.

By the localization property, for every smooth map $f : M \to N$, the restriction $E_x f : E_x M \to E_{f(x)} N$ of Ef to the fiber $E_x M$ of EM over $x \in M$ depends on $germ_x f$ only.

Definition 5. Let E be a bundle functor on $\mathcal{M}f$. We say that two smooth maps $f, g : M \to N$ determine the same E-jet at $x \in M$, if $E_x f = E_x g$. The corresponding equivalence class will be denoted by $j_x^E f$ or $j^E(germ_x f)$.

Write $J^E(M, N)$ for the set of all E-jets of the smooth maps of M into N.

Since E is a functor, $j_x^E(g \circ f)$ is characterized by $E_x(g \circ f) = E_{f(x)} g \circ E_x f$, so that the composability property II always holds. However, the following example demonstrates that the product property IV need not to be satisfied.

Consider the k-th exterior power $\bigwedge^k T : \mathcal{M}f \to \mathcal{FM}$ of the tangent functor T. Clearly, $\bigwedge^k T$ does not preserve products for $k \geq 2$. If $\dim M < k$, then $\bigwedge^k TM$ is identified with M. Hence $J^{\bigwedge^k T}(M, N)$ is identified with $M \times N$ whenever $\dim M < k$ or $\dim N < k$. Therefore, if $\dim M \geq k$, $\dim N_1 < k$, $\dim N_2 < k$ and $\dim N_1 + \dim N_2 \geq k$, then $(M \times N_1) \times_M (M \times N_2) = M \times N_1 \times N_2$, which is not isomorphic to $J^{\bigwedge^k T}(M, N_1 \times N_2)$.

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