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ON CLASSIFICATION OF 4-MANIFOLDS ACCORDING TO GENUS

by Alberto CAVICCHIOLI and Mauro MESCHIARI

Résumé. Nous étudions la structure topologique des 4-variétés fermées (compactes et sans bord) par rapport au genre. En particulier, $\mathbb{S}^2 \times \mathbb{S}^2$ (resp. $\mathbb{R}P^4$) est démontré être l'unique 4-variété fermée, orientable (resp. non-orientable) et indivisible (respectant la somme connexe) de genre 4 (resp.6).

1. Introduction

It is known that a closed connected smooth (or PL) n-manifold M can be represented by suitable edge-coloured graphs (for details see [1], [7], [18]). This allows to define new topological invariants for M as for example its genus. We briefly recall the definition. An n-dimensional pseudocomplex (see [10], p. 49) K is said to be a contracted *triangulation* of M if it has exactly $n+1$ vertices, v_0, v_1, ... , v_n say. This notion is strictly related to a graph theoretic one as follows. An $(n+1)$-*coloured graph* (G, c) is a multigraph $G = (\mathbf{V}(G), \mathbf{E}(G))$, regular of degree $n+1$, together with an edge-colouring $c\colon \mathbf{E}(G) \to \{0, 1, \dots, n\}$ such that incident edges have different colours.

A *crystallization* of M is the $(n+1)$-coloured graph obtained by taking the 1-skeleton of the dual complex of K and by labelling the dual of each $(n-1)$-simplex by the colour i if it does not contain the vertex v_i. The *genus* $g(M)$ of M is the minimum genus of a closed connected surface into which an arbitrary crystallization of M regularly imbeds (also compare [19]). Clearly this genus is just the classical one in dimension two. Further, it is not difficult to show that the genus of a 3-manifold equals (resp. twice) its

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Heegaard genus in the orientable (resp. non-orientable) case and that the genus is even for any non-orientable n-manifold.

For the classification of all orientable (non-orientable) closed 4-manifolds of genus ≤ 2 (≤ 4) we refer to [2], [3]. Here we go on with the classification.

Besides general results, we characterize the topological product $\mathbb{S}^2 \times \mathbb{S}^2$ and the real projective 4-space $\mathbb{R}P^4$ among closed 4-manifolds. Indeed, $\mathbb{S}^2 \times \mathbb{S}^2$ (resp. $\mathbb{R}P^4$) is proved to be the unique prime closed connected orientable (resp. non-orientable) 4-manifold of genus 4 (resp. 6), up to (TOP) homeomorphism.

2. Main results

In order to state our results we need some preliminaries and formulae first proved in [2] and [3]. From now on, let us denote by

(1) M^4 a smooth (or PL) closed connected orientable (resp. non-orientable) 4-manifold of genus g (resp. h).
(2) (G, c) a crystallization of M.
(3) $K = K(G)$ the contracted triangulation represented by (G, c).
(4) $\mathcal{C}_G = \{0, 1, 2, 3, 4\}$ the colour-set of (G, c), $\{v_i : i \in \mathcal{C}_G\}$ the vertex-set of K and (i, j, r, s, t) an arbitrary permutation of $\mathcal{C}_G$.

We may always suppose that (G, c) regularly imbeds into the closed connected orientable (resp. non-orientable) surface of genus g (resp. h) and that v_i corresponds to the subgraph $G_{\hat{\imath}}$ ($i \in \mathcal{C}_G$) obtained by deleting all i-coloured edges from G (for details see [2] and [3]).

Let $K(i, j)$ (resp. $K(r, s, t)$) be the one-dimensional (resp. two-dimensional) subcomplex of K generated by the vertices v_i and v_j (resp. v_r, v_s and v_t). Let γ_{ij} (resp. γ_{rst}) denote the number of edges (resp. triangles) of $K(i, j)$ (resp. $K(r, s, t)$).

If $N = N(i, j)$ and $N' = N(r, s, t)$ are regular neighborhoods of $K(i, j)$ and $K(r, s, t)$ respectively, then N and N' are complementary bordered 4-manifolds, i.e. $M = N \cup N'$ and $N \cap N' = \partial N = \partial N'$.

Now the Mayer–Vietoris sequence of the triple (M, N, N') gives

$$0 \longrightarrow H_4(M) \longrightarrow H_3(\partial N) \longrightarrow 0,$$

hence M is orientable (resp. non-orientable) if and only if ∂N is.

Setting $\epsilon = g$ (resp. $\epsilon = h/2$) for the orientable (resp. non-orientable) case, we have the following relations (see [2] and [3]);

$$\gamma_{ij} = 1 + \epsilon - g_{\hat{\imath}} - g_{\hat{\jmath}} \tag{1}$$

$(i \in C_G,\ j \equiv i+2 \pmod 5)$

(2) $$\gamma_{rst} = \gamma_{rs} + \epsilon - g_{\hat{t}}$$

$(r \in C_G,\ s \equiv r+1 \pmod 5),\ t \equiv r+3 \pmod 5))$

(3) $$\chi(M) = 2 - 2\epsilon + \Delta, \qquad \Delta = \sum_{i=0}^{4} g_{\hat{i}}$$

(4) $$\gamma_{24} + \gamma_{14} + \gamma_{13} + \gamma_{03} + \gamma_{02} = 5 + 5\epsilon - 2\Delta$$

where $g_{\hat{i}}$ is the genus of an orientable closed connected surface into which $G_{\hat{i}}$ regularly imbeds and $\chi(M)$ is the Euler–Poincaré characteristic of M.

Relations (2) and (4) directly imply that

(5) $$0 \le g_{\hat{i}} \le \epsilon \quad i \in C_G, \qquad 0 \le \Delta \le \left[\frac{5}{2}\epsilon\right]$$

where $[x]$ denotes the integer part of the real non-negative number x.

By (3) it follows that

(6) $$\beta_2 = 2\beta_1 - 2g + \Delta \quad \text{(orientable case)}, \qquad \beta_2^{(2)} = 2\beta_1^{(2)} - h + \Delta \quad \text{(non-orientable case)},$$

hence

(7) $$\beta_1 \ge g - \frac{\Delta}{2}, \qquad \beta_1^{(2)} \ge \frac{h}{2} - \frac{\Delta}{2}$$

where β_k (resp. $\beta_k^{(2)}$) is the k-th integral (resp mod 2) Betti number of M.

Finally relation (1) and the inequalities (compare also [1] and [7])

(8) $$\max\{\beta_1, \beta_1^{(2)}\} \le \operatorname{rk} H_1(M) \le \operatorname{rk} \pi_1(M) \le \gamma_{ij} - 1$$

imply that

$$\max\{\beta_1, \beta_1^{(2)}\} \leq \epsilon - g_{\hat{i}} - g_{\widehat{i+2}} \leq \epsilon \tag{9}$$

($i \in \mathcal{C}_G$, indices $\mod 5$).

In sect. 3 and 4 we will study the possible values that the sum Δ may assume and classify the corresponding 4-manifolds.

Now we state the main results of the paper. Here $\mathbb{S}^n$ and $\mathbb{R}\mathrm{P}^n$ (resp. $\mathbb{C}\mathrm{P}^n$) denote the n-sphere and the real (resp. complex) projective n-space; $\mathbb{S}^1 \otimes \mathbb{S}^n$ represents either the topological product $\mathbb{S}^1 \times \mathbb{S}^n$ or $\mathbb{S}^1 \underset{\sim}{\times} \mathbb{S}^n$ the twisted $\mathbb{S}^n$-bundle over $\mathbb{S}^1$. Further let us define $\#_p\mathbb{S}^1 \otimes \mathbb{S}^n$ as the connected sum of p copies of $\mathbb{S}^1 \otimes \mathbb{S}^n$ if $p > 0$ and as $\mathbb{S}^{n+1}$ if $p = 0$.

For the orientable case we have:

Theorem 1. *Let M^4 be a smooth (or PL) closed orientable connected 4-manifold of genus g. If $\Delta = 0$, then M is (PL) homeomorphic to the connected sum $\#_g\mathbb{S}^1 \times \mathbb{S}^3$. If $\Delta = 5$, then M is (PL) homeomorphic to the connected sum $(\#_{g-2}\mathbb{S}^1 \times \mathbb{S}^3)\#\mathbb{C}P^2$.*

Then we prove that there are no 4-manifolds of genus g for which the sum Δ satisfies $1 \leq \Delta \leq 9$, $\Delta \neq 5$. Therefore we classify all closed orientable 4-manifolds of genus $g \leq 4$ (for $g \leq 2$ see [2] and [3]).

Theorem 2. *Let M^4 be a smooth (or PL) closed orientable connected 4-manifold of genus g. If $g = 3$, then M is (PL) homeomorphic to either $\#_3\mathbb{S}^1 \times \mathbb{S}^3$ or $\mathbb{C}P^2\#\mathbb{S}^1 \times \mathbb{S}^3$. If $g = 4$ and $\Delta \leq 9$, then M is (PL) homeomorphic to either $\#_4\mathbb{S}^1 \times \mathbb{S}^3$ or $(\#_2\mathbb{S}^1 \times \mathbb{S}^3)\#\mathbb{C}P^2$. If $g = 4$ and $\Delta = 10$, then M is (TOP) homeomorphic to one of the following manifolds: $\mathbb{C}P^2\#\mathbb{C}P^2$, $\mathbb{S}^2 \underset{\sim}{\times} \mathbb{S}^2$ (the twisted $\mathbb{S}^2$-bundle over $\mathbb{S}^2$) and $\mathbb{S}^2 \times \mathbb{S}^2$.*

This characterizes $\mathbb{S}^2 \times \mathbb{S}^2$ among closed orientable 4-manifolds, i.e. $\mathbb{S}^2 \times \mathbb{S}^2$ is the unique prime smooth (or PL) closed orientable 4-manifold of genus four, up to (TOP) homeomorphism.

The above results and [16] also imply that $g(\mathbb{R}\mathrm{P}^3 \times \mathbb{S}^1) = 6$. We conjecture that this manifold is the unique prime closed connected orientable 4-manifold of genus six.

For the non-orientable case, we have

Theorem 3. *Let M^4 be a smooth (or PL) closed non-orientable connected 4-manifold of genus h. If $\Delta = 0$, then M is (PL) homeomorphic to the connected sum $\#_{h/2}\mathbb{S}^1 \otimes \mathbb{S}^3$.*

If $\Delta = 5$ and $H_2(M)$ has no 2-torsion, then M is homeomorphic to either $\#_{(h-4)/2}\mathbb{S}^1 \otimes \mathbb{S}^3 \# \mathbb{C}P^2$ or $\#_{(h-6)/2}\mathbb{S}^1 \otimes \mathbb{S}^3 \# \mathbb{R}P^4$.

If $\Delta = 5$ and $H_2(M)$ has 2-torsion, then the homology groups of M are:

$$H_1(M) \simeq \bigoplus_{(h-4)/2} \mathbb{Z}, \qquad H_2(M) \simeq \mathbb{Z}_{2n} \quad (n \geq 1),$$

$$H_3(M) \simeq \bigoplus_{(h-8)/2} \mathbb{Z} \oplus \mathbb{Z}_2$$

(hence $h \geq 8$) and

$$H_q(M) \simeq 0 \qquad (q \geq 4).$$

Then we prove that there are no 4-manifolds of genus h for which the sum Δ satisfies $1 \leq \Delta \leq 4$. Therefore we classify all closed non-orientable 4-manifolds of genus $h \leq 6$ (for $h \leq 4$ see [3]).

Theorem 4. *Let M^4 be a smooth (or PL) closed non-orientable connected 4-manifold of genus h. If $h = 6$ and $\Delta \neq 5$, then M is (PL) homeomorphic to $\#_3\mathbb{S}^1 \otimes \mathbb{S}^3$. If $h = 6$ and $\Delta = 5$, then M is (TOP) homeomorphic to either $\mathbb{S}^1 \underset{\sim}{\times} \mathbb{S}^3 \# \mathbb{C}P^2$ or $\mathbb{R}P^4$.*

This characterizes $\mathbb{R}P^4$ among closed non-orientable 4-manifolds as the unique prime smooth (or PL) closed non-orientable 4-manifold of genus six, up to (TOP) homeomorphism.

Finally we summarize our knowledge about the classification, in table I for orientable 4-manifolds and in table II for non-orientable 4-manifolds.

Open problem.

Fill in some of the places of the tables marked with a question mark. We conjecture that if g is odd, then M is (PL) homeomorphic to the connected sum $\tilde{M} \# \mathbb{S}^1 \times \mathbb{S}^3$, $\tilde{M}$ being a closed connected orientable 4-manifold of genus $g - 1$.

We also observe that cases $\Delta \geq 10$ and $g \geq 5$ can not be treated as the previous ones since it may not be possible to apply the Gordon–Luecke results (see [9]).

Finally we note that it might exist a closed prime non-orientable 4-manifold M such that $g(M) = 8$, $\chi(M) = -1$ (i.e. $\Delta = 5$) and its homology groups are: $H_1(M) \simeq \mathbb{Z} \oplus \mathbb{Z}$, $H_2(M) \simeq \mathbb{Z}_{2n}$ $(n \geq 1)$, $H_3(M) \simeq \mathbb{Z}_2$ and $H_q(M) \simeq 0$ $(q \geq 4)$.

TABLE I. Orientable 4-manifolds

	$g=0$	$g=1$	$g=2$	$g=3$	$g=4$	$g=5$
$\Delta=0$	$\mathbb{S}^4$	$\mathbb{S}^1\times\mathbb{S}^3$	$\#_2\mathbb{S}^1\times\mathbb{S}^3$	$\#_3\mathbb{S}^1\times\mathbb{S}^3$	$\#_4\mathbb{S}^1\times\mathbb{S}^3$	$\#_5\mathbb{S}^1\times\mathbb{S}^3$
$\Delta=1$		empty	empty	empty	empty	empty
$\Delta=2$		empty	empty	empty	empty	empty
$\Delta=3$			empty	empty	empty	empty
$\Delta=4$			empty	empty	empty	empty
$\Delta=5$			$\mathbb{C}P^2$	$\mathbb{S}^1\times\mathbb{S}^3\#\mathbb{C}P^2$	$\#_2\mathbb{S}^1\times\mathbb{S}^3\#\mathbb{C}P^2$	$\#_3\mathbb{S}^1\times\mathbb{S}^3\#\mathbb{C}P^2$
$\Delta=6$				empty	empty	empty
$\Delta=7$				empty	empty	empty
$\Delta=8$					empty	empty
$\Delta=9$					empty	empty
$\Delta=10$					$\mathbb{S}^2\times\mathbb{S}^2$ $\mathbb{S}^2\tilde{\times}\mathbb{S}^2$ $\mathbb{C}P^2\#\mathbb{C}P^2$	?

3. Proofs: the orientable case

$\Delta = 0$.

If $\Delta = 0$ (recall that $\epsilon = g$), then relations (7) and (9) imply that $\beta_1 = g$, hence $\beta_2 = 0$ by (6). Thus we have $FH_2(M) \simeq 0$ and

$$H_3(M) \simeq H^1(M) \simeq FH_1(M) \simeq \bigoplus_g \mathbb{Z}.$$

Now we consider the complementary bordered 4-manifolds $N = N(2,4)$ and $N' = N(0,1,3)$. Because $\gamma_{24} = 1 + g$ (use (1)), the pseudocomplex $K(2,4)$ consists of exactly $1+g$ edges, hence N is (PL) homeomorphic to the boundary connected sum $\#_g\mathbb{S}^1\times\mathbb{B}^3$, $\mathbb{B}^3$ being a closed 3-ball. Further $K(0,3)$ and $K(1,3)$ are also formed by $1+g$ edges eachone as $\gamma_{03} = \gamma_{13} = 1+g$ by formula (1). Because $\gamma_{013} = \gamma_{01} + g$ (use (2)), the complex $K(0,1,3)$ has

TABLE II. Non-orientable 4-manifolds

	$h=2$	$h=4$	$h=6$	$h=8$
$\Delta=0$	$\mathbb{S}^1 \underset{\sim}{\times} \mathbb{S}^3$	$\#_2\mathbb{S}^1 \otimes \mathbb{S}^3$	$\#_3\mathbb{S}^1 \otimes \mathbb{S}^3$	$\#_4\mathbb{S}^1 \otimes \mathbb{S}^3$
$\Delta=1$	empty	empty	empty	empty
$\Delta=2$	empty	empty	empty	empty
$\Delta=3$		empty	empty	empty
$\Delta=4$		empty	empty	empty
$\Delta=5$		empty	$\mathbb{S}^1 \underset{\sim}{\times} \mathbb{S}^3 \# \mathbb{C}\mathrm{P}^2$ $\mathbb{R}\mathrm{P}^4$	?
$\Delta=6$			empty	empty
$\Delta=7$			empty	empty

many triangles but g as there are edges in $K(0,1)$. We observe that $H_2(N')$ is free since $N' = N(0,1,3)$ collapses to the 2-dimensional pseudocomplex $K(0,1,3)$. Thus the Mayer-Vietoris sequence of the triple (M,N,N') gives

$$0 \longrightarrow H_3(M) \simeq \bigoplus_g \mathbb{Z} \longrightarrow H_2(\partial N) \simeq \bigoplus_g \mathbb{Z} \longrightarrow H_2(N') \longrightarrow 0,$$

hence $H_2(N') \simeq 0$. Therefore it does not exist two triangles in $K(0,1,3)$ with common boundary (notice that any r-ball of a pseudocomplex is abstractly isomorphic to the standard r-simplex). Thus any triangle of $K(0,1,3)$ can be retracted, by deformation, on a one-dimensional subcomplex. This implies that the regular neighbourhood N' of $K(0,1,3)$ is (PL) homeomorphic to a boundary connected sum $\#_h\mathbb{S}^1 \times \mathbb{B}^3$. Since $\partial N' \simeq \partial N \simeq \#_g\mathbb{S}^1 \times \mathbb{S}^2$, it follows that $h = g$. Therefore the manifold M must be $\#_g\mathbb{S}^1 \times \mathbb{S}^3$ by theorem 2 of [15]. Now the result follows as the genus of $\#_g\mathbb{S}^1 \times \mathbb{S}^3$ is really g by corollary 2 of [3].

$\Delta = 1$.

If $\Delta = 1$, then at least one of the g_i's in the sum Δ equals 1, hence relation (9) implies that $\beta_1 \leq g - 1$. On the other hand, we have $\beta_1 \geq g$ by (7), i.e. a contradiction.

$\Delta = 2$.

If $\Delta = 2$, then the addendum $g_{\hat{\imath}}$ of Δ may assume (up to circular permutations) the values listed in the following table:

case	$g_{\hat{0}}$	$g_{\hat{1}}$	$g_{\hat{2}}$	$g_{\hat{3}}$	$g_{\hat{4}}$
2.1	1	1	0	0	0
2.2	1	0	1	0	0
2.3	2	0	0	0	0

Indeed, doing the above-mentioned change of names in the colour-set $\mathcal{C}_G$ the permutation of $\mathcal{C}_G$ giving the regular imbedding of G is the same.

By (7) we have $\beta_1 \geq g - 1$. Thus cases 2.2 and 2.3 give a contradiction since $\beta_1 \leq g - 2$ by (9).

For case 2.1, it follows that $\beta_1 = g - 1$ (use (9)), hence $\beta_2 = 0$ by (6). Now relations (1) and (2) give $\gamma_{02} = \gamma_{13} = \gamma_{14} = g$ and $\gamma_{134} = \gamma_{34} + g - 1$. Then we can repeat the same arguments of the case $\Delta = 0$ by replacing g and $(K(2,4), K(0,1,3))$ with $g - 1$ and $(K(0,2), K(1,3,4))$ respectively. It follows that M is (PL) homeomorphic to $\#_{g-1}\mathbb{S}^1 \times \mathbb{S}^3$, which is a contradiction because this manifold has genus $g - 1$ by corollary 2 of [3].

$\Delta = 3$.

By (7) we have $\beta_1 \geq g - 3/2$, hence relation (8) gives $\gamma_{ij} \geq g - 1/2$, i.e. $\gamma_{ij} \geq g$. Thus (4) implies the inequality $5 + 5g - 2\Delta \geq 5g$, which is a contradiction.

$\Delta = 4$.

Relation (7) becomes $\beta_1 \geq g - 2$ so (9) implies that $g_{\hat{\imath}} + g_{\widehat{i+2}} \leq 2$ for each colour $i \in \mathcal{C}_G$. Thus the addendum of Δ may assume (up to circular permutations) the following values:

case	$g_{\hat{0}}$	$g_{\hat{1}}$	$g_{\hat{2}}$	$g_{\hat{3}}$	$g_{\hat{4}}$
4.1	2	2	0	0	0
4.2	0	1	1	1	1
4.3	1	0	0	1	2

In any case, relations (7) and (9) give $\beta_1 = g - 2$ so $\beta_2 = 0$ by (6), i.e. $H_3(M) \simeq \oplus_{g-2}\mathbb{Z}$ and $FH_2(M) \simeq 0$.

(case 4.1). Since $\gamma_{02} = \gamma_{03} = \gamma_{14} = g - 1$ and $\gamma_{023} = \gamma_{23} + g - 2$ (use (1) and (2)), we can repeat the same arguments of case $\Delta = 0$ by replacing g and $(K(2,4), K(0,1,3))$ with $g - 2$ and $(K(1,4), K(0,2,3))$ respectively. Then M is (PL) homeomorphic to $\#_{g-2}\mathbb{S}^1 \times \mathbb{S}^3$, which is a contradiction as usual.

(case 4.2). Relations (1) and (2) imply that $\gamma_{13} = \gamma_{24} = g-1, \gamma_{03} = g$ and $\gamma_{013} = \gamma_{01}+g-1$. Then $N = N(2,4)$ is (PL) homeomorphic to $\#_{g-2}\mathbb{S}^1 \times \mathbb{B}^3$.

Since $H_3(M) \simeq \oplus_{g-2}\mathbb{Z}$, $FH_2(M) \simeq 0$ and $H_2(N) \simeq 0$, the Mayer–Vietoris sequence of the triple (M, N, N'), $N' = N(0,1,3)$, implies that $H_1(N') \simeq \oplus_{g-2}\mathbb{Z}$ and $H_2(N') \simeq 0$ (compare also $\Delta = 0$). Thus we obtain the contradiction of the previous case too.

(case 4.3). Since $\gamma_{03} = \gamma_{14} = \gamma_{24} = g - 1$ and $\gamma_{124} = \gamma_{12} + g - 2$ (use (1) and (2)), we can repeat the same arguments of case 4.1 by replacing the pair $(K(1,4), K(0,2,3))$ with $(K(0,3), K(1,2,4))$.

$\Delta = 5$.

If $\Delta = 5$, then (7) implies that $\beta_1 \geq g - 2$. Since $\gamma_{ij} \geq \beta_1 + 1 \geq g - 1$ by (8), relation (4) gives $\gamma_{24} = \gamma_{14} = \gamma_{13} = \gamma_{03} = \gamma_{02} = g - 1$. Thus by (1) we obtain $g_{\hat{i}} + g_{\widehat{i+2}} = 2$ (indices mod 5), and whence $g_{\hat{i}} = 1$ for each $i \in \mathcal{C}_G$. Now relations (6) and (9) imply that $\beta_1 = g - 2$ and $\beta_2 = 1$, i.e. $H_3(M) \simeq FH_1(M) \simeq \oplus_{g-2}\mathbb{Z}$ and $FH_2(M) \simeq \mathbb{Z}$. Since $\gamma_{13} = g - 1$, the complex $K(1,3)$ is formed by two vertices joined by exactly $g - 1$ edges, hence $N = N(1,3)$ is (PL) homeomorphic to $\#_{g-2}\mathbb{S}^1 \times \mathbb{B}^3$.

Further $K(0,2)$ and $K(2,4)$ consist of exactly $g - 1$ edges eachone as $\gamma_{02} = \gamma_{24} = g - 1$. Because $\gamma_{024} = \gamma_{04} + g - 1$ (use (2)), the complex $K(0,2,4)$ has many triangles but $g - 1$ as there are edges in $K(0,4)$. Since $FH_2(M) \simeq \mathbb{Z}$, $H_2(N) \simeq 0$, $H_3(M) \simeq H_2(\partial N) \simeq \oplus_{g-2}\mathbb{Z}$ and $H_2(N')$ is free $(N' = N(0,2,4))$ the Mayer–Vietoris sequence of the triple (M, N, N')

$$0 \longrightarrow H_3(M) \simeq \bigoplus_{g-2}\mathbb{Z} \xrightarrow{\text{iso}} H_2(\partial N) \simeq \bigoplus_{g-2}\mathbb{Z} \to$$

$$\to H_2(N') \longrightarrow H_2(M) \simeq \mathbb{Z} \oplus TH_2(M) \longrightarrow H_1(\partial N) \simeq \bigoplus_{g-2}\mathbb{Z} \to$$

$$\to H_1(N) \oplus H_1(N') \simeq \bigoplus_{g-2}\mathbb{Z} \oplus H_1(N') \longrightarrow H_1(M) \simeq \bigoplus_{g-2}\mathbb{Z} \oplus TH_1(M) \longrightarrow 0$$

implies that $TH_2(M) \simeq 0$, hence $H_1(M) \simeq H^3(M) \simeq FH_3(M) \oplus TH_2(M) \simeq \oplus_{g-2}\mathbb{Z}$, $H_1(N') \simeq \oplus_{g-2}\mathbb{Z}$ and $H_2(N') \simeq \mathbb{Z}$. Thus $K(0,2,4)$ collapses to a 2-dimensional subcomplex formed by a combinatorial 2-sphere $\mathbb{S}^2$ and by $g - 2$ edges $e_1, e_2, \ldots, e_{g-2}$ such that $e_j \cap \mathbb{S}^2 = \partial e_j$. Further $\mathbb{S}^2$ consists of exactly two triangles $\sigma_1, \sigma_2 \in K(0,2,4)$ with common boundary as $\gamma_{02} = \gamma_{24} = g-1$, $\gamma_{024} = \gamma_{04} + g - 1$ and $H_1(N') \simeq \oplus_{g-2}\mathbb{Z}$.

By isotopy we can always assume that σ_1 is the standard 2-simplex in M. Let $\hat{\sigma}_1$ be the barycenter of σ_1 and $\mathrm{Sd}^2 K$ be the second barycentric subdivision of $K = K(G)$. Then N' is the orientable bordered 4-manifold obtained by adding a 2-handle (a regular neighborhood of $\hat{\sigma}_1$ in $\mathrm{Sd}^2 K$) to $\#_{g-2}\mathbb{S}^1 \times \mathbb{B}^3 \# \mathbb{B}^4$ along a knot $L \subset \partial\mathbb{B}^4$, $\mathbb{B}^4$ being a small neighborhood of σ_2 in M.

Since the surgery is given by attaching 2-handles in dimension 4, the surgery coefficient associated to L must be an integer and by homological reasons equals to ± 1 (use $H_2(N') \simeq \mathbb{Z}$). Since $\partial N' \simeq \partial N \simeq \#_{g-2}\mathbb{S}^1 \times \mathbb{S}^2 \# \mathbb{S}^3$, by theorem 2 of [9] (also compare [17]) L must be the trivial knot so the manifold N' is (PL) homeomorphic to $\#_{g-2}\mathbb{S}^1 \times \mathbb{B}^3 \# (\pm \mathbb{C}\mathrm{P}^2 \setminus \text{open 4-ball})$. Thus M is the connected sum $\#_{g-2}\mathbb{S}^1 \times \mathbb{S}^3 \# \mathbb{C}\mathrm{P}^2$ as requested. Now the result follows by the "subadditivity" of the genus as $g(\mathbb{C}\mathrm{P}^2) = 2$ and $g(\#_{g-2}\mathbb{S}^1 \times \mathbb{S}^3) = g - 2$ by [3].

Here we recall that $g(M_1 \# M_2) \leq g(M_1) + g(M_2)$ for any two orientable (resp. non-orientable) closed manifolds. On the contrary, if M_1 is orientable and M_2 is non-orientable, then $g(M_1 \# M_2) \leq 2g(M_1) + g(M_2)$.

$\Delta = 6$.

Since $\beta_1 \geq g-3$ by (7), it follows that $\gamma_{ij} \geq \beta_1 + 1 \geq g-2$ (see (8)). Thus it is easily seen that the γ_{ij}'s in (4) must assume (up to circular permutations) the values listed in the following table:

case	γ_{24}	γ_{14}	γ_{13}	γ_{03}	γ_{02}
6.1	$g+1$	$g-2$	$g-2$	$g-2$	$g-2$
6.2	g	$g-1$	$g-2$	$g-2$	$g-2$
6.3	g	$g-2$	$g-1$	$g-2$	$g-2$
6.4	$g-1$	$g-1$	$g-1$	$g-2$	$g-2$
6.5	$g-2$	$g-1$	$g-2$	$g-1$	$g-1$

Now by (1) we have:

(6.1) $$g_{\hat{0}} = g_{\hat{1}} = 3 \qquad g_{\hat{2}} = g_{\hat{3}} = g_{\hat{4}} = 0$$

(6.2) $$g_{\hat{0}} = g_{\hat{1}} = 2 \qquad g_{\hat{2}} = g_{\hat{3}} = 1 \qquad g_{\hat{4}} = 0$$

(6.3) $$g_{\hat{0}} = 3 \qquad g_{\hat{1}} = 2 \qquad g_{\hat{2}} = g_{\hat{3}} = 0 \qquad g_{\hat{4}} = 1$$

(6.4) $$g_{\hat{0}} = 2 \qquad g_{\hat{1}} = g_{\hat{2}} = g_{\hat{3}} = g_{\hat{4}} = 1$$

(6.5) $g_{\hat{0}} = 0 \qquad g_{\hat{1}} = g_{\hat{4}} = 1 \qquad g_{\hat{2}} = g_{\hat{3}} = 2,$

hence relations (6), (7) and (9) give $\beta_1 = g - 3$ and $\beta_2 = 0$ for any case. Therefore we obtain the contradiction

$$M \underset{\text{PL}}{\cong} \#_{g-3}\mathbb{S}^1 \times \mathbb{S}^3.$$

For conciseness we only sketch the proof in case 6.1. Here relations $\gamma_{14} = \gamma_{02} = \gamma_{03} = g - 2$ and $\gamma_{023} = \gamma_{23} + g - 3$ hold (use (1) and (2)). Then we can repeat the same arguments of case 4.1 by replacing $g - 2$ with $g - 3$.

$\Delta = 7$.

Since $\beta_1 \geq g - 3$ and $\gamma_{ij} \geq g - 2$ by (7) and (8), we must have (up to circular permutations) $\gamma_{24} = g - 1$ and $\gamma_{14} = \gamma_{13} = \gamma_{03} = \gamma_{02} = g - 2$ (see (4)). Then relation (1) implies that $g_{\hat{0}} = g_{\hat{1}} = 2$ and $g_{\hat{2}} = g_{\hat{3}} = g_{\hat{4}} = 1$, hence $\beta_1 = g - 3$ and $\beta_2 = 1$ by (6), (7) and (9). Since $\gamma_{14} = \gamma_{03} = \gamma_{02} = g - 2$ and $\gamma_{023} = \gamma_{23} + g - 2$ by (2), we can repeat the same arguments of case $\Delta = 5$ by replacing $g - 1$ and $(K(1,3), K(0,2,4))$ with $g - 2$ and $(K(1,4), K(0,2,3))$ respectively. Thus, we obtain the contradiction

$$M \underset{\text{PL}}{\cong} \#_{g-3}\mathbb{S}^1 \times \mathbb{S}^3 \# \mathbb{C}\mathrm{P}^2.$$

$\Delta = 8$.

Since $\beta_1 \geq g - 4$ by (7), it follows that $\gamma_{ij} \geq \beta_1 + 1 \geq g - 3$ (see (8)). Thus it is easily seen that the γ_{ij}'s in (4) may assume (up to circular permutations) the values listed in the following table:

case	γ_{24}	γ_{14}	γ_{13}	γ_{03}	γ_{02}
8.1	$g + 1$	$g - 3$	$g - 3$	$g - 3$	$g - 3$
8.2	g	$g - 2$	$g - 3$	$g - 3$	$g - 3$
8.3	g	$g - 3$	$g - 2$	$g - 3$	$g - 3$
8.4	$g - 1$	$g - 2$	$g - 2$	$g - 3$	$g - 3$
8.5	$g - 1$	$g - 2$	$g - 3$	$g - 2$	$g - 3$
8.6	$g - 1$	$g - 3$	$g - 2$	$g - 2$	$g - 3$
8.7	$g - 1$	$g - 2$	$g - 3$	$g - 3$	$g - 2$
8.8	$g - 2$	$g - 3$	$g - 2$	$g - 2$	$g - 2$

Now by (1) we have:

(8.1) $g_{\hat{0}} = g_{\hat{1}} = 4 \qquad g_{\hat{2}} = g_{\hat{3}} = g_{\hat{4}} = 0$

(8.2) $$g_{\hat{0}} = g_{\hat{1}} = 3 \qquad g_{\hat{2}} = g_{\hat{3}} = 1 \qquad g_{\hat{4}} = 0$$

(8.3) $$g_{\hat{0}} = 4 \qquad g_{\hat{1}} = 3 \qquad g_{\hat{2}} = g_{\hat{3}} = 0 \qquad g_{\hat{4}} = 1$$

(8.4) $$g_{\hat{0}} = 3 \qquad g_{\hat{1}} = 2 \qquad g_{\hat{2}} = g_{\hat{3}} = g_{\hat{4}} = 1$$

(8.5) $$g_{\hat{0}} = g_{\hat{2}} = 2 \qquad g_{\hat{1}} = 3 \qquad g_{\hat{3}} = 1 \qquad g_{\hat{4}} = 0$$

(8.6) $$g_{\hat{0}} = g_{\hat{1}} = 3 \qquad g_{\hat{2}} = g_{\hat{4}} = 1 \qquad g_{\hat{3}} = 0$$

(8.7) $$g_{\hat{0}} = g_{\hat{1}} = 2 \qquad g_{\hat{2}} = g_{\hat{4}} = 1 \qquad g_{\hat{3}} = 2$$

(8.8) $$g_{\hat{0}} = g_{\hat{1}} = g_{\hat{4}} = 2 \qquad g_{\hat{2}} = g_{\hat{3}} = 1,$$

hence relations (6), (7), and (9) give $\beta_1 = g - 4$ and $\beta_2 = 0$ for any case. Therefore we obtain the contradiction

$$M \underset{\text{PL}}{\simeq} \#_{g-4}\mathbb{S}^1 \times \mathbb{S}^3$$

(also compare with $\Delta = 6$).

$\Delta = 9$.

Since $\beta_1 \geq g - 4$ and $\gamma_{ij} \geq g - 3$ by (7) and (8), we must have (up to circular permutations) the following cases (use (4)):

case	γ_{24}	γ_{14}	γ_{13}	γ_{03}	γ_{02}
9.1	$g-1$	$g-3$	$g-3$	$g-3$	$g-3$
9.2	$g-2$	$g-2$	$g-3$	$g-3$	$g-3$
9.3	$g-2$	$g-3$	$g-2$	$g-3$	$g-3$

Now by (1) we obtain:

(9.1) $$g_{\hat{0}} = g_{\hat{1}} = 3 \qquad g_{\hat{2}} = g_{\hat{3}} = g_{\hat{4}} = 1$$

(9.2) $$g_{\hat{0}} = g_{\hat{1}} = g_{\hat{2}} = g_{\hat{3}} = 2 \qquad g_{\hat{4}} = 1$$

(9.3) $$g_{\hat{0}} = 3 \qquad g_{\hat{1}} = g_{\hat{4}} = 2 \qquad g_{\hat{2}} = g_{\hat{3}} = 1,$$

hence $\beta_1 = g - 4$ and $\beta_2 = 1$ for any case (use (6), (7) and (9)). Now we can repeat the same arguments of case $\Delta = 5$ (or $\Delta = 7$) to obtain the contradiction

$$M \underset{\text{PL}}{\simeq} \#_{g-4}\mathbb{S}^1 \times \mathbb{S}^3\#\mathbb{C}\text{P}^2.$$

Now we have only to consider case $\Delta = 10$ and $g = 4$ to complete the proof of theorem 2. Indeed, if $g \leq 4$, then $\Delta \leq 10$.

$\Delta = 10$, $g = 4$.

For this case, we have $\gamma_{24} = \gamma_{14} = \gamma_{13} = \gamma_{03} = \gamma_{02} = 1$ by (4), hence $\pi_1(M) \simeq 0$ and $\beta_2 = 2$ (use (6) and (8)). Since $H_2(M) \simeq H^2(M) \simeq FH_2(M)$ is free, it follows that $H_2(M) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

According to [5], [6] and [8], closed simply-connected smooth (or PL) 4-manifolds are classified (up to homeomorphism) by their intersection forms. Since Poincaré duality identifies $H_2(M)$ with $H^2(M)$, we can consider the intersection form λ_M as a pairing $H^2(M) \otimes H^2(M) \to \mathbb{Z}$ so defined: $\lambda_M(x, y) = (x \cup y)[M]$, where $\cup$ and $[M]$ denote the cup product and the fundamental class of M respectively.

Combining Donaldson's theorem [5], [6] and Freedman's classification, we have the following cases:

(1) If λ_M is positive (resp. negative) definite, then λ_M is isomorphic over the integers to $(1) \oplus (1)$ (resp. $(-1) \oplus (-1)$) by [5] and [6] (use the fact that $H_2(M) \simeq \mathbb{Z} \oplus \mathbb{Z}$). Thus M is (TOP) homeomorphic to either $\mathbb{C}P^2 \# \mathbb{C}P^2$ or $(-\mathbb{C}P^2) \# (-\mathbb{C}P^2)$ respectively (use [8]).

(2) If λ_M is an odd indefinite form, then λ_M is isomorphic to $(1) \oplus (-1)$ (see for example [14]), hence $M \underset{\text{TOP}}{\simeq} \mathbb{C}P^2 \# (-\mathbb{C}P^2) \simeq \mathbb{S}^2 \underset{\sim}{\times} \mathbb{S}^2$ by [8] and [14].

(3) If λ_M is an even indefinite form, then λ_M is isomorphic to the form

$$\omega = 2aE_8 + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\operatorname{rank}(\omega) = 16|a| + 2|b|$.

Since $\operatorname{rank}(\lambda_M) = \operatorname{rank}(\omega) = \operatorname{rank}(H_2(M)) = 2$, we obtain $a = 0$ and $b = 1$, i.e.

$$\lambda_M \underset{\text{ISO}}{\simeq} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now the Freedman theorem implies that M is (TOP) homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$ as M is simply connected. Now the proof is completed because $g(\mathbb{S}^2 \times \mathbb{S}^2) = g(\mathbb{S}^2 \underset{\sim}{\times} \mathbb{S}^2) = g(\mathbb{C}P^2 \# \mathbb{C}P^2) = 4$ by corollary 2 of [3].

To conclude the section we now prove that the genus of $\mathbb{R}P^3 \times \mathbb{S}^1$ is 6. Since the Euler–Poincaré characteristic of $\mathbb{R}P^3 \times \mathbb{S}^1$ is 0, relation (3) becomes

$0 = 2 - 2g + \Delta$. Hence the inequality $\Delta \geq 10$ implies that $g \geq 6$. Now the proof is completed because a crystallization of $\mathbb{RP}^3 \times \mathbb{S}^1$ with genus 6 is really constructed in [16].

4. Proofs: the non-orientable case

$\Delta = 0$.

If $\Delta = 0$ (recall that $\epsilon = h/2$), then relation (1) implies that $\gamma_{03} = \gamma_{24} = \gamma_{14} = \gamma_{13} = \gamma_{02} = 1 + h/2$. Since $\gamma_{24} = 1 + h/2$, the pseudo-complex $K(2,4)$ consists of exactly $1 + h/2$ edges, hence $N = N(2,4)$ is (PL) homeomorphic to the boundary connected sum $\#_{h/2}\mathbb{S}^1 \otimes \mathbb{B}^3$. Here $\mathbb{S}^1 \otimes \mathbb{B}^3$ represents either $\mathbb{S}^1 \times \mathbb{B}^3$ or the twisted $\mathbb{B}^3$-bundle over $\mathbb{S}^1$. Since M is non-orientable (and whence ∂N is non-orientable), we have $H_1(\partial N) \simeq \oplus_{h/2}\mathbb{Z}$, $H_2(\partial N) \simeq \oplus_{(h-2)/2}\mathbb{Z} \oplus \mathbb{Z}_2$ and $H_q(\partial N) \simeq 0$ for any $q \geq 3$. By (7) and (9) it follows that $\beta_1^{(2)} = h/2$, hence (6) gives $\beta_2^{(2)} = 0$. Since $H_2(M;\mathbb{Z}_2) \simeq H_2(M) \otimes \mathbb{Z}_2 \oplus \mathrm{Tor}(H_1(M),\mathbb{Z}_2) \simeq 0$, we have $FH_2(M) \simeq 0$ and $H_1(M)$ has no 2-torsion. Since $H_1(M;\mathbb{Z}_2) \simeq H_1(M) \otimes \mathbb{Z}_2 \simeq \oplus_{h/2}\mathbb{Z}_2$ and $\mathrm{Tor}(H_1(M);\mathbb{Z}_2) \simeq 0$, we also obtain $H_1(M) \simeq FH_1(M) \simeq \oplus_{h/2}\mathbb{Z}$, i.e. $\beta_1 = h/2$. This implies that $\chi(M) = 2 - h = 1 - \beta_1 + \beta_2 - \beta_3 = 1 - h/2 - \beta_3$ ($\beta_2 = 0$ as $FH_2(M) \simeq 0$), and whence $\beta_3 = h/2 - 1$. Moreover $H_3(M;\mathbb{Z}_2) \simeq H_3(M) \otimes \mathbb{Z}_2 \oplus \mathrm{Tor}(H_2(M);\mathbb{Z}_2) \simeq H_3(M) \otimes \mathbb{Z}_2 \simeq \oplus_{h/2}\mathbb{Z}_2$ as $\beta_1^{(2)} = \beta_3^{(2)} = h/2$, hence $H_3(M) \simeq \oplus_{(h-2)/2}\mathbb{Z} \oplus \mathbb{Z}_2$.

Thus the Mayer–Vietoris sequence of the triple (M, N, N'), $N' = N(0,1,3)$, gives

$$0 \longrightarrow FH_3(M) \simeq \bigoplus_{(h-2)/2} \mathbb{Z} \longrightarrow FH_2(\partial N) \simeq \bigoplus_{(h-2)/2} \mathbb{Z} \to$$
$$\to H_2(N') \longrightarrow FH_2(M) \simeq 0$$

and

$$0 \longrightarrow H_1(\partial N) \simeq \bigoplus_{h/2} \mathbb{Z} \to$$
$$\to H_1(N) \oplus H_1(N') \simeq \bigoplus_{h/2} \mathbb{Z} \oplus H_1(N') \longrightarrow H_1(M) \simeq \bigoplus_{h/2} \mathbb{Z} \longrightarrow 0,$$

hence $H_2(N') \simeq FH_2(N') \simeq 0$ and $H_1(N') \simeq \oplus_{h/2}\mathbb{Z}$. Further $K(0,3)$ and $K(1,3)$ are also formed by $1 + h/2$ edges eachone as $\gamma_{03} = \gamma_{13} = 1 + h/2$

by (1). Because $\gamma_{013} = \gamma_{01} + h/2$ (see (2)), the complex $K(0,1,3)$ has many triangles but $h/2$ as there are edges in $K(0,1)$.

Since $H_2(N') \simeq 0$ and $H_1(N') \simeq \oplus_{h/2}\mathbb{Z}$ there are no two triangles in $K(0,1,3)$ with common boundary. Now it follows that $K(0,1,3)$ collapses to an one-dimensional subcomplex. Hence the regular neighbourhood N' of $K(0,1,3)$ is (PL) homeomorphic to a boundary connected sum $\#_p\mathbb{S}^1 \otimes \mathbb{B}^3$. Since $\partial N' \simeq \partial N \simeq \#_{h/2}\mathbb{S}^1 \otimes \mathbb{S}^2$, it follows that $p = h/2$. Thus M is $\#_{h/2}\mathbb{S}^1 \otimes \mathbb{S}^3$ by theorem 2 of [15] and lemma 1 of [4] (see also [13]).

Now the result follows as $g(\mathbb{S}^1 \underset{\sim}{\times} \mathbb{S}^3) = 2$ (see [3]) and the genus is "subadditive".

$\Delta = 1$.

Relations (7) and (9) give $\beta_1^{(2)} \geq h/2$ and $\beta_1^{(2)} \leq h/2 - 1$ respectively, i.e. a contradiction.

$\Delta = 2$.

Using the same arguments shown in $\Delta = 2$ (orientable case) and $\Delta = 0$ (non-orientable case), we prove that M is (PL) homeomorphic to $\#_{(h-2)/2}\mathbb{S}^1 \otimes \mathbb{S}^3$, which is a contradiction.

$\Delta = 3$.

By (7) we have $\beta_1^{(2)} \geq (h-3)/2$, hence $\gamma_{ij} \geq \beta_1^{(2)} + 1 \geq (h-1)/2$, i.e. $\gamma_{ij} \geq h/2$ by (9). Thus relation (4) gives $5 + \frac{5}{2}h - 2\Delta \geq \frac{5}{2}h$, i.e. a contradiction.

$\Delta = 4$.

Using the same arguments shown in $\Delta = 4$ (orientable case) and $\Delta = 0$ (non-orientable case), one obtains the contradiction

$$M \underset{\mathrm{PL}}{\simeq} \#_{(h-4)/2}\mathbb{S}^1 \otimes \mathbb{S}^3.$$

$\Delta = 5$.

If $\Delta = 5$, then relation (7) implies that $\beta_1^{(2)} \geq h/2 - 2$. By (8) we have $\gamma_{ij} \geq \beta_1^{(2)} + 1 \geq h/2 - 1$, hence $\gamma_{24} = \gamma_{14} = \gamma_{13} = \gamma_{03} = \gamma_{02} = h/2 - 1$ by (4). Now relation (1) implies that $g_{\hat{\imath}} = 1$ for each colour $i \in \mathcal{C}_G$.

Since $\gamma_{24} = h/2 - 1$, $K(2,4)$ consists of two vertices joined by $h/2 - 1$ edges, hence $N = N(2,4) \underset{\mathrm{PL}}{\simeq} \#_{(h-4)/2}\mathbb{S}^1 \otimes \mathbb{B}^3$, i.e. $\partial N \underset{\mathrm{PL}}{\simeq} \#_{(h-4)/2}\mathbb{S}^1 \otimes \mathbb{S}^2$.

Since $g_{\hat{\imath}} = 1$ ($i \in \mathcal{C}_G$), we obtain $\beta_1^{(2)} \leq h/2 - 2$ by (9), and whence $\beta_1^{(2)} = h/2 - 2$. Thus relation (6) gives $\beta_2^{(2)} = 1$.

Furthermore relation (8) also implies that $\operatorname{rk}\pi_1(M) = \operatorname{rk} H_1(M) = h/2 - 2$.

Since $\mathbb{Z}_2 \simeq H_2(M;\mathbb{Z}_2) \simeq H_2(M) \otimes \mathbb{Z}_2 \oplus \operatorname{Tor}(H_1(M);\mathbb{Z}_2)$, it may occur three cases:

(1) $FH_2(M) \simeq 0$, $\operatorname{Tor}(H_1(M);\mathbb{Z}_2) \simeq \mathbb{Z}_2$ and $H_2(M)$ has no 2-torsion.
(2) $FH_2(M) \simeq \mathbb{Z}$, $\operatorname{Tor}(H_1(M);\mathbb{Z}_2) \simeq 0$ and $H_2(M)$ has no 2-torsion.
(3) $FH_2(M) \simeq 0$, $\operatorname{Tor}(H_1(M);\mathbb{Z}_2) \simeq 0$ and $H_2(M)$ has one 2-torsional factor.

Case 1. Since $\operatorname{Tor}(H_1(M);\mathbb{Z}_2) \simeq \mathbb{Z}_2$ and $\beta_1^{(2)} = h/2 - 2$, it follows that $\oplus_{(h-4)/2}\mathbb{Z}_2 \simeq H_1(M) \otimes \mathbb{Z}_2$, hence $\beta_1 = h/2 - 3$. Now relations $\beta_2 = 0$ (use $FH_2(M) \simeq 0$) and $\chi(M) = 7 - h = 1 - \beta_1 - \beta_3$ imply that $\beta_3 = h/2 - 3$, hence $H_3(M) \simeq \oplus_{(h-6)/2}\mathbb{Z} \oplus \mathbb{Z}_2$ as

$$\bigoplus_{(h-4)/2} \mathbb{Z}_2 \simeq H_3(M;\mathbb{Z}_2) \simeq H_3(M) \otimes \mathbb{Z}_2 \oplus \operatorname{Tor}(H_2(M);\mathbb{Z}_2) \simeq H_3(M) \otimes \mathbb{Z}_2$$

(use $\beta_1^{(2)} = \beta_3^{(2)} = h/2 - 2$ and $\operatorname{Tor}(H_2(M);\mathbb{Z}_2) \simeq 0$).

Since $\operatorname{rk} H_1(M) = h/2 - 2$ and $FH_1(M) \simeq \oplus_{(h-6)/2}\mathbb{Z}$, we obtain $H_1(M) \simeq \oplus_{(h-6)/2}\mathbb{Z} \oplus \mathbb{Z}_{2n}$ for some integer $n \geq 1$ as $\operatorname{Tor}(H_1(M);\mathbb{Z}_2) \simeq \mathbb{Z}_2$.

Now the Mayer–Vietoris sequence of the triple (M, N, N'), $N = N(2,4)$, $N' = N(0,1,3)$,

$$0 \longrightarrow H_3(M) \simeq \bigoplus_{(h-6)/2} \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{\text{iso}} H_2(\partial N) \simeq \bigoplus_{(h-6)/2} \mathbb{Z} \oplus \mathbb{Z}_2 \rightarrow$$
$$\rightarrow H_2(N') \longrightarrow H_2(M) \longrightarrow H_1(\partial N) \simeq \bigoplus_{(h-4)/2} \mathbb{Z} \rightarrow$$
$$\rightarrow H_1(N) \oplus H_1(N') \simeq \bigoplus_{(h-4)/2} \mathbb{Z} \oplus H_1(N') \rightarrow$$
$$\rightarrow H_1(M) \simeq \bigoplus_{(h-6)/2} \mathbb{Z} \oplus \mathbb{Z}_{2n} \longrightarrow 0$$

splits in the following exact sequences

$$(*) \qquad 0 \longrightarrow H_2(N') \longrightarrow FH_2(M) \longrightarrow 0$$

and

$$(**) \quad 0 \longrightarrow H_2(M) \longrightarrow H_1(\partial N) \simeq \bigoplus_{(h-4)/2} \mathbb{Z} \rightarrow$$
$$\rightarrow \bigoplus_{(h-4)/2} \mathbb{Z} \oplus H_1(N') \longrightarrow H_1(M) \simeq \bigoplus_{(h-6)/2} \mathbb{Z} \oplus \mathbb{Z}_{2n} \longrightarrow 0,$$

hence $H_2(N') \simeq 0$ by (*) and $H_2(M) \simeq FH_2(M) \simeq 0$, $H_1(N') \simeq \oplus_{(h-6)/2}\mathbb{Z} \oplus \mathbb{Z}_{2n}$ by (**).

Since $\gamma_{13} = \gamma_{03} = h/2-1$, $\gamma_{013} = \gamma_{01} + h/2 - 1$ (see (2)), $H_q(N') \simeq 0$ ($q \geq 2$) and $H_1(N') \simeq \oplus_{(h-6)/2}\mathbb{Z} \oplus \mathbb{Z}_{2n}$, the manifold N' is (PL) homeomorphic to the boundary connected sum $\#_{(h-6)/2}\mathbb{S}^1 \otimes \mathbb{B}^3 \# W$. Here W is a bordered 4-manifold homotopy equivalent to $e^0 \cup e^1 \cup e^2$ (e^i i-cell) with $\partial e^2 = 2ne^1$, i.e. $H_q(W) \simeq 0$, $q \geq 2$, and $H_1(W) \simeq \mathbb{Z}_{2n}$. Moreover e^2 must be formed by exacly two triangles of $K(0,1,3)$ since $\gamma_{013} = \gamma_{01} + h/2 - 1$ and $FH_1(N') \simeq \oplus_{(h-6)/2}\mathbb{Z}$. Thus it follows that $n = 1$, i.e. M is (PL) homeomorphic to $\#_{(h-6)/2}\mathbb{S}^1 \otimes \mathbb{S}^3 \# V^4$, where V^4 is a closed connected non-orientable 4-manifold with $H_1(V) = \pi_1(V) \simeq \mathbb{Z}_2$, $H_2(V) \simeq 0$, $H_3(V) \simeq \mathbb{Z}_2$ and $H_q(V) \simeq 0$, $q \geq 4$ (use $\operatorname{rk}\pi_1(M) = h/2 - 2$, $\pi_1(M) \simeq *_{(h-6)/2}\mathbb{Z} * \pi_1(V)$, i.e. $\pi_1(V)$ is cyclic). Since $\chi(V) = 1$ and $\pi_1(V) \simeq \mathbb{Z}_2$, the universal covering $\tilde{V}$ of V is a simply connected closed 4-manifold of Euler–Poincaré characteristic 2, hence $\tilde{V} \underset{\text{TOP}}{\simeq} \mathbb{S}^4$ by the Freedman theorem (see [8]). This implies that V is homotopy equivalent to the real projective 4-space $\mathbb{RP}^4$, hence $V \underset{\text{TOP}}{\simeq} \mathbb{RP}^4$ by [8] and [12] (see problem 4.13 of [12]). Now the proof is complete because a crystallization of $\mathbb{RP}^4$ with genus 6 is really constructed in [1] (see also [3])

Case 2. Since $\beta_1^{(2)} = h/2 - 2$ and $\operatorname{Tor}(H_1(M);\mathbb{Z}_2) \simeq 0$, we obtain $\oplus_{(h-4)/2}\mathbb{Z}_2 \simeq H_1(M;\mathbb{Z}_2) \simeq H_1(M) \otimes \mathbb{Z}_2$, hence $FH_1(M) \simeq \oplus_{(h-4)/2}\mathbb{Z}$, i.e. $\beta_1 = h/2 - 2$. Thus $\operatorname{rk} H_1(M) = h/2 - 2 = \beta_1$ gives $TH_1(M) \simeq 0$. Since $H_2(M)$ has no 2-torsion and $\beta_1^{(2)} = \beta_3^{(2)}$, we also have $\oplus_{(h-4)/2}\mathbb{Z}_2 \simeq H_3(M;\mathbb{Z}_2) \simeq H_3(M) \otimes \mathbb{Z}_2$. Further $\beta_2 = 1$ (use $FH_2(M) \simeq \mathbb{Z}$) and $\beta_1 = h/2 - 2$ imply that $\chi(M) = 7 - h = 1 - h/2 + 2 + 1 - \beta_3$, hence $\beta_3 = h/2 - 3$ so $H_3(M) \simeq \oplus_{(h-6)/2}\mathbb{Z} \oplus \mathbb{Z}_2$.

Now the Mayer–Vietoris sequence of the triple (M, N, N'), $N = N(2,4) \simeq \#_{(h-4)/2}\mathbb{S}^1 \otimes \mathbb{B}^3$, $N' = N(0,1,3)$, gives

$$0 \longrightarrow H_3(M) \simeq \bigoplus_{(h-6)/2} \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{\text{iso}} H_2(\partial N) \simeq \bigoplus_{(h-6)/2} \mathbb{Z} \oplus \mathbb{Z}_2 \rightarrow$$

$$\rightarrow H_2(N') \longrightarrow H_2(M) \simeq \mathbb{Z} \oplus TH_2(M) \longrightarrow H_1(\partial N) \simeq \bigoplus_{(h-4)/2} \mathbb{Z} \rightarrow$$

$$\rightarrow H_1(N) \oplus H_1(N') \simeq \bigoplus_{(h-4)/2} \mathbb{Z} \oplus H_1(N') \longrightarrow H_1(M) \simeq \bigoplus_{(h-4)/2} \mathbb{Z} \longrightarrow 0$$

hence $H_2(N') \simeq \mathbb{Z}$, $H_2(M) \simeq FH_2(M) \simeq \mathbb{Z}$ and $H_1(N') \simeq \oplus_{(h-4)/2}\mathbb{Z}$. Thus we can repeat the arguments used in $\Delta = 5$ (orientable case) to obtain

$$M \underset{\text{PL}}{\simeq} \#_{(h-4)/2}\mathbb{S}^1 \otimes \mathbb{S}^3 \# \mathbb{C}\mathrm{P}^2.$$

Case 3. Since $\mathrm{Tor}(H_1(M);\mathbb{Z}_2) \simeq 0$ and $\oplus_{(h-4)/2}\mathbb{Z}_2 \simeq H_1(M;\mathbb{Z}_2) \simeq H_1(M) \otimes \mathbb{Z}_2$, we have $FH_1(M) \simeq \oplus_{(h-4)/2}\mathbb{Z}$, i.e. $\beta_1 = h/2 - 2$.

Now relations $\mathrm{rk}\, H_1(M) = h/2 - 2 = \beta_1$ imply that $TH_1(M) \simeq 0$. Since $\beta_2 = 0$ (use $FH_2(M) \simeq 0$) and $\chi(M) = 7 - h = 1 - h/2 + 2 - \beta_3$, it follows that $\beta_3 = h/2 - 4$. Thus we have

$$\bigoplus_{(h-4)/2} \mathbb{Z}_2 \simeq H_3(M;\mathbb{Z}_2) = H_3(M)\otimes\mathbb{Z}_2 \oplus \mathrm{Tor}(H_2(M);\mathbb{Z}_2) \simeq H_3(M)\otimes\mathbb{Z}_2\oplus\mathbb{Z}_2,$$

i.e. $H_3(M) \simeq \oplus_{(h-8)/2}\mathbb{Z} \oplus \mathbb{Z}_2$.

Now the Mayer–Vietoris sequence of the triple (M, N, N'), $N = N(2,4) \simeq \#_{(h-4)/2}\mathbb{S}^1 \otimes \mathbb{B}^3$, $N' = N(0,1,3)$, becomes

$$0 \longrightarrow H_3(M) \simeq \bigoplus_{(h-8)/2} \mathbb{Z} \oplus \mathbb{Z}_2 \overset{\text{mono}}{\longrightarrow} H_2(\partial N) \simeq \bigoplus_{(h-6)/2} \mathbb{Z} \oplus \mathbb{Z}_2 \rightarrow$$
$$\rightarrow H_2(N') \longrightarrow H_2(M) \simeq TH_2(M) \longrightarrow H_1(\partial N) \simeq \bigoplus_{(h-4)/2} \mathbb{Z} \rightarrow$$
$$\rightarrow H_1(N) \oplus H_1(N') \simeq \bigoplus_{(h-4)/2} \mathbb{Z} \oplus H_1(N') \longrightarrow H_1(M) \simeq \bigoplus_{(h-4)/2} \mathbb{Z} \longrightarrow 0,$$

hence $H_1(N') \simeq \oplus_{(h-4)/2}\mathbb{Z}$ and

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow TH_2(M) \longrightarrow 0,$$

i.e. $H_2(M) \simeq \mathbb{Z}_{2n}$ for some integer $n \geq 1$ as $H_2(M)$ has one 2-torsional factor. Thus M is (PL) homeomorphic to $\#_{(h-8)/2}\mathbb{S}^1 \otimes \mathbb{S}^3 \# M'$, where M' is a closed connected non-orientable prime 4-manifold (if exists) such that $H_1(M') \simeq \mathbb{Z} \oplus \mathbb{Z}$, $H_2(M') \simeq \mathbb{Z}_{2n}$, $H_3(M') \simeq \mathbb{Z}_2$, $H_q(M') \simeq 0$, $\chi(M') = -1$ and $g(M') = h = 8$.

Now we prove theorem 4.

If $\Delta \leq 5$ and $h = 6$, then M is homeomorphic to either $\#_3\mathbb{S}^1 \otimes \mathbb{S}^3$ or $\mathbb{C}\mathrm{P}^2 \# \mathbb{S}^1 \underset{\sim}{\times} \mathbb{S}^3$ or $\mathbb{R}\mathrm{P}^4$, hence $M \underset{TOP}{\simeq} \mathbb{R}\mathrm{P}^4$ if M is prime. If $h = 6$ and $6 \leq \Delta \leq 7$, relation (4) becomes

(***) $$\gamma_{24} + \gamma_{14} + \gamma_{13} + \gamma_{03} + \gamma_{02} = 20 - 2\Delta,$$

hence $20 - 2\Delta \in \{8, 6\}$, i.e. at least one of the γ_{ij}'s in (***) must be 1.

This implies that $\pi_1(M) \simeq 0$ (use (8)) so M is simply-connected; but this is a contradiction as M is non-orientable.

Indeed we can also prove that $\Delta \in \{6, 7\}$ and $h \geq 6$ give a contradiction as we obtain the manifolds $\#_{(h-6)/2}\mathbb{S}^1 \otimes \mathbb{S}^3$ and $\#_{(h-6)/2}\mathbb{S}^1 \otimes \mathbb{S}^3 \# \mathbb{C}P^2$ respectively. □

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