CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

MURRAY HEGGIE

Homotopy colimits in presheaf categories

Cahiers de topologie et géométrie différentielle catégoriques, tome 34, nº 1 (1993), p. 13-36

<http://www.numdam.org/item?id=CTGDC_1993__34_1_13_0>

CAHIERS DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE CATÉGORIQUES
VOL. XXXIV-1 (1993)

HOMOTOPY COLIMITS IN PRESHEAF CATEGORIES

by Murray HEGGIE

RESUME. On construit des colimites homotopiques dans les catégories de préfaisceaux, dont la théorie d'homotopie est équivalente à la théorie d'homotopie de la catégorie des catégories.

1 Introduction

In an unpublished but widely circulated manuscript, Grothendieck proposes to base homotopy theory on $\mathcal{CAT}$, the category of (small) categories [2]. He initiates this program by proving a theorem that equates the homotopy category of certain presheaf categories $(\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$ and the homotopy category of $\mathcal{CAT}$. This paper describes the construction of homotopy colimits in the afore-mentioned presheaf categories and compares these to homotopy colimits in $\mathcal{CAT}$.

Before giving a more detailed account of the contents of this paper, some fundamental notions for the homotopy theory of categories will be briefly recalled. The first is that of a *Grothendieck fibration* [1]. Let $\mathbf{F} : \mathbf{C} \to \mathbf{D}$ be a functor and let D be an object of $\mathbf{D}$. The comma category $D/\mathbf{F}$ is the category with objects all pairs (C, f) where $C \in \mathbf{C}$ and $f : D \to \mathbf{F}(C) \in \mathbf{D}$. A map from one object (C, f) to another (C', f') consists of a map $g : C \to C' \in \mathbf{C}$ satisfying $f' = \mathbf{F}(g) \circ f$. Let $\mathbf{F}^{-1}(D)$ denote the fibre of $\mathbf{F}$ over $D \in \mathbf{D}$. Evidently, there is an inclusion

$$\iota : \mathbf{F}^{-1}(D) \hookrightarrow D/\mathbf{F}.$$

If ι has a right adjoint, left inverse for every object $D \in \mathbf{D}$, $\mathbf{F}$ is called a *Grothendieck fibration.*

Let $Y : \mathbf{C}^{op} \to \mathcal{CAT}$ be a contravariant $\mathcal{CAT}$-valued diagram on $\mathbf{C}$. There is a fibration

$$\pi : Y \textstyle\int \mathbf{C} \to \mathbf{C}$$

where the domain category $Y \int \mathbf{C}$ is defined as follows: Objects of $Y \int \mathbf{C}$ are pairs (C, y) where $C \in \mathbf{C}$ and $y \in Y(C)$. A map $(C, y) \to (C', y')$ is a pair $(c : C \to C' \in \mathbf{C}, g : y \to Y(c)(y'))$. $Y \int \mathbf{C}$ is known variously as the *fibred category associated to* Y or the *Grothendieck construction of* Y.

A functor $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ is called a *weak equivalence* if its image Nerve($\mathbf{F}$) under the functor

$$\mathrm{Nerve} : \mathcal{CAT} \to \mathcal{SS}$$

is a weak equivalence of simplicial sets, i.e. induces an isomorphism between the homotopy groups of **Nerve(A)** and **Nerve(B)**. The class of weak equivalences has several saturation properties:

(1) Isomorphisms are weak equivalences.
(2) The composite of two weak equivalences is a weak equivalence.
(3) If $\mathbf{G} \circ \mathbf{F} = \mathbf{H}$ and any two of **F**,**G**, or **H** are weak equivalences, then the remaining map is also a weak equivalence.
(4) If **P** is a split epimorphism with section **S** and the composite $\mathbf{S} \circ \mathbf{P}$ is homotopic to the identity then **P** and **S** are weak equivalences.

A category **A** is called *weakly contractible* if the unique map $\mathbf{A} \to \mathbf{1}$ to the terminal category **1** is a weak equivalence. Weak equivalences in the functor category $(\mathbf{A}, \mathcal{CAT})$ are defined pointwise: A natural transformation $\theta : X \Rightarrow Y$ is a weak equivalence if $\theta(A) : X(A) \Rightarrow Y(A)$ is a weak equivalence in $\mathcal{CAT}$ for all $A \in \mathbf{A}$.

Let $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ and $\mathbf{G} : \mathbf{B} \to \mathbf{A}$ be functors and let $\theta : \mathbf{F} \Rightarrow \mathbf{G}$ be a natural transformation. Let 2 denote the category

$$0 \longrightarrow 1$$

with two objects and one non-identity arrow. θ determines a map

$$\hat{\theta} : \mathbf{A} \times 2 \longrightarrow \mathbf{B}$$

by the prescription

$$\begin{aligned} \hat{\theta}(a : A \to A', 0 \to 1) &= \mathbf{G}(a) \circ \theta(A) \\ &= \theta(A') \circ \mathbf{F}(a). \end{aligned}$$

Since $\mathbf{Nerve}(2) = \Delta[1]$, the simplicial interval, and **Nerve** preserves finite limits, $\mathbf{Nerve}(\hat{\theta})$ is a homotopy between **Nerve(F)** and **Nerve(G)**. It follows that if

$$\mathbf{F} \dashv \mathbf{G} : \mathbf{A} \to \mathbf{B}$$

are adjoints, **A** and **B** are homotopy equivalent. For the adjunctions

$$\eta : id_{\mathbf{A}} \Longrightarrow \mathbf{G} \circ \mathbf{F}$$

and

$$\varepsilon : \mathbf{F} \circ \mathbf{G} \Longrightarrow id_{\mathbf{B}}$$

map to homotopy inverses via **Nerve**.

If the category **A** has a terminal object 1, then the identity $id_{\mathbf{A}}$ is homotopic to the composite

$$\mathbf{A} \to 1 \hookrightarrow \mathbf{A}$$

of the unique map to the terminal category followed by the inclusion of the terminal object in **A**. The homotopy is the map whose value at $A \in \mathbf{A}$ is the unique map $A \to 1$. Similarly, if **A** has an initial object, **A** is weakly contractible.

The Grothendieck construction, $X \int \mathbf{C}$, has already been described. The property of $X \int \mathbf{C}$ that is crucial to the sequel is Thomason's observation that, for $\mathcal{CAT}$-based homotopy theory,

$$X \int \mathbf{C} \equiv \mathrm{hocolim}_{\mathbf{C}},$$

the *homotopy colimit of* **C** [6]. In more detail, let $\Sigma^{-1}\mathcal{CAT}$ denote the category of fractions of $\mathcal{CAT}$ with respect to the class of weak equivalences and let $\Sigma^{-1}(\mathbf{C},\mathcal{CAT})$ denote the category of fractions of $(\mathbf{C},\mathcal{CAT})$ with respect to the class of point-wise weak equivalences. Let

$$\pi_{\mathbf{C}}: \mathbf{C} \to 1$$

denote the unique functor to the terminal category and let

$$\pi^*_{\mathbf{C}}: \mathcal{CAT} \to (\mathbf{C},\mathcal{CAT})$$

denote the functor induced by $\pi_{\mathbf{C}}$. It is evident that $\pi^*_{\mathbf{C}}$ preserves weak equivalences and therefore

$$\Sigma^{-1}\pi^*_{\mathbf{C}} : \Sigma^{-1}\mathcal{CAT} \to \Sigma^{-1}(\mathbf{C},\mathcal{CAT})$$

exists. Frequent reference will be made to the following fundamental result:

1.1 Theorem (Thomason's Theorem) *The Grothendieck construction preserves weak equivalences and therefore*

$$\Sigma^{-1}(\cdot)\int \mathbf{C}: \Sigma^{-1}(\mathbf{C},\mathcal{CAT}) \to \Sigma^{-1}\mathcal{CAT}$$

exists. Moreover, there is an adjunction

$$\Sigma^{-1}(\cdot)\int \mathbf{C} \dashv \Sigma^{-1}\pi^*_{\mathbf{C}}.$$

Proof. [6] □

A second cornerstone of $\mathcal{CAT}$-based homotopy theory is *Quillen's Theorem* A [5]:

1.2 Theorem *Let* $\mathbf{F}: \mathbf{A} \to \mathbf{B}$ *be a functor. Suppose that* $\mathbf{F}/B$ *is weakly contractible for all objects* $B \in \mathbf{B}$. *Then* **F** *is a weak equivalence.* □

The property of fibrations enunciated in the lemma which follows will play a key role in the proof of many of the results contained in this paper.

1.3 Lemma *Let*

$$\begin{array}{ccc} \mathbf{C}' & \xrightarrow{\mathbf{G}'} & \mathbf{C} \\ {\scriptstyle \mathbf{F}'}\downarrow & & \downarrow{\scriptstyle \mathbf{F}} \\ \mathbf{D}' & \xrightarrow{\mathbf{G}} & \mathbf{D} \end{array} \tag{1}$$

be a pullback in $\mathcal{CAT}$. Suppose that $\mathbf{G}$ *is a fibration and that, for all* $D \in \mathbf{D}$, $\mathbf{F}/D$ *is weakly contractible. Then* $\mathbf{F}'$ *is a weak equivalence.*

Proof. For every object $D' \in \mathbf{D}'$ there are functors

$$\mathbf{H}\colon \mathbf{F}'/D' \to \mathbf{F}/\mathbf{G}'(D')$$

and

$$\mathbf{K}\colon \mathbf{F}/\mathbf{G}'(D') \to \mathbf{F}'/D'$$

such that

$$\mathbf{H} \circ \mathbf{K} = \mathrm{id} \quad \text{and} \quad \mathbf{H} \dashv \mathbf{K}.$$

In particular, $\mathbf{H}$ is a homotopy equivalence. Therefore, $\mathbf{F}'/D'$ is weakly contractible for all objects $D' \in \mathbf{D}'$. By Quillen's Theorem A, $\mathbf{F}'$ is a weak equivalence. Additional details can be found in [3] □

For any category $\mathbf{A}$, there is a canonical functor

$$\iota_{\mathbf{A}} : (\mathbf{A}^{\mathrm{op}}, \mathrm{Sets}) \to \mathcal{CAT}$$

Grothendieck proposes to relate the homotopy theory of $(\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$ to the homotopy theory of $\mathcal{CAT}$ *via* $\iota_{\mathbf{A}}$. His main result gives sufficient conditions for $\iota_{\mathbf{A}}$ to induce an equivalence of homotopy theories. This result is recapitulated in §2. In §3, a class of categories meeting the conditions of Grothendieck's theorem is introduced. Properties of categories belonging to this class are explored in §3 and §4. Let $\mathbf{A}$ be a category belonging to the class. Let

$$X\colon \mathbf{C} \to (\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$$

be a diagram. The *homotopy colimit* of X,

$$\mathrm{hocolim}_{\mathbf{C}} X \in (\mathbf{A}^{\mathrm{op}}, \mathrm{Sets}),$$

is constructed in §5. Various properties of the homotopy colimit are described in §6 and §7.

It should be noted that the theory sketched by Grothendieck applies to the standard example, *viz.* $(\Delta^{\mathrm{op}}, \mathrm{Sets})$, the category of simplicial sets.

2 Grothendieck's theorem

Let $\mathbf{A}$ denote a (small) category and let $\mathbf{A}/\cdot : \mathbf{A} \to \mathcal{CAT}$ denote the functor which assigns the comma category $\mathbf{A}/A$ to each object $A \in \mathbf{A}$.

2.1 Definition $\iota_{\mathbf{A}} : (\mathbf{A}^{\mathrm{op}}, \mathrm{Sets}) \to \mathcal{CAT}$ is the left Kan extension of $\mathbf{A}/\cdot$ along the Yoneda embedding $\mathbf{A} \to (\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$.

For $X \in (\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$,$\iota_{\mathbf{A}}(X)$ is the category whose objects are all pairs $\{(A, x) \mid A \in \mathbf{A} \text{ and } x \in X(A)\}$. A map $(A, x) \to (B, y)$ from one object to another is a map $f : A \to B$ such that $X(f)(y) = x$. $\iota_{\mathbf{A}}(X)$ is variously known as the *diagram of* X, the *category of elements of* X, or the *fibred category associated to* X. $\iota_{\mathbf{A}}$ is left adjoint to the functor

$$\mathcal{CAT}(\mathbf{A}/\cdot, \cdot) : \mathcal{CAT} \to (\mathbf{A}^{\mathrm{op}}, \mathrm{Sets}).$$

Let $j_{\mathbf{A}}$ denote this functor. It will be useful in the sequel to have an explicit description of the counit ε of the adjunction $\iota_{\mathbf{A}} \dashv j_{\mathbf{A}}$. To this end, let $\mathbf{C}$ be a category and let $(A, \mathbf{F} : \mathbf{A}/A \to \mathbf{C})$ be an object of $\iota_{\mathbf{A}} \circ j_{\mathbf{A}}(\mathbf{C})$. Then $\varepsilon_{\mathbf{C}}(A, \mathbf{F}) = \mathbf{F}(\mathrm{id}_A : A \to A) = \mathrm{id}_{\mathbf{F}(A)}$.The extension to maps is immediate.

2.2 Lemma *The forgetful functor* $\iota_{\mathbf{A}}(X) \to \mathbf{A}$ *is a (Grothendieck) fibration with discrete fibres* □

2.3 Definition [2] A map $\theta : X \to Y$ in $(\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$ is a *weak equivalence* if the functor $\iota_{\mathbf{A}}(\theta)$ is a weak equivalence in $\mathcal{CAT}$. The collection of weak equivalences in $(\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$ will be denoted $\Sigma_{\mathbf{A}}$.

$\Sigma_{\mathbf{A}}$ inherits closure properties from the class of weak equivalences in $\mathcal{CAT}$:

(1) Isomorphisms belong to $\Sigma_{\mathbf{A}}$.
(2) If θ and ψ are a composable pair of maps and any two of θ,ψ, or $\psi \circ \theta$ are in $\Sigma_{\mathbf{A}}$, so is the third.
(3) If ρ is a split epimorphism with section σ and the composite $\sigma \circ \rho \in \Sigma_{\mathbf{A}}$,then $\rho \in \Sigma_{\mathbf{A}}$.

2.4 Definition Let $I \in (\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$ have two disjoint global sections: $\delta_0 : 1 \to I$ and $\delta_1 : 1 \to I$. A map $\rho : Y \to X \in (\mathbf{A}^{\mathrm{op}}, \mathrm{sets})$ is *the dual of a deformation retract with respect to* I if

(1) ρ has a section $\sigma : X \to Y$: $\rho \circ \sigma = \mathrm{id}_X$.
(2) There is a map $\mathbf{H} : Y \times I \to Y$ such that (2) commutes:

$$\begin{array}{ccccc} Y & \xrightarrow{\mathrm{id}\times\delta_0} & Y \times I & \xleftarrow{\mathrm{id}\times\delta_1} & Y \\ \rho\downarrow & & \mathbf{H}\downarrow & & \mathrm{id}\downarrow \\ X & \xrightarrow{\sigma} & Y & \xleftarrow{\mathrm{id}} & Y. \end{array} \qquad (2)$$

This definition is motivated by considerations from axiomatic homotopy theory.

2.5 Definition [2] An object I which has two disjoint global sections, $\delta_0 : 1 \to I$ and $\delta_1 : 1 \to I$, is called a *homotopy interval* if duals of deformation retracts with respect to I belong to $\Sigma_{\mathbf{A}}$.

2.6 Theorem (Grothendieck's Theorem) *Let* $\mathbf{A}$ *be a weakly contractible category. Assume that* $(\mathbf{A}^{\mathrm{op}}, Sets)$ *contains a homotopy interval. Then the functors* $\iota_{\mathbf{A}}$ *and* $j_{\mathbf{A}}$ *induce adjoint equivalences:*

$$\Sigma^{-1}\iota_{\mathbf{A}} \dashv \Sigma^{-1}j_{\mathbf{A}} : \Sigma^{-1}(\mathbf{A}^{\mathrm{op}}, Sets) \to \Sigma^{-1}\mathcal{CAT}.$$

Proof. [2] □

3 Categories of models

3.1 Definition [2] A category $\mathbf{A}$ is *acyclic* if the diagonal functor has weakly contractible homotopy fibres, i.e., the pullback

$$\begin{array}{ccc} \Delta/(A,B) & \longrightarrow & \mathbf{A} \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ \mathbf{A} \times \mathbf{A}/(A,B) & \longrightarrow & \mathbf{A} \times \mathbf{A} \end{array} \tag{3}$$

is weakly contractible for all pairs $(A, B) \in \mathbf{A} \times \mathbf{A}$.

3.2 Proposition $\mathbf{A}$ *is an acyclic category if and only if the projection* $\pi_B : \mathbf{A}(\cdot, A) \times \mathbf{A}(\cdot, B) \to \mathbf{A}(\cdot, B) \in \Sigma_{\mathbf{A}}$ *for all objects* A *and* B *of* $\mathbf{B}$.

Proof. Assume that $\pi_B \in \Sigma_{\mathbf{A}}$. By definition, $\iota_{\mathbf{A}}(\pi_B)$ is a weak equivalence in $\mathcal{CAT}$. But $\iota_{\mathbf{A}}(\mathbf{A}(\cdot, A) \times \mathbf{A}(\cdot, B)) = \Delta/(A, B)$, the homotopy fibre of Δ over $(A, B) \in \mathbf{A} \times \mathbf{A}$, and $\iota_{\mathbf{A}}(\pi_B) = \Delta/(A, B) \to \mathbf{A}/B$, the evident functor. Since $\mathbf{A}/B$ is contractible, $\Delta/(A, B)$ is weakly contractible,i.e., $\mathbf{A}$ is acyclic. By running this argument in reverse, one obtains a proof of the converse □

3.3 Proposition *An acyclic category is weakly contractible.*

Proof. Let $\mathbf{A}$ be an acylic category. The diagonal functor Δ induces isomorphisms

$$\pi_0(\mathbf{Nerve}(\mathbf{A})) \to \pi_0(\mathbf{Nerve}(\mathbf{A})) \times \pi_0(\mathbf{Nerve}(\mathbf{A}))$$

and

$$\pi_n(\mathbf{Nerve}(\mathbf{A}), A) \to \pi_n(\mathbf{Nerve}(\mathbf{A}), A) \times \pi_n(\mathbf{Nerve}(\mathbf{A}), A)$$

for all choices of $A \in \mathbf{A}$. □

Acyclicity of $\mathbf{A}$ imposes additional requirements on the collection $\Sigma_{\mathbf{A}}$.

3.4 Proposition *Assume that* $\mathbf{A}$ *is an acyclic category. Then* $\theta \times \theta' \in \Sigma_{\mathbf{A}}$ *whenever* $\theta \in \Sigma_{\mathbf{A}}$ *and* $\theta' \in \Sigma_{\mathbf{A}}$.

Proof. Let $\theta\colon X \to Y$ and $\theta'\colon X' \to Y'$ be maps in $\Sigma_{\mathbf{A}}$. By definition, $\iota_{\mathbf{A}}(\theta)$ and $\iota_{\mathbf{A}}(\theta')$ are weak equivalences in $\mathcal{CAT}$. In virtue of the saturation properties of the collection of weak equivalences in $\mathcal{CAT}$, $\iota_{\mathbf{A}}(\theta) \times \iota_{\mathbf{A}}(\theta')$ is a weak equivalence. For any two objects X and X' of $(\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$, there is a natural map

$$\mathbf{F}\colon \iota_{\mathbf{A}}(X \times X') \to \iota_{\mathbf{A}}(X) \times \iota_{\mathbf{A}}(X')$$

defined on objects by $(A,(x,x')) \mapsto ((A,x),(A,x'))$. By inspection,$\mathbf{F}$ fits into a pullback diagram

$$\begin{array}{ccc} \iota_{\mathbf{A}}(X \times X') & \xrightarrow{\mathbf{F}} & \iota_{\mathbf{A}}(X) \times \iota_{\mathbf{A}}(X') \\ \downarrow & & \downarrow \\ \mathbf{A} & \xrightarrow{\Delta} & \mathbf{A} \times \mathbf{A} \end{array} \qquad (4)$$

where the vertical arrows are the forgetful functors. By 2.2, the rightmost vertical arrow is a fibration. Since Δ satisfies the hypotheses of Quillen's Theorem A, $\mathbf{F}$ is a weak equivalence. Given two maps $\theta\colon X \to Y$ and $\theta'\colon X' \to Y'$, construct the commuting square

$$\begin{array}{ccc} \iota_{\mathbf{A}}(X \times X') & \longrightarrow & \iota_{\mathbf{A}}(X) \times \iota_{\mathbf{A}}(X') \\ \downarrow{\scriptstyle \iota_{\mathbf{A}}(\theta \times \theta')} & & \downarrow{\scriptstyle \iota_{\mathbf{A}}(\theta) \times \iota_{\mathbf{A}}(\theta')} \\ \iota_{\mathbf{A}}(Y \times Y') & \longrightarrow & \iota_{\mathbf{A}}(Y) \times \iota_{\mathbf{A}}(Y'). \end{array} \qquad (5)$$

Since $\iota_{\mathbf{A}}(\theta) \times \iota_{\mathbf{A}}(\theta')$ is a weak equivalence and both horizontal arrows are weak equivalences, $\iota_{\mathbf{A}}(\theta \times \theta')$ is also a weak equivalence ☐

The preceding proposition has a partial converse:

3.5 Proposition *Let* $\mathbf{A}$ *be a weakly contractible category. Assume that* $\theta \times \theta' \in \Sigma_{\mathbf{A}}$ *whenever* θ *and* θ' *are in* $\Sigma_{\mathbf{A}}$. *Then* $\mathbf{A}$ *is acyclic.*

Proof. Let 1 denote the terminal object of $(\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$. Note that $\iota_{\mathbf{A}}(1) = \mathbf{A}$.Let A be an object of $\mathbf{A}$. $\iota_{\mathbf{A}}(\mathbf{A}(\cdot, A)) = \mathbf{A}/A$. Since $\mathbf{A}$ is weakly contractible and $\mathbf{A}/A$ is contractible, the forgetful map $\mathbf{A}/A \to \mathbf{A} = \iota_{\mathbf{A}}(\mathbf{A}(\cdot, A) \to 1)$ is a weak equivalence. By definition, $\mathbf{A}(\cdot, A) \to 1 \in \Sigma_{\mathbf{A}}$. Consequently, for each $B \in \mathbf{A}$, $\mathbf{A}(\cdot, A) \times \mathbf{A}(\cdot, B) \to 1 \times \mathbf{A}(\cdot, B) \in \Sigma_{\mathbf{A}}$. But this map is isomorphic to π_B. By 3.2, $\mathbf{A}$ is acyclic ☐

Let $\Omega_{\mathbf{A}}$ denote the subobject classifier in $(\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$.

3.6 Proposition *Let* $\mathbf{A}$ *be an acyclic category. Then* $\iota_{\mathbf{A}}$ *and* $j_{\mathbf{A}}$ *induce adjoint equivalences*

$$\Sigma^{-1}\iota_{\mathbf{A}} \dashv \Sigma^{-1}j_{\mathbf{A}}$$

if and only if $\Omega_{\mathbf{A}}$ *is a homotopy interval.*

Proof. One direction is immediate from Grothendieck's Theorem. To prove the converse, let $2 \in \mathcal{CAT}$ denote the category with two objects, 0 and 1, and one non-identity arrow $0 \to 1$. By assumption, $j_{\mathbf{A}}$ preserves weak equivalences. Since $2 \to 1$ is a weak equivalence in $\mathcal{CAT}$, $j_{\mathbf{A}}(2) \to 1 \in \Sigma_{\mathbf{A}}$. But $j_{\mathbf{A}}(2) = \Omega_{\mathbf{A}}$. Let $\delta_0 \colon 1 \to \Omega_{\mathbf{A}}$ denote the classifying map for $0 \to 1$ and let $\delta_1 \colon 1 \to \Omega_{\mathbf{A}}$ denote the classifying map for $1 \to 1$. Then δ_0 and δ_1 are disjoint global sections. As $\Omega_{\mathbf{A}} \to 1 \in \Sigma_{\mathbf{A}}$ and

$$\begin{array}{ccc} 1 & \xrightarrow{\delta_i} & \Omega_{\mathbf{A}} \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & 1 \end{array} \tag{6}$$

commutes for $i = 0, 1$, $\delta_i \in \Sigma_{\mathbf{A}}$ for $i = 0, 1$. Let $X \in (\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$. As $\mathbf{A}$ is acyclic,id $\times\, \delta_i \colon X \to X \times \Omega_{\mathbf{A}} \in \Sigma_{\mathbf{A}}$ by 3.4.To finish the argument that $\Omega_{\mathbf{A}}$ is a homotopy interval, let $\rho \colon Y \to X$ be a map that has a section $\sigma \colon X \to Y$. Let $\mathbf{H}$ be a homotopy between id_Y and $\sigma \circ \rho$:

$$\begin{array}{ccccc} Y & \xrightarrow{\mathrm{id}\times\delta_0} & \Omega_{\mathbf{A}} \times Y & \xleftarrow{\mathrm{id}\times\delta_1} & Y \\ {\scriptstyle\rho}\downarrow & & {\scriptstyle\mathbf{H}}\downarrow & & {\scriptstyle\mathrm{id}}\downarrow \\ X & \xrightarrow{\sigma} & Y & \xleftarrow{\mathrm{id}} & Y. \end{array} \tag{7}$$

By the closure properties of $\Sigma_{\mathbf{A}}$, $\mathbf{H} \in \Sigma_{\mathbf{A}}$. By the same reasoning, $\sigma \circ \rho \in \Sigma_{\mathbf{A}}$. The closure properties of $\Sigma_{\mathbf{A}}$ imply that $\rho \in \Sigma_{\mathbf{A}}$ □

3.7 Definition A category A is *a category of models* if

(1) A is acyclic.
(2) $\Omega_{\mathbf{A}}$ is a homotopy interval.

A substantial portion of the homotopy theory of spaces can be instantiated in $(\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$ if $\mathbf{A}$ is a category of models. The propositions which follow provide partial support for this claim. Further details can be found in [3].

3.8 Proposition *Assume that* $\mathbf{A}$ *is a category of models. Let* $p \colon Y \to X$ *be a map in* $(\mathbf{A}^{\mathrm{op}}, Sets)$*. Suppose that* p *has the right lifting property with respect to all monomorphisms: there is a lifting* $h \colon X' \to Y$ *in all commuting diagrams of the* *form*

$$\begin{array}{ccc} Y' & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{g} & X \end{array} \tag{8}$$

where $i\colon Y' \to X'$ is a monomorphism. Then p is the dual of a deformation retract with respect to $\Omega_{\mathbf{A}}$.

Proof. Let 0 denote the initial object of $(\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$. The unique map $0 \to X$ is a monomorphism. By assumption, there is a lifting $s\colon X \to Y$ in

$$\begin{array}{ccc} 0 & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{\mathrm{id}} & Y. \end{array} \tag{9}$$

Let $Y+Y$ denote the coproduct of Y with itself and let $\delta_0 + \delta_1 : Y + Y \to Y \times \Omega_{\mathbf{A}}$ be the map which agrees with δ_0 on the first summand of the coproduct and with δ_1 on the second. Similarly, let $s \circ p + \mathrm{id}_Y : Y + Y \to Y$ be the map which coincides with $s \circ p$ on the first summand and with id_Y on the second. Let $\pi_Y : Y \times \Omega_{\mathbf{A}} \to Y$ denote the projection. Then there is a lifting $\mathbf{H}\colon Y \times \Omega_{\mathbf{A}} \to Y$ in

$$\begin{array}{ccc} Y+Y & \xrightarrow{s\circ p+\mathrm{id}_Y} & Y \\ {\scriptstyle \delta_0+\delta_1}\downarrow & & \downarrow{\scriptstyle p} \\ Y \times \Omega_{\mathbf{A}} & \xrightarrow{p\circ\pi_Y} & X \end{array} \tag{10}$$

as $\delta_0 + \delta_1$ is a monomorphism. $\mathbf{H}$ exhibits p as the dual of a deformation retract with respect to $\Omega_{\mathbf{A}}$ □

3.8 Corollary *Let $f\colon Y \to X$ be a map in $(\mathbf{A}^{\mathrm{op}}, Sets)$. Then f can be factored in the form $f = p \circ i$ where p is the dual of a deformation retract with respect to $\Omega_{\mathbf{A}}$ and i is a monomorphism.*

Proof. As the comma category

$$(\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})/X$$

is a topos, it has enough injectives. Embed f in an injective object of $(\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})/X$:

$$\begin{array}{ccc} Y & \xrightarrow{i} & Y' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{\mathrm{id}} & X. \end{array} \tag{11}$$

By virtue of the definition of injectivity, p has the right lifting property with respect to all monomorphisms. By 3.8, p is the dual of a deformation retract □

3.9 Corollary *Let $p\colon Y \to X$ be a map which has the right lifting property with respect to monomorphisms which are in $\Sigma_{\mathbf{A}}$. Assume in addition that $p \in \Sigma_{\mathbf{A}}$. Then p is the dual of a deformation retract with respect to $\Omega_{\mathbf{A}}$.*

Proof. As before, embed p in an injective object of the topos $(\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})/X$:

$$\begin{array}{ccc} Y & \xrightarrow{i} & Y' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow{\mathrm{id}} & X. \end{array} \tag{12}$$

As both p and p' are in $\Sigma_{\mathbf{A}}$, $i \in \Sigma_{\mathbf{A}}$ also. By assumption, there is a lifting $r\colon Y' \to Y$ in

$$\begin{array}{ccc} Y & \xrightarrow{\mathrm{id}} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{p'} & X. \end{array} \tag{13}$$

But then p is the retract of a map which has the right lifting property with respect to all monomorphisms. It is easily verified that p must have the same property □

3.10 Proposition *Suppose that $f\colon Y \to X$ has the left lifting property with respect to all duals of deformation retracts: there is a lifting in every commutative square of the form*

$$\begin{array}{ccc} Y & \xrightarrow{q} & Y' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{r} & X' \end{array} \tag{14}$$

where p is the dual of a deformation retract with respect to $\Omega_{\mathbf{A}}$. Then f is a monomorphism.

Proof. By 3.8, the unique map $Y \to 1$ can be factored in the form $p \circ i\colon Y \to Y' \to 1$ where i is a monomorphism and p is the dual of a deformation retract. By assumption, there is a lifting $g\colon X \to Y'$ in

$$\begin{array}{ccc} Y & \xrightarrow{i} & Y' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ X & \longrightarrow & 1. \end{array} \tag{15}$$

As $i = g \circ f$ is a monomorphism, f is a monomorphism □

Therefore, monomorphisms are characterized by the left lifting property with respect to duals of deformation retracts and duals of deformation retracts are characterized by the right lifting property with respect to monomorphisms. In the axiomatic development of homotopy theory in $(\mathbf{A}^{op}, \text{Sets})$ using Quillen's notion of a model category structure [4], cofibrations are monomorphisms and acyclic fibrations are duals of deformation retracts with respect to $\Omega_{\mathbf{A}}$. This line of thought is pursued in [3].

4 Auxiliary results on acyclic categories

The results of this § are needed for the construction of homotopy colimits in $(\mathbf{A}^{op}, \text{Sets})$ in case $\mathbf{A}$ is a category of models.

4.1 Lemma *Let* $\iota_{\mathbf{A}*} : (\mathbf{B}^{op} \times \mathbf{A}^{op}, Sets) \to (\mathbf{B}^{op}, \mathcal{CAT})$ *be the functor on diagrams induced by* $\iota_{\mathbf{A}} : \iota_{\mathbf{A}*}(X)(B) = \iota_{\mathbf{A}}(X(B, \cdot))$ *for* $B \in \mathbf{B}$. *The the following diagram commutes:*

$$\begin{array}{ccc} (\mathbf{B}^{op} \times \mathbf{A}^{op}, Sets) & \xrightarrow{\iota_{\mathbf{A}*}} & (\mathbf{B}^{op}, \mathcal{CAT}) \\ {\scriptstyle \iota_{\mathbf{A}\times\mathbf{B}}} \downarrow & & \downarrow {\scriptstyle (\cdot)\int \mathbf{B}} \\ \mathcal{CAT} & \xrightarrow{\text{id}} & \mathcal{CAT} \end{array} \qquad \square \tag{16}$$

4.2 Lemma *Let* $\theta \colon X \to Y$ *be a* pointwise weak equivalence *in the category of* $\mathbf{B}$*-indexed diagrams*

$$(\mathbf{B}^{op}, (\mathbf{A}^{op}, Sets)),$$

i.e. $\theta_B : X(B, \cdot) \to Y(B, \cdot) \in \Sigma_{\mathbf{A}}$ *for each* $B \in \mathbf{B}$. *Then, making the identification of*

$$(\mathbf{B}^{op}, (\mathbf{A}^{op}, Sets))$$

with

$$(\mathbf{B}^{op} \times \mathbf{A}^{op}, Sets)$$

it follows that $\theta \in \Sigma_{\mathbf{B}\times\mathbf{A}}$.

Proof. By assumption, $\iota_{\mathbf{A}}(\theta_B)$ is a weak equivalence in $\mathcal{CAT}$ for all $B \in \mathbf{B}$. By the homotopy invariance of the Grothendieck construction, this implies that

$$((\iota_{\mathbf{A}})_* \theta) \int \mathbf{B}$$

is a weak equivalence. But, by 4.1, $((\iota_{\mathbf{A}})_* \theta) \int \mathbf{B} = \iota_{\mathbf{B}\times\mathbf{A}}(\theta)$ $\square$

Let $X \in (\mathbf{A}^{op} \times \mathbf{A}^{op}, \text{Sets})$. Let

$$\Delta^* : (\mathbf{A}^{op} \times \mathbf{A}^{op}, \text{Sets}) \to (\mathbf{A}^{op}, \text{Sets})$$

be the functor induced by the diagonal $\Delta\colon \mathbf{A} \to \mathbf{A}\times\mathbf{A}$. For each $X \in (\mathbf{A}^{\mathrm{op}} \times \mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$ there is a natural map $\mathbf{F}\colon \iota_{\mathbf{A}}(\Delta^* X) \to \iota_{\mathbf{A}\times\mathbf{A}}(X)$ defined on objects by $(A, x) \mapsto ((A, A), x)$. Evidently,

$$\begin{array}{ccc} \iota_{\mathbf{A}}(\Delta^* X) & \xrightarrow{\mathbf{F}} & \iota_{\mathbf{A}\times\mathbf{A}}(X) \\ \downarrow & & \downarrow \\ \mathbf{A} & \xrightarrow{\Delta} & \mathbf{A}\times\mathbf{A} \end{array} \tag{17}$$

commutes.

4.3 Lemma (17) *is a pullback* □

4.4 Proposition *Let* $\mathbf{A}$ *be an acyclic category. For all* $X \in (\mathbf{A}^{\mathrm{op}} \times \mathbf{A}^{\mathrm{op}}, Sets)$, *the map* $\mathbf{F}\colon \iota_{\mathbf{A}}(\Delta^* X) \to \iota_{\mathbf{A}\times\mathbf{A}}(X)$ *is a weak equivalence.*

Proof. The pullback of a map with homotopically trivial fibres along a Grothendieck fibration is a weak equivalence □

4.5 Corollary *Let* $\mathbf{A}$ *be an acyclic category. Then* $\Delta^*(\Sigma_{\mathbf{A}\times\mathbf{A}}) \subseteq \Sigma_{\mathbf{A}}$.

Proof. Let $\theta\colon X \to Y \in \Sigma_{\mathbf{A}\times\mathbf{A}}$. There is a commutative diagram

$$\begin{array}{ccc} \iota_{\mathbf{A}}(\Delta^* X) & \longrightarrow & \iota_{\mathbf{A}\times\mathbf{A}}(X) \\ {\scriptstyle \iota_{\mathbf{A}}(\Delta^*\theta)}\downarrow & & {\scriptstyle \iota_{\mathbf{A}\times\mathbf{A}}(\theta)}\downarrow \\ \iota_{\mathbf{A}}(\Delta^* Y) & \longrightarrow & \iota_{\mathbf{A}\times\mathbf{A}}(Y) \end{array} \tag{18}$$

where the horizontal maps are the natural maps considered previously. By assumption $\iota_{\mathbf{A}\times\mathbf{A}}(\theta)$ is a weak equivalence. By 4.4, both horizontal arrows are weak equivalences. It follows from the saturation properties of the class of weak equivalences that $\iota_{\mathbf{A}}(\Delta^*\theta)$ is a weak equivalence □

4.6 Corollary *Let* $\mathbf{A}$ *be an acyclic category. If* $\theta\colon X \to Y$ *is a pointwise weak equivalence in* $(\mathbf{A}^{\mathrm{op}} \times \mathbf{A}^{\mathrm{op}}, Sets)$, $\Delta^*(\theta) \in \Sigma_{\mathbf{A}}$.

Proof. By 4.2, $\theta \in \Sigma_{\mathbf{A}\times\mathbf{A}}$. Now apply 4.5 □

4.7 Proposition *Let* $\mathbf{A}$ *be weakly contractible. If* $\Delta^*(\Sigma_{\mathbf{A}\times\mathbf{A}}) \subseteq \Sigma_{\mathbf{A}}$, *then* $\mathbf{A}$ *is acyclic.*

Proof. In virtue of the weak contractibility of $\mathbf{A}$, $\mathbf{A}\times\mathbf{A}$ is also weakly contractible. Let $\pi_1 : \mathbf{A}\times\mathbf{A} \to \mathbf{A}$ ($\pi_2 : \mathbf{A}\times\mathbf{A} \to \mathbf{A}$) denote the projection on the first (second) factor. Let X and $Y \in (\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$. It is readily verified that $\iota_{\mathbf{A}\times\mathbf{A}}(\pi_1^*(X) \times \pi_2^*(Y))$ is naturally isomorphic to $\iota_{\mathbf{A}}(X) \times \iota_{\mathbf{A}}(Y)$. Let $X = \mathbf{A}(\cdot, A)$ and $Y = \mathbf{A}(\cdot, B)$ be representable presheaves. Then $\pi_1^*(\mathbf{A}(\cdot,A)) \times \pi_2^*(\mathbf{A}(\cdot,B)) \to 1 \in \Sigma_{\mathbf{A}\times\mathbf{A}}$.For, by the previous remark, $\iota_{\mathbf{A}\times\mathbf{A}}(\pi_1^*(\mathbf{A}(\cdot,A)) \times \pi_2^*(\mathbf{A}(\cdot,B)) \to 1)$ is isomorphic to

$$\iota_{\mathbf{A}}(\mathbf{A}(\cdot,A)) \times \iota_{\mathbf{A}}(\mathbf{A}(\cdot,B)) \to \mathbf{A}\times\mathbf{A}) = \mathbf{A}/A \times \mathbf{A}/B \to \mathbf{A}\times\mathbf{A}$$

and $\mathbf{A}\times\mathbf{A}$ is weakly contractible. By hypothesis, $\Delta^*(\pi_1^*(\mathbf{A}(\cdot,A)) \times \pi_2^*(\mathbf{A}(\cdot,B)) \to 1) \in \Sigma_{\mathbf{A}}$. Since Δ^* is a right adjoint, $\Delta^*(1)$ is isomorphic to 1 and $\Delta^*(\pi_1^*(\mathbf{A}(\cdot,A)) \times \pi_2^*(\mathbf{A}(\cdot,B)))$ is isomorphic to

$$\Delta^*(\pi_1^*(\mathbf{A}(\cdot,A))) \times \Delta^*(\pi_2^*(\mathbf{A}(\cdot,B))) = \mathbf{A}(\cdot,A) \times \mathbf{A}(\cdot,B).$$

Hence $\mathbf{A}(\cdot,A) \times \mathbf{A}(\cdot,B) \to 1 \in \Sigma_{\mathbf{A}}$ for all objects A and B of $\mathbf{A}$. Consequently, by 3.5, $\mathbf{A}$ is acyclic provided it is weakly contractible □

5 Homotopy colimits in $(\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$

Let $\mathbf{C}$ be a (small) category. Let $\mathbf{R}_{\mathbf{C}}\colon (\mathbf{C},(\mathbf{A}^{\mathrm{op}},\mathrm{Sets})) \to (\mathbf{A}^{\mathrm{op}},\mathrm{Sets})$ denote the composite $\Delta^* \circ (j_{\mathbf{A}})_* \circ (\iota_{\mathbf{C}^{\mathrm{op}}})_*$:

$$\begin{array}{ccc} (\mathbf{C},(\mathbf{A}^{\mathrm{op}},\mathrm{Sets})) & \xrightarrow{\mathbf{R}_{\mathbf{C}}} & (\mathbf{A}^{\mathrm{op}},\mathrm{Sets}) \\ {\scriptstyle (\iota_{\mathbf{C}^{\mathrm{op}}})_*}\downarrow & & \uparrow{\scriptstyle \Delta^*} \\ (\mathbf{A}^{\mathrm{op}},\mathcal{CAT}) & \xrightarrow{(j_{\mathbf{A}})_*} & (\mathbf{A}^{\mathrm{op}}\times\mathbf{A}^{\mathrm{op}},\mathrm{Sets}). \end{array} \qquad (19)$$

5.1 Theorem *Let* $\mathbf{A}$ *be a category of models. If* $\theta\colon X \to Y \in (\mathbf{C},(\mathbf{A}^{\mathrm{op}},\mathit{Sets}))$ *is a pointwise weak equivalence, i.e.* $\theta_C\colon X(C,\cdot) \to Y(C,\cdot) \in \Sigma_{\mathbf{A}}$ *for every object* $C \in \mathbf{C}$*, then* $\mathbf{R}_{\mathbf{C}}(\theta) \in \Sigma_{\mathbf{A}}$.

Proof. It must be shown that $\iota_{\mathbf{A}}(\mathbf{R}_{\mathbf{C}}(\theta))$ is a weak equivalence in $\mathcal{CAT}$. Consider the diagram:

$$
\begin{array}{ccc}
\iota_{\mathbf{A}}\Delta^*(j_{\mathbf{A}})_*(\iota_{\mathbf{C}^{\mathrm{op}}})_*X & \xrightarrow{\mathbf{R}_{\mathbf{C}}(\theta)} & \iota_{\mathbf{A}}\Delta^*(j_{\mathbf{A}})_*(\iota_{\mathbf{C}^{\mathrm{op}}})_*Y \\
\downarrow & & \downarrow \\
\iota_{\mathbf{A}\times\mathbf{A}}(j_{\mathbf{A}})_*(\iota_{\mathbf{C}^{\mathrm{op}}})_*X & \longrightarrow & \iota_{\mathbf{A}\times\mathbf{A}}(j_{\mathbf{A}})_*(\iota_{\mathbf{C}^{\mathrm{op}}})_*Y \\
\downarrow & & \downarrow \\
(\iota_{\mathbf{A}})_*(j_{\mathbf{A}})_*(\iota_{\mathbf{C}^{\mathrm{op}}})_*X) \int \mathbf{A} & \longrightarrow & (\iota_{\mathbf{A}})_*(j_{\mathbf{A}})_*(\iota_{\mathbf{C}^{\mathrm{op}}})_*Y) \int \mathbf{A} \\
\varepsilon_*\downarrow & & \varepsilon_*\downarrow \\
(\iota_{\mathbf{C}^{\mathrm{op}}})_*X \int \mathbf{A} & \xrightarrow{(\iota_{\mathbf{C}^{\mathrm{op}}})_*\theta \int \mathbf{A}} & (\iota_{\mathbf{C}^{\mathrm{op}}})_*X \int \mathbf{A} \\
\downarrow & & \downarrow \\
\iota_{\mathbf{A}\times\mathbf{C}^{\mathrm{op}}}X & \xrightarrow{\iota_{\mathbf{A}\times\mathbf{C}^{\mathrm{op}}}\theta} & \iota_{\mathbf{A}\times\mathbf{C}^{\mathrm{op}}}Y \\
\uparrow & & \uparrow \\
(\iota_{\mathbf{A}})_*X \int \mathbf{C}^{\mathrm{op}} & \xrightarrow{(\iota_{\mathbf{A}})_*\theta \int \mathbf{C}^{\mathrm{op}}} & (\iota_{\mathbf{A}})_*Y \int \mathbf{C}^{\mathrm{op}}.
\end{array}
\tag{20}
$$

By assumption, $\theta_C\colon X(C,\cdot) \to Y(C,\cdot) \in \Sigma_{\mathbf{A}}$ for every object $C \in \mathbf{C}$. By definition, $(\iota_{\mathbf{A}})_*(\theta)$ is a pointwise weak equivalence in $(\mathbf{C}, \mathcal{CAT})$. By the homotopy invariance of the Grothendieck construction, this implies that $(\iota_{\mathbf{A}})_*(\theta) \int \mathbf{C}^{\mathrm{op}}$ is a weak equivalence. By 4.1,

$$
\begin{aligned}
((\iota_{\mathbf{A}})_*(\theta)) \int \mathbf{C}^{\mathrm{op}} &= (\iota_{\mathbf{A}\times\mathbf{C}^{\mathrm{op}}})(\theta) \\
&= ((\iota_{\mathbf{C}^{\mathrm{op}}})_*(\theta)) \int \mathbf{A}.
\end{aligned}
$$

Consequently, $((\iota_{\mathbf{C}^{\mathrm{op}}})_*(\theta)) \int \mathbf{A}$ is a weak equivalence. As $\mathbf{A}$ is acyclic and $\Omega_{\mathbf{A}}$ is a homotopy interval, the components of the counit $\varepsilon\colon \iota_{\mathbf{A}} \circ j_{\mathbf{A}} \to 1$ are weak equivalences. This implies that the components of $\varepsilon_*\colon (\iota_{\mathbf{A}})_* \circ (j_{\mathbf{A}})_* \to 1$ are pointwise weak equivalences. By the homotopy invariance of the Grothendieck construction, $(\varepsilon_*((\iota_{\mathbf{C}^{\mathrm{op}}})_*(X))) \int \mathbf{A}$ and $(\varepsilon_*((\iota_{\mathbf{C}^{\mathrm{op}}})_*(Y))) \int \mathbf{A}$ are weak equivalences. By saturation, $((\iota_{\mathbf{A}})_* \circ (j_{\mathbf{A}})_* \circ (\iota_{\mathbf{C}^{\mathrm{op}}})_*(\theta)) \int \mathbf{A}$ is a weak equivalence. By 4.1, $\iota_{\mathbf{A}\times\mathbf{A}}((j_{\mathbf{A}})_* \circ (\iota_{\mathbf{C}^{\mathrm{op}}})_*(\theta))$ is a weak equivalence. Since $\mathbf{A}$ is acyclic, 4.4 applies: there are natural weak equivalences

$$
\iota_{\mathbf{A}} \circ \Delta^*((j_{\mathbf{A}})_* \circ (\iota_{\mathbf{C}^{\mathrm{op}}})_*(X) \to \iota_{\mathbf{A}\times\mathbf{A}}((j_{\mathbf{A}})_* \circ (\iota_{\mathbf{C}^{\mathrm{op}}})_*(X))
$$

and

$$\iota_{\mathbf{A}} \circ \Delta^*((j_{\mathbf{A}})_* \circ (\iota_{\mathbf{C}^{op}})_*(Y) \rightarrow \iota_{\mathbf{A}\times\mathbf{A}}((j_{\mathbf{A}})_* \circ (iota_{\mathbf{C}^{op}})_*(Y)).$$

By saturation, $\mathbf{R_C}(\theta)$ is a weak equivalence □

Assume for the remainder of this § that **A** is a category of models. By the argument used to establish 5.1, the square

$$\begin{array}{ccc} (\mathbf{C},(\mathbf{A}^{op},\mathrm{Sets})) & \xrightarrow{\mathbf{R_C}} & (\mathbf{A}^{op},\mathrm{Sets}) \\ {\scriptstyle (\iota_{\mathbf{A}})_*}\downarrow & & \downarrow{\scriptstyle \iota_{\mathbf{A}}} \\ (\mathbf{C},\mathcal{CAT}) & \xrightarrow{(\cdot)\int \mathbf{C}^{op}} & \mathcal{CAT} \end{array} \tag{21}$$

commutes up to natural weak equivalence. As $\mathbf{R_C}$ preserves weak equivalences, the corresponding diagram of homotopy categories

$$\begin{array}{ccc} \Sigma^{-1}(\mathbf{C},(\mathbf{A}^{op},\mathrm{Sets})) & \xrightarrow{\Sigma^{-1}\mathbf{R_C}} & \Sigma^{-1}(\mathbf{A}^{op}) \\ {\scriptstyle \Sigma^{-1}(\iota_{\mathbf{A}})_*}\downarrow & & \downarrow{\scriptstyle \Sigma^{-1}\iota_{\mathbf{A}}} \\ \Sigma^{-1}(\mathbf{C},\mathcal{CAT}) & \xrightarrow{\Sigma^{-1}(\cdot)\int \mathbf{C}^{op}} & \Sigma^{-1}\mathcal{CAT} \end{array} \tag{22}$$

commutes up to natural isomorphism. As $\iota_{\mathbf{A}}$ and $j_{\mathbf{A}}$ induce adjoint equivalences

$$\Sigma^{-1}\iota_{\mathbf{A}} \dashv \Sigma^{-1}j_{\mathbf{A}} : \Sigma^{-1}(\mathbf{A}^{op},\mathrm{Sets}) \rightarrow \Sigma^{-1}\mathcal{CAT},$$

$(\iota_{\mathbf{A}})_*$ and $(j_{\mathbf{A}})_*$ induce adjoint equivalences

$$\Sigma^{-1}(\iota_{\mathbf{A}})_* \dashv \Sigma^{-1}(j_{\mathbf{A}})_* : \Sigma^{-1}(\mathbf{C},(\mathbf{A}^{op},\mathrm{Sets})) \rightarrow \Sigma^{-1}(\mathbf{C},\mathcal{CAT})$$

for every category **C**. For any diagram $X \in (\mathbf{C},\mathcal{CAT})$, $\mathbf{C}\int X = (X \int \mathbf{C}^{op})^{op}$. Since, for any category **A**, **Nerve(A)** is weakly equivalent to $\mathbf{Nerve(A^{op})}$, there is a natural isomorphism in $\Sigma^{-1}\mathcal{CAT}$ between $\mathbf{C}\int X$ and $X \int \mathbf{C}^{op}$. By Thomason's theorem, there is an adjunction

$$\Sigma^{-1}(\mathbf{C}\int \cdot) \dashv \Sigma^{-1}(\pi_{\mathbf{C}}^*) : (\mathbf{C},\mathcal{CAT}) \rightarrow \mathcal{CAT}$$

where $\pi_{\mathbf{C}}^*$ is the functor induced by the unique map $\mathbf{C} \rightarrow 1 : \pi_{\mathbf{C}}^*(\mathbf{B})(X) = \mathbf{B}$ for all categories **B** and diagrams X. Consequently, there is an adjunction $\Sigma^{-1}(\cdot \int \mathbf{C}^{op}) \dashv \Sigma^{-1}(\pi_{\mathbf{C}}^*)$. It is readily verified that the following diagram commutes on the nose:

$$\begin{array}{ccc} (\mathbf{C},(\mathbf{A}^{op},\mathrm{Sets})) & \xleftarrow{\pi_{\mathbf{C}}^*} & (\mathbf{A}^{op},\mathrm{Sets}) \\ {\scriptstyle (j_{\mathbf{A}})_*}\uparrow & & \uparrow{\scriptstyle j_{\mathbf{A}}} \\ (\mathbf{C}^{op},\mathcal{CAT}) & \xleftarrow{\pi\mathbf{C}^*} & \mathcal{CAT}. \end{array} \tag{23}$$

Since **A** is a category of models, the vertical functors in (23) preserve weak equivalences. Consequently, (23) descends to the derived categories:

$$\begin{array}{ccc} \Sigma^{-1}(\mathbf{C},(\mathbf{A}^{\mathrm{op}},\mathrm{Sets})) & \xleftarrow{\Sigma^{-1}\pi^*_{\mathbf{C}}} & \Sigma^{-1}(\mathbf{A}^{\mathrm{op}},\mathrm{Sets}) \\ {\scriptstyle \Sigma^{-1}(j_{\mathbf{A}})_*}\uparrow & & {\scriptstyle \Sigma^{-1}j_{\mathbf{A}}}\uparrow \\ \Sigma^{-1}(\mathbf{C}^{\mathrm{op}},\mathcal{CAT}) & \xleftarrow{\Sigma^{-1}\pi\mathbf{C}^*} & \Sigma^{-1}\mathcal{CAT}. \end{array} \tag{24}$$

The next lemma is required for the proof of the main result of this §.

5.2 Lemma *Given diagrams*

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\mathbf{F}} & \mathbf{B} \\ {\scriptstyle \mathbf{G}}\downarrow & & {\scriptstyle \mathbf{H}}\downarrow \\ \mathbf{C} & \xrightarrow{\mathbf{K}} & \mathbf{D} \end{array} \tag{25}$$

and

$$\begin{array}{ccc} \mathbf{A} & \xleftarrow{\mathbf{N}} & \mathbf{B} \\ {\scriptstyle \mathbf{L}}\uparrow & & {\scriptstyle \mathbf{M}}\uparrow \\ \mathbf{C} & \xleftarrow{\mathbf{J}} & \mathbf{D} \end{array} \tag{26}$$

in $\mathcal{CAT}$ subject to the following conditions:

(1) **L** *and* **G** *are quasi-inverse equivalences.*
(2) **M** *and* **H** *are quasi-inverse equivalences.*
(3) $\mathbf{K} \dashv \mathbf{J}$.
(4) *There is a natural isomorphism* $\mathbf{K} \circ \mathbf{G} \Rightarrow \mathbf{H} \circ \mathbf{F}$.
(5) $\mathbf{L} \circ \mathbf{J} = \mathbf{N} \circ \mathbf{M}$.

Then **F** *is left adjoint to* **G** □

5.3 Theorem $\Sigma^{-1}(\mathbf{R_C}) \dashv \Sigma^{-1}(\pi^*_{\mathbf{C}})$.

Proof. Apply 5.2 to (22) and (24) □

The preceding theorem validates the identification

$$\mathbf{R_C}(X) \equiv \mathrm{hocolim}_{\mathbf{C}}(X)$$

for any diagram $X : \mathbf{C} \to (\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$.

6 Properties of homotopy colimits

From the proof of 5.1, it is immediate that, if **A** is a category of models, $\iota_{\mathbf{A}}$ commutes with $\mathrm{hocolim}_{\mathbf{C}}$: there is a natural isomorphism

$$\iota_{\mathbf{A}}(\mathrm{hocolim}_{\mathbf{C}}(X)) \to ((\iota_{\mathbf{A}})_*(X)) \int \mathbf{C}^{\mathrm{op}}$$

in $\Sigma^{-1}\mathcal{CAT}$. The next proposition establishes the analogous property of $j_{\mathbf{A}}$.

6.1 Proposition *Let* $\mathbf{A}$ *be a category of models. The following square*

$$\begin{array}{ccc} (\mathbf{C}, \mathcal{CAT}) & \xrightarrow{(\cdot)\int \mathbf{C}^{\mathrm{op}}} & \mathcal{CAT} \\ {\scriptstyle (j_{\mathbf{A}})_*}\downarrow & & \downarrow{\scriptstyle j_{\mathbf{A}}} \\ (\mathbf{C}, (\mathbf{A}^{\mathrm{op}}, Sets)) & \xrightarrow{\mathrm{hocolim}_{\mathbf{C}}} & (\mathbf{A}^{\mathrm{op}}, Sets) \end{array} \qquad (27)$$

commutes up to natural weak equivalence.

Proof. Let $X \in (\mathbf{C}, (\mathbf{A}^{\mathrm{op}}, \mathrm{Sets}))$. It has already been noted that there is a natural weak equivalence

$$\iota_{\mathbf{A}}\mathrm{hocolim}_{\mathbf{C}}(j_{\mathbf{A}})_*(X) \to (\iota_{\mathbf{A}})_*(j_{\mathbf{A}})_*(X) \int \mathbf{C}^{\mathrm{op}}.$$

As $\varepsilon_*(X)$ is a weak equivalence, $\varepsilon_*(X) \int \mathbf{C}^{\mathrm{op}}$ is also. By composition, this yields a weak equivalence

$$\iota_{\mathbf{A}}\mathrm{hocolim}_{\mathbf{C}}(j_{\mathbf{A}})_*(X) \to X \int \mathbf{C}^{\mathrm{op}}.$$

As $j_{\mathbf{A}}$ preserves weak equivalences, application of $j_{\mathbf{A}}$ to the preceding map yields a natural weak equivalence

$$j_{\mathbf{A}}\iota_{\mathbf{A}}\mathrm{hocolim}_{\mathbf{C}}(j_{\mathbf{A}})_*(X) \to j_{\mathbf{A}}(X \int \mathbf{C}^{\mathrm{op}}).$$

As the components of the unit $\eta\colon 1 \to j_{\mathbf{A}}\iota_{\mathbf{A}}$ are weak equivalences, one obtains the desired weak equivalence

$$\mathrm{hocolim}_{\mathbf{C}}(j_{\mathbf{A}})_*(X) \to j_{\mathbf{A}}(X \int \mathbf{C}^{\mathrm{op}}) \quad \Box$$

Let $\mathbf{F}\colon \mathbf{C} \to \mathbf{D}$ be a functor. For every $X \in (\mathbf{D}, (\mathbf{A}^{\mathrm{op}}, \mathrm{Sets}))$ and $A \in \mathbf{A}$ there is a natural map $\mathbf{G}\colon \iota_{\mathbf{C}^{\mathrm{op}}}X(\mathbf{F}(\cdot), A) \to \iota_{\mathbf{D}^{\mathrm{op}}}X(\cdot, A)$ defined on objects by $(C, x) \mapsto (\mathbf{F}(X), x)$.

6.2 Lemma $\mathbf{G}$ *fits into a pullback diagram*

$$\begin{array}{ccc} \iota_{\mathbf{C}^{\mathrm{op}}}(X(\mathbf{F}(\cdot), A)) & \xrightarrow{\mathbf{G}} & \iota_{\mathbf{C}^{\mathrm{op}}}(X(\cdot, A)) \\ \downarrow & & \downarrow \\ \mathbf{C}^{\mathrm{op}} & \xrightarrow{\mathbf{F}^{\mathrm{op}}} & \mathbf{D}^{\mathrm{op}} \end{array} \qquad (28)$$

where the vertical arrows are the projections □

6.3 Lemma *Let* $\mathbf{F}: \mathbf{C} \to \mathbf{D}$ *be a functor. If* $\mathbf{F}$ *has weakly contractible homotopy fibres, i.e.* $D/\mathbf{F}$ *is weakly contractible for all objects* $D \in \mathbf{D}$*, then the induced functor* $\iota_{\mathbf{C}^{\mathrm{op}}} X(\mathbf{F}(\cdot), A) \to \iota_{\mathbf{D}^{\mathrm{op}}} X(\cdot, A)$ *is a weak equivalence for all objects* $A \in \mathbf{A}$.

Proof. By assumption, $\mathbf{F}^{\mathrm{op}}/D = D/\mathbf{F}$ is weakly contractible for all objects $D \in \mathbf{D}$. As $\iota_{\mathbf{D}^{\mathrm{op}}} X(\cdot, A) \to \mathbf{D}^{\mathrm{op}}$ is a fibration and (28) is a pullback, $\mathbf{G}$ is a weak equivalence □

6.4 Proposition *Let* $\mathbf{A}$ *be a category of models. Let* $\mathbf{F}: \mathbf{C} \to \mathbf{D}$ *be a functor with weakly contractible homotopy fibres. Then the diagram*

$$\begin{array}{ccc} (\mathbf{C}, (\mathbf{A}^{\mathrm{op}}, \mathit{Sets})) & \xrightarrow{\mathbf{F}^*} & (\mathbf{D}, (\mathbf{A}^{\mathrm{op}}, \mathit{Sets})) \\ {\scriptstyle \mathrm{hocolim}_{\mathbf{C}}} \downarrow & & \downarrow {\scriptstyle \mathrm{hocolim}_{\mathbf{D}}} \\ (\mathbf{A}^{\mathrm{op}}, \mathit{Sets}) & \xrightarrow{\mathrm{id}} & (\mathbf{A}^{\mathrm{op}}, \mathit{Sets}) \end{array} \tag{29}$$

commutes up to natural weak equivalence.

Proof. By 6.3, there is a natural weak equivalence

$$(\iota_{\mathbf{C}^{\mathrm{op}}})_* \mathbf{F}^*(X) \to (\iota_{\mathbf{D}^{\mathrm{op}}})_*(X)$$

in $(\mathbf{A}^{\mathrm{op}}, \mathcal{CAT})$ for every diagram $X \in (\mathbf{D}, (\mathbf{A}^{\mathrm{op}}, \mathrm{Sets}))$. Hence there is a pointwise weak equivalence

$$(j_{\mathbf{A}})_*(\iota_{\mathbf{C}^{\mathrm{op}}})_* \mathbf{F}^*(X) \to (j_{\mathbf{A}})_*(\iota_{\mathbf{D}^{\mathrm{op}}})_*(X)$$

in $(\mathbf{A}^{\mathrm{op}} \times \mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$. By 4.6, since $\mathbf{A}$ is acyclic, Δ^* carries pointwise weak equivalences into $\Sigma_{\mathbf{A}}$. By definition,

$$\mathrm{hocolim}_{\mathbf{C}} \equiv \Delta^*(j_{\mathbf{A}})_*(\iota_{\mathbf{C}^{\mathrm{op}}})_*$$

and

$$\mathrm{hocolim}_{\mathbf{D}} \equiv \Delta^*(j_{\mathbf{A}})_*(\iota_{\mathbf{D}^{\mathrm{op}}})_*.$$

This establishes the commutatvity of (29) up to natural weak equivalence □

In the special case that the indexing category $\mathbf{C} = \mathbf{A}$, the homotopy colimit admits a simple description.

6.5 Proposition *Let* $\mathbf{A}$ *be a category of models. Then for each presheaf*

$$X \in (\mathbf{A}^{op} \times \mathbf{A}^{op}, Sets)$$

there is a natural weak equivalence $\Delta^*(X) \to hocolim_{\mathbf{A}}(X)$.

Proof. By 4.1 and the remark following the proof of 5.1 ,there is a natural isomorphism $\iota_{\mathbf{A}}\text{hocolim}_{\mathbf{A}}(X) \to \iota_{\mathbf{A}\times\mathbf{A}}(X)$ in $\Sigma^{-1}\mathcal{CAT}$. By 4.4, there is a natural weak equivalence $\iota_{\mathbf{A}}\Delta^*(X) \to \iota_{\mathbf{A}\times\mathbf{A}}(X)$ □

The next proposition concerns transport of homotopy colimits along certain functors $\mathbf{P}\colon \mathbf{A} \to \mathbf{B}$ between categories of models $\mathbf{A}$ and $\mathbf{B}$. Let

$$\mathbf{P}^*\colon (\mathbf{C}, (\mathbf{B}^{op}, \text{Sets})) \to (\mathbf{C}, (\mathbf{A}^{op}, \text{Sets}))$$

be the functor induced by $\mathbf{P} : \mathbf{P}^*(X)(C, A) = X(C, \mathbf{P}(A))$. There is a natural map

$$(\iota_{\mathbf{C}^{op}})_*\mathbf{P}^*X \int \mathbf{A} \to (\iota_{\mathbf{C}^{op}})_*X \int \mathbf{B}$$

defined on objects by $(A \in \mathbf{A}, C \in \mathbf{C}, x \in X(C, \mathbf{P}(A))) \mapsto (\mathbf{P}(A), C, x)$. It is easily verified that

$$\begin{array}{ccc} (\mathbf{C}, (\mathbf{B}^{op}, \text{Sets})) & \xrightarrow{\mathbf{P}^*} & (\mathbf{C}, (\mathbf{A}^{op}, \text{Sets})) \\ {\scriptstyle (\iota_{\mathbf{C}^{op}})_*}\downarrow & & {\scriptstyle (\iota_{\mathbf{C}^{op}})_*}\downarrow \\ (\mathbf{B}^{op}, \mathcal{CAT}) & \xrightarrow{\mathbf{P}^*} & (\mathbf{A}^{op}, \mathcal{CAT}) \end{array} \qquad (30)$$

commutes. Consequently,

$$\begin{array}{ccc} (\iota_{\mathbf{C}^{op}})_*\mathbf{P}^*X \int \mathbf{A} & \longrightarrow & (\iota_{\mathbf{C}^{op}})_*\mathbf{P}^*X \int \mathbf{B} \\ \downarrow & & \downarrow \\ \mathbf{A} & \xrightarrow{\mathbf{P}} & \mathbf{B} \end{array} \qquad (31)$$

is a pullback for every $X \in (\mathbf{C}, (\mathbf{B}^{op}, \text{Sets})$. For, by commutativity of (30),

$$(\iota_{\mathbf{C}^{op}})_*\mathbf{P}^*(X) \int \mathbf{A} = \mathbf{P}^*((\iota_{\mathbf{C}^{op}})_*(X)) \int \mathbf{A}$$

and

$$\begin{array}{ccc} \mathbf{P}^*((\iota_{\mathbf{C}^{op}})_*X) \int \mathbf{A} & \longrightarrow & (\iota_{\mathbf{C}^{op}})_*X \int \mathbf{B} \\ \downarrow & & \downarrow \\ \mathbf{A} & \xrightarrow{\mathbf{P}} & \mathbf{A} \end{array} \qquad (32)$$

is a pullback.

6.6 Proposition *Let* $\mathbf{P}\colon \mathbf{A} \to \mathbf{B}$ *be a functor with weakly contractible homotopy fibres. Assume that* $\mathbf{A}$ *and* $\mathbf{B}$ *are model categories. Then there is a natural weak equivalence*

$$\iota_{\mathbf{A}}(hocolim_{\mathbf{C}}\mathbf{P}^*(X)) \to \iota_{\mathbf{B}}(hocolim_{\mathbf{C}}(X)).$$

Proof. Because (31) is a pullback, $(\iota_{\mathbf{C}^{\mathrm{op}}})_*(X)\int\mathbf{B} \to \mathbf{B}$ is a fibration, and $\mathbf{P}$ is a functor satisfying the hypotheses of Quillen's Theorem A,

$$(\iota_{\mathbf{C}^{\mathrm{op}}})_*\mathbf{P}^*(X)\int\mathbf{A} \to (\iota_{\mathbf{C}^{\mathrm{op}}})_*X\int\mathbf{B}$$

is a weak equivalence. By the argument that established 5.1, there are natural weak equivalences

$$\iota_{\mathbf{A}}\mathrm{hocolim}_{\mathbf{C}}\mathbf{P}^*(X) \to (\iota_{\mathbf{C}^{\mathrm{op}}})_*\mathbf{P}^*(X)\int\mathbf{A}$$

and

$$\iota_{\mathbf{B}}\mathrm{hocolim}_{\mathbf{C}}(X) \to (\iota_{\mathbf{C}^{\mathrm{op}}})_*X\int\mathbf{B} \quad \square$$

7 Homotopy colimits and coends

The homotopy colimit of a diagram $X\colon \mathbf{C} \to (\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$ can be described as a coend:

$$\begin{aligned}\mathrm{hocolim}_{\mathbf{C}}(X) &= \int^{\mathbf{C}} j_{\mathbf{A}}(\cdot/\mathbf{C})^{\mathrm{op}} \times X \\ &\equiv j_{\mathbf{A}}(\cdot/\mathbf{C})^{\mathrm{op}} \otimes X.\end{aligned}$$

The purpose of the present § is to establish this identity. Let $A \in \mathbf{A}, C \in \mathbf{C}$, and $X\colon \mathbf{C} \to (\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$.For each object $(\mathbf{F}, x)$ of

$$j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}(A) \times X(C,A) = \mathcal{CAT}(\mathbf{A}/A, (C/\mathbf{C})^{\mathrm{op}}) \times X(C,A),$$

define

$$\sigma_{(C,A)}(\mathbf{F})(x)\colon \mathbf{A}/A \to (\iota_{\mathbf{C}^{\mathrm{op}}})_*(X)(A)$$

on objects by

$$(f : B \to A) \mapsto (\mathrm{codomain}(\mathbf{F}(f)), X(\mathbf{F}(f), \mathrm{id}_A)(x)).$$

Extending to maps in the obvious way defines a functor $\sigma_{(C,A)}(\mathbf{F})(x)$. By definition, $\sigma_{(C,A)}(\mathbf{F})(x) \in \mathrm{hocolim}_{\mathbf{C}}(X)(A)$.

7.1 Lemma *For each $C \in \mathbf{C}$ and $f: A \to B \in \mathbf{A}$,*

$$\begin{array}{ccc} j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}(A) \times X(C,A) & \xrightarrow{\sigma(C,A)} & hocolim_{\mathbf{C}}X(A) \\ {\scriptstyle \mathrm{id}\times X(\mathrm{id}_C,f)}\uparrow & & \uparrow{\scriptstyle \mathrm{hocolim}_{\mathbf{C}}X(f)} \\ j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}(B) \times X(C,B) & \xrightarrow{\sigma(C,B)} & hocolim_{\mathbf{C}}X(B) \end{array} \qquad (33)$$

commutes. In other words,

$$\sigma_C : j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}(\cdot) \times X(C,\cdot) \to hocolim_{\mathbf{C}}X(\cdot)$$

is natural in $A \in \mathbf{A}$ □

7.2 Lemma *Let $g: C \to D \in \mathbf{C}$. Then*

$$\begin{array}{ccc} j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}(A) \times X(C,A) & \xrightarrow{\sigma(C,A)} & \mathrm{hocolim}_{\mathbf{C}}X(A) \\ {\scriptstyle j_{\mathbf{A}}(g/\mathbf{C})^{\mathrm{op}}\times\mathrm{id}}\uparrow & & \uparrow{\scriptstyle \sigma(D,A)} \\ j_{\mathbf{A}}(D/\mathbf{C})^{\mathrm{op}}(A) \times X(C,A) & \xrightarrow{\mathrm{id}\times X(g,\mathrm{id}_A)} & j_{\mathbf{A}}(D/\mathbf{C})^{\mathrm{op}}(A) \times X(D,A) \end{array} \qquad (34)$$

commutes for every $A \in \mathbf{A}$ □

For $C \in \mathbf{C}$, let $\alpha_C : j_{\mathbf{A}}(\cdot/\mathbf{C})^{\mathrm{op}} \times X(C,\cdot) \to \int^{\mathbf{C}} j_{\mathbf{A}}(\cdot/\mathbf{C})^{\mathrm{op}} \times X$ denote the canonical map. As a consequence of 7.1, 7.2, and the universal property of the coend

$$\int^{\mathbf{C}} j_{\mathbf{A}}(\cdot/\mathbf{C})^{\mathrm{op}} \times X,$$

there is a unique map $\sigma : \int^{\mathbf{C}} j_{\mathbf{A}}(\cdot/\mathbf{C})^{\mathrm{op}} \times X \to \mathrm{hocolim}_{\mathbf{C}}(X)$ such that

$$\begin{array}{ccc} \int^{\mathbf{C}} j_{\mathbf{A}}(\cdot/\mathbf{C})^{\mathrm{op}} \times X & \xrightarrow{\sigma} & \mathrm{hocolim}_{\mathbf{C}}X \\ {\scriptstyle \alpha_C}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ j_{\mathbf{A}}(\cdot/\mathbf{C})^{\mathrm{op}} \times X(C,\cdot) & \xrightarrow{\sigma_C} & \mathrm{hocolim}_{\mathbf{C}}X \end{array} \qquad (35)$$

commutes for every $C \in \mathbf{C}$.

7.3 Theorem $\{\sigma_C \mid C \in \mathbf{C}\}$ *is a universal dinatural transformation, i.e.*

$$hocolim_{\mathbf{C}}X$$

has the universal property of the coend

$$\int^{\mathbf{C}} j_{\mathbf{A}}(\cdot/\mathbf{C})^{\mathrm{op}} \times X.$$

Proof. The proof is a (lengthy) verification of the universal property. Details can be found in [3] □

Consequently, $\text{hocolim}_{\mathbf{C}}(X)$ admits two descriptions:

$$\begin{aligned}\text{hocolim}_{\mathbf{C}}X(A) &= \mathcal{CAT}(\mathbf{A}/A, \iota_{\mathbf{C}^{\text{op}}}(X(\cdot, A)) \\ &= \int^{\mathbf{C}} j_{\mathbf{A}}(\cdot/\mathbf{C})^{\text{op}} \times X(\cdot, A).\end{aligned}$$

The first description of $\text{hocolim}_{\mathbf{C}}$ is more amenable to direct calculation. This was illustrated in the previous § when, e.g., the connection between $\text{hocolim}_{\mathbf{C}}$ and the diagonal functor Δ^* was described. The second description also has its uses. For example,it is better adapted to theoretical investigations. By way of illustration, the adjunction

$$\Sigma^{-1}\text{hocolim}_{\mathbf{C}} \dashv \Sigma^{-1}\pi^*_{\mathbf{C}}$$

will be rederived using the second description of $\text{hocolim}_{\mathbf{C}}$. For objects X and Y of $(\mathbf{A}^{\text{op}}, \text{Sets})$,let $\mathcal{HOM}(X,Y)$ denote the internal hom–functor in $(\mathbf{A}^{\text{op}}, \text{Sets})$:

$$\mathcal{HOM}(X,Y)(A) = (\mathbf{A}^{\text{op}}, \text{Sets})(X \times \mathbf{A}(\cdot, A), Y).$$

The functor

$$\int^{\mathbf{C}} j_{\mathbf{A}}(\cdot/\mathbf{C})^{\text{op}} \times (\cdot) \colon (\mathbf{C}, (\mathbf{A}^{\text{op}}, \text{Sets}) \to (\mathbf{A}^{\text{op}}, \text{Sets})$$

is left adjoint to $\mathcal{HOM}(j_{\mathbf{A}}(\cdot/\mathbf{C})^{\text{op}}, \cdot)$.

7.4 Proposition *Let* $\mathbf{A}$ *be a category of models. Then, for each* $C \in \mathbf{C}$ *and* $X \in (\mathbf{A}^{\text{op}}, \textit{Sets})$ *,there is a natural weak equivalence*

$$\mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\text{op}}, X) \to X.$$

Proof. It will be shown that there is a map

$$\mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\text{op}}, X) \to X$$

which is the dual of a strong deformation retract with respect to $\Omega_{\mathbf{A}}$. Hence $\mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\text{op}}, X) \to X \in \Sigma_{\mathbf{A}}$. For each $C \in \mathbf{C}$,the comma category $(C/\mathbf{C})^{\text{op}}$ has a terminal object $(\text{id}_C \colon C \to C)$. Consequently, there is a homotopy

$$\mathbf{H} \colon (C/\mathbf{C})^{\text{op}} \times 2 \to (C/\mathbf{C})^{\text{op}}$$

such that

$$\begin{array}{ccccc}
(C/\mathbf{C})^{\mathrm{op}} & \xrightarrow{\mathrm{id}\times\delta_0} & (C/\mathbf{C})^{\mathrm{op}}\times 2 & \xleftarrow{\mathrm{id}\times\delta_1} & (C/\mathbf{C})^{\mathrm{op}} \\
{\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathbf{H}}\downarrow & & \downarrow \\
(C/\mathbf{C})^{\mathrm{op}} & \xrightarrow{\mathrm{id}} & (C/\mathbf{C})^{\mathrm{op}} & \longleftarrow & 1
\end{array} \tag{36}$$

commutes ($1 \to (C/\mathbf{C})^{\mathrm{op}}$ picks out the terminal object). Since $j_{\mathbf{A}}$ is a right adjoint, application of $j_{\mathbf{A}}$ to (36) gives a homotopy

$$\begin{array}{ccccc}
j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}} & \xrightarrow{\mathrm{id}\times\delta_0} & j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}\times \Omega_{\mathbf{A}} & \xleftarrow{\mathrm{id}\times\delta_1} & j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}} \\
{\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathbf{H}}\downarrow & & \downarrow \\
j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}} & \xrightarrow{\mathrm{id}} & j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}} & \longleftarrow & 1
\end{array} \tag{37}$$

with respect to $\Omega_{\mathbf{A}} = j_{\mathbf{A}}(2)$. Apply $\mathcal{HOM}(\cdot, X)$ to this diagram giving

$$\begin{array}{ccc}
\mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}, X) & \xleftarrow{\mathcal{HOM}(\mathrm{id}\times\delta_0, X)} & \mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}\times\Omega_{\mathbf{A}}, X) \\
{\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle \mathcal{HOM}(\mathbf{H},X)}\uparrow \\
\mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}, X) & \xleftarrow{\mathrm{id}} & \mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}, X)
\end{array} \tag{38}$$

and

$$\begin{array}{ccc}
\mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}\times\Omega_{\mathbf{A}}, X) & \xrightarrow{\mathcal{HOM}(\mathrm{id}\times\delta_1, X)} & \mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}, X) \\
{\scriptstyle \mathcal{HOM}(\mathbf{H},X)}\uparrow & & {\scriptstyle \sigma}\uparrow \\
\mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}, X) & \xrightarrow{\rho} & \mathcal{HOM}(1, X).
\end{array} \tag{39}$$

Since the composite $1 \to j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}} \to 1$ is evidently the identity, by functoriality,

$$\rho\circ\sigma\colon \mathcal{HOM}(1,X) \to \mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}, X) \to \mathcal{HOM}(1,X)$$

is also the identity. Let

$$\mathcal{HOM}(\mathbf{H},X)^{\sim}\colon \mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}, X)\times\Omega_{\mathbf{A}} \to \mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}, X)$$

correspond to $\mathcal{HOM}(\mathbf{H}, X)$ via the adjunction

$$\mathcal{HOM}(\Omega_{\mathbf{A}}, \cdot) \dashv \cdot\times\Omega_{\mathbf{A}}.$$

Using the natural isomorphisms

$$\mathcal{HOM}(1, X) \to X$$

and

$$\mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}} \times \Omega_{\mathbf{A}}, X) \to \mathcal{HOM}(\Omega_{\mathbf{A}}, \mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}, X)),$$

it is readily verified that

$$\begin{array}{ccc} \mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}, X) & \xrightarrow{\mathrm{id}} & \mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}, X) \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathcal{HOM}(\mathbf{H},X)^{\sim}} \\ \mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}, X) & \xrightarrow{\mathrm{id}\times\delta_0} & \mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}, X) \times \Omega_{\mathbf{A}} \end{array} \tag{40}$$

and

$$\begin{array}{ccc} \mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}, X) & \xleftarrow{\sigma} & X \\ {\scriptstyle \mathcal{HOM}(\mathbf{H},X)^{\sim}}\uparrow & & \uparrow{\scriptstyle \rho} \\ \mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}, X) \times \Omega_{\mathbf{A}} & \xleftarrow{\mathrm{id}\times\delta_1} & \mathcal{HOM}(j_{\mathbf{A}}(C/\mathbf{C})^{\mathrm{op}}, X) \end{array} \tag{41}$$

commute □

7.5 **Corollary** *Let* **A** *be a category of models. Then*

$$\Sigma^{-1} \int^{\mathbf{C}} j_{\mathbf{A}}(\cdot/\mathbf{C})^{\mathrm{op}} \times X \dashv \Sigma^{-1}\pi_{\mathbf{C}}^{*}.$$

Proof. By 7.4, there is a pointwise weak equivalence

$$\mathcal{HOM}(j_{\mathbf{A}}(\cdot/\mathbf{C})^{\mathrm{op}}, X) \to \pi_{\mathbf{C}}^{*}(X)$$

in $(\mathbf{C}, (\mathbf{A}^{\mathrm{op}}, \mathrm{Sets}))$. Thus $\mathcal{HOM}(j_{\mathbf{A}}(\cdot/\mathbf{C})^{\mathrm{op}}, \cdot)$ preserves weak equivalences and there is a natural isomorphism in $\Sigma^{-1}(\mathbf{A}^{\mathrm{op}}, \mathrm{Sets})$,

$$\Sigma^{-1}\mathcal{HOM}(j_{\mathbf{A}}(\cdot/\mathbf{C})^{\mathrm{op}}, \cdot) \simeq \Sigma^{-1}\pi_{\mathbf{C}}^{*} \quad \square$$

REFERENCES

1. J. Gray, *Fibred and Co-Fibred Categories*, Conference on Categorical Algebra, Springer Verlag, Berlin, 1965, pp. 21-83.
2. A. Grothendieck, *Pursuing Stacks*, unpublished letter to D. Quillen.
3. M. Heggie, *Tensor Products in Homotopy Theory*, Ph.D. Thesis, McGill University, 1986.
4. D. Quillen, *Homotopical Algebra*, Lecture Notes in Mathematics, vol. 43, Springer Verlag, Berlin, 1967.
5. ———, *Higher Algebraic K-Theory I*, Lecture Notes in Mathematics, vol. 341, Springer Verlag, Berlin, 1973, pp. 85–147.
6. R. Thomason, *Homotopy Colimits in the Category of Small Categories*, Math. Proc. Camb. Phil. Soc. **85** (1979), 91–109.

SYDNEY, NOVA SCOTIA, B1P6L2

E-mail: heggie@cad.uccb.ns.ca