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MURRAY HEGGIE

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HOMOTOPY COFIBRATIONS IN CAT

by Murray HEGGIE

RESUME. Une classe de foncteurs ayant les principales propriétés des cofibrations au sens de la théorie de l'homotopie est définie et étudiée. En particulier, on montre que le "pushout" d'une équivalence faible le long d'un foncteur appartenant à cette classe est une équivalence faible.

1 Introduction

The purpose of this paper is to isolate a class of functors that have properties analoguous to cofibrations in the sense of homotopy theory. In particular, the class of functors defined herein can serve as the basis for an axiomatic development of homotopy theory in $\mathcal{CAT}$ using Anderson's notion of a cofibration category [1].

Before describing in more detail the contents of this paper, some fundamental notions for the homotopy theory of categories will be briefly recalled. Let $X\colon \mathbf{C} \to \mathcal{CAT}$ be a $\mathcal{CAT}$- valued diagram.Define a category

$$\mathbf{C}\int X$$

as follows: Objects of $\mathbf{C}\int X$ are pairs (C, x) where $C \in \mathbf{C}$ and $x \in X(C)$. Maps $(C, x) \to (C', x')$ are pairs

$$(c\colon C \to C', f\colon X(c)(x) \to x') \in \mathbf{C}(C, C') \times X(C)(X(c)(x), x').$$

The composite of two maps (c, f) and (c', f') is defined by

$$(c', f') \circ (c, f) = (c' \circ c, f' \circ X(c')(f)).$$

There is an evident projection $\pi\colon \mathbf{C}\int X \to \mathbf{C}$. This construction enjoys a universal property which will not be recapitulated here [2]. $\mathbf{C}\int X$ is known variously as the *Grothendieck construction* or the *op-fibred category associated to X*.

A functor $\mathbf{F}\colon \mathbf{A} \to \mathbf{B}$ is called a *weak equivalence* if its image Nerve(F) under Nerve is a weak equivalence of simplicial sets [3]. A category $\mathbf{A}$ is called *weakly contractible* if the unique map $\mathbf{A} \to 1$ from $\mathbf{A}$ to the terminal category is a weak equivalence. Weak equivalences in the functor category $(\mathbf{A}, \mathcal{CAT})$ are defined pointwise.

The property of the Grothendieck construction that is crucial in the sequel is its *homotopy invariance*: If $\theta\colon X \Rightarrow Y$ is weak equivalence in $(\mathbf{C}, \mathcal{CAT})$ then the induced map $\mathbf{C} \int X \to \mathbf{C} \int Y$ is a weak equivalence in $\mathcal{CAT}$ [5,10].

The definition of homotopy cofibrations in $\mathcal{CAT}$ given here is an adaptation to the categorical context of the notion of a *neighbourhood deformation retract* in the category of compactly generated spaces [11,p.22]. Closed inclusions are modelled by *coideals* (§2) and deformation retracts by *inclusions which admit coretractions from the ideal generated by the image* (§3). Functors of both types admit natural categorical characterizations (cf. propositions 2.6 and 3.3). Once in possession of a categorical characterization, it is a simple matter to establish stability under pushout of each class (cf. corollaries 2.7 and 3.4). *Strong coideals*, the proposed candidates for cofibrations in $\mathcal{CAT}$, are functors in the intersection. It is shown in §4–6 that strong coideals have the distinguishing features of homotopy cofibrations:

- The pushout of a weak equivalence along a strong coideal is a weak equivalence.
- The pushout of a strong coideal that is at the same time a weak equivalence along an arbitrary map is a weak equivalence.
- Pushouts along strong coideals are homotopy invariant.
- Every map is isomorphic in the homotopy category to a strong coideal.
- Modulo strong coideals, every map induces a long exact sequence in homotopy.

The properties of the class of strong coideals were developed in order to facilitate the construction of homotopy colimits in presheaf categories. This application is described in [6].

2 Coideals

2.1 Definition A functor $\mathbf{I}\colon \mathbf{A} \to \mathbf{B}$ is a *coideal* if

(1) $\mathbf{I}$ is a full inclusion.
(2) Whenever $b\colon B \to \mathbf{I}(A)$ is a map in $\mathbf{B}$, b is in the image of $\mathbf{I}$.

Dually, a functor $\mathbf{J}\colon \mathbf{A} \to \mathbf{B}$ is an *ideal* if $\mathbf{J}$ is a full inclusion with the property that every map $\mathbf{J}(A) \to B$ in $\mathbf{B}$ whose domain is in the image of $\mathbf{J}$ is itself in the image of $\mathbf{J}$. If $\mathbf{A}$ and $\mathbf{B}$ are posets and $\mathbf{I}$ is an order preserving map, then $\mathbf{I}(\mathbf{A}) \subseteq \mathbf{B}$ is an coideal in the usual sense.

2.2 Lemma *The class of coideals in $\mathcal{CAT}$ is closed under strong retracts: if $\mathbf{I}\colon \mathbf{A} \to \mathbf{B}$ is a coideal which fits into a commuting diagram*

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "id_{\mathbf{A}}"] \arrow[d, "\mathbf{J}"'] & \mathbf{A} \arrow[r, "id_{\mathbf{A}}"] \arrow[d, "\mathbf{I}"'] & \mathbf{A} \arrow[d, "\mathbf{J}"'] \\
\mathbf{B} \arrow[r, "\mathbf{R}"] & \mathbf{A} \arrow[r, "\mathbf{P}"] & \mathbf{B}
\end{tikzcd}$$

with $\mathbf{P} \circ \mathbf{R} = id_{\mathbf{B}}$*, then* $\mathbf{J}\colon \mathbf{A} \to \mathbf{C}$ *is a coideal.*

Proof. A simple diagram chase establishes the claim □

2.3 Definition Let $\mathbf{F}\colon \mathbf{A} \to \mathbf{B}$ be a functor. The *cylinder* of $\mathbf{F}$, CYL($\mathbf{F}$), is the opfibred category associated to the diagram $\mathbf{2} \to \mathcal{CAT}$ which sends the unique non–identity arrow $0 \to 1 \in \mathbf{2}$ to $\mathbf{F}$.

Explicitly, objects of CYL($\mathbf{F}$) are pairs
(1) $(A, 0)$ where $A \in \mathbf{A}$
(2) $(B, 1)$ where $B \in \mathbf{B}$

Maps in CYL($\mathbf{F}$) are defined by

$$\begin{array}{lll} (1)\ \mathrm{CYL}(\mathbf{F})((A,0),(A',0)) = \mathbf{A}(A,A') & & \text{if } A \text{ and } A' \in \mathbf{A} \\ (2)\ \mathrm{CYL}(\mathbf{F})((B,1),(B',1)) = \mathbf{B}(B,B') & & \text{if } B \text{ and } B' \in \mathbf{B} \\ (3)\ \mathrm{CYL}(\mathbf{F})((A,0),(B,1)) = \mathbf{B}(\mathbf{F}(A),B) & & \text{if } A \in \mathbf{A} \text{ and } B \in \mathbf{B} \\ (4)\ \mathrm{CYL}(\mathbf{F})((B,1),(A,0)) = \phi & & \text{if } A \in \mathbf{A} \text{ and } B \in \mathbf{B} \end{array}$$

There are evident functors $\iota_{\mathbf{A}}\colon \mathbf{A} \to \mathrm{CYL}(\mathbf{F})$ and $\rho_{\mathbf{B}}\colon \mathrm{CYL}(\mathbf{F}) \to \mathbf{B}$. On objects, $\iota_{\mathbf{A}}(A) = (A, 0)$. $\rho_{\mathbf{B}}(A, 0) = \mathbf{F}(A)$ and $\rho_{\mathbf{B}}(B, 1) = B$. In each case, the extension to maps is straightforward.

2.4 Lemma $\iota_{\mathbf{A}}$ *is a coideal* □

2.5 Lemma $\rho_{\mathbf{B}}$ *is a left adjoint, left inverse.*

Proof. The right adjoint, right inverse to $\rho_{\mathbf{B}}$ sends $B \to B'$ to $(B, 1) \to (B', 1)$ □

2.6 Proposition $\mathbf{I}\colon \mathbf{A} \to \mathbf{B}$ *is a coideal if and only if* $\mathbf{I}$ *has the left lifting property (*LLP*) with respect to all left adjoint, left inverses: there is a lifting* $\mathbf{K}\colon \mathbf{B} \to \mathbf{C}$ *in all diagrams of the form*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\mathbf{F}"] \arrow[d, "\mathbf{I}"'] & \mathbf{C} \arrow[d, "\mathbf{P}"] \\ \mathbf{B} \arrow[r, "\mathbf{G}"] & \mathbf{D} \end{tikzcd}$$

where $\mathbf{P}$ *is a left adjoint, left inverse.*

Proof. Let $\mathbf{R}$ denote the right adjoint, right inverse of $\mathbf{P}$ and let $\eta\colon id_{\mathbf{C}} \to \mathbf{R} \circ \mathbf{P}$ denote the unit of the adjunction. Assume that $\mathbf{I}$ is a coideal. Define a map $\mathbf{K}\colon \mathbf{B} \to \mathbf{C}$ by

$$\mathbf{K}(b\colon B_0 \to B_1) = \begin{cases} \mathbf{F}(b), & \text{if } b \in \mathbf{I}(\mathbf{A}) \\ \mathbf{RG}(b), & \text{if } B_0 \notin \mathbf{I}(\mathbf{A}) \\ \mathbf{RG}(b) \circ \eta(\mathbf{F}(B_0)), & \text{if } B_0 \in \mathbf{I}(\mathbf{A}) \text{ and } b \notin \mathbf{I}(\mathbf{A}) \end{cases}$$

As $\mathbf{I}$ is a coideal by assumption, $\mathbf{K}$ is well-defined. It is readily verified that K is a functor and that $\mathbf{P} \circ \mathbf{K} = \mathbf{G}$. By construction, $\mathbf{K} \circ \mathbf{I} = \mathbf{F}$.

For the converse, assume that $\mathbf{J}: \mathbf{A} \to \mathbf{B}$ is a functor that has the LLP with respect to all left adjoint, left inverses. Consider the commutative diagram

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\iota_{\mathbf{A}}"] \arrow[d, "\mathbf{J}"'] & \mathrm{CYL}(\mathbf{J}) \arrow[d, "\rho_{\mathbf{B}}"] \\ \mathbf{B} \arrow[r, "id_{\mathbf{B}}"] & \mathbf{B} \end{tikzcd}$$

By assumption, there is a lifting $\mathbf{K}: \mathbf{B} \to \mathrm{CYL}(\mathbf{J})$. Therefore, $\mathbf{J}$ is a strong retract of the coideal $\iota_{\mathbf{A}}$. By a prior lemma, $\mathbf{J}$ is a coideal □

2.7 Corollary *The class of coideals is stable under pushout: if*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\mathbf{F}"] \arrow[d, "\mathbf{I}"'] & \mathbf{A}' \arrow[d, "\mathbf{I}'"] \\ \mathbf{B} \arrow[r, "\mathbf{F}'"] & \mathbf{B}' \end{tikzcd}$$

is a pushout in $\mathcal{CAT}$ and $\mathbf{I}$ *is a coideal, then* $\mathbf{I}'$ *is a coideal.*

Proof. Using the universal property of the pushout, it is readily verified that $\mathbf{I}'$ has the LLP with respect to all left adjoint, left inverses □

3 Strong coideals

3.1 Definition Let $\mathbf{I}: \mathbf{A} \to \mathbf{B}$ be a functor. $\mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$ denotes the ideal in $\mathbf{B}$ generated by the image of $\mathbf{I}$.

In more detail, $\mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$ is the full subcategory of $\mathbf{B}$ generated by arrows of the form $\mathbf{I}(A) \to B$.

3.2 Definition Let $\mathbf{A} \in \mathcal{CAT}$. The *cone of* $\mathbf{A}$, $\mathrm{Cone}(\mathbf{A})$, is the category

$$\mathrm{Cone}(\mathbf{A}) \equiv \mathrm{CYL}(\mathbf{A}^{\mathrm{op}} \to 1)^{\mathrm{op}}$$

where $1 \in \mathcal{CAT}$ denotes the terminal category.

$\mathrm{Cone}(\mathbf{A})$ is the category obtained from $\mathbf{A}$ by free attachment of a strict initial object.There is an evident inclusion

$$j_A : \mathbf{A} \to \mathrm{Cone}(\mathbf{A})$$

which, on objects, is defined by sending $A \in \mathbf{A}$ to $(A, 0)$. Evidently, for any functor $\mathbf{F} : \mathbf{A} \to \mathbf{B}$,

$$\begin{tikzcd} \mathbf{A} \arrow[r, "j_{\mathbf{A}}"] \arrow[d, "\mathbf{F}"'] & \mathrm{Cone}(\mathbf{A}) \arrow[d, "\mathrm{Cone}(\mathbf{F})"] \\ \mathbf{B} \arrow[r, "j_{\mathbf{B}}"] & \mathrm{Cone}(\mathbf{B}) \end{tikzcd}$$

commutes.

3.3 Proposition *Let* $\mathbf{I}\colon \mathbf{A} \to \mathbf{B}$ *be a functor. There is a coreflection* $\mathbf{R}\colon \mathbf{I}(\mathbf{A})\cdot\mathbf{B} \to \mathbf{A}$ *if and only if for all categories* $\mathbf{C}$ *having a strict initial object* 0*, the induced functor*

$$\mathbf{I}^*: \mathcal{CAT}(\mathbf{B}, \mathbf{C}) \to \mathcal{CAT}(\mathbf{A}, \mathbf{C})$$

has a left adjoint, right inverse

$$\mathbf{I}_!: \mathcal{CAT}(\mathbf{A}, \mathbf{C}) \to \mathcal{CAT}(\mathbf{B}, \mathbf{C})$$

($\mathcal{CAT}(\cdot, \cdot)$ denotes the $\mathcal{CAT}$- valued hom).

Proof. Before beginning the proof proper, I will spell out the meaning of the adjunction

$$\mathbf{I}_! \dashv \mathbf{I}^*: \mathcal{CAT}(\mathbf{A}, \mathbf{C}) \to \mathcal{CAT}(\mathbf{B}, \mathbf{C})$$

where, in addition, $\mathbf{I}^* \circ \mathbf{I}_! = id$. Explicity, for every functor $\mathbf{F}\colon \mathbf{A} \to \mathbf{C}$ there is a functor $\mathbf{I}_!(\mathbf{F})\colon \mathbf{B} \to \mathbf{C}$ such that $\mathbf{I}_!(\mathbf{F}) \circ \mathbf{I} = \mathbf{F}$ and whenever there is a natural transformation $\theta\colon \mathbf{F} \Rightarrow \mathbf{G} \circ \mathbf{I}$ there is a unique natural transformation $\phi\colon \mathbf{I}_!(\mathbf{F}) \Rightarrow \mathbf{G}$ such that $\phi.\mathbf{I} = \theta$.

Assume first that $\mathbf{I}_!$ exists for any category $\mathbf{C}$ with strict initial object 0. Let $\mathbf{J}$ denote the composite

$$\mathbf{A} \xrightarrow{\iota_{\mathbf{A}}} \mathrm{CYL}(\mathbf{I}) \xrightarrow{j_{\mathrm{CYL}(\mathbf{I})}} \mathrm{Cone}(\mathrm{CYL}(\mathbf{I}))\,.$$

Let $\mathbf{S} = \mathbf{I}_!(\mathbf{J})$ and let $\mathbf{R} = \mathbf{I}_!(j_{\mathbf{A}})$. Let $\mathbf{T}\colon \mathbf{B} \to \mathrm{Cone}(\mathrm{CYL}(\mathbf{I}))$ denote the composite

$$\mathbf{B} \xrightarrow{\mathbf{R}} \mathrm{Cone}(\mathbf{A}) \xrightarrow{\mathrm{Cone}(\iota_{\mathbf{A}})} \mathrm{Cone}(\mathrm{CYL}(\mathbf{I}))\,.$$

Since

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\iota_{\mathbf{A}}"] \arrow[d, "j_{\mathbf{A}}"'] & \mathrm{CYL}(\mathbf{I}) \arrow[d, "j_{\mathrm{CYL}(\mathbf{I})}"] \\ \mathrm{Cone}(\mathbf{A}) \arrow[r, "\mathrm{Cone}(\iota_{\mathbf{A}})"] & \mathrm{Cone}(\mathrm{CYL}(\mathbf{I})) \end{tikzcd}$$

commutes,

$$\begin{aligned}\mathbf{T}\circ\mathbf{I} &= \mathrm{Cone}(\iota_{\mathbf{A}})\circ\mathbf{R}\circ\mathbf{I}\\ &= \mathrm{Cone}(\iota_{\mathbf{A}})\circ j_{\mathbf{A}}\\ &= j_{\mathrm{CYL(I)}}\circ\iota_{\mathbf{A}}.\end{aligned}$$

By the universal property of $\mathbf{S} = \mathbf{I}_!(j_{\mathrm{CYL(I)}}\circ\iota_{\mathbf{A}})$, there is a unique natural transformation

$$\theta\colon \mathbf{S}\Rightarrow\mathbf{T}$$

satisfying

$$\theta.\mathbf{I} = id.$$

For all $B\in\mathbf{B}$, $\mathbf{T}(B)$ is in the image of $\mathrm{Cone}(\mathbf{A})$ under $\mathrm{Cone}(\iota_{\mathbf{A}})$. But $\mathrm{Cone}(\iota_{\mathbf{A}})$ is a coideal. Since, for every $B\in\mathbf{B}$, there is a map $\theta_B : \mathbf{S}(B)\to\mathbf{T}(B)$ in $\mathrm{Cone}(\mathrm{CYL}(\mathbf{I}))$, $\mathbf{S}(B)$ is also in the image of $\mathrm{Cone}(\mathbf{A})$ under $\mathrm{Cone}(\iota_{\mathbf{A}})$. As $\mathrm{Cone}(\iota_{\mathbf{A}})$ is a full inclusion, $\mathbf{S}$ factors through $\mathrm{Cone}(\iota_{\mathbf{A}})$. It follows that θ can be viewed as a natural transformation $\theta\colon\mathbf{S}\Rightarrow\mathbf{R}$. On the other hand, by the universal property of $\mathbf{R} = \mathbf{I}_!(j_{\mathbf{A}})$, there is a unique natural transformation

$$\phi\colon\mathbf{R}\Rightarrow\mathbf{S}$$

satisfying

$$\phi.\mathbf{I} = id.$$

θ and ϕ are inverses. For, on the one hand,

$$\begin{aligned}(\theta\circ\phi).\mathbf{I} &= (\theta.\mathbf{I})\circ(\phi.\mathbf{I})\\ &= id.\end{aligned}$$

By the universal property of $\mathbf{R}$, the foregoing implies that $\theta\circ\phi = id$. On the other hand, by the same type of reasoning,

$$((\mathrm{Cone}(\iota_{\mathbf{A}}).\phi)\circ\theta).\mathbf{I} = id.$$

Consequently,

$$\mathrm{Cone}(\iota_{\mathbf{A}})\circ\mathbf{R}\colon\mathbf{B}\to\mathrm{Cone}(\mathrm{CYL}(\mathbf{I}))$$

is the universal extension of $j_{\mathrm{CYL(I)}}\circ\iota_{\mathbf{A}}$ along $\mathbf{I}\colon\mathbf{A}\to\mathbf{B}$. For each $A\in\mathbf{A}$, let $\chi_A : (A,0)\to(\mathbf{I}(A),1)\in\mathrm{Cone}(\mathrm{CYL}(\mathbf{I}))$ be the map representing $id\colon\mathbf{I}(A)\to\mathbf{I}(A)$. Then $\{\chi_A \mid A\in\mathbf{A}\}$ are the components of a natural transformation

$$\chi\colon j_{\mathrm{CYL(I)}}\circ\iota_{\mathbf{A}}\Rightarrow\mathrm{Cone}(\iota_{\mathbf{B}})\circ j_{\mathbf{B}}\circ\mathbf{I}.$$

By the universal property of $\mathrm{Cone}(\iota_{\mathbf{A}}) \circ \mathbf{R}$, there is a unique natural transformation

$$\psi\colon \mathrm{Cone}(\iota_{\mathbf{A}}) \circ \mathbf{R} \Rightarrow \mathrm{Cone}(\iota_{\mathbf{B}}) \circ j_{\mathbf{B}}$$

satisfying

$$\psi.\mathbf{I} = \chi.$$

Let $\rho_{\mathbf{B}}\colon \mathrm{CYL}(\mathbf{I}) \to \mathbf{B}$ denote the projection. $\rho_{\mathbf{B}}$ satisfies

(1) $\rho_{\mathbf{B}} \circ \iota_{\mathbf{A}} = \mathbf{I}$
(2) $\rho_{\mathbf{B}} \circ \iota_{\mathbf{B}} = id_{\mathbf{B}}$.

Then

$$\mathrm{Cone}(\rho_{\mathbf{B}}) \circ \mathrm{Cone}(\iota_{\mathbf{B}}) \circ j_{\mathbf{B}} = j_{\mathbf{B}}$$

and

$$\mathrm{Cone}(\rho_{\mathbf{B}}) \circ \mathrm{Cone}(\iota_{\mathbf{A}}) \circ \mathbf{R} = \mathrm{Cone}(\mathbf{I}) \circ \mathbf{R}.$$

Therefore, $\mathrm{Cone}(\rho_{\mathbf{B}}).\psi$ is a natural transformation $\mathrm{Cone}(\mathbf{I}) \circ \mathbf{R} \Rightarrow j_{\mathbf{B}}$. As $\mathbf{R} \circ \mathbf{I} = j_{\mathbf{A}}\colon \mathbf{A} \to \mathrm{Cone}(\mathbf{A})$, $\mathbf{R}(B) \neq 0$ if $B \in \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$. For, suppose to the contrary that $\mathbf{R}(B) = 0$ for some $B \in \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$. Then there is a map $\mathbf{I}(A) \to B$ in $\mathbf{B}$. Since 0 is a strict initial object in $\mathrm{Cone}(\mathbf{A})$, this would imply that $j_{\mathbf{A}}(A) = \mathbf{R} \circ \mathbf{I}(A) = 0$. It follows that $\mathbf{R}$ yields a functor $\mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \to \mathbf{A}$ by restriction and that $\mathrm{Cone}(\rho_{\mathbf{B}}).\psi$ restricts to a natural transformation

$$\epsilon\colon \mathbf{I} \circ \mathbf{R} \Rightarrow id_{\mathbf{I}(\mathbf{A})\cdot\mathbf{B}}.$$

Note that if $B \notin \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$, $\mathbf{R}(B) = 0$. For suppose to the contrary that $\mathbf{R}(B_0) \neq 0$ for some $B_0 \notin \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$. Define $\mathbf{S}\colon \mathbf{B} \to \mathrm{Cone}(\mathbf{A})$ on objects by

$$\mathbf{S} = \begin{cases} \mathbf{R}(B), & \text{if } B \in \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \\ 0, & \text{if } B \notin \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \end{cases}$$

Define $\mathbf{S}$ on arrows by

$$\mathbf{S}(b\colon B_1 \to B_2) = \begin{cases} \mathbf{R}(b) & \text{if } b \in \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \\ 0 \to \mathbf{R}(B_2) & \text{if } B_1 \notin \mathbf{I}(\mathbf{A}) \cdot \mathbf{B},\ B_2 \in \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \\ id\colon 0 \to 0 & \text{if } B_1 \notin \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \end{cases}$$

It is readily verified that $\mathbf{S}$ is a functor. By construction, $\mathbf{S} \circ \mathbf{I} = j_{\mathbf{A}}$. As 0 is a strict initial object in $\mathrm{Cone}(\mathbf{A})$, there is no map in $\mathrm{Cone}(\mathbf{A})$,

$$0 \neq \mathbf{R}(B_0) \to \mathbf{S}(B_0) = 0.$$

A fortiori, there is no natural transformation $\mathbf{R} \Rightarrow \mathbf{S}$, in violation of the universal property of $\mathbf{R}$. An important consequence of the observation that $\mathbf{R}(B) = 0$ if

$B \notin \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$ is that the restriction of $\mathbf{R}$ to $\mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$ is the universal extension of $\mathbf{I}\colon \mathbf{A} \to \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$ along the identity $id_{\mathbf{A}}$. Since

$$\begin{aligned} \mathrm{Cone}(\rho_{\mathbf{B}})(\psi_{\mathbf{I}(A)}) &= \mathrm{Cone}(\rho_{\mathbf{B}})(\chi_A) \\ &= \mathrm{Cone}(id_{\mathbf{I}(A)}), \end{aligned}$$

$\epsilon_{\mathbf{I}(A)} = id_{\mathbf{I}(A)}$. Consequently, $\mathbf{R}.\epsilon\colon \mathbf{R} \Rightarrow \mathbf{R}$ is the identity by the universal property of the restriction of $\mathbf{R}$. As $\mathbf{R} \circ \mathbf{I} = id_{\mathbf{A}}$, this proves that $\{\epsilon_B \mid B \in \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}\}$ is the counit of an adjunction

$$\mathbf{I} \dashv \mathbf{R}, \quad \mathbf{R} \circ \mathbf{I} = id.$$

To prove the converse, assume that $\mathbf{I}\colon \mathbf{A} \to \mathbf{B}$ admits a coreflection $\mathbf{R}\colon \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \to \mathbf{A}$. Let $\epsilon\colon \mathbf{I} \circ \mathbf{R} \Rightarrow id_{\mathbf{I}(\mathbf{A}) \cdot \mathbf{B}}$ be the counit of the adjunction. Let $\mathbf{C}$ be a category with a strict initial object 0. Let $\mathbf{F}\colon \mathbf{A} \to \mathbf{C}$ be any functor. Define a functor $\mathbf{G}\colon \mathbf{B} \to \mathbf{C}$ on objects by

$$\mathbf{G}(B) = \begin{cases} 0, & \text{if } B \notin \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \\ \mathbf{F} \circ \mathbf{R}(B), & \text{if } B \in \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \end{cases}$$

On arrows, $\mathbf{G}$ is defined by

$$\mathbf{G}(b : B \to B') = \begin{cases} id_0, & \text{if } B, B' \notin \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \\ \mathbf{FR}(b), & \text{if } B \in \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \\ 0 \to \mathbf{FR}(B'), & \text{if } B \notin \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \text{ and } B' \in \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \end{cases}$$

It is readily verified that $\mathbf{G}$ is indeed a functor. Evidently, $\mathbf{G} \circ \mathbf{I} = \mathbf{F}$. Note that, if $B \in \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$,

$$\begin{aligned} \mathbf{G}(\epsilon_B \colon \mathbf{I} \circ \mathbf{R}(B) \to B) &= \mathbf{F} \circ \mathbf{R}(\epsilon_B) \\ &= \mathbf{F}(id_{\mathbf{R}(B)}) \\ &= id_{\mathbf{F} \circ \mathbf{R}(B)} \end{aligned}$$

since $\mathbf{R} \circ \mathbf{I} = id_{\mathbf{A}}$ and $\mathbf{I} \dashv \mathbf{R}$. I claim that $\mathbf{G}$ has the universal property defining $\mathbf{I}_!(\mathbf{F})$. To see this, assume that $\mathbf{H}\colon \mathbf{B} \to \mathbf{C}$ is a functor and $\theta\colon \mathbf{G} \circ \mathbf{I} \Rightarrow \mathbf{H} \circ \mathbf{I}$ a natural transformation. Define a natural transformation $\phi\colon \mathbf{G} \Rightarrow \mathbf{H}$ by

$$\phi_B = \begin{cases} 0 \to \mathbf{H}(B), & \text{if } B \notin \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \\ \mathbf{H}(\epsilon_B).\theta_{\mathbf{R}(B)}, & \text{if } B \in \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \end{cases}$$

The first clause in the definition of ϕ is dictated by the definition of $\mathbf{G}$. The second clause is dictated by the requirement that $\phi.\mathbf{I} = \theta$. For, assume that ϕ is a natural transformation satisfying $\phi.\mathbf{I} = \theta$. Since ϕ is a natural transformation,

$$\begin{tikzcd} \mathbf{G} \circ \mathbf{I} \circ \mathbf{R}(B) \arrow[r, "\phi_{\mathbf{IR}(B)}"] \arrow[d, "\mathbf{G}(\epsilon_B)"'] & \mathbf{H} \circ \mathbf{I} \circ \mathbf{R}(B) \arrow[d, "\mathbf{H}(\epsilon_B)"'] \\ \mathbf{G}(B) \arrow[r, "\phi_B"] & \mathbf{H}(B) \end{tikzcd}$$

commutes. But, by assumption, $\phi_{\mathbf{I}\circ\mathbf{R}(B)} = \theta_{\mathbf{R}(B)}$ and $\mathbf{G}(\epsilon_B) = id_{\mathbf{F}\circ\mathbf{R}(B)}$ when $B \in \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$. Therefore, $\phi_B = \mathbf{H}(\epsilon_B) \circ \theta_{\mathbf{R}(B)}$. Consequently, $\mathbf{G}$ has the universal property defining $\mathbf{I}_!(\mathbf{F})$ $\square$

3.4 Corollary *Let* $\mathbf{I}\colon \mathbf{A} \to \mathbf{B}$ *be a functor for which there is a coreflection* $\mathbf{R}\colon \mathbf{I}(A)\cdot \mathbf{B} \to \mathbf{A}$. *Let*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\mathbf{F}"] \arrow[d, "\mathbf{I}"'] & \mathbf{A}' \arrow[d, "\mathbf{I}'"'] \\ \mathbf{B} \arrow[r, "\mathbf{F}'"] & \mathbf{B}' \end{tikzcd}$$

be a pushout in $\mathcal{CAT}$. *Then there is a coreflection* $\mathbf{R}'\colon \mathbf{I}'(\mathbf{A}') \cdot \mathbf{B}' \to \mathbf{A}'$ *and*

$$\begin{tikzcd} \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \arrow[r, "\mathbf{F}'"] \arrow[d, "\mathbf{R}"'] & \mathbf{I}(\mathbf{A}') \cdot \mathbf{B}' \arrow[d, "\mathbf{R}'"'] \\ \mathbf{A} \arrow[r, "\mathbf{F}"] & \mathbf{A}' \end{tikzcd}$$

commutes up to a natural isomorphism

$$\mathbf{R}' \circ \mathbf{F}' \Rightarrow \mathbf{F} \circ \mathbf{R}.$$

Proof. Let $\mathbf{C}$ be a category with a strict initial object 0. Applying the internal hom $\mathcal{CAT}(\cdot, \mathbf{C})$ to the pushout diagram produces a pullback:

$$\begin{tikzcd} \mathcal{CAT}(\mathbf{B}', \mathbf{C}) \arrow[r, "(\mathbf{F}')^*"] \arrow[d, "(\mathbf{I}')^*"'] & \mathcal{CAT}(\mathbf{B}, \mathbf{C}) \arrow[d, "\mathbf{I}^*"'] \\ \mathcal{CAT}(\mathbf{A}', \mathbf{C}) \arrow[r, "\mathbf{F}^*"] & \mathcal{CAT}(\mathbf{A}, \mathbf{C}) \end{tikzcd}$$

By the previous proposition, $\mathbf{I}^*$ is a right adjoint, left inverse $\mathbf{I}_!$. But it is easly proved that the pullback of a right adjoint, left inverse is a right adjoint, left inverse. Therefore, $(\mathbf{I}')^*$ has a left adjoint, right inverse $(\mathbf{I}')_!$. Moreover, the following square is commutative:

$$\begin{tikzcd} \mathcal{CAT}(\mathbf{A}', \mathbf{C}) \arrow[r, "\mathbf{F}^*"] \arrow[d, "(\mathbf{I}')_!"'] & \mathcal{CAT}(\mathbf{A}, \mathbf{C}) \arrow[d, "\mathbf{I}_!"'] \\ \mathcal{CAT}(\mathbf{B}', \mathbf{C}) \arrow[r, "(\mathbf{F}')^*"] & \mathcal{CAT}(\mathbf{B}, \mathbf{C}) \end{tikzcd}$$

By the previous proposition, the existence of $(\mathbf{I}')_!$ for all categories $\mathbf{C}$ having a strict initial object is equivalent to the existence of a coreflection

$$\mathbf{R}'\colon \mathbf{I}'(\mathbf{A}') \cdot \mathbf{B}' \to \mathbf{A}'.$$

By the argument of the previous proposition, the coreflection $\mathbf{R}'$ is the restriction to $\mathbf{I}'(\mathbf{A}') \cdot \mathbf{B}'$ of the universal extension $(\mathbf{I}')_!(j_{\mathbf{A}'})$. It follows that, if $B \in \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$,

$$\begin{aligned}\mathbf{R}' \circ \mathbf{F}'(B) &= ((\mathbf{I}')_!(j_{\mathbf{A}'})\mathbf{F}'(B) \\ &= (\mathbf{F}')^*(\mathbf{I}')_!(j_{\mathbf{A}'}))(B) \\ &= \mathbf{I}_!(\mathbf{F}^*(j_{\mathbf{A}}))(B) \\ &= \mathbf{I}_!(\mathrm{Cone}(\mathbf{F})j_{\mathbf{A}})(B)\end{aligned}$$

Let $\mathbf{G}: \mathbf{B} \to \mathrm{Cone}(\mathbf{A}')$ denote the composite $(\mathbf{I}')_!(\mathrm{Cone}(\mathbf{F}) \circ j_{\mathbf{A}})$. Then the following diagram commutes:

$$\begin{tikzcd} \mathbf{A} \arrow[r, "j_{\mathbf{A}}"] \arrow[d, "\mathbf{I}"'] & \mathrm{Cone}(\mathbf{A}) \arrow[d, "\mathrm{Cone}(\mathbf{F})"] \\ \mathbf{B} \arrow[r, "\mathbf{G}"] & \mathrm{Cone}(\mathbf{A}') \end{tikzcd}$$

Since $\mathbf{R} \circ \mathbf{I} = j_{\mathbf{A}}$, $\mathrm{Cone}(\mathbf{F}) \circ \mathbf{R} \circ \mathbf{I} = \mathrm{Cone}(\mathbf{F}) \circ j_{\mathbf{A}} = \mathbf{G} \circ \mathbf{I}$. By the universal property of $\mathbf{G}$, there is a unique natural transformation $\theta: \mathbf{G} \Rightarrow \mathrm{Cone}(\mathbf{F}) \circ \mathbf{R}$ such that $\theta.\mathbf{I} = id$. Recall that, if $B \notin \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$, $\mathbf{R}(B) = 0$ and that the restriction of $\mathbf{R}$ to $\mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$ is the universal extension of $\mathbf{I}: \mathbf{A} \to \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$ along the identity $id_{\mathbf{A}}$. $\mathrm{Cone}(\mathbf{F})$ preserves 0, hence for $B \notin \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$, $\mathrm{Cone}(\mathbf{F}) \circ \mathbf{R}(B) = 0$. In virtue of the natural transformation $\theta: \mathbf{G} \Rightarrow \mathrm{Cone}(\mathbf{F}) \circ \mathbf{R}$, this implies that $\mathbf{G}(B) = 0$ if $B \notin \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$. An important consequence of the observation that $\mathbf{R}(B) = 0$ if $B \notin \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$ is that the restriction of $\mathbf{R}$ to $\mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$ is the universal extension of $\mathbf{I}: \mathbf{A} \to \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$ along the identity $id_{\mathbf{A}}$. Arguing as for $\mathbf{R}$, the restriction of $\mathbf{G}$ enjoys the same universal property, *mutatis mutandis*, that $\mathbf{G}$ does. Next, assume that $B \in \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$. Let $\epsilon_B : \mathbf{I} \circ \mathbf{R}(B) \to B$ denote the counit of the adjunction

$$\mathbf{I} \dashv \mathbf{R}: \mathbf{A} \to \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}.$$

By naturality,

$$\begin{tikzcd} \mathbf{G} \circ \mathbf{I} \circ \mathbf{R}(B) \arrow[r, "\theta_{\mathbf{I} \circ \mathbf{R}(B)}"] \arrow[d, "\mathbf{G}(\epsilon_B)"'] & \mathrm{Cone}(\mathbf{F}) \circ \mathbf{R} \circ \mathbf{I} \circ \mathbf{R}(B) \arrow[d, "\mathrm{Cone}(\mathbf{F})(\mathbf{R}(\epsilon_B))"] \\ \mathbf{G}(B) \arrow[r, "\theta_B"] & \mathrm{Cone}(\mathbf{F})(\mathbf{R}(B)) \end{tikzcd}$$

commutes. As $\mathbf{R}$ is a coreflection, $\mathbf{R}.\epsilon = id$. Therefore, $\mathrm{Cone}(\mathbf{F}) \circ \mathbf{R}(\epsilon_B) = id$. Also, $\theta.\mathbf{I} = id$. Therefore $\theta_{\mathbf{I} \circ \mathbf{R}(B)} = id$. It follows that $\theta_B \circ \mathbf{G}(\epsilon_B) = id$ for $B \in \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$. On the other hand, $((\mathbf{G}.\epsilon) \circ \theta).\mathbf{I})_A = \mathbf{G}(\epsilon_{\mathbf{I}(A)}) \circ \theta_{\mathbf{I}(A)} = id$ for all $A \in \mathbf{A}$. By the universal property of the restriction of $\mathbf{G}$ to $\mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$, this implies that $(\mathbf{G}.\epsilon) \circ \theta = id$. Therefore, θ is a natural isomorphism

$$\mathbf{I}_!(\mathrm{Cone}(\mathbf{F}) \circ j_{\mathbf{A}}) \to \mathrm{Cone}(\mathbf{F}) \circ \mathbf{R}.$$

But $\mathrm{Cone}(\mathbf{F}) \circ \mathbf{R} = j_{\mathbf{A}} \circ \mathbf{F} \circ \mathbf{R}$. Thus θ is a natural isomorphism

$$j_{\mathbf{A}'} \circ \mathbf{R}' \circ \mathbf{F}' \to j_{\mathbf{A}'} \circ \mathbf{R} \circ \mathbf{F}.$$

By restriction, θ yields a natural isomorphism

$$\mathbf{R}' \circ \mathbf{F}' \Rightarrow \mathbf{F} \circ \mathbf{R}$$

as required □

3.5 Definition $\mathbf{I}\colon \mathbf{A} \to \mathbf{B}$ is a *strong coideal* if a) $\mathbf{I}$ is a coideal and b) there is a coreflection $\mathbf{R}\colon \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \to \mathbf{A}$.

By corollaries 2.7 and 3.4, the class of strong coideals is stable under pushout.

4 Homotopy pushouts

The lemmas which follow will be used in the derivation of the main theorem of the present §. Let $\mathbf{X}\colon \mathbf{C} \to \mathcal{CAT}$ be a $\mathcal{CAT}$– valued diagram. Let $\mathbf{F}\colon \mathbf{C} \int \mathbf{X} \to \mathbf{B}$ be a functor. For each $C \in \mathbf{C}$, let $\iota(C)\colon \mathbf{X}(C) \hookrightarrow \mathbf{C} \int \mathbf{X}$ be the inclusion of the fibre over C. Let $B \in \mathbf{B}$. The "homotopy fibre" of $\mathbf{F} \circ \iota(C)$ over B is the pullback

$$\begin{tikzcd} B/(\mathbf{F} \circ \iota(C)) \arrow[r] \arrow[d] & \mathbf{X}(C) \arrow[d, "\mathbf{F}\circ\iota(C)"'] \\ B/\mathbf{B} \arrow[r] & \mathbf{B} \end{tikzcd}$$

By varying C over $\mathbf{C}$ one obtains a diagram

$$B/\mathbf{F} \circ \iota(\cdot)\colon \mathbf{C} \to \mathcal{CAT}.$$

4.1 Lemma *There is a natural isomorphism*

$$\mathbf{C} \int (B/\mathbf{F} \circ \iota(\cdot)) \to B/\mathbf{F}$$

Proof. Unpack the definitions □

4.2 Lemma *Let* $\mathbf{F} \dashv \mathbf{G}\colon \mathbf{A} \to \mathbf{B}$. *Let* $A \in \mathbf{A}$. *There is a natural isomorphism*

$$\mathbf{F}(A)/\mathbf{B} \to A/\mathbf{G}.$$

Proof. Let $\eta\colon id \Rightarrow \mathbf{G} \circ \mathbf{F}$ be the unit of the adjunction. Define a mapping on objects of $\mathbf{F}(A)/\mathbf{B}$ by sending $(b\colon \mathbf{F}(A) \to B, B)$ to $(\mathbf{G}(b) \circ \eta_A\colon A \to \mathbf{G}(B), B)$. Extending this mapping in the obvious way yields the required isomorphism □

4.3 Corollary *In the situation of the previous lemma, $A/\mathbf{G}$ is contractible.*

Proof. $A/\mathbf{G}$ is isomorphic to the contractible category $\mathbf{F}(A)/\mathbf{B}$ □

4.4 Lemma *If*

$$\begin{array}{ccc} \mathbf{C}' & \xrightarrow{\mathbf{G}'} & \mathbf{C} \\ {\scriptstyle \mathbf{F}'}\downarrow & & {\scriptstyle \mathbf{F}}\downarrow \\ \mathbf{D}' & \xrightarrow{\mathbf{G}} & \mathbf{D} \end{array}$$

is a pullback, $\mathbf{G}$ *is an opfibration, and* $\mathbf{F}$ *is a functor whose homotopy fibres* $D/\mathbf{F}$ *are weakly equivalent to a point,then* $\mathbf{F}'$ *is a weak equivalence.*

Proof. [4,p.9] □

4.5 Definition The *homotopy pushout* of the diagram

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\mathbf{F}} & \mathbf{A}' \\ {\scriptstyle \mathbf{I}}\downarrow & & \\ \mathbf{B} & & \end{array} \tag{1}$$

is the opfibred category associated to (1).

Let $\mathbf{X}$ denote the homotopy pushout of (1). The set of objects of $\mathbf{X}$ is the union

$$|\mathbf{A}| \cup |\mathbf{B}| \cup |\mathbf{A}'|$$

of the objects of $\mathbf{A}$,$\mathbf{B}$, and $\mathbf{A}'$.Maps in $\mathbf{X}$ are defined by

$$\mathbf{X}(X,Y) = \begin{cases} \mathbf{A}(X,Y) & \text{if } X,Y \in \mathbf{A} \\ \mathbf{B}(X,Y) & \text{if } X,Y \in \mathbf{B} \\ \mathbf{A}'(X,Y) & \text{if } X,Y \in \mathbf{A}' \\ \mathbf{B}(\mathbf{I}(X),Y) & \text{if } X \in \mathbf{A} \text{ and } Y \in \mathbf{B} \\ \mathbf{A}'(\mathbf{F}(X),Y) & \text{if } X \in \mathbf{A} \text{ and } Y \in \mathbf{A}' \\ \phi & \text{otherwise} \end{cases}$$

Let

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\mathbf{F}} & \mathbf{A}' \\ {\scriptstyle \mathbf{I}}\downarrow & & {\scriptstyle \mathbf{I}'}\downarrow \\ \mathbf{B} & \xrightarrow{\mathbf{F}'} & \mathbf{B}' \end{array}$$

be the pushout of (1). By the universal property of $\mathbf{X}$ there is a canonical map

$$\mathbf{G}\colon \mathbf{X} \to \mathbf{B}'.$$

4.6 **Theorem** *If, in* (1), *the functor* $\mathbf{I}$ *is a strong coideal, then the canonical functor* $\mathbf{G}$ *is a weak equivalence.*

Proof. For each $B' \in \mathbf{B}'$, let $B'/\mathbf{G}$ denote the "homotopy fibre of $\mathbf{G}$ over B'". The objects of $B'/\mathbf{G}$ are pairs

$$\{(x\colon B' \to \mathbf{G}(X), X) \mid X \in \mathbf{X}\}.$$

A map from $(x\colon B' \to \mathbf{G}(X), X)$ to $(y\colon B' \to \mathbf{G}(Y), Y)$ is an arrow $z\colon X \to Y \in \mathbf{X}$ such that $\mathbf{G}(z) \circ x = y$. By Quillens Theorem A [9,p.85], to show that $\mathbf{G}$ is a weak equivalence, it is enough to prove that the unique map

$$B'/\mathbf{G} \to 1$$

to the terminal category 1 is a weak equivalence. By the stability of strong coideals under pushout, $\mathbf{I}'$ is a strong coideal. Moreover, the coretraction

$$\mathbf{R}'\colon \mathbf{I}'(\mathbf{A}') \cdot \mathbf{B}' \to \mathbf{B}'$$

can be chosen so that

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\mathbf{F}"] & \mathbf{A}' \\ \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \arrow[u, "\mathbf{R}"] \arrow[r, "\mathbf{F}'"] & \mathbf{I}'(\mathbf{A}') \cdot \mathbf{B}' \arrow[u, "\mathbf{R}'"'] \end{tikzcd} \qquad (2)$$

commutes.

Objects of the pullback $\mathbf{B}'$ are the equivalence classes for the equivalence relation $\sim$ on the union $|\mathbf{B}| \cup |\mathbf{A}'|$ of the set of objects of $\mathbf{B}$ and $\mathbf{A}'$ generated by $B \sim A'$ if $\mathbf{I}(A) = B$ and $\mathbf{F}(A) = A'$ for some $A \in \mathbf{A}$. Thus every object in $\mathbf{B}'$ is either of the form $\mathbf{I}'(A')$ for some (necessarily unique) $A' \in \mathbf{A}'$ or of the form $\mathbf{F}'(B)$ for some $B \in \mathbf{B}, B \notin \mathrm{Image}(\mathbf{I})$. First consider the case that $B = \mathbf{I}'(A')$. It will be shown that there is a chain of weak equivalences connecting $\mathbf{I}'(A')/\mathbf{G}$ and the contractible category $A'/\mathbf{A}'$. By the lemma, there is an isomorphism between $\mathbf{I}'(A')/\mathbf{G}$ and the homotopy pushout of

$$\begin{tikzcd} \mathbf{I}'(A')/\mathbf{I}' \circ \mathbf{F} \arrow[r] \arrow[d] & \mathbf{I}'(A')/\mathbf{I}' \\ \mathbf{I}'(A')/\mathbf{F}' & \end{tikzcd}$$

since $\mathbf{G} \circ \iota(A') = \mathbf{I}'$, $\mathbf{G} \circ \iota(B) = \mathbf{F}'$, and $\mathbf{G} \circ \iota(A) = \mathbf{I}' \circ \mathbf{F}$. Let $\mathbf{J} : \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \to \mathbf{B}$ denote the inclusion. Then there is an isomorphism

$$\mathbf{I}'(A')/\mathbf{F}' \to \mathbf{I}'(A')/\mathbf{F}' \circ \mathbf{J}. \tag{3}$$

To see this, let $(u : \mathbf{I}'(A' \to \mathbf{F}'(B), B)$ be an object of $\mathbf{I}'(A')/\mathbf{F}'$. Then $B \in \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$. Let $(u_0, \dots, u_n)$ is a representative of u. If $B \in \mathrm{Image}(\mathbf{I})$, there is nothing to show. Suppose that $B \notin \mathrm{Image}(\mathbf{I})$. As $\mathbf{I}$ is an inclusion, $\mathbf{F}'(B) = \{B\}$. Thus u_n is a map in $\mathbf{B}$, say $u_n : B_0 \to B$. If $n = 0$, then B_0 is equivalent to A'. But then $B_0 = \mathbf{I}(A)$ for some $A \in \mathbf{A}$. Hence,$B \in \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$. If $n > 0$ and $B_0 \notin \mathrm{Image}(\mathbf{I})$, $(u_0, \dots, u_{n-1})$ is a representative of a map $\mathbf{I}'(A') \to \mathbf{F}'(B_0)$. By induction, $B_0 \in \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$. But then $B \in \mathbf{I}(\mathbf{A}) \cdot \mathbf{B}$ as well.

There is a weak equivalence

$$\mathbf{I}'(A')/\mathbf{F}' \circ \mathbf{J} \to A'/\mathbf{F}. \tag{4}$$

To see this, note that there is an isomorphism

$$\mathbf{I}'(A')/\mathbf{I}'(\mathbf{A}') \cdot \mathbf{B}' \to A'/\mathbf{R}'$$

in virtue of the adjunction

$$\mathbf{I}' \dashv \mathbf{R}'.$$

Thus there is a commuting diagram of pullback squares

$$\begin{tikzcd}
\mathbf{I}'(A')/\mathbf{F}' \circ \mathbf{J} \arrow[r] \arrow[d] & \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \arrow[d, "\mathbf{F}'"'] \\
\mathbf{I}'(A')/\mathbf{R}' \arrow[r] \arrow[d] & \mathbf{I}'(\mathbf{A}') \cdot \mathbf{B}' \arrow[d, "\mathbf{R}'"'] \\
A'/\mathbf{A}' \arrow[r] & \mathbf{A}'
\end{tikzcd} \tag{5}$$

As $\mathbf{R}' \circ \mathbf{F}' = \mathbf{F} \circ \mathbf{R}$ by (2), there is a unique map

$$\mathbf{Q} : \mathbf{I}'(A')/\mathbf{F}' \circ \mathbf{J} \to A'/\mathbf{F}$$

satisfying the commutativity properties required for satisfaction of the universal property of the pullback

$$\begin{tikzcd}
A'/\mathbf{F} \arrow[r] \arrow[d] & \mathbf{A} \arrow[d, "\mathbf{F}"'] \\
A'/\mathbf{A}' \arrow[r] & \mathbf{A}'
\end{tikzcd}$$

Since the outermost square of (5) is equal to the outermost square of

$$\begin{tikzcd} \mathbf{I}'(A')/\mathbf{F}' \circ \mathbf{J} \arrow[r] \arrow[d, "\mathbf{Q}"'] & \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \arrow[d, "\mathbf{R}"] \\ A'/\mathbf{F}' \arrow[r] \arrow[d] & \mathbf{A} \arrow[d, "\mathbf{F}"] \\ A'/\mathbf{A}' \arrow[r] & \mathbf{A}' \end{tikzcd} \tag{6}$$

and all inner squares of (5) and (6) save possibly

$$\begin{tikzcd} \mathbf{I}'(A')/\mathbf{F}' \circ \mathbf{J} \arrow[r] \arrow[d, "\mathbf{Q}"'] & \mathbf{I}(\mathbf{A}) \cdot \mathbf{B} \arrow[d, "\mathbf{R}"] \\ A'/\mathbf{F} \arrow[r] & \mathbf{A} \end{tikzcd} \tag{7}$$

are pullbacks, (7) is also a pullback. Since $\mathbf{R}$ is a right adjoint, the homotopy fibres $A/\mathbf{R}$ are contractible. As the forgetful functor $A'/\mathbf{F} \to \mathbf{A}$ is an opfibration, Q is a weak equivalence, proving (4).

As $\mathbf{I}'$ is full and faithful, there are isomorphisms

$$\begin{aligned} \mathbf{I}'(A')/\mathbf{I}' \circ \mathbf{F} &\to A'/\mathbf{F} \\ \mathbf{I}'(A')/\mathbf{I}' &\to A'/\mathbf{A}' \end{aligned} \tag{8}$$

By the homotopy invariance of the Grothendieck construction [5,10], the weak equivalences (3),(4), and (8) induce a weak equivalence from $\mathbf{V}$ to the homotopy pushout $\mathbf{W}$ of the diagram

$$\begin{tikzcd} A'/\mathbf{F} \arrow[r] \arrow[d, "id"'] & A'/\mathbf{A}' \\ A'/\mathbf{F} & \end{tikzcd}$$

The collapsing functor which identifies the two copies of $A'/\mathbf{F} \subset \mathbf{W}$ induces a homotopy equivalence with the cylinder

$$\mathrm{CYL}(A'/\mathbf{F} \to A'/\mathbf{A}').$$

The latter is in turn homotopy equivalent to $A'/\mathbf{A}'$, a contractible category.

Next consider the case that $B' = \mathbf{F}'(B)$ for some $B \notin \mathbf{B}, B \in \mathrm{Image}(\mathbf{I})$. As before, there is an isomorphism

$$B'/\mathbf{G} \to \mathbf{V}$$

where $\mathbf{V}$ is the homotopy pushout of

$$\begin{tikzcd} \mathbf{F}'(B)/\mathbf{I}' \circ \mathbf{F} \arrow[r] \arrow[d] & \mathbf{F}'(B)/\mathbf{I}' \\ \mathbf{F}'(B)/\mathbf{F}' & \end{tikzcd}$$

As $B \notin \mathrm{Image}(\mathbf{I})$ by assumption, $\mathbf{F}'(B) \notin \mathrm{Image}(\mathbf{I}')$. As $\mathbf{I}'$ is a coideal, this implies that $\mathbf{F}'(B)/\mathbf{I}' = \phi$, the initial category. Likewise, $\mathbf{F}'(B)/\mathbf{I}' \circ \mathbf{F} = \phi$. Consequently, $\mathbf{V}$ is the homotopy pushout of

$$\begin{tikzcd} \phi \arrow[r] \arrow[d] & \phi \\ \mathbf{F}'(B)/\mathbf{F}' & \end{tikzcd}$$

By inspection, the homotopy pushout of the latter diagram is isomorphic to $\mathbf{F}'(B)/\mathbf{F}'$. But $\mathbf{F}'(B)/\mathbf{F}'$ is isomorphic to the contractible category $B/\mathbf{B}$. Suppose first that $(u\colon \mathbf{F}'(B) \to \mathbf{F}'(B_0), B_0)$ is an object of $\mathbf{F}'(B)/\mathbf{F}'$. Then $B_0 \notin \mathrm{Image}(\mathbf{I})$. To prove this, assume that $B_0 \in \mathrm{Image}(\mathbf{I})$, say $B_0 = \mathbf{I}(A)$ for $A \in \mathbf{A}$. Then

$$\begin{aligned} \mathbf{F}'(B_0) &= \mathbf{F}' \circ \mathbf{I}(A) \\ &= \mathbf{I}' \circ \mathbf{F}(A). \end{aligned}$$

As $\mathbf{I}'$ is a coideal, this implies that $\mathbf{F}'(B) \in \mathrm{Image}(\mathbf{I}')$. It follows that $B \in \mathrm{Image}(\mathbf{I})$, contradicting the choice of B. As $B_0 \notin \mathrm{Image}(\mathbf{I})$, $\mathbf{F}'(B_0) = \{B_0\}$. Let $u\colon \{B\} \to \{B_0\}$ be a map in $\mathbf{B}'$. Using the fact that $\mathbf{I}$ and $\mathbf{I}'$ are coideals, it is readily verified that u lifts to a unique map $B \to B_0$ in $\mathbf{B}$. $\square$

The proof of the next theorem is, *mutatis mutandis*, identical to the proof of the preceding theorem.

4.7 Theorem *Let*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\mathbf{J}"] \arrow[d, "\mathbf{I}"'] & \mathbf{A}' \arrow[d, "\mathbf{I}'"] \\ \mathbf{B} \arrow[r, "\mathbf{J}'"] & \mathbf{B}' \end{tikzcd}$$

be a pushout. Assume that $\mathbf{I}$ *and* $\mathbf{J}$ *are coideals. Then the canonical map from the homotopy pushout of*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\mathbf{J}"] \arrow[d, "\mathbf{I}"'] & \mathbf{A}' \\ \mathbf{B} & \end{tikzcd}$$

to $\mathbf{B}'$ *is a weak equivalence* $\square$

5 Applications

This § enumerates the principal corollaries of the two theorems on homotopy pushouts in $\mathcal{CAT}$. Most notably, pushouts along strong coideals are homotopy invariant.

5.1 Proposition *Let*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\mathbf{F}"] \arrow[d, "\mathbf{I}"'] & \mathbf{A}' \arrow[d, "\mathbf{I}'"'] \\ \mathbf{B} \arrow[r, "\mathbf{F}'"] & \mathbf{B}' \end{tikzcd}$$

be a pushout. Assume that $\mathbf{I}$ *is a strong coideal. Then if* $\mathbf{I}$ *(*$\mathbf{F}$*) is a weak equivalence,* $\mathbf{I}'$ *(*$\mathbf{F}'$ *resp.) is a weak equivalence.*

Proof. Let $\mathbf{X}$ be the homotopy pushout of the diagram

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\mathbf{F}"] \arrow[d, "id_{\mathbf{A}}"'] & \mathbf{A}' \\ \mathbf{A} & \end{tikzcd}$$

and let $\mathbf{Y}$ be the homotopy pushout of

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\mathbf{F}"] \arrow[d, "\mathbf{I}"'] & \mathbf{F}' \\ \mathbf{A}'. & \end{tikzcd}$$

There is a natural transformation ϕ from the first diagram to the second whose components are the vertical arrows in the following diagram:

$$\begin{tikzcd} \mathbf{A} \arrow[d, "\mathbf{I}"'] & \mathbf{A} \arrow[l, "id_{\mathbf{A}}"'] \arrow[r, "\mathbf{F}"] \arrow[d, "id_{\mathbf{A}}"'] & \mathbf{A}' \arrow[d, "id_{\mathbf{A}'}"'] \\ \mathbf{B} & \mathbf{A} \arrow[l, "\mathbf{I}"'] \arrow[r, "\mathbf{F}"] & \mathbf{A}' \end{tikzcd}$$

ϕ induces a map $\mathbf{H}\colon \mathbf{X} \to \mathbf{Y}$. By the homotopy invariance of the Grothendieck construction , $\mathbf{H}$ is a weak equivalence. Consider the diagram

$$\begin{tikzcd} \mathbf{Y} \arrow[d, "\mathbf{G}"'] & \mathbf{X} \arrow[l, "\mathbf{H}"'] \arrow[r, "id_{\mathbf{X}}"] \arrow[d, "\rho_{\mathbf{A}'} \circ \mathbf{K}"'] & \mathbf{X} \arrow[d, "\mathbf{K}"'] \\ \mathbf{B}' & \mathbf{A}' \arrow[l, "\mathbf{I}'"'] & \mathrm{CYL}(\mathbf{F}) \arrow[l, "\rho_{\mathbf{A}'}"'] \end{tikzcd}$$

where $\mathbf{G}$ is the canonical map. By the main result on homotopy push outs, $\mathbf{G}$ is a weak equivalence. $\rho_{\mathbf{A}'}$, the projection on $\mathbf{A}'$ is an adjoint. *A fortiori*, $\rho_{\mathbf{A}'}$ is a weak equivalence. Likewise, $\mathbf{K}$, the collapsing functor which identifies the two copies of $\mathbf{A} \subset \mathrm{CYL}(\mathbf{F})$, is right adjoint to the evident inclusion $\mathrm{CYL}(\mathbf{F}) \hookrightarrow \mathbf{X}$. Hence $\mathbf{K}$ is also a weak equivalence. A straightforward diagram chase shows that

$$\mathbf{G} \circ \mathbf{H} = \mathbf{I}' \circ \rho_{\mathbf{A}'} \circ \mathbf{K}.$$

By saturation of the class of weak equivalences, $\mathbf{I}'$ is a weak equivalence. An exactly analoguous argument can be used to show that if $\mathbf{F}$ is a weak equivalence, $\mathbf{F}'$ is weak equivalence □

The second theorem on homotopy pushouts has an analoguous corollary.

5.2 **Proposition** *Let*

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\mathbf{I}} & \mathbf{A}' \\ {\scriptstyle \mathbf{J}}\downarrow & & \downarrow{\scriptstyle \mathbf{J}'} \\ \mathbf{B} & \xrightarrow{\mathbf{I}'} & \mathbf{B}' \end{array}$$

be a pushout diagram. Assume that both $\mathbf{I}$ *and* $\mathbf{J}$ *are coideals. Then, if* $\mathbf{I}$ *is a weak equivalence,* $\mathbf{I}'$ *is a weak equivalence.*

Proof. The proof parallels the proof of the previous proposition □

6 Factorizations

The purpose of this § is to establish the ubiquity, up to homotopy, of strong coideals. A second aim is to show that, modulo strong coideals which are weak equivalences, every functor induces a long exact sequence in homotopy.

6.1 **Proposition** *Let* $\mathbf{F}: \mathbf{A} \to \mathbf{B}$ *be a functor. Then* $\mathbf{F}$ *admits a factorization of the form* $\mathbf{F} = \mathbf{P} \circ \mathbf{I}$ *where* $\mathbf{P}$ *is a homotopy equivalence and* $\mathbf{I}$ *is a strong coideal.*

The proof proceeds in stages. Let $\mathbf{F}: \mathbf{A} \to \mathbf{B}$ be a functor. Let $\mathbf{C}$ be the opfibred category associated with the diagram

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\mathbf{F}} & \mathbf{B} \\ {\scriptstyle id_{\mathbf{A}}}\downarrow & & \\ \mathbf{A} & \xleftarrow{id_{\mathbf{A}}} & \mathbf{A} \end{array}$$

Explicitly, objects of $\mathbf{C}$ are pairs (A, i) where $A \in \mathbf{A}$ and $i \in \{0, 1, 2\}$ or are of the form $B \in \mathbf{B}$. Maps are defined by

$$
\begin{array}{lll}
(1) & \mathbf{C}((A,0),(A,i)) = \begin{cases} \mathbf{A}(A,A') & i=0,1 \\ \phi & i=2 \end{cases} \\
(2) & \mathbf{C}((A,1),(A',i)) = \begin{cases} \mathbf{A}(A,A') & i=1 \\ \phi & i=0,2 \end{cases} \\
(3) & \mathbf{C}((A,2),(A',i)) = \begin{cases} \mathbf{A}(A,A') & i=1,2 \\ \phi & i=0 \end{cases} \\
(4) & \mathbf{C}((A,i),B) = \begin{cases} \mathbf{B}(\mathbf{F}(A),B) & i=2 \\ \phi & i=0,1 \end{cases} \\
(5) & \mathbf{C}(B,B') = \mathbf{B}(B,B') \\
(6) & \mathbf{C}(B,(A,i)) = \phi \qquad i=0,1,2
\end{array}
$$

Let $\mathbf{I}\colon \mathbf{A} \to \mathbf{C}$ be the evident inclusion defined on objects by $\mathbf{I}(A) = (A,0)$.

6.2 Lemma **I** *is a coideal* □

6.3 Lemma **I** *admits a coretraction*

$$\mathbf{R}\colon \mathbf{I}(\mathbf{A}) \cdot \mathbf{C} \to \mathbf{A}$$

from the ideal it generates in $\mathbf{C}$.

Proof. Let $\mathbf{I}(A) \to C$ be a map in $\mathbf{I}(\mathbf{A}) \cdot \mathbf{C}$. Then, by the definition of $\mathbf{C}$, $C = (A',0)$ or $C = (A',1)$ for some $A' \in \mathbf{A}$. In either case, $\mathbf{I}(A) \to C$ is represented by a unique map $a\colon A \to A' \in \mathbf{A}$. Define $\mathbf{R}(\mathbf{I}(A) \to C) = a$.The verification that

$$\mathbf{I} \dashv \mathbf{R}, \quad \mathbf{R} \circ \mathbf{I} = id$$

is straightforward □

In other words, **I** is a strong coideal. This proves the first part of the proposition. Let **D** be the opfibred category associated with the diagram

$$
\begin{array}{ccc}
\mathbf{A} & \xrightarrow{\mathbf{F}} & \mathbf{B} \\
{\scriptstyle id_{\mathbf{A}}}\downarrow & & \\
\mathbf{A} & &
\end{array}
$$

Define $\mathbf{Q}\colon \mathbf{C} \to \mathbf{D}$ on objects by

$$
\mathbf{Q}(C) = \begin{cases} B & \text{if } C = B \in \mathbf{B} \\ (A,2) & \text{if } C = (A,2) \\ (A,1) & \text{if } C = (A,0) \text{ or } C = (A,1) \end{cases}
$$

6.4 Lemma *$\mathbf{Q}$ is a left adjoint, left inverse.*

Proof. Let $\mathbf{J}: \mathbf{D} \to \mathbf{C}$ be the evident inclusion. Then $\mathbf{Q} \dashv \mathbf{J}$ and $\mathbf{Q} \circ \mathbf{J} = id$ □

In particular, $\mathbf{Q}$ is a homotopy equivalence. Let $\mathbf{R}: \mathbf{D} \to \mathrm{CYL}(\mathbf{F})$ be the functor which identifies $(A, 1)$ and $(A, 2)$ for all $A \in \mathbf{A}$.

6.5 Lemma *$\mathbf{R}$ is a right adjoint, left inverse.*

Proof. Let $\mathbf{K}: \mathrm{CYL}(\mathbf{F}) \to \mathbf{D}$ be the evident inclusion. It is readily verified that

$$\mathbf{K} \dashv \mathbf{R} \quad \textit{and} \quad \mathbf{R} \circ \mathbf{K} = id \quad \Box$$

In particular, $\mathbf{R}$ is a homotopy equivalence.

6.6 Lemma *Let $\rho_{\mathbf{B}}$: $CYL(\mathbf{F}) \to \mathbf{B}$ be the projection introduced earlier. $\rho_{\mathbf{B}}$ is a homotopy equivalence.*

Proof. As before, let $\iota_B : \mathbf{B} \to \mathrm{CYL}(\mathbf{F})$ denote the inclusion. For each $C \in \mathrm{CYL}(\mathbf{F})$, define θ_C by

$$\theta_C = \begin{cases} id_B : (B,1) \to (B,1) & \text{if } C = (B,1) \\ id_{\mathbf{I}(A)} : (A,0) \to (\mathbf{I}(A),1) & \text{if } C = (A,0) \end{cases}$$

$\{\theta_C\}$ are the components of a natural transformation

$$\theta : id \to \iota_B \circ \rho_{\mathbf{B}}.$$

As,in addition, $\rho_{\mathbf{B}} \circ \iota_B = id$,$\rho_{\mathbf{B}}$ and ι_B are homotopy inverses □

As each of $\rho_{\mathbf{B}}$,$\mathbf{Q}$, and $\mathbf{R}$ is a homotopy equivalence, the composite $\rho_{\mathbf{B}} \circ \mathbf{Q} \circ \mathbf{R}$ is also a homotopy equivalence. Let $A \in \mathbf{A}$. Then

$$\begin{aligned} \rho_{\mathbf{B}} \circ \mathbf{Q} \circ \mathbf{R} \circ \mathbf{I}(A) &= \rho_{\mathbf{B}} \circ \mathbf{Q} \circ \mathbf{R}(A,0) \\ &= \rho_{\mathbf{B}} \circ \mathbf{Q}(A,1) \\ &= \rho_{\mathbf{B}}(A,0) \\ &= \mathbf{F}(A). \end{aligned}$$

This finishes the proof of the second part of the proposition □

Strong coideals can be used to show that, in the homotopy category, every map has the distinguishing property of a fibration. Specifically, let $\mathbf{Sd}$: $(\Delta^{\mathrm{op}}, Sets) \to (\Delta^{\mathrm{op}}, Sets)$ denote Kan's simplicial subdivision functor [K]. Let $\mathbf{Cat}$: $(\Delta^{\mathrm{op}}, Sets) \to CAT$ denote the left adjoint to **Nerve** [3]. The proof of the main result makes use of the next three lemmas.

6.7 Lemma *Let $i\colon Y \to X \in (\Delta^{\mathrm{op}}, Sets)$ be a monomorphism. Then* $\mathbf{Cat} \circ \mathrm{Sd}^2(i)$ *is a strong coideal. If, in addition, i is a weak equivalence, then* $\mathbf{Cat} \circ \mathrm{Sd}^2(i)$ *is also a weak equivalence.*

Proof. [4,p.93] □

6.8 Lemma *Let $\{\mathbf{I}_\alpha\colon \mathbf{C}_\alpha \to \mathbf{D}_\alpha \mid \alpha \in A\}$ be a collection of coideals which admit coreflections*

$$\mathbf{R}_\alpha\colon \mathbf{I}_\alpha(\mathbf{C}_\alpha) \cdot \mathbf{D}_\alpha \to \mathbf{C}_\alpha.$$

Then the induced map

$$\coprod_{\alpha\in A} \mathbf{I}_\alpha\colon \coprod_{\alpha\in A} \mathbf{C}_\alpha \to \coprod_{\alpha\in A} \mathbf{D}_\alpha$$

has the same property. If, for each $\alpha \in A$, $\mathbf{I}_\alpha$ is a weak equivalence, then $\coprod \mathbf{I}_\alpha$ is a weak equivalence.

Proof. ,[4,p.79] □

6.9 Lemma *Let $\{\mathbf{I}_n\colon \mathbf{A}_n \to \mathbf{A}_{n+1} \mid n \geq 0\}$ be a collection of strong coideals. Let*

$$\mathbf{I}\colon \mathbf{A}_0 \to \lim_{n\to\infty} \mathbf{A}_n$$

be the map induced by

$$\{\mathbf{I}_{n-1} \circ \cdots \circ \mathbf{I}_0 \mid n \geq 1\}.$$

Then $\mathbf{I}$ *is a strong coideal. If each* $\mathbf{I}_n$ *ia a weak equivalence,* $\mathbf{I}$ *is a weak equivalence.*

Proof. [4,p.80] □

Let $\mathbf{F}\colon \mathbf{A} \to \mathbf{B}$ be a functor. Let $\mathbf{F}^{-1}(B)$ denote the pullback

$$\begin{array}{ccc} \mathbf{F}^{-1}(B) & \xrightarrow{\mathbf{J}} & \mathbf{A} \\ \downarrow & & \downarrow{\scriptstyle \mathbf{F}} \\ 1 & \xrightarrow{B} & \mathbf{B} \end{array}$$

where $B\colon 1 \to \mathbf{B}$ picks out $B \in \mathbf{B}$. In other words, $\mathbf{F}^{-1}(B)$ is the fibre of $\mathbf{F}$ over B.

6.10 Definition $\mathbf{F}$ is a *quasifibration* if for all $B \in \mathbf{B}$, the sequence

$$\mathbf{F}^{-1}(B) \xrightarrow{\mathbf{J}} \mathbf{A} \xrightarrow{\mathbf{F}} \mathbf{B}$$

induces a long exact sequence of homotopy groups

$$\cdots \to \pi_n(\mathbf{Nerve}(\mathbf{F})^{-1}(B), A) \xrightarrow{\mathbf{J}_*} \pi_n(\mathbf{Nerve}(\mathbf{A}), A) \xrightarrow{\mathbf{F}_*}$$
$$\pi_n(\mathbf{Nerve}(\mathbf{B}), B) \xrightarrow{\partial} \pi_{n-1}(\mathbf{Nerve}(\mathbf{F})^{-1}(B), A) \to \cdots$$

In other words, $\mathbf{Nerve(F)}$ is a simplicial quasifibration.

6.11 Proposition *Every functor* $\mathbf{F}\colon \mathbf{A} \to \mathbf{B}$ *admits a factorization* $\mathbf{F} = \mathbf{P} \circ \mathbf{I}$ *where* $\mathbf{I}$ *is a strong coideal and a weak equivalence and* $\mathbf{P}$ *is a quasifibration.*

Proof. Let

$$\Lambda^k(n) \hookrightarrow \Delta(n) \quad (n > 0, 0 \leq k \leq n)$$

denote the k-horn [3,p.60]. As $\mathbf{Cat} \circ \mathbf{Sd}^2(\Lambda^k(n))$ is finite and connected, the hom–functor

$$\mathcal{CAT}(\mathbf{Cat} \circ \mathbf{Sd}^2(\Lambda^k(n)), \cdot)\colon \mathcal{CAT} \to Sets$$

commutes with sequential limits,i.e. $\mathbf{Cat} \circ \mathbf{Sd}^2(\Lambda^k(n))$ is *small* to borrow Quillen's terminology [9,p.34]. By Quillen's small object argument [9,p.34], it follows that any functor $\mathbf{F}\colon \mathbf{A} \to \mathbf{B}$ has a factorization of the form $\mathbf{F} = \mathbf{P} \circ \mathbf{I}$ where $\mathbf{P}$ has the right lifting property (RLP) with respect to the collection

$$\{\mathbf{Cat} \circ \mathbf{Sd}^2(\Lambda^k(n)) \hookrightarrow \mathbf{Cat} \circ \mathbf{Sd}^2(\Lambda^k(n))\}.$$

By Quillen's argument, $\mathbf{I}$ is in the closure of the collection

$$\mathbf{Cat} \circ \mathbf{Sd}^2(\Lambda^k(n)) \hookrightarrow \mathbf{Cat} \circ \mathbf{Sd}^2(\Lambda^k(n))\}$$

under pushout, formation of coproducts, and formation of countable sequential limits. By previous results, $\mathbf{I}$ is a strong ideal and a weak equivalence. It remains to prove that $\mathbf{P}$ induces a long exact sequence in homotopy. Since $\mathbf{Sd}$ has a right adjoint $\mathbf{Ex}$ [7,p. 460],$\mathbf{Cat} \circ \mathbf{Sd}^2$ has a right adjoint $\mathbf{Ex}^2 \circ \mathbf{Nerve}$. In virtue of the adjunction, $\mathbf{Ex}^2 \circ \mathbf{Nerve(P)}$ has the RLP with respect to the collection

$$\{\Lambda^k(n) \hookrightarrow \Delta(n)\}.$$

In other words,$\mathbf{Ex}^2 \circ \mathbf{Nerve(P)}$ is a Kan fibration [3,p.65]. By definition,

$$\mathbf{Ex}(X)(n) = HOM(\mathbf{Sd}(\Delta(n)), X)$$

where $X \in (\Delta^{\mathrm{op}}, Sets)$ and $HOM(\cdot,\cdot)$ is the $(\Delta^{\mathrm{op}}, Sets)$–valued hom. In particular, $\mathbf{Ex}(X)(0) = HOM(\mathbf{Sd}(\Delta(n)), X) = X(0)$, the set of 0–simplices of X. Consequently, $\mathbf{Ex}^2 \circ \mathbf{Nerve(B)}(0) = \mathbf{Nerve(B)}(0)$. Let $B \in \mathbf{B}$. Then B is a 0–simplex of $\mathbf{Nerve(B)}$ and hence of $\mathbf{Ex}^2 \circ \mathbf{Nerve(B)}$. As $\mathbf{Ex}^2 \circ \mathbf{Nerve}$ is a right adjoint, $\mathbf{Ex}^2 \circ \mathbf{Nerve}$ commutes with inverse limits. In particular, the pullback diagram defining the fibre $\mathbf{P}^{-1}(B)$ is carried into a pullback

$$\begin{array}{ccc}
\mathbf{Ex}^2 \circ \mathbf{Nerve}(\mathbf{P}^{-1}(B)) & \xrightarrow{\mathbf{Ex}^2\circ\mathbf{Nerve(J)}} & \mathbf{Ex}^2 \circ \mathbf{Nerve(A)} \\
\downarrow & & \downarrow{\scriptstyle \mathbf{Ex}^2\circ\mathbf{Nerve(F)}} \\
\Delta(0) & \xrightarrow{\mathbf{Ex}^2\circ\mathbf{Nerve}(B)} & \mathbf{Ex}^2 \circ \mathbf{Nerve(B)}
\end{array}$$

In short,

$$\mathbf{Ex}^2 \circ \mathbf{Nerve}(\mathbf{P}^{-1}(B)) \simeq (\mathbf{Ex}^2 \circ \mathbf{Nerve}(\mathbf{P}))^{-1}(B).$$

There is a natural transformation

$$\phi\colon id \Rightarrow \mathbf{Ex}^2$$

whose components

$$\{\phi_X \mid X \in (\Delta^{\mathrm{op}}, Sets\}$$

are weak equivalences [7,p.455]. Let $C \in \mathbf{P}^{-1}(B)$. Consider the diagram:

$$\begin{tikzcd}
(\mathbf{Nerve}(\mathbf{P}^{-1}(B)), C) \arrow[r, "\phi"] \arrow[d, "\mathbf{Nerve(J)}"'] & (\mathbf{Ex}^2 \circ \mathbf{Nerve}(\mathbf{P}^{-1}(B)), C) \arrow[d, "\mathbf{Ex}^2\circ\mathbf{Nerve(J)}"'] \\
(\mathbf{Nerve}(\mathbf{C}), C) \arrow[r, "\phi"] \arrow[d, "\mathbf{Nerve(P)}"'] & (\mathbf{Ex}^2 \circ \mathbf{Nerve}(\mathbf{C}), C) \arrow[d, "\mathbf{Ex}^2\circ\mathbf{Nerve(P)}"'] \\
(\mathbf{Nerve}(\mathbf{B}), B) \arrow[r, "\phi"] & (\mathbf{Ex}^2 \circ \mathbf{Nerve}(\mathbf{B}), B)
\end{tikzcd}$$

As $\mathbf{Ex}^2 \circ \mathbf{Nerve}(\mathbf{P}^{-1}(B)) \simeq (\mathbf{Ex}^2 \circ \mathbf{Nerve}(\mathbf{P}))^{-1}(B)$, the right hand column is a fibre sequence. As the horizontal arrows are weak equivalences, the left hand column induces a long exact sequence in homotopy. This finishes the proof that $\mathbf{P}$ is a quasfibration □

References

1. D. Anderson, *Geometric Realization and Fibrations*, Bulletin of the A.M.S. **84** (1978), 765–788.
2. J. Gray, *Fibred and Co-Fibred Categories*, Conference on Categorical Algebra, Springer Verlag, Berlin, 1965, pp. 21-83.
3. P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory*, Springer Verlag, Berlin, 1967.
4. M. Heggie, *Tensor Products in Homotopy Theory*, Ph.D. Thesis, McGill University, 1986.
5. ———, *The Left Derived Tensor Product of CAT- valued Diagrams*, preprint.
6. ———, *Homotopy Colimits in Presheaf Categories*, preprint.
7. D. Kan, *On C.S.S. Complexes*, American Journal of Mathematics **79** (1957), 449–476.
8. D. Quillen, *Homotopical Algebra*, Lecture Notes in Mathematics, vol. 43, Springer Verlag, Berlin, 1967.
9. ———, *Higher Algebraic K-Theory I*, Lecture Notes in Mathematics, vol. 341, Springer Verlag, Berlin, 1973, pp. 85–147.
10. R. Thomason, *Homotopy Colimits in the Category of Small Categories*, Math. Proc. Camb. Phil. Soc. **85** (1979), 91–109.
11. G.W.Whitehead, *Elements of Homotopy Theory*, Springer Verlag, Berlin, 1978.

SYDNEY, NOVA SCOTIA, CANADA, B1P6L2

E-mail: heggie@cad.uccb.ns.ca