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EPIREFLECTIONS WHICH ARE COMPLETIONS

by G.C.L. BRUMMER, E. GIULI and H. HERRLICH

*We dedicate this paper to the
memory of Siegfried Grässer*

Résumé. Nous axiomatisons la situation où tout objet d'une catégorie \mathbf{X} a un complété et où tout plongement dense dans un objet complet quelconque est une réflexion dans la sous-catégorie pleine des objets complets. On dit alors que \mathbf{X} admet une sous-catégorie \mathcal{S} -fermement \mathcal{E} -réflexive. Ici, \mathcal{S} est une classe de morphismes de \mathbf{X} ayant des propriétés analogues aux plongements, et la classe \mathcal{E} représente la densité appropriée. Pour le cas $\mathcal{E} = \text{Epi}\mathbf{X}$ nous relierons cette notion avec celles de fermeture \mathcal{S} -absolue, de \mathcal{S} -saturation, et de $(\mathcal{E} \cap \mathcal{S})$ -injectivité; nous en donnons plusieurs caractérisations, en particulier la préservation des \mathcal{S} -morphismes; et nous considérons beaucoup d'exemples topologiques et algébriques. Quand \mathbf{X} est une catégorie topologique on a un contexte naturel pour lequel \mathcal{E} est plus large que la classe des épomorphismes.

0. Introduction

Among the various kinds of extensions that an object can have, compactifications and completions of spaces exhibit two very different forms of behaviour. A Tychonoff space, say, can have many mutually inequivalent Hausdorff compactifications, among which there is the

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Čech-Stone compactification which is the reflection to compact spaces. On the other hand, a metric space, say, admits, up to isometry, just one completion, and that completion is the reflection to complete spaces.

The latter form of behaviour is paradigmatic for the kind of completion of objects that we study in this paper. When it occurs in a category \mathbf{X} , we say that the “complete” objects form an \mathcal{S} -firm \mathcal{E} -reflective subcategory of \mathbf{X} . Here \mathcal{S} is a class of morphisms in \mathbf{X} of which we like to think as embeddings, and \mathcal{E} represents the kind of density that an object should have in its completion.

Our chosen setting then is an $(\mathcal{E}, \mathcal{M})$ -category \mathbf{X} with a designated class \mathcal{S} of morphisms. Mild assumptions on the interplay between the $(\mathcal{E}, \mathcal{M})$ -factorization and the class \mathcal{S} determine the results. These become trivial when \mathcal{S} coincides with \mathcal{M} .

For the sake of clarity, the main body of the paper is developed for the case that $(\mathcal{E}, \mathcal{M}) = (\text{epi}, \text{extremal mono})$. The category \mathbf{X} with the given \mathcal{S} admits at most one \mathcal{S} -firmly epireflective subcategory \mathbf{R} . In this case, \mathbf{R} consists of those \mathbf{X} -objects which are injective with respect to epimorphisms in \mathcal{S} , and \mathbf{R} also coincides with the subcategory of \mathcal{S} -saturated objects; under an additional assumption, \mathbf{R} is the subcategory of absolutely \mathcal{S} -closed objects. It is clear that \mathbf{X} admits an \mathcal{S} -firm epireflection if and only if every \mathbf{X} -object admits an epic \mathcal{S} -morphism into an object which is $(\mathcal{S} \cap \text{Epi}\mathbf{X})$ -injective. We show (Theorem 1.6) that this is equivalent to \mathbf{X} being \mathcal{S} -cogenerated by a class of $(\mathcal{S} \cap \text{Epi}\mathbf{X})$ -injectives. The main result (Theorem 2.5) is that a given \mathcal{S} -epireflector is \mathcal{S} -firm if and only if it preserves \mathcal{S} -morphisms (equivalently: \mathcal{S} -sources) and reflects into the class of \mathcal{S} -saturated objects. We give examples in topology and algebra. The applications to topological categories also provide a context for firm \mathcal{E} -reflections with \mathcal{E} other than epi.

Injectives with respect to epimorphic embeddings were studied by P.D. Bacsich [2], with results partly of the same intent as the present paper. R.-E. Hoffmann [38] defined and investigated firm reflections in topological categories. The latter paper was the first, to our knowledge, which proved the firmness of the sobrification epireflector in the

T_0 -topological spaces. It may be observed that the precategory paper [10] by G. Birkhoff can still serve as a source of ideas connected with the notion of completeness.

1. Firm epireflections

Throughout the paper we consider a category \mathbf{X} which is complete and well-powered and a fixed class \mathcal{S} of morphisms of \mathbf{X} satisfying the following properties:

- (s₁) $\text{Iso}\mathbf{X} \subset \mathcal{S} \subset \text{Mono}\mathbf{X}$;
- (s₂) \mathcal{S} is closed under composition;
- (s₃) If $me \in \mathcal{S}$ and $e \in \text{Epi}\mathbf{X}$, $m \in \text{ExMono}\mathbf{X}$, then $e \in \mathcal{S}$.

$\text{Mor}\mathbf{X}$, $\text{Epi}\mathbf{X}$, $\text{Mono}\mathbf{X}$, $\text{ExMono}\mathbf{X}$, $\text{RegMono}\mathbf{X}$, $\text{Iso}\mathbf{X}$ denote the class of all morphisms, epimorphisms, monomorphisms, extremal monomorphisms, regular monomorphisms and isomorphisms of \mathbf{X} , respectively. If \mathbf{X} is concrete then $\text{Emb}\mathbf{X}$ denotes the class of embeddings of \mathbf{X} (i.e. initial maps whose underlying maps are mono); it satisfies conditions (s₁)-(s₃).

Note that every \mathcal{M} which is part of an $(\mathcal{E}, \mathcal{M})$ -factorization structure of \mathbf{X} , with $\text{Epi}\mathbf{X} \subset \mathcal{E}$, has the properties (s₁)-(s₃) and in fact also satisfies the additional assumptions (s₄) and (s₅) which we impose later.

All subcategories will be taken as full and isomorphism-closed.

Since \mathbf{X} is complete and well-powered, then (cf. [36, 34A]):

(x₁) \mathbf{X} is an $(\text{Epi}\mathbf{X}, \text{ExMono}\mathbf{X})$ -category.

Definitions 1.1. *Let X be an \mathbf{X} -object.*

- (1) X is said to be \mathcal{S} -injective if, for each $e : Y \rightarrow Z$ in \mathcal{S} and each \mathbf{X} -morphism $f : Y \rightarrow X$, there is an \mathbf{X} -morphism $g : Z \rightarrow X$ such that $ge = f$. Then g is called (an) extension of f (to \mathbf{Z}).
- (2) X is said to be weakly \mathcal{S} -injective if it is $(\text{Epi}\mathbf{X} \cap \mathcal{S})$ -injective. $\text{Inj}(\mathcal{S})$ ($\text{WInj}(\mathcal{S})$) denotes the class of all (weakly) \mathcal{S} -injective objects of \mathbf{X} .
- (3) X is said to be \mathcal{S} -saturated if an \mathbf{X} -morphism $f : X \rightarrow Y$ is an isomorphism whenever $f \in \text{Epi}\mathbf{X} \cap \mathcal{S}$. $\text{Sat}(\mathcal{S})$ denotes the

class of all \mathcal{S} -saturated objects of \mathbf{X} .

- (4) *X is said to be absolutely \mathcal{S} -closed if an \mathbf{X} -morphism $f : X \rightarrow Y$ is a regular monomorphism whenever $f \in \mathcal{S}$. $AC(\mathcal{S})$ denotes the class of all absolutely \mathcal{S} -closed objects of \mathbf{X} .*

We refer to [36] and [32] for other categorical terms not defined here.

Weakly \mathcal{S} -injective implies \mathcal{S} -saturated and absolutely \mathcal{S} -closed implies \mathcal{S} -saturated (see Proposition 1.3 below).

For the case \mathbf{X} a category of algebras (in particular of rings or semigroups) and \mathcal{S} the class of all monomorphisms, \mathcal{S} -saturated and absolutely \mathcal{S} -closed objects were introduced and investigated by Isbell [42] (see also [39]). It is shown in [42] that an \mathcal{S} -saturated algebra need not be absolutely \mathcal{S} -closed.

For the case \mathbf{X} an epireflective subcategory of \mathbf{Top} and $\mathcal{S} = \mathbf{EmbX}$, \mathcal{S} -saturated and absolutely \mathcal{S} -closed objects were studied by Dikranjan and Giuli in [16] and [19]. It was observed in [16] that absolute \mathcal{S} -closedness coincides with \mathbf{X} -closedness whenever the \mathbf{X} -epimorphisms coincide with the dense continuous maps. The latter notion was introduced by Alexandroff and Urysohn [1] for the category of Hausdorff spaces (H -closedness) and was investigated by many authors (see e.g. [9] and [51]). In such a case also absolutely \mathcal{S} -closed = \mathcal{S} -saturated. It was also proved in [19] that, for \mathbf{X} the category of Urysohn spaces (or, more generally for \mathbf{X} the category of $S(n)$ -spaces), an \mathcal{S} -saturated space need not be absolutely \mathcal{S} -closed and absolutely \mathcal{S} -closed need not be \mathbf{X} -closed.

Weak \mathcal{S} -injectivity vanishes in some contexts in which the \mathcal{S} -injectivity is consistent (e.g. in abelian categories \mathbf{X} with $\mathcal{S} = \mathbf{MonoX}$). In \mathbf{Unif}_0 , with $\mathcal{S} = \{\text{Embeddings}\}$, \mathcal{S} -injectivity is a much stronger property than weak \mathcal{S} -injectivity [41]. The latter notion is equivalent to completeness. Analogously in \mathbf{Top}_0 , with $\mathcal{S} = \{\text{Embeddings}\}$, \mathcal{S} -injective space means retract of a product of Sierpinski spaces [49] while weakly \mathcal{S} -injective space means sober space [38].

\mathcal{S} -injectivity with no conditions on \mathbf{X} and on \mathcal{S} , is investigated by Herrlich [35] (see also [34]).

Let X be a topological space and let \mathbf{X} be the epireflective hull of X in \mathbf{Top} . Sobral [50] showed that the Eilenberg-Moore factorization of the functor $\text{Hom}(-, X): \mathbf{Top}^{\text{op}} \rightarrow \mathbf{Set}$ can be obtained via the corresponding factorization of its restriction to the subcategory $AC(\text{Emb}\mathbf{X})$, provided \mathbf{X} satisfies an injectivity condition. This injectivity condition is weaker than the $(\text{Emb}\mathbf{X})$ -injectivity notion but it is not comparable with our weak $(\text{Emb}\mathbf{X})$ -injectivity.

If \mathbf{R} is an epireflective subcategory of \mathbf{X} , $R: \mathbf{X} \rightarrow \mathbf{X}$ denotes the reflection functor, and, for each \mathbf{X} -object X , $r_X: X \rightarrow R(X)$ denotes the \mathbf{R} -reflection morphism. For each \mathbf{X} -morphism $f: X \rightarrow Y$, $Y \in \text{Ob}\mathbf{R}$, $f^*: R(X) \rightarrow Y$ denotes the unique \mathbf{X} -morphism such that $f^*r_X = f$.

Definitions 1.2. *Let \mathbf{R} be a reflective subcategory of \mathbf{X} .*

- (1) \mathbf{R} is said to be \mathcal{S} -epireflective if for each \mathbf{X} -object X , $r_X: X \rightarrow R(X)$ belongs to $\text{Epi}\mathbf{X} \cap \mathcal{S}$.
- (2) \mathbf{R} is said to be an \mathcal{S} -firm epireflective subcategory if it is \mathcal{S} -epireflective and, for each $f: X \rightarrow Y$, with $Y \in \mathbf{R}$, f^* is an isomorphism whenever $f \in \text{Epi}\mathbf{X} \cap \mathcal{S}$.

Proposition 1.3. *Let \mathbf{R} be a subcategory of \mathbf{X} and for an \mathbf{X} -object X consider the following conditions:*

- (i) X is \mathcal{S} -saturated;
- (ii) X belongs to \mathbf{R} ;
- (iii) X is weakly \mathcal{S} -injective;
- (iv) X is absolutely \mathcal{S} -closed.

Then the following hold:

- (a) Always (iii) \Rightarrow (i) and (iv) \Rightarrow (i);
- (b) (i) \Rightarrow (iv) whenever $\text{ExMono}\mathbf{X} = \text{RegMono}\mathbf{X}$;
- (c) (i) \Rightarrow (ii) whenever \mathbf{R} is \mathcal{S} -epireflective;
- (d) (ii) \Rightarrow (iii) whenever \mathbf{R} is \mathcal{S} -firmly epireflective in \mathbf{X} .

Proof.

(a) (iii) \Rightarrow (i): Since \mathbf{X} is weakly \mathcal{S} -injective, then for each $(e: X \rightarrow$

$Z) \in \text{Epi}\mathbf{X} \cap \mathcal{S}$, the identity 1_X has an extension g , so $ge = 1_X$, consequently $e \in \text{Iso}\mathbf{X}$. (iv) \Rightarrow (i): Every \mathbf{X} -morphism which is both epi and regular mono is an isomorphism.

(b) If an \mathbf{X} -morphism $f : X \rightarrow Y$ belongs to \mathcal{S} then, by property (s_3) , in the (epi, extremal mono)-factorization $me = f$, $e \in \mathcal{S}$. If $X \in \text{Sat}(\mathcal{S})$ then e is an isomorphism, so f is an extremal mono, hence, by assumption, it is a regular mono, consequently $X \in \text{AC}(\mathcal{S})$.

(c) If \mathbf{R} is \mathcal{S} -epireflective then every \mathbf{R} -reflection $r_X : X \rightarrow R(X)$ belongs to $\text{Epi}\mathbf{X} \cap \mathcal{S}$. If in addition $X \in \text{Sat}(\mathcal{S})$, then r_X must be an isomorphism, consequently $X \in \mathbf{R}$.

(d) Whenever $e : Y \rightarrow Z$ belongs to $\text{Epi}\mathbf{X} \cap \mathcal{S}$, since \mathbf{R} is \mathcal{S} -epireflective and since \mathcal{S} has property (s_2) , then $r_Z e : Y \rightarrow Z \rightarrow R(Z)$ belongs to $\text{Epi}\mathbf{X} \cap \mathcal{S}$. Since \mathbf{R} is supposed to be \mathcal{S} -firm, then $(r_Z e)^*$ in the commutative diagram below is an isomorphism.

$$\begin{array}{ccccc}
 Y & \xrightarrow{e} & Z & \xrightarrow{r_Z} & R(Z) \\
 \downarrow f & & \searrow r_Y & & \uparrow (r_Z e)^* \\
 X & \xleftarrow{f^*} & & & R(Y)
 \end{array}$$

Thus, for every \mathbf{X} -morphism $f : Y \rightarrow X$, the \mathbf{X} -morphism $g = f^*((r_Z e)^*)^{-1}r_Z$ is the needed extension of f .

We refer to [2, (Theorem 2.2)] for other conditions under which $\text{Sat}(\mathcal{S})$, $\text{AC}(\mathcal{S})$ and $\text{WInj}(\mathcal{S})$ coincide.

Corollary 1.4. \mathbf{X} admits at most one \mathcal{S} -firm epireflective subcategory \mathbf{R} . In such a case $\mathbf{R} = \text{Sat}(\mathcal{S}) = \text{WInj}(\mathcal{S})$.

Question A. Prove or disprove that, for each (concrete) category \mathbf{X} (and $\mathcal{S} = \text{Emb}\mathbf{X}$), $\text{Sat}(\mathcal{S}) = \text{AC}(\mathcal{S})$ holds whenever \mathbf{X} admits an \mathcal{S} -firm epireflective subcategory.

Question B. Prove or disprove that, for each (concrete) category \mathbf{X} (and $\mathcal{S} = \text{Emb}\mathbf{X}$), $\text{WInj}(\mathcal{S}) \subset \text{AC}(\mathcal{S})$.

Definition 1.5. Let \mathbf{P} be a class of \mathbf{X} -objects. We say that \mathbf{X} is \mathcal{S} -cogenerated by \mathbf{P} if every \mathbf{X} -object is an \mathcal{S} -subobject (X is \mathcal{S} -subobject of Y if there is $s : X \rightarrow Y$, with $s \in \mathcal{S}$) of a product of objects in \mathbf{P} .

The category \mathbf{X} admits an \mathcal{S} -firm epireflection if and only if each \mathbf{X} -object is the domain of some $\mathcal{S} \cap \text{Epi}\mathbf{X}$ -morphism into some object of $W\text{Inj}(\mathcal{S})$. (This is immediately clear from Proposition 1.3 (d).)

$W\text{Inj}(\mathcal{S})$ is closed for the taking of products in \mathbf{X} (again clear, with no assumptions on \mathbf{X} or \mathcal{S} other than $\mathcal{S} \subset \text{Mor}\mathbf{X}$).

$W\text{Inj}(\mathcal{S})$ is closed for the taking of extremal subobjects in \mathbf{X} (this immediately follows from merely the $(\text{Epi}\mathbf{X}, \text{ExMono}\mathbf{X})$ -diagonalization property).

If \mathbf{X} is assumed to be complete, well-powered and co-(well-powered), it follows by [36, (Theorem 37.1)] that $W\text{Inj}(\mathcal{S})$ is epireflective in \mathbf{X} . Alternatively, it was shown in [6, (Proposition 1 and pp. 157-158)] that $W\text{Inj}(\mathcal{S})$ is epireflective in \mathbf{X} , provided \mathbf{X} is an $(\text{Epi}, \text{ExMonoSource})$ -category (cf. [34, (p. 331)]).

It is worth noting that the following result does not assume \mathbf{X} to be co-(well-powered).

Theorem 1.6. *The category \mathbf{X} admits an \mathcal{S} -firm epireflective subcategory \mathbf{R} iff there exists a class $\mathbf{P} \subset W\text{Inj}(\mathcal{S})$ which \mathcal{S} -cogenerates \mathbf{X} . In this case, \mathbf{R} is the epireflective hull in \mathbf{X} of \mathbf{P} .*

Proof. Suppose $r_X : X \rightarrow R(X)$ is the firm \mathcal{S} -epireflection of X in \mathbf{X} . By definition, $r_X \in \mathcal{S}$, and by Proposition 1.3, $R(X) \in W\text{Inj}(\mathcal{S})$. Hence $W\text{Inj}(\mathcal{S})$ \mathcal{S} -cogenerates \mathbf{X} . Conversely, suppose there is a class $\mathbf{P} \subset W\text{Inj}(\mathcal{S})$ which \mathcal{S} -cogenerates \mathbf{X} . Let \mathbf{R} be the class of all extremal subobjects of products of \mathbf{P} -objects. By the above remarks, $\mathbf{R} \subset W\text{Inj}(\mathcal{S})$. consider any \mathbf{X} -object X . Since \mathbf{X} is \mathcal{S} -cogenerated by \mathbf{P} , there exists an \mathcal{S} -morphism $s : X \rightarrow \prod P_i$ for some set of $P_i \in \mathbf{P}$. Let $X \xrightarrow{e} M \xrightarrow{m} \prod P_i$ be the $(\text{Epi}\mathbf{X}, \text{ExMono}\mathbf{X})$ -factorization of s . Then $e \in \text{Epi}\mathcal{E} \cap \mathcal{S}$ by (s_3) , and $M \in \mathbf{R}$ by definition of \mathbf{R} . Consider any $f : X \rightarrow Y$ with $Y \in \mathbf{R}$. The weak \mathcal{S} -injectivity of Y then gives a (unique) $f^* : M \rightarrow Y$ with $f^*e = f$. Thus $e : X \rightarrow M$ is an \mathbf{R} -reflection of X ; we choose an equivalent $r_X : X \rightarrow R(X)$, whence by (s_1) and (s_2) $r_X \in \text{Epi}\mathbf{X} \cap \mathcal{S}$. To see that \mathbf{R} is \mathcal{S} -firm, consider $g : X \rightarrow Z$ in $\text{Epi}\mathbf{X} \cap \mathcal{S}$ with $Z \in \mathbf{R}$. We have $g^* : R(X) \rightarrow Z$ with $g^*r_X = g$; the weak \mathcal{S} -injectivity of $R(X)$ also gives $h : Z \rightarrow R(X)$ with $hg = r_X$, and clearly h is inverse to g^* . Finally it is clear that \mathbf{R}

is the epireflective hull of \mathbf{P} , also in the sense that \mathbf{R} is the smallest epireflective subcategory of \mathbf{X} that contains \mathbf{P} .

Corollary 1.7. *$W\text{Inj}(\mathcal{S})$ is \mathcal{S} -firmly epireflective in \mathbf{X} if and only if \mathbf{X} is \mathcal{S} -cogenerated by $W\text{Inj}(\mathcal{S})$.*

A result by Bacsich [2, (Theorem 3.1)] partly overlaps with the content of Theorem 1.6, but under different assumptions. The corresponding results mentioned by Kiss et al. [44, (pp. 94-95)] and given by Tholen [52, (esp. Lemma 7)] are essentially different.

Example 1.8. In all examples below we assume that \mathcal{S} is the class of embeddings and we drop “ \mathcal{S} ” in all terms in which it previously appeared.

(1) Firmness becomes trivial in concrete categories \mathbf{X} in which epimorphisms are onto maps (in particular in topological categories as well as in abelian categories). Indeed in such a case, $\mathbf{R} = \mathbf{X}$.

(2) No non-trivial ($\neq \mathbf{Singl} = \{\text{spaces with at most one point}\}$) epireflective subcategory of \mathbf{Top} consisting of Hausdorff spaces admits a firm epireflection. In fact none of the subcategories above admits a class of weakly injective cogenerators: if \mathbf{X} is as above, then the two-point discrete space $D_2 (= \{0, 1\})$, the discrete space of natural numbers \mathbf{N} and its Alexandrov compactification \mathbf{N}^* belong to \mathbf{X} . Let $f : \mathbf{N} \rightarrow D_2$ be the continuous map defined by $f(2n) = 0$ and $f(2n + 1) = 1$, $n \in \mathbf{N}$. Then f cannot be extended to \mathbf{N}^* , while the inclusion $e : \mathbf{N} \rightarrow \mathbf{N}^*$ is a dense, hence \mathbf{X} -epi, embedding. We conclude that no space with more than one point is weakly injective in \mathbf{X} , so, by Theorem 1.6, \mathbf{X} does not admit a firm epireflection. The same proof also establishes that if \mathbf{Y} is any epireflective subcategory of \mathbf{Haus} containing \mathbf{N} , then \mathbf{Y} has no firm epireflective subcategory.

(3) in \mathbf{Top}_0 , the category of T_0 topological space, the Sierpinski two-point space is a cogenerator of \mathbf{Top}_0 and it is weakly injective in \mathbf{Top}_0 . The epireflective hull in \mathbf{Top}_0 of the Sierpinski space is the category \mathbf{Sob} of sober spaces, so, in virtue of Theorem 1.6, \mathbf{Sob} is the firm epireflective subcategory of \mathbf{Top}_0 . This restates a result which Hoffmann [38] established by an internal argument. In virtue

of Proposition 1.3, sober, weakly injective, saturated and absolutely closed coincide in \mathbf{Top}_0 .

(4) By \mathbf{Unif}_0 we mean the category of separated (i.e. T_0 , hence Hausdorff) uniform spaces and uniformly continuous maps. It is well known [53] that the complete spaces in \mathbf{Unif}_0 form the firm epireflective subcategory of \mathbf{Unif}_0 .

(5) \mathbf{Prox}_0 denotes the category of separated proximity spaces and proximity maps [41]. The complete (which are here the same as compact) spaces in \mathbf{Prox}_0 form the firm epireflective subcategory of \mathbf{Prox}_0 . This is an instance of Theorem 1.6, the required weakly injunctive cogenerator being the compact unit interval.

(6) The complete metric spaces form a firm epireflective subcategory of the category of metric spaces and non-expansive maps. (An interesting extended setting for this classical result is given by Hoffmann [38, (p. 321 example 3.4)].)

(7) \mathbf{Qun} will denote the category of quasi-uniform spaces and quasi-uniform maps (see e.g. [15] or [24]). Its subcategory \mathbf{Qun}_0 of separated objects consists of those quasi-uniform spaces for which the join of the two induced topologies is T_0 (hence Tychonoff), or equivalently the first topology is T_0 , or equivalently the second topology is T_0 (by the join of two structures we shall always mean the least fine structure finer than both). Császár ([14], [15]) showed that \mathbf{Qun}_0 has a firm epireflective subcategory whose objects he named “doubly complete”. These are the spaces whose uniform coreflection, formed by joining a quasi-uniformity with its inverse, is a complete uniform space. A convenient construction, and proof of the firmness, is given in [24], where the objects are called “bicomplete” (cf. [13]).

(8) \mathbf{Qprox} denotes the category of quasi-proximity spaces, known to be isomorphic to the full subcategory of totally bounded spaces in \mathbf{Qun} (cf. [47] or [24]). \mathbf{Qprox}_0 denotes the corresponding subcategory of separated objects. \mathbf{Qprox}_0 has a cogenerator I which is injective with respect to \mathbf{Qprox}_0 -epi embeddings; I is the closed unit interval with the quasi-proximity relation δ given by $A\delta B$ iff, for each $a > 0$, there exist $x \in A$ and $y \in B$ with $y > x - a$. Thus by Theorem 1.6, \mathbf{Qprox}_0 has a firm epireflective subcategory. The spaces in

this subcategory are those for which the join of the two underlying topologies is compact; equivalently, they are the bicomplete (in the sense of (7) above) totally bounded quasi-uniform spaces.

(9) $\mathbf{2Top}$ denotes the category of bitopological spaces (more briefly, “bispaces”) in the sense of [43]. Objects are triples of the form $X = (|X|, \mathbf{O}_1X, \mathbf{O}_2X)$ where $|X|$ is a set and $\mathbf{O}_1X, \mathbf{O}_2X$ are topologies on $|X|$; a morphism $f : X \rightarrow Y$ is a function with $f : (|X|, \mathbf{O}_iX) \rightarrow (|Y|, \mathbf{O}_iY)$ continuous for $i = 1, 2$. The full subcategory $\mathbf{2Top}_o$ consists of those X with $(|X|, \mathbf{O}_1X \vee \mathbf{O}_2X) \in \mathbf{Top}_o$. $\mathbf{2Top}_o$ has a cogenerator Q which is weakly injective in $\mathbf{2Top}_o$. This was shown in [28], where the smallest such object was named “the quad”:

$$Q = (\{0, 1, 2, 3\}, \{\emptyset, \{0, 1\}, |Q|\}, \{\emptyset, \{0, 3\}, |Q|\}) .$$

Thus by Theorem 1.6 $\mathbf{2Top}_o$ has a firm epireflective subcategory \mathbf{R} whose objects are precisely the absolutely closed objects and also precisely the weakly injective objects of $\mathbf{2Top}_o$. (In [28] the coincidence $AC(\mathcal{S}) = WInj(\mathcal{S})$ was proved though the firmness was not observed). It was also shown in [28] that \mathbf{R} is contained in, but differs from, the subcategory of sober bispaces in $\mathbf{2Top}_o$. The sober bispaces are given by the largest duality between bispaces and biframes [4]; they are those bispaces for which the join of the two topologies is a sober topological space. It is noteworthy that in \mathbf{Qun}_o and \mathbf{QProx}_o the “complete” objects are precisely those whose symmetrization is “complete” in \mathbf{Unif}_o and \mathbf{Prox}_o respectively, a phenomenon which fails for $\mathbf{2Top}_o$ versus \mathbf{Top}_o .

(10) The category $\mathbf{RegNear}$ of regular nearness spaces has a firm epireflective subcategory [8], whose objects are called complete. In fact the completion functor in $\mathbf{RegNear}$ restricts to the one in \mathbf{Unif}_o . In the larger category $\mathbf{SepNear}$ of separated nearness spaces, completeness misbehaves in interesting ways [7], [8].

(11) \mathbf{PTop} (\mathbf{PTop}_o) will denote the category of (T_o) pretopological spaces (= Čech spaces). It is shown in [20] that \mathbf{PTop}_o -epis are onto. So \mathbf{PTop}_o trivially admits a firm epireflective subcategory.

If \mathbf{X} is a non trivial epireflective subcategory of \mathbf{PTop} consisting

of Hausdorff pretopological spaces, then the argument in (2) can be used to show that \mathbf{X} does not admit a firm epireflection.

Notice that the class of saturated (= absolutely closed = compact) Hausdorff pretopological spaces does not even form (the object class of) a reflective subcategory of \mathbf{PTop} (cf. [12]).

The negative results above remain valid in other categories of filter convergence spaces which properly contain \mathbf{Top} (e.g. in the category \mathbf{PsTop} of pseudotopological spaces and in the category \mathbf{Lim} of limit spaces).

It is also easy to see that firmness becomes trivial in all epireflective subcategories of both \mathbf{Born} (the category of bornological spaces [40]) and \mathbf{Simp} (the category of abstract simplicial complexes [46]).

(12) The category of normed vector spaces over a fixed subfield \mathbf{K} of \mathbf{C} , with non-decreasing \mathbf{K} -linear maps, has the Banach spaces over \mathbf{K} as firm epireflective subcategory. This fact is placed in the setting of topological categories in [38].

(13) \mathbf{TopGrp}_o denotes the category of topological groups with T_o - (hence Hausdorff) topology; the morphisms are continuous homomorphisms. Completion with respect to the two-sided uniformity would provide the firm epireflection if we knew that every epimorphic embedding $G \rightarrow H$ with H complete in the two-sided uniformity was dense. Since the epimorphism problem for \mathbf{TopGrp}_o is unsolved, we only know that a full subcategory \mathbf{X} of \mathbf{TopGrp}_o admits a firm epireflection if the epis of the stated kind in \mathbf{X} are dense. This is the case for the subcategory \mathbf{TopAb}_o of abelian topological groups with T_o -topology [11].

(14) Let \mathbf{D} be the category of bounded distributive lattices, with lattice homomorphisms preserving 0 and 1. $\mathbf{EmbD} = \mathbf{MonoD}$. The 2-chain is an injective cogenerator of \mathbf{D} [5]. Hence by Theorem 1.6 \mathbf{D} has a firm epireflective subcategory, the epireflective hull of $\{2\}$, consisting of the Boolean algebras. Thus (cf. [2]) the reflection embeds any object of \mathbf{D} in its Boolean envelope.

(15) The category of cancellative abelian monoids, with homomorphisms preserving neutral element, has the category of abelian groups as firm epireflective subcategory, the reflection being the group of dif-

ferences [10], [2]. The corresponding fact is true for the category of integral domains with ring homomorphisms preserving 1, the reflection giving the field of fractions (that this category does not satisfy our initial assumption of completeness is immaterial for the present purpose). Similarly, the category of torsion-free abelian groups admits a firm epireflection, which gives the divisible hulls of these groups [2]. This is a particular case of the following example.

(16) Let $\mathbf{R}\text{-Mod}$ be the category of all left modules over a fixed unitary ring \mathbf{R} and, for a given radical \mathbf{r} in $\mathbf{R}\text{-Mod}$, let $\mathcal{F}_{\mathbf{r}}$ be the corresponding torsion free class, i.e. $\mathcal{F}_{\mathbf{r}} = \{X \in \mathbf{R}\text{-Mod} \mid \mathbf{r}(X) = 0\}$. $\mathcal{F}_{\mathbf{r}}$ is epireflective in $\mathbf{R}\text{-Mod}$ and every epireflective subcategory of $\mathbf{R}\text{-Mod}$ is of the form $\mathcal{F}_{\mathbf{r}}$ for a unique radical \mathbf{r} (cf. [21]). Denote by $\mathcal{FJ}_{\mathbf{r}}$ the class of all \mathbf{r} -torsion free modules which are \mathbf{U} -injective in $\mathbf{R}\text{-Mod}$, where \mathbf{U} is the class of all \mathbf{r} -dense (= $\mathcal{F}_{\mathbf{r}}$ -dense) monomorphisms. It is shown in [21] that $\mathcal{FJ}_{\mathbf{r}}$ is firmly epireflective in $\mathcal{F}_{\mathbf{r}}$ if and only if \mathbf{r} is a hereditary radical.

(17) No category of algebras (as defined by Isbell [42]), in particular semigroups or rings, admits a firm epireflective subcategory. In fact it is shown in [42] Example 3.2, that the saturation of an algebra is not unique (and there is not a universal saturation which maps onto all others; so the saturated algebras do not even form an epireflective subcategory).

(18) A *separated projection space* [23] is a pair $(X, (\alpha_n : n \in \mathbf{N}))$ consisting of a set X and a sequence of maps $\alpha_n : X \rightarrow X$, subject to the following conditions:

$$\text{(pro)} \quad \alpha_n \circ \alpha_p = \alpha_{\text{Min}(n,p)} \quad \text{for all } n, p \in \mathbf{N} ;$$

$$\text{(sep)} \quad \text{if } \alpha_n(x) = \alpha_n(y) \quad \text{for each } n \in \mathbf{N} , \quad \text{then } x = y .$$

Abbreviated notation for $(X, (\alpha_n : n \in \mathbf{N}))$ is (X, α_n) .

A *projection morphism* $f : (X, \alpha_n) \rightarrow (Y, \beta_n)$ is a map $f : X \rightarrow Y$ satisfying the condition

$$\text{(mor)} \quad \beta_n \circ f = f \circ \alpha_n \quad \text{for each } n \in \mathbf{N} .$$

$\underline{\text{PRO}}$, will denote the (concrete) category of separated projection spaces and projection morphisms.

A sequence (x_m) in (X, α_n) is called a **Cauchy sequence** if $\alpha_n(x_{n+1})$

$= x_n$, for each $n \in \mathbf{N}$. A Cauchy sequence (x_m) converges to a point x if $\alpha_n(x) = x_n$, for each $n \in \mathbf{N}$. A separated projection space is called *complete* if each Cauchy sequence converges.

It is shown in [26] that the subcategory of all complete separated projection spaces is firmly epireflective in $\underline{\text{PRO}}$.

2. Preservation of \mathcal{S} -sources

We now have to extend the notion of \mathcal{S} -morphism to sources. We emphasize that we consider sources indexed over arbitrary classes.

Definitions 2.1. (1) A source $(f_i : X \rightarrow Y_i | i \in I)$ is called an \mathcal{S} -source if for some set $J \subset I$ the \mathbf{X} -morphism $\langle f_j \rangle : X \rightarrow \Pi(Y_j | j \in J)$ belongs to \mathcal{S} .

(2) We say that a functor $F : \mathbf{X} \rightarrow \mathbf{X}$ preserves \mathcal{S} -morphisms if $F(s) : F(X) \rightarrow F(Y)$ belongs to \mathcal{S} whenever $s : X \rightarrow Y$ is in \mathcal{S} . The definition for the preservation of \mathcal{S} -sources is analogous.

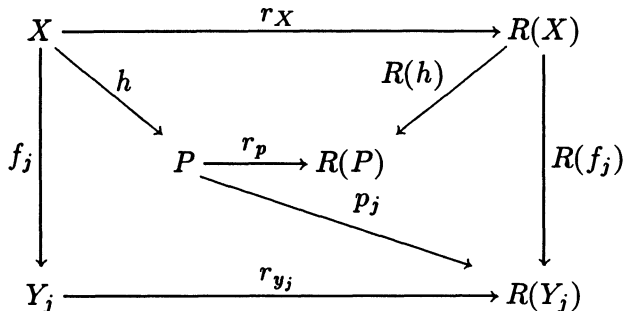
We shall sometimes need the following additional conditions on \mathcal{S} :

(s₄) Whenever $f_i : X_i \rightarrow Y_i$ is in \mathcal{S} for each $i \in I$, I any set, the product morphism $\Pi f_i : \Pi(X_i | i \in I) \rightarrow \Pi(Y_i | i \in I)$ is in \mathcal{S} ;

(s₅) $\text{ExMonoX} \subset \mathcal{S}$.

Proposition 2.2. Assume condition (s₁)-(s₄). Let \mathbf{R} be an \mathcal{S} -epireflective subcategory of \mathbf{X} with $R : \mathbf{X} \rightarrow \mathbf{X}$ the reflector to \mathbf{R} . If R preserves \mathcal{S} -morphism, then R preserves \mathcal{S} -sources.

Proof. Let $(f_i : X \rightarrow Y_i | i \in I)$ be an \mathcal{S} -source. Thus there is a set $J \subset I$ such that the morphism $\langle f_j \rangle : X \rightarrow \Pi(Y_j | j \in J)$ is in \mathcal{S} . Let $P = \Pi(R(Y_j) | j \in J)$ with projection morphisms p_j . By properties (s₂) and (s₄), the morphism $h = \langle r_{Y_j} f_j \rangle : X \rightarrow P$ is in \mathcal{S} , since each r_{Y_j} is in \mathcal{S} .



Since \mathbf{R} is epireflective $P \in \mathbf{R}$, so that r_P is an isomorphism. We have $R(f_j)r_X = r_{Y_j}f_j = p_jh = p_jr_P^{-1}R(h)r_X$, whence $R(f_j) = p_jr_P^{-1}R(h)$, in other words $r_P^{-1}R(h) = \langle R(f_j) \rangle$; but this morphism is in \mathcal{S} since both r_P^{-1} and $R(h)$ are in \mathcal{S} . Thus $(R(f_i) : R(X) \rightarrow R(Y_i) | i \in I)$ is an \mathcal{S} -source, as required.

Proposition 2.3. *Assume conditions (s_1) - (s_3) and (s_5) . If \mathbf{R} is an \mathcal{S} -firm epireflective subcategory of \mathbf{X} , the \mathbf{R} -reflection functor R preserves \mathcal{S} -morphisms.*

Proof. Let $f : X \rightarrow Y$ be an \mathcal{S} -morphism and let $X \xrightarrow{e} M \xrightarrow{m} R(Y)$ be an (epi, extremal mono)-factorization of $r_Y f : X \rightarrow Y \rightarrow R(Y)$ (see property (x_1) in §1). Then $e \in \text{Epi}\mathbf{X} \cap \mathcal{S}$ by (s_3) . Since M is an extremal subobject of $R(Y) \in \mathbf{R}$, $M \in \mathbf{R}$. Since \mathbf{R} is \mathcal{S} -firm and $e \in \text{Epi}\mathbf{X} \cap \mathcal{S}$ the morphism $e^* : R(X) \rightarrow M$ is an isomorphism, so it belongs to \mathcal{S} , by (s_1) . We have $me^*r_X = me = r_Y f = R(f)r_X$, whence $me^* = R(f)$. Since $m \in \text{ExMono}\mathbf{X} (\subset \mathcal{S}$ by assumption) and $e^* \in \mathcal{S}$, $R(f) = me^* \in \mathcal{S}$ by (s_2) and the proof is complete.

Corollary 2.4. *Assume conditions (s_1) - (s_5) . If \mathbf{R} is \mathcal{S} -firmly epireflective in \mathbf{X} , then the \mathbf{R} -reflection functor preserves \mathcal{S} -sources.*

Theorem 2.5. *Let \mathbf{X} be a complete and well-powered category and \mathcal{S} a class of \mathbf{X} -morphisms satisfying conditions (s_1) - (s_5) . Let \mathbf{R} be an \mathcal{S} -epireflective subcategory of \mathbf{X} with \mathbf{R} -reflector R .*

Then, the following conditions are equivalent:

- (i) \mathbf{R} is \mathcal{S} -firmly epireflective in \mathbf{X} ;

- (ii) R preserves \mathcal{S} -morphisms and $\mathbf{R} \subset \text{Sat}(\mathcal{S})$;
- (iii) R preserves \mathcal{S} -sources and $\mathbf{R} \subset \text{Sat}(\mathcal{S})$;
- (iv) $\mathbf{R} \subset \text{WInj}(\mathcal{S})$;
- (v) There is a class $\mathbf{P} \subset \text{WInj}(\mathcal{S})$ such that \mathbf{R} is the epireflective hull of \mathbf{P} in \mathbf{X} ;
- (vi) R preserves \mathcal{S} -morphisms and $\mathcal{S} \cap \text{Epi}\mathbf{R} = \text{Iso}\mathbf{R}$.

When one of these conditions holds, then $\mathbf{R} = \text{Sat}(\mathcal{S}) = \text{WInj}(\mathcal{S})$.

Proof. (i) \Rightarrow (ii) follows from Proposition 2.3 and from (ii) \Rightarrow (i) of Proposition 1.3.

(ii) \Rightarrow (iii) follows from Proposition 2.2.

(iii) \Rightarrow (i): If $f : X \rightarrow Y$ is in $\text{Epi}\mathbf{X} \cap \mathcal{S}$ and $Y \in \mathbf{R}$ then $f^* \in \mathcal{S}$ and $f^* \in \text{Epi}\mathbf{X}$, so $\mathbf{R} \subset \text{Sat}(\mathcal{S})$ gives f^* iso.

(i) \Rightarrow (iv) is precisely (ii) \Rightarrow (iii) of Proposition 1.3.

(iv) \Rightarrow (i): In such case \mathbf{X} is trivially cogenerated by a class of weak \mathcal{S} -injective objects so Theorem 1.6 applies.

(iv) \Rightarrow (v) is trivial and (v) \Rightarrow (i) follows from Theorem 1.6.

(vi) \Rightarrow (iv): Let $A \in \mathbf{R}$. To show $A \in \text{WInj}(\mathcal{S})$, consider $f : X \rightarrow A$ and $e : X \rightarrow Y$ with $e \in \text{Epi}\mathbf{X} \cap \mathcal{S}$. since $(R(e))r_X = r_Y e$, $R(e)$ is an \mathbf{X} -epimorphism and hence an \mathbf{R} -epimorphism. By (vi), $R(e) \in \mathcal{S}$, hence $R(e) \in \text{Epi}\mathbf{R} \cap \mathcal{S}$, so that $R(e) \in \text{Iso}\mathbf{R}$. With $f = f^*r_X$ we then have $f = f^*(R(e))^{-1}r_Y e$, which proves that $A \in \text{WInj}(\mathcal{S})$.

(ii) \Rightarrow (vi): Observe that always $\text{Epi}\mathbf{R} = \text{Epi}\mathbf{X} \cap \text{Mor}\mathbf{R}$, which follows at once from the reflection maps being monic (by (s_1)). Consider $f : X \rightarrow Y$ in $\text{Epi}\mathbf{R} \cap \mathcal{S}$. Then $f \in \text{Epi}\mathbf{X} \cap \mathcal{S}$ with $X, Y \in \mathbf{R}$. Assuming (ii) we have $X \in \text{Sat}(\mathcal{S})$, so that f is an isomorphism. Thus (vi) holds.

The following example shows that the condition $\mathbf{R} \subset \text{Sat}(\mathcal{S})$ cannot be deleted from conditions (ii) and (iii) in Theorem 2.5.

Example 2.6. Let \mathbf{Alex} be the category of Alexandroff spaces [31], [25]. These are the same as the zero-set spaced of [30]. It is shown in [25] that \mathbf{Alex} admits no $(\text{Emb}\mathbf{Alex})$ -firm epireflection, but that the realcompact reflection maps $X \rightarrow \nu X$ in \mathbf{Alex} are essential embeddings. This implies that the realcompact epireflector ν preserves

embeddings, so that by Proposition 2.2 ν also preserves embeddings-sources (i.e. initial sources). Moreover ν preserves arbitrary products in **Alex** [30]. (Christopher Gilmour kindly reminded us of this example).

It is worth noting that **Alex** is isomorphic to the full subcategory **SMF** of separable metric-fine spaces in **Unif_o** [31]. The completion epireflector in **Unif_o** then restricts to a firm epireflector in **SMF** with respect to uniform embeddings (note that **SMF** is stable under completion). But, being isomorphic to **Alex**, **SMF** admits no firm epireflector with respect to the larger class of **SMF**-embeddings.

Similarly the category **Tych**, which by Example 1.8. (2) admits no firm epireflector with respect to embeddings, has many completion-stable full embeddings into **Unif_o**. Each of the embedded subcategories of **Unif_o** then admits a firm epireflector with respect to uniform embeddings.

Proposition 2.7. *Let \mathbf{X} admit an \mathcal{S} -firm epireflector R . If $\text{Epi}\mathbf{X} \cap \mathcal{S}$ is closed under the formation of products, then R preserves products.*

Proof. By assumption $\prod r_{X_j} \in \text{Epi}\mathbf{X} \cap \mathcal{S}$. Also $R(X_i) \in \mathbf{R}$. Then by the definition of \mathcal{S} -firmness, the extension of $\prod r_{X_j}$ to $R(\prod X_i)$ is an isomorphism.

Definition 2.8. ([3], [52]) *An \mathbf{X} -morphism $f \in \mathcal{S}$ is called \mathcal{S} -essential if $g \in \mathcal{S}$ whenever $gf \in \mathcal{S}$. If $f : X \rightarrow Y$ is \mathcal{S} -essential and Y is \mathcal{S} -injective, then f is called an \mathcal{S} -injective hull of \mathbf{X} .*

Remarks 2.9. (1) If \mathcal{S} satisfies (s_5) , then an \mathbf{X} -epimorphism is \mathcal{S} -essential if and only if it is $\text{Epi}\mathbf{X} \cap \mathcal{S}$ -essential.

(2) Let $r : 1 \rightarrow R$ be an \mathcal{S} -epireflection in \mathbf{X} .

If each r_X is \mathcal{S} -essential, then R preserves \mathcal{S} -morphisms. The converse implication holds under a slight strengthening of condition (s_3) , e.g. that $gf \in \mathcal{S}$ implies $f \in \mathcal{S}$ which is satisfied by e.g. extremal monomorphisms, initial morphisms, and embeddings.

Proposition 2.10. *Let condition (s_5) hold and let \mathbf{X} admit an \mathcal{S} -firm epireflection $r : 1 \rightarrow R$. Then, for each $X \in \mathbf{X}$, $r_X : X \rightarrow R(X)$ is an $\text{Epi}\mathbf{X} \cap \mathcal{S}$ -injective hull of X .*

Proof. To see that r_X is $\text{Epi}\mathbf{X} \cap \mathcal{S}$ -essential, consider $f : R(X) \rightarrow Y$ such that $fr_X \in \text{Epi}\mathbf{X} \cap \mathcal{S}$. Since $f \in \text{Epi}X$, we only have to prove $f \in \mathcal{S}$. By Proposition 2.3 $R(fr_X) \in \mathcal{S}$. Since $R(fr_X) = r_Y f$, in fact $R(fr_X) \in \text{Epi}\mathbf{X} \cap \mathcal{S}$. With $R(X)$ being \mathcal{S} -saturated, this implies that $R(fr_X)$ is an isomorphism. Thus f is a section, hence an isomorphism, so that $f \in \mathcal{S}$ (and r_X is maximally $\text{Epi}X \cap \mathcal{S}$ -essential).

3. Applications to topological categories

We consider a topological category \mathbf{A} over \mathbf{Set} in the sense of [33] and [45], cf. [34]. An object A of \mathbf{A} is a T_o -object in the sense of [45] iff each \mathbf{A} -morphism from the indiscrete two-point object of \mathbf{A} into A is constant.

Let $T_o\mathbf{A}$ denote the category of T_o -objects of \mathbf{A} (our examples 1.8 (3)(4)(5)(7)(8)(9)(11) contain instances of this notion).

Proposition 3.1. ([45]) *$T_o\mathbf{A}$ is extremal-epireflective in \mathbf{A} , and $T_o\mathbf{A}$ is the largest epireflective, non-bireflective subcategory of \mathbf{A} .*

Proposition 3.2. *If $f : X \rightarrow Y$ is an initial morphism in \mathbf{A} and $X \in T_o\mathbf{A}$, then f is injective.*

However, if $X \in \mathbf{A}$ is such that each initial morphism with domain X is injective, then X need not be in $T_o\mathbf{A}$ [54], [55].

Proposition 3.3. *Let \mathbf{X} be an epireflective subcategory of the topological category \mathbf{A} . Then \mathbf{X} is a well-powered and complete $(\text{Epi}\mathbf{X}, \text{ExMono}\mathbf{X})$ -category. The class $\text{Mor}\mathbf{X} \cap \text{Emb}\mathbf{A}$ satisfies conditions (s_1) - (s_5) .*

Proof. All properties, except maybe (s_5) , are well known (see e.g. [36]). To prove (s_5) consider $f : X \rightarrow Y$ in $\text{ExMono}\mathbf{X}$. Let $X \xrightarrow{e} A \xrightarrow{m} Y$ be its $(\text{Epi}\mathbf{A}, \text{ExMono}\mathbf{A})$ -factorization. Then $A \in \mathbf{X}$ since \mathbf{X} is epireflective in \mathbf{A} . Therefore $e \in \text{Epi}\mathbf{X}$, and since $f \in \text{ExMono}\mathbf{X}$, e is an isomorphism, whence $f \in \text{ExMono}\mathbf{A}$. But $\text{ExMono}\mathbf{A} = \text{Emb}\mathbf{A}$ since \mathbf{A} is topological [33], [32].

Thus, the hypotheses of Theorem 2.5 are satisfied by any epireflective subcategory \mathbf{X} of a topological category, with $\mathcal{S} = \text{Emb}\mathbf{X}$. There

are examples of such \mathbf{X} which are not co-(well-powered) ([48], [27], [17], [18], [22], [29]).

Question C. *Find an example of a category \mathbf{X} which is not co-(well-powered) and yet admits a non-trivial firm epireflective subcategory.*

For a subcategory \mathbf{X} of \mathbf{A} , morphisms or sources which are initial in \mathbf{X} need not be so in \mathbf{A} ; we have the following version of Theorem 2.5:

Proposition 3.4. *Let \mathbf{X} be an epireflective subcategory of the topological category \mathbf{A} , and let $\mathcal{S} = \text{Mor}\mathbf{X} \cap \text{Emb}\mathbf{A}$. If \mathbf{X} admits an \mathcal{S} -firm epireflector R , then R preserves \mathbf{A} -initial sources in \mathbf{X} .*

Proof. In case \mathbf{X} is bireflective in \mathbf{A} , \mathbf{X} is itself a topological category, \mathbf{X} -epimorphisms are onto, and R is the identity functor on \mathbf{X} . In case \mathbf{X} is non-bireflective in \mathbf{A} , then by Propositions 3.1 and 3.2, every \mathbf{A} -initial source in \mathbf{X} is an embedding-source, and the required result follows from Proposition 3.3 and Theorem 2.5.

There is an occasion to extend the theory of firm epireflections to firm \mathcal{E} -reflections in a category \mathbf{X} which has a factorization structure $(\mathcal{E}, \mathcal{M})$ on $\text{Mor}\mathbf{X}$, where \mathcal{E} need not necessarily consist of epimorphisms.

Definition 3.5. *Let $\text{Mor}\mathbf{X}$ have a factorization structure $(\mathcal{E}, \mathcal{M})$ and let $\mathcal{S} \subset \text{Mor}\mathbf{X}$. Let \mathbf{R} be a reflective subcategory of \mathbf{X} with reflection $r : 1_{\mathbf{X}} \rightarrow R$. We say that \mathbf{R} is \mathcal{S} -firmly \mathcal{E} -reflective in \mathbf{X} iff*

- (1) *Each $r_X : X \rightarrow R(X)$ is in $\mathcal{E} \cap \mathcal{S}$;*
- (2) *Whenever $f : X \rightarrow Y$ is in $\mathcal{E} \cap \mathcal{S}$ and $Y \in \mathbf{R}$, the morphism $f^* : X \rightarrow Y$ for which $f^*r_X = f$, is an isomorphism.*

Most of the results in this paper can be restated in this context, with suitable proviso's. We have the following concrete occurrence of \mathcal{S} -firm \mathcal{E} -reflections.

Let \mathbf{A} be a topological category which is universal in the sense of [37] and [45]; this means that \mathbf{A} is the bireflective hull of $T_o\mathbf{A}$, equivalently that the $T_o\mathbf{A}$ -reflection morphisms, say $t_A : A \rightarrow T(A)$,

are initial.

Examples of such \mathbf{A} are **Top**, **2Top**, **Unif**, **Prox**, **Qun**, **QProx**, **Near**; but **PTop** (1.8 (11) above) is a non-example; more examples are given in [45] and [54], [55].

Let $\mathbf{X} = T_o\mathbf{A}$ and let $t : 1_{\mathbf{A}} \rightarrow T$ be the reflection to \mathbf{X} . A morphism $f : A \rightarrow B$ in \mathbf{A} is called **relatively \mathbf{X} -epi** if and only if, whenever $r, s : B \rightarrow C$ are \mathbf{A} -morphisms with $rf = sf$ and $C \in \mathbf{X}$, then $r = s$. Let \mathcal{E} be the class of relative \mathbf{X} -epimorphisms in \mathbf{A} . There exists a class \mathcal{M} such that $(\mathcal{E}, \mathcal{M})$ is a factorization structure on $\text{Mor}\mathbf{A}$ precisely when the regular closure operator induced by \mathbf{X} is weakly hereditary [18]. Also, a source in \mathbf{X} is \mathbf{X} -initial if and only if it is \mathbf{A} -initial. Let $\text{Init}\mathbf{A}$ denote just the class of \mathbf{A} -initial morphisms.

Proposition 3.6. *let \mathbf{R} be a reflective subcategory of a universally topological category \mathbf{A} with reflection $r : 1_{\mathbf{A}} \rightarrow R$, and let $\mathbf{X} = T_o\mathbf{A}$. Let \mathcal{E} be the class of relative \mathbf{X} -epimorphisms in \mathbf{A} , and assume that the regular closure operator induced by \mathbf{X} is weakly hereditary. Then, \mathbf{R} is $\text{Init}\mathbf{A}$ -firmly \mathcal{E} -reflective in \mathbf{A} if and only if \mathbf{R} is $\text{Emb}\mathbf{X}$ -firmly epi-reflective in \mathbf{X} . In this case, $\mathbf{R} = \text{Inj}(\text{Epi}\mathbf{X} \cap \text{Emb}\mathbf{X}) = \text{Inj}(\mathcal{E} \cap \text{Init}\mathbf{A})$, and R preserves \mathbf{A} -initial sources.*

The correspondence in the above result arises by factoring the morphism r_A through the $T_o\mathbf{A}$ -reflection t_A . Thus, our examples of $\text{Emb}\mathbf{X}$ -firm epi-reflections in 1.8. (3)-(5), (7)-(9) give us $\text{Init}\mathbf{A}$ -firm \mathcal{E} -reflections in $\mathbf{A} = \mathbf{Top}$, **2Top**, **Unif**, **Prox**, **Qun**, **QProx**.

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