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THE LEFT DERIVED TENSOR PRODUCT OF CAT VALUED DIAGRAMS

by Murray HEGGIE

Résumé. **The tensor product $\otimes$ of $\mathcal{CAT}$–valued diagrams is left derived in the sense of Quillen's homotopical algebra. The calculus of the resulting operation on categories of diagrams is developed and its principal properties enumerated. Thomason's identification of homotopy colimits in $\mathcal{CAT}$ with the Grothendieck construction is exhibited as an important special case.**

1 Introduction

The work reported here places in a general setting the insight due to Thomason [14] that the Grothendieck construction plays a role in $\mathcal{CAT}$, the category of small categories, precisely analoguous to that played by homotopy colimits in $\mathcal{SS}$, the category of simplicial sets. This is achieved by left deriving in the sense of homotopical algebra [12] the tensor product of functors ([1],[10,p.222]).

Before describing in more detail the contents of this paper, some fundamental notions for the homotopy theory of categories will be briefly recalled. The first is that of a *Grothendieck (op–)fibration*[4]. Let $\mathbf{F} : \mathbf{C} \to \mathbf{D}$ be a functor and let D be an object of $\mathbf{D}$. The comma category $\mathbf{F}/D$ is the category with objects all pairs (C, f) where $C \in \mathbf{C}$ and $f : \mathbf{F}(C) \to D \in \mathbf{D}$. A map from one object (C, f) to another (C', f') consists of a map $g : C \to C' \in \mathbf{C}$ satisfying $f' \circ \mathbf{F}(g) = f$. The comma category $D/\mathbf{F}$ is defined in a similar fashion. Let $\mathbf{F}^{-1}(D)$ denote the fibre of $\mathbf{F}$ over $D \in \mathbf{D}$. Evidently, there is an inclusion

$$\iota : \mathbf{F}^{-1}(D) \hookrightarrow \mathbf{F}/D.$$

If ι has a left adjoint, left inverse for every object $D \in \mathbf{D}$, $\mathbf{F}$ is called a *Grothendieck opfibration.* Dually, if the inclusion $\iota : \mathbf{F}^{-1}(D) \hookrightarrow D/\mathbf{F}$ has a right adjoint, left inverse for every object $D \in \mathbf{D}$, $\mathbf{F}$ is called a *Grothendieck fibration.* The stability of Grothendieck fibrations and opfibrations under pullback will be used several times in the sequel.

Let $X : \mathbf{C} \to \mathcal{CAT}$ be a $\mathcal{CAT}$–valued diagram. Define a category

$$\mathbf{C} \int X$$

as follows: Objects of $\mathbf{C} \int X$ are pairs (C, x) where $C \in \mathbf{C}$ and $x \in X(C)$. Maps $(C, x) \to (C', x')$ consist of pairs $(c : C \to C' \in \mathbf{C}, f : X(c)(x) \to x' \in X(C))$. The composite of two such maps (c, f) and (c', f') is defined to be $(c' \circ c, f' \circ X(c')(f))$. The evident projection $\pi : \mathbf{C} \int X \rightarrow \mathbf{C}$ is an opfibration. This construction enjoys a universal property which will not be recapitulated here[G1]. $\mathbf{C} \int X$ is known variously as the *Grothendieck construction* or *the opfibred category associated to* X. Dually, let $Y : \mathbf{C}^{op} \rightarrow \mathcal{CAT}$ be a contravariant $\mathcal{CAT}$-valued diagram on $\mathbf{C}$. There is a fibration

$$\pi : Y \int \mathbf{C} \rightarrow \mathbf{C}$$

where the domain category $Y \int \mathbf{C}$ is defined as follows: Objects of $Y \int \mathbf{C}$ are pairs (C, y) where $C \in \mathbf{C}$ and $y \in Y(C)$. A map $(C, y) \rightarrow (C', y')$ is a pair $(c : C \rightarrow C' \in \mathbf{C}, g : y \rightarrow Y(c)(y'))$. $Y \int \mathbf{C}$ is sometimes called the *fibred category associated to* Y.

A functor $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ is called a *weak equivalence* if its image $\mathbf{Nerve(F)}$ under the functor

$$\mathbf{Nerve} : \mathcal{CAT} \rightarrow \mathcal{SS}$$

is a weak equivalence of simplicial sets, i.e. induces an isomorphism between the homotopy groups of $\mathbf{Nerve(A)}$ and $\mathbf{Nerve(B)}$ [2]. The class of weak equivalences has several saturation properties:

(1) Isomorphisms are weak equivalences.
(2) The composite of two weak equivalences is a weak equivalence.
(3) If $\mathbf{G} \circ \mathbf{F} = \mathbf{H}$ and any two of $\mathbf{F}$,$\mathbf{G}$, or $\mathbf{H}$ are weak equivalences, then the remaining map is also a weak equivalence.

A category $\mathbf{A}$ is called *weakly contractible* if the unique map $\mathbf{A} \rightarrow \mathbf{1}$ to the terminal category $\mathbf{1}$ is a weak equivalence. Weak equivalences in the functor category $(\mathbf{A}, \mathcal{CAT})$ are defined pointwise: A natural transformation $\theta : X \Rightarrow Y$ is a weak equivalence if $\theta(A) : X(A) \Rightarrow Y(A)$ is a weak equivalence in $\mathcal{CAT}$ for all $A \in \mathbf{A}$.

Let $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{G} : \mathbf{B} \rightarrow \mathbf{A}$ be functors and let $\theta : \mathbf{F} \Rightarrow \mathbf{G}$ be a natural transformation. Let 2 denote the category

$$0 \longrightarrow 1$$

with two objects and one non-identity arrow. θ determines a map

$$\hat{\theta} : \mathbf{A} \times 2 \longrightarrow \mathbf{B}$$

by the prescription

$$\begin{aligned} \hat{\theta}(a : A \rightarrow A', 0 \rightarrow 1) &= \mathbf{G}(a) \circ \theta(A) \\ &= \theta(A') \circ \mathbf{F}(a). \end{aligned}$$

Since $\mathbf{Nerve}(2) = \Delta[1]$, the simplicial interval, and $\mathbf{Nerve}$ preserves finite limits, $\mathbf{Nerve}(\hat{\theta})$ is a homotopy between $\mathbf{Nerve(F)}$ and $\mathbf{Nerve(G)}$. It follows that if

$$\mathbf{F} \dashv \mathbf{G} : \mathbf{A} \rightarrow \mathbf{B}$$

are adjoints, **A** and **B** are homotopy equivalent. For the adjunctions

$$\eta : id_{\mathbf{A}} \Longrightarrow \mathbf{G} \circ \mathbf{F}$$

and

$$\varepsilon : \mathbf{F} \circ \mathbf{G} \Longrightarrow id_{\mathbf{B}}$$

map to homotopy inverses via **Nerve**. In particular, if $\mathbf{F} : \mathbf{C} \longrightarrow \mathbf{D}$ is an opfibration, the inclusion

$$\iota : \mathbf{F}^{-1}(D) \hookrightarrow \mathbf{F}/D$$

is a weak equivalence for every object $D \in \mathbf{D}$. Likewise, if $\mathbf{F}$ is a fibration, the inclusion $\iota : \mathbf{F}^{-1}(D) \hookrightarrow D/\mathbf{F}$ is a weak equivalence.

If the category **A** has a terminal object 1, then the identity $id_{\mathbf{A}}$ is homotopic to the composite

$$\mathbf{A} \longrightarrow 1 \hookrightarrow \mathbf{A}$$

of the unique map to the terminal category followed by the inclusion of the terminal object in **A**. The homotopy is the map whose value at $A \in \mathbf{A}$ is the unique map $A \longrightarrow 1$. Similarly, if **A** has an initial object, **A** is weakly contractible.

The Grothendieck construction,$\mathbf{C} \int X$, has already been described. This construction admits a natural generalization. Let X be a $\mathcal{CAT}$-valued diagram on **C** and let Y be a $\mathcal{CAT}$-valued diagram on $\mathbf{C}^{op}$, the opposite of **C**. There is a category $X//Y$ associated to X and Y which specializes to $\mathbf{C} \int X$ in case $Y = \mathbf{1}$, the terminal diagram. Likewise, $X//Y$ yields $Y \int \mathbf{C}$ in the special case that $X = \mathbf{1}$. This generalization of the Grothendieck construction will prove to be extremely useful in extending Thomason's identification of homotopy colimits in $\mathcal{CAT}$ with the Grothendieck construction. The crucial property of $X//Y$, it's homotopy invariance, is derived from a slight extension of Quillen's well-known Theorem A [13]. Roughly speaking, the extension established here asserts that a fibre-wise weak equivalence in $\mathcal{CAT}$ is a weak equivalence.

Theorem *Let* **P** *and* **Q** *be two elements of the comma category* $\mathcal{CAT}/\mathbf{C}$ *for some category* **C** *and let*

$$\theta : \mathbf{P} \Longrightarrow \mathbf{Q}$$

be a map from **P** *to* **Q**. *If the map*

$$\theta/C : \mathbf{P}/C \longrightarrow \mathbf{Q}/C$$

induced by θ *is a weak equivalence for all objects* C *of* **C** *then* θ *is a weak equivalence.*

With this generalization of Quillen's Theorem A in hand, the homotopy invariance of $X//Y$ can be established:

Theorem *Let $\alpha : X \Longrightarrow X'$ be a weak equivalence in the functor category $(\mathbf{C}, \mathcal{CAT})$ and let $\beta : Y \Longrightarrow Y'$ be a weak equivalence in $(\mathbf{C}^{op}, \mathcal{CAT})$. Then the induced map*

$$\alpha//\beta : X//Y \longrightarrow X'//Y'$$

is a weak equivalence in $\mathcal{CAT}$.

The principal reason for introducing the construction $X//Y$ is to show that it is *the* homotopy-theoretical substitute for the ordinary tensor product $X \otimes Y$ of two diagrams $X : \mathbf{C} \longrightarrow \mathcal{CAT}$ and $Y : \mathbf{C}^{op} \longrightarrow \mathcal{CAT}$. Validating the substitution is facilitated by introducing the notions of *free diagrams* and *free resolutions of a diagram.* It is shown that every diagram $X : \mathbf{C} \longrightarrow \mathcal{CAT}$ has a canonical free resolution $\mathbf{F}X \Longrightarrow X$. Free diagrams are analoguous to *cofibrant objects for a model category structure* and free resolutions are analoguous to *cofibrant resolutions* [12]. For example, although the functor

$$X \otimes \cdot : (\mathbf{C}^{op}, \mathcal{CAT}) \longrightarrow \mathcal{CAT}$$

does not preserve weak equivalences in general, preservation is guaranteed if $X : \mathbf{C} \longrightarrow \mathcal{CAT}$ is free. This circumstance permits the definition of the left derived tensor product

$$X \otimes^{\mathbf{L}} Y$$

of $X : \mathbf{C} \longrightarrow \mathcal{CAT}$ and $Y : \mathbf{C}^{op} \longrightarrow \mathcal{CAT}$. Namely, let $\mathbf{F}X \Longrightarrow X$ be a free resolution of X and let $\mathbf{F}Y \Longrightarrow Y$ be a free resolution of Y. By definition,

$$X \otimes^{\mathbf{L}} Y \equiv \mathbf{F}X \otimes \mathbf{F}Y.$$

Let $\Sigma^{-1}\mathcal{CAT}$ denote the category of fractions of $\mathcal{CAT}$ with respect to the class of weak equivalences and let $\Sigma^{-1}(\mathbf{C}, \mathcal{CAT})$ denote the category of fractions of $(\mathbf{C}, \mathcal{CAT})$ with respect to the class of point–wise weak equivalences [2]. Up to natural isomorphism in $\Sigma^{-1}\mathcal{CAT}$, the left derived tensor product of X and Y is independent of the free resolutions $\mathbf{F}X$ and $\mathbf{F}Y$ used. As a consequence, there is a well defined functor

$$\otimes^{\mathbf{L}} : \Sigma^{-1}(\mathbf{C}, \mathcal{CAT}) \times \Sigma^{-1}(\mathbf{C}^{op}, \mathcal{CAT}) \longrightarrow \Sigma^{-1}\mathcal{CAT}.$$

$\otimes^{\mathbf{L}}$ is the right Kan extension of the composite

$$\rho \circ \otimes : (\mathbf{C}, \mathcal{CAT}) \times (\mathbf{C}^{op}, \mathcal{CAT}) \longrightarrow \Sigma^{-1}\mathcal{CAT}$$

along the localization functor

$$\rho \times \rho : (\mathbf{C}, \mathcal{CAT}) \times (\mathbf{C}^{op}, \mathcal{CAT}) \longrightarrow \Sigma^{-1}(\mathbf{C}, \mathcal{CAT}) \times \Sigma^{-1}(\mathbf{C}^{op}, \mathcal{CAT}).$$

Consequently, ***the left derived tensor product is the left derived functor of the tensor product in the sense of homotopical algebra*** [13]. Moreover, there is a natural isomorphism in $\Sigma^{-1}\mathcal{CAT}$,

$$X//Y \longrightarrow X \otimes^{\mathbf{L}} Y.$$

This isomorphism in the derived category explains the ubiquity of the Grothendieck construction in $\mathcal{CAT}$-based homotopy theory. As the left derived tensor product of $X : \mathbf{C} \longrightarrow \mathcal{CAT}$ with the terminal diagram $\mathbf{1}$ is a representative in $\Sigma^{-1}\mathcal{CAT}$ of the Grothendieck construction $\mathbf{C} \int X$, Thomason's insight has been placed in a more general setting.

By developing the calculus of the left derived tensor product, it is seen that it possesses many of the properties of the ordinary tensor product provided that these are interpreted in the derived categories $\Sigma^{-1}\mathcal{CAT}$ and $\Sigma^{-1}(\mathbf{C}, \mathcal{CAT})$.

2 Definition and basic properties of $X//Y$

Let $X : \mathbf{C} \longrightarrow \mathcal{CAT}$ and $Y : \mathbf{C}^{op} \longrightarrow \mathcal{CAT}$ be $\mathcal{CAT}$-valued diagrams on $\mathbf{C}$ and $\mathbf{C}^{op}$ respectivley.

2.1 Definition $X//Y$ denotes the pullback

$$\begin{array}{ccc} X//Y & \longrightarrow & Y \int \mathbf{C} \\ \downarrow & & \downarrow {\scriptstyle p_Y} \\ \mathbf{C} \int X & \xrightarrow{p_X} & \mathbf{C} \end{array}$$

Objects of $X//Y$ are triples (x, C, y) where C is an object of $\mathbf{C}$, x is an object of $X(C)$ and y is an object of $Y(C)$. Maps $(x, C, y) \longrightarrow (x', C', y')$ are triples (f, c, g) where $c : C \to C'$ is a map in $\mathbf{C}$, $f : X(c)(x) \to x'$ is a map in $X(C')$, and $g : y \to Y(c)(y')$ is a map in $Y(C)$. It is readily verified that the assignment of $X//Y$ to X and Y is functorial.

$X//Y$ has previously figured in work of Guitart [6,7]. Guitart's uses $X//Y$ in order to present $X \otimes Y$ as a category of fractions of $X//Y$. The exact relationship between his work and the work reported here is a subject for future investigation.

The property of $X//Y$ which will be crucial in the sequel is its homotopy invariance: If $\alpha : X \Rightarrow X'$ and $\beta : Y \Rightarrow Y'$ are pointwise weak equivalences then the induced map $\alpha//\beta : X//Y \to X'//Y'$ is a weak equivalence in $\mathcal{CAT}$. The proof relies on the following extension of Quillen's Theorem A [12,p.85].

2.2 Theorem *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be two elements of the comma category* $\mathcal{CAT}/\mathbf{C}$ *for some category* $\mathbf{C}$ *and let*

$$\theta : \mathbf{P} \Longrightarrow \mathbf{Q}$$

be a map from $\mathbf{P}$ *to* $\mathbf{Q}$. *If the map*

$$\theta/C : \mathbf{P}/C \longrightarrow \mathbf{Q}/C$$

induced by θ *is a weak equivalence for all objects* C *of* $\mathbf{C}$ *then* θ *is a weak equivalence.*

Proof. The proof closely parallels that given by Quillen [12,p.87]. Assume that $\mathbf{P} : \mathbf{A} \to \mathbf{C}$ and that $\mathbf{Q} : \mathbf{B} \to \mathbf{C}$. Then θ is functor $\mathbf{A} \to \mathbf{B}$ satisfying $\mathbf{Q} \circ \theta = \mathbf{P}$. Define a bisimplicial set $\mathcal{S}(\mathbf{P})$ by

$$\mathcal{S}(\mathbf{P})(m,n) = \{(A_m \to \cdots \to A_0, \mathbf{P}(A_0) \to C_0, C_0 \to \cdots \to C_n\}$$

where $A_m \to \cdots \to A_0 \in \mathbf{Nerve(A)}(m)$, $\mathbf{P}(A_0) \to C_0$ is map in $\mathbf{C}$ and $C_0 \to \cdots \to C_n \in \mathbf{Nerve(C)}(n)$. Define $\mathcal{S}(\mathbf{Q} \circ \theta)$ in the same way. There is an induced map of bisimplicial sets

$$\mathcal{S}(\theta) : \mathcal{S}(\mathbf{P}) \longrightarrow \mathcal{S}(\mathbf{Q} \circ \theta)$$

defined on vertices by

$$\begin{gathered}\mathcal{S}(\theta)(A_m \to \cdots \to A_0, \mathbf{P}(A_0) \to C_0, C_0 \to \cdots \to C_n) = \\ (\theta(A_m) \to \cdots \to \theta(A_0), \mathbf{Q} \circ \theta(A_0) \to C_0, C_0 \to \cdots \to C_n)\end{gathered}$$

By fixing the second variable one obtains a commutative diagram of simplicial sets:

$$\begin{array}{ccc} \mathcal{S}(\mathbf{P})(\cdot,n) & \longrightarrow & \mathcal{S}(\mathbf{Q} \circ \theta)(\cdot,n) \\ \downarrow & & \downarrow \\ \coprod_{C_0 \to \cdots \to C_n} \mathbf{Nerve}(\mathbf{P}/C_0) & \longrightarrow & \coprod_{C_0 \to \cdots \to C_n} \mathbf{Nerve}(\mathbf{Q}/C_0) \end{array}$$

By assumption, for every object C of $\mathbf{C}$,

$$\mathbf{Nerve}(\mathbf{P}/C) \longrightarrow \mathbf{Nerve}(\mathbf{Q}/C)$$

is a weak equivalence. Consequently, the horizontal arrow on the bottom is a weak equivalence. But both vertical arrows are isomorphisms. By saturation of the collection of weak equivalences, the horizontal arrow on the top is a weak equivalence for each n. This implies, by a well-known theorem, that $diag\mathcal{S}(\theta)$ is a weak equivalence. By fixing the first variable one obtains a map of simplicial sets

$$\mathcal{S}(\mathbf{P})(m,\cdot) \simeq \coprod_{A_m \to \cdots \to A_0} \mathbf{Nerve}(\mathbf{P}(A_0)/\mathbf{C}) \longrightarrow \coprod_{A_m \to \cdots \to A_0} 1.$$

Because $\mathbf{P}(A)/\mathbf{C}$ has an initial object for every object A of $\mathbf{A}$, $\mathbf{P}(A)/\mathbf{C} \to 1$ is a weak equivalence. Consequently, $\mathcal{S}(\mathbf{P}(m,\cdot) \to \coprod 1$ is a weak equivalence. Thus $diag\mathcal{S}(\mathbf{P}) \to \mathbf{Nerve(A)}$ is a weak equivalence. In the same fashion, there is a weak equivalence $\mathcal{S}(\mathbf{Q} \circ \theta) \to \mathbf{Nerve(B)}$. By commutativity of

$$\begin{array}{ccc} diag\mathcal{S}(\mathbf{P}) & \xrightarrow{diag\mathcal{S}(\theta)} & diag\mathcal{S}(\mathbf{Q} \circ \theta) \\ \downarrow & & \downarrow \\ \mathbf{Nerve(A)} & \xrightarrow{\mathbf{Nerve}(\theta)} & \mathbf{Nerve(B)} \end{array}$$

and saturation of the class of weak equivalences, $\mathbf{Nerve}(\theta)$ is a weak equivalence □

2.3 Corollary (Quillen's Theorem A) *Let $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ be a functor such that the unique map $\mathbf{F}/B \to \mathbf{1}$ is a weak equivalence for all objects B of $\mathbf{B}$. Then $\mathbf{F}$ is a weak equivalence.*

Proof. View $\mathbf{F}$ as a map $\mathbf{F} \Rightarrow id$ in $\mathcal{CAT}/\mathbf{B}$. The previous theorem applies since the map $\mathbf{F}/B \to 1/B$ is a weak equivalence □

2.4 Corollary *Let $\theta : \mathbf{P} \Rightarrow \mathbf{Q}$ be a map in $\mathcal{CAT}/\mathbf{C}$ for some category $\mathbf{C}$ and assume that $\mathbf{P}$ and $\mathbf{Q}$ are opfibrations. If the restriction of θ to the fibres $\mathbf{P}^{-1}(C) \to \mathbf{Q}^{-1}(C)$ is a weak equivalence for all objects C of $\mathbf{C}$ then θ is a weak equivalence.*

Proof. Since $\mathbf{P}$ and $\mathbf{Q}$ are opfibrations, the inclusions

$$\mathbf{P}^{-1}(C) \hookrightarrow \mathbf{P}/C$$

and

$$\mathbf{Q}^{-1}(C) \hookrightarrow \mathbf{Q}/C$$

are weak equivalences for all objects C of $\mathbf{C}$ □

By taking opposites, there is an exactly analoguous result with fibrations in place of cofibrations.

With these preliminaries in hand, the homotopy invariance of $X//Y$ can be established.

2.5 Theorem (Homotopy Invariance of $X//Y$) *Let $\alpha : X \Rightarrow X'$ be a natural transformation of functors $\mathbf{C} \to \mathcal{CAT}$ and let $\beta : Y \Rightarrow Y'$ be a natural transformation of functors $\mathbf{C}^{op} \to \mathcal{CAT}$. If $\alpha(C)$ and $\beta(C)$ are weak equivalences for all objects C of $\mathbf{C}$ then the induced map*

$$\alpha//\beta : X//Y \longrightarrow X'//Y'$$

is a weak equivalence in $\mathcal{CAT}$.

Proof. Write $\alpha//\beta$ as the composite $id//\beta \circ \alpha//id$. Let $p : X//Y \to Y \int \mathbf{C}$ and $p' : X'//Y \to Y \int \mathbf{C}$ denote the projections. p and p' are opfibrations and $p' \circ \alpha//1 = p$. For each object (C, y) of $Y \int \mathbf{C}$, $p^{-1}(C, y) = X(C)$ and $p'^{-1}(C, y) = X'(C)$. Moreover, the induced map on the fibres is $\alpha(C)$. By assumption $\alpha(C)$ is a weak equivalence for each object C of $\mathbf{C}$. Therefore $\alpha//1$ is a weak equivalence. Similarly $1//\beta$ is a weak equivalence. As the composite of weak equivalences is a weak equivalence, $\alpha//\beta$ is a weak equivalence □

The next lemma is used in the derivation of further properties of $X//Y$

2.6 Lemma *Let*

$$\begin{array}{ccc} \mathbf{C}' & \xrightarrow{\mathbf{G}'} & \mathbf{C} \\ {\scriptstyle \mathbf{F}'}\downarrow & & {\scriptstyle \mathbf{F}}\downarrow \\ \mathbf{D}' & \xrightarrow{\mathbf{G}} & \mathbf{D} \end{array}$$

be a pullback in $\mathcal{CAT}$. Suppose that $\mathbf{G}$ is a fibration and that for all objects $D \in \mathbf{D}$, the comma category $\mathbf{F}/D$ is weakly contractible. Then, for all objects $D' \in \mathbf{D}'$, $\mathbf{F}'/D'$ is weakly contractible. In particular, $\mathbf{F}'$ is a weak equivalence.

Proof. [8,p.9] □

2.7 Corollary *If*

$$\begin{array}{ccc} \mathbf{C}' & \xrightarrow{\mathbf{G}'} & \mathbf{C} \\ {\scriptstyle \mathbf{F}'}\downarrow & & {\scriptstyle \mathbf{F}}\downarrow \\ \mathbf{D}' & \xrightarrow{\mathbf{G}} & \mathbf{D} \end{array}$$

is a pullback in $\mathcal{CAT}$, $\mathbf{G}$ is a fibration, and $\mathbf{F}$ is an opfibration with weakly contractible fibres, then $\mathbf{F}'$ is an opfibration with weakly contractible fibres.

Proof. Because $\mathbf{F}$ is an opfibration, there is a weak equivalence $\mathbf{F}/D \to \mathbf{F}^{-1}(D)$ for all objects D of $\mathbf{D}$ and the previous lemma applies □

The preceding lemma and the corollary which follows it admit evident dualizations.

(1) If

$$\begin{array}{ccc} \mathbf{C}' & \xrightarrow{\mathbf{G}'} & \mathbf{C} \\ {\scriptstyle \mathbf{F}'}\downarrow & & {\scriptstyle \mathbf{F}}\downarrow \\ \mathbf{D}' & \xrightarrow{\mathbf{G}} & \mathbf{D} \end{array}$$

is a pullback in $\mathcal{CAT}$, $\mathbf{G}$ is an opfibration, and for all $D \in \mathbf{D}$, $D/\mathbf{F}$ is weakly contractible, then for all $D' \in \mathbf{D}$, $D'/\mathbf{F}'$ is weakly contractible.

(2) If, in the above square, $\mathbf{F}$ is a fibration with weakly contractible fibres, then $\mathbf{F}'$ is a fibration with weakly contractible fibres.

Let $\cdot/\mathbf{C} : \mathbf{C}^{op} \to \mathcal{CAT}$ denote the functor which assigns the comma category $C/\mathbf{C}$ to objects C of $\mathbf{C}$.

2.8 Proposition *For every diagram $X : \mathbf{C} \to \mathcal{CAT}$, there is a natural weak equivalence*

$$X//(\cdot/\mathbf{C}) \to \mathbf{C} \textstyle\int X.$$

Proof. $X//(\cdot/\mathbf{C})$ is defined by the pullback diagram

$$\begin{array}{ccc} X//(\cdot/\mathbf{C}) & \longrightarrow & (\cdot/\mathbf{C})\textstyle\int \mathbf{C} \\ \downarrow & & \downarrow \mathbf{P} \\ \mathbf{C}\textstyle\int X & \xrightarrow{\mathbf{R}} & \mathbf{C}. \end{array}$$

$\mathbf{P}$ is a fibration with fibre $\mathbf{P}^{-1}(C)$ the comma category $C/\mathbf{C}$ and $\mathbf{R}$ is an opfibration. But since the identity arrow $id_C : C \to C$ is an initial object in $C/\mathbf{C}$, $C/\mathbf{C}$ is weakly contractible. Hence the corollary to the previous lemma applies: $X//(\cdot/\mathbf{C}) \to \mathbf{C} \int X$ is fibration with weakly contractible fibres □

2.9 Corollary *Let X and X' be $\mathcal{CAT}$-valued diagrams on $\mathbf{C}$ and let $\alpha : X \Rightarrow X'$ be a natural transformation. If α is a pointwise weak equivalence then $\mathbf{C} \int \alpha : \mathbf{C} \int X \to \mathbf{C} \int X'$ is a weak equivalence.*

Proof. In the following commutative diagram,

$$\begin{array}{ccc} X//(\cdot/\mathbf{C}) & \xrightarrow{\alpha//1} & X'//(\cdot/\mathbf{C}) \\ \downarrow & & \downarrow \\ \mathbf{C}\textstyle\int X & \xrightarrow{\mathbf{C}\int\alpha} & \mathbf{C}\textstyle\int X' \end{array}$$

the vertical arrows are weak equivalences by the previous proposition. By the homotopy invariance of $X//Y$, $\alpha//1$ is a weak equivalence. By saturation of the collection of weak equivalences in $\mathcal{CAT}$, $\mathbf{C} \int \alpha$ is a weak equivalence □

As before, the preceding proposition and the corollary which follows it admit evident dualizations. Let $\mathbf{C}/\cdot : \mathbf{C} \to \mathcal{CAT}$ be the functor which assigns the comma category $\mathbf{C}/C$ to $C \in \mathbf{C}$.

(1) For every diagram $Y \in (\mathbf{C}^{op}, \mathcal{CAT})$, there is a natural weak equivalence

$$(\mathbf{C}/\cdot)//Y \longrightarrow Y \textstyle\int \mathbf{C}.$$

(2) If $\beta : Y \Rightarrow Y'$ is a weak equivalence in $(\mathbf{C}^{op}, \mathcal{CAT})$, then

$$\beta \int \mathbf{C} : Y \int \mathbf{C} \longrightarrow Y' \int \mathbf{C}$$

is a weak equivalence.

3 Free resolutions of $\mathcal{CAT}$–valued diagrams

Let $\mathbf{P} : \mathbf{A} \to \mathbf{C}$. The assignment of the comma category $\mathbf{P}/C$ to objects C of $\mathbf{C}$ defines a functor $(\mathbf{P}/\cdot) : \mathbf{C} \to \mathcal{CAT}$. The correspondence $\mathbf{P} \mapsto \mathbf{P}/\cdot$ defines a functor $\mathcal{CAT}/\mathbf{C} \to (\mathbf{C}, \mathcal{CAT})$. Diagrams of the form $(\mathbf{P}/\cdot)$ for some $\mathbf{P} : \mathbf{A} \to \mathbf{C}$ are called *free*. A *free resolution* of a diagram $X : \mathbf{C} \to \mathcal{CAT}$ is a weak equivalence $\theta : Y \Rightarrow X$ from a free diagram Y.

3.1 Theorem *Every diagram $X : \mathbf{C} \to \mathcal{CAT}$ has a free resolution.*

Proof. Let $\pi : \mathbf{C} \int X \to \mathbf{C}$ denote the opfibred category associated to X. Suppressing the projection π, let $((\mathbf{C} \int X)/\cdot)$ denote $(\pi/\cdot)$. Define

$$\varepsilon_X : ((\mathbf{C} \int X)/\cdot) \Rightarrow X$$

as follows: Objects of $\mathbf{C} \int X/C$ are pairs $(x, c : C \to C')$ such that $c \in \mathbf{C}$ and $x \in X(C')$. Put $\varepsilon(C)(x, c) = X(c)(x)$. A map $(x, c' : C' \to C) \to (x', c'' : C'' \to C)$ is a pair (c, f) such that $c : C' \to C''$, $c'' o c = c'$, and $f : X(c)(x) \to x'$ in $X(C'')$. Define $\varepsilon(C)(c, f) = X(c'')(f)$. Is easily verified that $\varepsilon(C) : \mathbf{C} \int X/C \to X(C)$ is a functor for each object C of $\mathbf{C}$ which is natural in C.It remains to verify that $\varepsilon(C)$ is a weak equivalence for all objects C of $\mathbf{C}$. To this end, define $\phi(C) : X(C) \to \mathbf{C} \int X/C$ by $\phi(C)(x) = (x, id : C \to C)$ for objects x of $X(C)$ and $\phi(C)(f : x \to x') = (f, id)$ for maps f. Then $\varepsilon(C)$ is left adjoint,left inverse to $\phi(C)$ for every object C of $\mathbf{C}$. Therefore,

$$\varepsilon_X : ((\mathbf{C} \int X)/\cdot) \Rightarrow X$$

is a pointwise homotopy equivalence □

Free objects and free resolutions in $(\mathbf{C}^{op}, \mathcal{CAT})$ are defined in much the same way. Let $\mathbf{Q} : \mathbf{B} \to \mathbf{C}$ be a functor. The assignment of the comma category $C/\mathbf{Q}$ to each object C of $\mathbf{C}$ defines a diagram $(\cdot/\mathbf{Q}) : \mathbf{C}^{op} \to \mathcal{CAT}$. Diagrams of the form $(\cdot/\mathbf{Q})$ are called *free*. Let $Y : \mathbf{C}^{op} \to \mathcal{CAT}$. A *free resolution* of Y is a weak equivalence $\phi : Y' \Rightarrow Y$ with Y' free. For each Y as above, there is a free resolution

$$\varepsilon_Y : (\cdot/(Y \int \mathbf{C})) \Rightarrow Y$$

from the free object $(\cdot/(Y \int \mathbf{C}))$ derived from the fibred category

$$Y \int \mathbf{C} \to \mathbf{C}.$$

For diagrams $X : \mathbf{C} \to \mathcal{CAT}$ and $Y : \mathbf{C}^{op} \to \mathcal{CAT}$, let $X \otimes Y$ denote the coend $\int^{\mathbf{C}} X \times Y$ [9, p.102].

3.2 Lemma *Let $\mathbf{P} : \mathbf{A} \to \mathbf{C}$. Then, for all diagrams $Y : \mathbf{C}^{op} \to \mathcal{CAT}$, there is a natural isomorphism*

$$(\mathbf{P}/\cdot) \otimes Y \simeq \mathbf{P}^* Y \int \mathbf{A}$$

($\mathbf{P}^* : (\mathbf{C}^{op}, \mathcal{CAT}) \to (\mathbf{A}^{op}, \mathcal{CAT})$ is the lifting of $\mathbf{P}$).

Proof. The proof is a verification that $\mathbf{P}^* Y \int \mathbf{A}$ has the universal property of the coend [8,p.19] □

3.3 Corollary *Let $X : \mathbf{C} \to \mathcal{CAT}$ be free and let $\theta : Y \Rightarrow Y'$ be a pointwise weak equivalence in $(\mathbf{C}^{op}, \mathcal{CAT})$. Then the induced map*

$$X \otimes \theta : X \otimes Y \longrightarrow X \otimes Y'$$

is a weak equivalence.

Proof. Assume that X is the free object associated to $\mathbf{P} : \mathbf{A} \to \mathcal{CAT}$. By the lemma, $X \otimes \theta \simeq \mathbf{P}^* \theta \int \mathbf{A}$. As $\mathbf{P}^*$ evidently preserves weak equivalences, the claim follows from the homotopy invariance of $(\cdot \int \mathbf{A})$ □

I remark that the preceding lemma and the corollary which follows it have evident dualizations:

(1) Let $\mathbf{P} : \mathbf{A} \to \mathbf{C}$.For every diagram $X : \mathbf{C} \to \mathcal{CAT}$ there is a natural isomorphism

$$X \otimes (\cdot/\mathbf{P}) \simeq \mathbf{A} \int \mathbf{P}^* X.$$

(2) Let Y be a free object in $(\mathbf{C}^{op}, \mathcal{CAT})$ and let $\theta : X \Rightarrow X'$ be a pointwise weak equivalence in $(\mathbf{C}, \mathcal{CAT})$. Then the induced map

$$\theta \otimes Y : X \otimes Y \to X' \otimes Y$$

is a weak equivalence.

It follows that if $\theta : X \Rightarrow X'$ is a weak equivalence of free $\mathcal{CAT}$-valued diagrams on $\mathbf{C}$ and $\phi : Y \Rightarrow Y'$ is a weak equivalence of free $\mathcal{CAT}$-valued diagrams on $\mathbf{C}^{op}$, then

$$\theta \otimes \phi : X \otimes Y \longrightarrow X' \otimes Y'$$

is a weak equivalence.

4 The left derived tensor product

Let $X : \mathbf{C} \to \mathcal{CAT}$ and $Y : \mathbf{C}^{op} \to \mathcal{CAT}$ be $\mathcal{CAT}$-valued diagrams.

4.1 Definition Choose free resolutions $\mathbf{F}X \Rightarrow X$ and $\mathbf{F}Y \Rightarrow Y$ of X and Y respectively. Define

$$X \otimes^{\mathbf{L}} Y \equiv \mathbf{F}X \otimes \mathbf{F}Y.$$

4.2 Proposition *Up to natural isomorphism in $\Sigma^{-1}\mathcal{CAT}$, $X \otimes^{\mathbf{L}} Y$ is independent of the free resolutions chosen.*

Proof. Let $\theta : \mathbf{F}X \Rightarrow X$ be a free resolution of X. There is a commutative diagram

$$\begin{array}{ccc} ((\mathbf{C} \int \mathbf{F}X)/\cdot) & \xrightarrow{((\mathbf{C}\int\theta)/\cdot)} & ((\mathbf{C}\int X)/\cdot) \\ {\scriptstyle \varepsilon_{\mathbf{F}X}}\downarrow & & {\scriptstyle \varepsilon_X}\downarrow \\ \mathbf{F}X & \xrightarrow{\theta} & X \end{array}$$

where $\varepsilon_{\mathbf{F}X}$ and ε_X are the pointwise weak equivalences constructed in the previous §. As θ is a weak equivalence by assumption, $((\mathbf{C}\int\theta)/\cdot)$ is a weak equivalence. Similarly, if $\phi : \mathbf{F}Y \Rightarrow Y$ is a free resolution of Y, there is a commuting square of weak equivalences

$$\begin{array}{ccc} (\cdot/(\mathbf{F}Y \int \mathbf{C})) & \xrightarrow{(\cdot/(\phi\int\mathbf{C}))} & (\cdot/(Y\int\mathbf{C})) \\ {\scriptstyle \varepsilon_{\mathbf{F}Y}}\downarrow & & {\scriptstyle \varepsilon_Y}\downarrow \\ \mathbf{F}Y & \xrightarrow{\phi} & Y \end{array}$$

As $\otimes$ preserves weak equivalences between free objects, there is a chain of weak equivalences

$$\begin{array}{ccc} ((\mathbf{C}\int\mathbf{F}X)/\cdot)\otimes(\cdot/(\mathbf{F}Y\int\mathbf{C})) & \longrightarrow & \mathbf{F}X\otimes\mathbf{F}Y \\ \downarrow & & \\ ((\mathbf{C}\int X)/\cdot)\otimes(\cdot/(Y\int\mathbf{C})) & & \end{array}$$

That is, $\mathbf{F}X\otimes\mathbf{F}Y$ is naturally isomorphic in $\Sigma^{-1}\mathcal{CAT}$ to $((\mathbf{C}\int X)/\cdot)\otimes(\cdot/(Y\int\mathbf{C}))$ irrespective of the free resolutions $\mathbf{F}X$ and $\mathbf{F}Y$ chosen □

As a consequence, left deriving $\otimes$ defines a functor

$$\otimes^{\mathbf{L}} : \Sigma^{-1}(\mathbf{C},\mathcal{CAT}) \times \Sigma^{-1}(\mathbf{C}^{op},\mathcal{CAT}) \longrightarrow \Sigma^{-1}\mathcal{CAT}$$

Let $\rho : \mathcal{CAT} \to \Sigma^{-1}\mathcal{CAT}$ denote the localization functor.

4.3 Theorem *$\otimes^{\mathbf{L}}$ is the right Kan extension of*

$$\rho\circ\otimes : (\mathbf{C},\mathcal{CAT}) \times (\mathbf{C}^{op},\mathcal{CAT}) \to \mathcal{CAT} \to \Sigma^{-1}\mathcal{CAT}$$

along the localization functor

$$\rho\times\rho : (\mathbf{C},\mathcal{CAT}) \times (\mathbf{C}^{op},\mathcal{CAT}) \to \Sigma^{-1}(\mathbf{C},\mathcal{CAT}) \times \Sigma^{-1}(\mathbf{C}^{op},\mathcal{CAT})$$

Proof. As a representative of $X \otimes^{\mathbf{L}} Y$ in $\Sigma^{-1}\mathcal{CAT}$ choose $((\mathbf{C} \int X)/\cdot) \otimes (\cdot/(Y \int \mathbf{C}))$. The maps ε_X and ε_Y are the components of a natural transformation $\otimes^{\mathbf{L}} \Rightarrow \rho \circ \otimes$. The pair $(\otimes^{\mathbf{L}}, \varepsilon)$ has the universal property of the right Kan extension. Namely, if

$$\mathbf{G} : \Sigma^{-1}(\mathbf{C}, \mathcal{CAT}) \times \Sigma^{-1}(\mathbf{C}^{op}, \mathcal{CAT}) \to \Sigma^{-1}\mathcal{CAT}$$

is a functor and

$$\vartheta : \mathbf{G} \circ (\rho \times \rho) \Rightarrow \rho \circ \otimes$$

a natural transformation, then there is a unique natural transformation $\sigma : \mathbf{G} \Rightarrow \otimes^{\mathbf{L}}$ such that $\varepsilon \circ (\sigma.(\rho \times \rho)) = \vartheta$. For $(X, Y) \in \Sigma^{-1}(\mathbf{C}, \mathcal{CAT}) \times \Sigma^{-1}(\mathbf{C}^{op}, \mathcal{CAT})$, define $\sigma(X, Y)$ to be the composite

$$\vartheta((\mathbf{C} \int X)/\cdot, \cdot/(Y \int \mathbf{C})) \circ \mathbf{G}(\varepsilon_X^{-1}, \varepsilon_Y^{-1}).$$

It is readily verified that σ is the unique natural transformation $\sigma : \mathbf{G} \Rightarrow \otimes^{\mathbf{L}}$ satisfying $\varepsilon \circ (\sigma.(\rho \times \rho)) = \vartheta$ □

In other words, $\otimes^{\mathbf{L}}$ *is the left derived functor of* $\otimes$ *in the sense of Quillen's homotopical algebra* [12,4.1].

4.4 Theorem *Let* $X : \mathbf{C} \to \mathcal{CAT}$ *and* $Y : \mathbf{C}^{op} \to \mathcal{CAT}$ *be* $\mathcal{CAT}$*-valued diagrams. Then* $X//Y$ *is a representative in* $\Sigma^{-1}\mathcal{CAT}$ *of* $X \otimes^{\mathbf{L}} Y$.

Proof. Let $\mathbf{P} : \mathbf{C} \int X \to \mathbf{C}$ denote the projection. By direct calculation,

$$X//Y \simeq \mathbf{P}^* Y \int (\mathbf{C} \int X).$$

But $\mathbf{P}^* Y \int (\mathbf{C} \int X) = ((\mathbf{C} \int X)/\cdot) \otimes Y \simeq X \otimes^{\mathbf{L}} Y$ □

In particular, $X \otimes^{\mathbf{L}} 1 \simeq \mathbf{C} \int X$ in $\Sigma^{-1}\mathcal{CAT}$. The identification of the Grothendieck construction with $(\cdot) \otimes^{\mathbf{L}} 1$ explains its importance in the homotopy theory of categories.

5 The calculus of $\otimes^{\mathbf{L}}$

Homotopy left Kan extensions, and thus homotopy colimits, can be constructed using the left derived tensor product.

5.1 Definition Let $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ be a functor. The functor

$$\mathbf{LF} : \Sigma^{-1}(\mathbf{A}, \mathcal{CAT}) \to \Sigma^{-1}(\mathbf{B}, \mathcal{CAT})$$

is defined by

$$\mathbf{LF}(X)(B) = X \otimes^{\mathbf{L}} \mathcal{H}om_{\mathbf{B}}(\mathbf{F}(\cdot), B)$$

for objects B of $\mathbf{B}$ and diagrams $X : \mathbf{A} \to \mathcal{CAT}$. The extension of $\mathbf{LF}$ to maps is evident.

Let $\cdot/\mathbf{F}/\cdot : \mathbf{A}^{op} \times \mathbf{B} \to \mathcal{CAT}$ denote the diagram whose value at $(A, B) \in \mathbf{A}^{op} \times \mathbf{B}$ is the pullback

$$\begin{array}{ccc} A/\mathbf{F}/B & \longrightarrow & \mathbf{B}/B \\ \downarrow & & \downarrow{\scriptstyle \pi_B} \\ A/\mathbf{A} & \xrightarrow{\mathbf{F}\circ\pi_A} & \mathbf{B} \end{array}$$

where $\pi_A : A/\mathbf{A} \to \mathbf{A}$ and $\pi_B : \mathbf{B}/B \to \mathbf{B}$ are the projections from the comma categories.

5.2 Proposition *$X \otimes (\cdot/\mathbf{F}/\cdot)$ is a representative of $X \otimes^{\mathbf{L}} \mathcal{H}om_{\mathbf{B}}(\mathbf{F}(\cdot), B)$.*

Proof. By inspection, $\mathcal{H}om_{\mathbf{B}}(\mathbf{F}(\cdot), B) \int \mathbf{A} = \mathbf{F}/B$. As

$$\cdot/(\mathcal{H}om_{\mathbf{B}}(\mathbf{F}(\cdot), B) \textstyle\int \mathbf{A}) \Rightarrow \mathcal{H}om_{\mathbf{B}}(\mathbf{F}(\cdot), B)$$

is a free resolution,

$$\begin{aligned} X \otimes^{\mathbf{L}} \mathcal{H}om_{\mathbf{B}}(\mathbf{F}(\cdot), B) &\simeq X \otimes ((\cdot/\mathcal{H}om_{\mathbf{B}}(\mathbf{F}(\cdot), B)) \textstyle\int \mathbf{A}) \\ &= X \otimes (\cdot/(\mathbf{F}/B)) \\ &= X \otimes (\cdot/\mathbf{F}/B) \quad \square \end{aligned}$$

Let $L\mathbf{F} : (\mathbf{A}, \mathcal{CAT}) \to (\mathbf{B}, \mathcal{CAT})$ denote the functor whose value at $X : \mathbf{A} \to \mathcal{CAT}$ is the diagram defined on objects $B \in \mathbf{B}$ by

$$L\mathbf{F}(X)(B) = X \otimes (\cdot/\mathbf{F}/\cdot).$$

That is,

$$\begin{array}{ccc} (\mathbf{A}, \mathcal{CAT}) & \xrightarrow{L\mathbf{F}} & (\mathbf{B}, \mathcal{CAT}) \\ {\scriptstyle \rho}\downarrow & & {\scriptstyle \rho}\downarrow \\ \Sigma^{-1}(\mathbf{A}, \mathcal{CAT}) & \xrightarrow{\mathbf{LF}} & \Sigma^{-1}(\mathbf{B}, \mathcal{CAT}) \end{array}$$

commutes up to natural isomorphism.

5.3 Lemma *$\mathbf{LF}$ is left adjoint to $\mathcal{HOM}(\cdot/\mathbf{F}/\cdot, \cdot)$.*

Proof. $\mathcal{HOM}$ denotes the category valued hom in $(\mathbf{B}, \mathcal{CAT})$ [5,p.8]. A proof of this lemma can be found in [9,p.63] $\square$

5.4 Proposition *Let $Y : \mathbf{C}^{op} \to \mathcal{CAT}$. There is a natural weak equivalence*

$$\mathbf{F}^*Y \longrightarrow \mathcal{HOM}(\cdot/\mathbf{F}/\cdot, Y)$$

in $(\mathbf{A}, \mathcal{CAT})$.

Proof. Before sketching the proof, I recall the category structure on

$$\mathcal{HOM}(A/\mathbf{F}/\cdot, Y)$$

, where $(A \in \mathbf{A})$.An object of $\mathcal{HOM}(A/\mathbf{F}/\cdot, Y)$ is a natural transformation $\theta : A/\mathbf{F}/\cdot \Rightarrow Y$. A map $v : \theta \to \theta'$ from one object θ to another θ' is a *modification*,i.e. a family of natural transformations $\{v(B) \mid B \in \mathbf{B}\}$ such that for all maps $b : B \to B' \in \mathbf{B}$,

$$Y(b).v(B) = v(B'.(A/\mathbf{F}/b).$$

Let $\theta \in \mathcal{HOM}(A/\mathbf{F}/\cdot, Y)$. Define $\mathcal{F}(A)(\theta) \in \mathbf{F}^*Y(A)$ by

$$\mathcal{F}(A)(\theta) = \theta(\mathbf{F}(A))(id : A \to A, id : \mathbf{F}(A) \to \mathbf{F}(A)).$$

If $v : \theta \to \theta'$ is a modification, define $\mathcal{F}(A)(v) : \mathcal{F}(A)(\theta) \to \mathcal{F}(A)(\theta')$ by

$$\mathcal{F}(A)(v) = v(\mathbf{F}(A))(id_A, id_{\mathbf{F}(A)}).$$

For each $A \in \mathbf{A}$,

$$\mathcal{F}(A) : \mathcal{HOM}(A/\mathbf{F}/\cdot, Y) \longrightarrow \mathbf{F}^*Y(A)$$

is a functor. For each object $A \in \mathbf{A}$, define

$$\mathcal{G}(A) : \mathbf{F}^*Y(A) \to \mathcal{HOM}(A/\mathbf{F}/\cdot, Y)$$

by

$$\mathcal{G}(A)(x)(b) = Y(B \circ \mathbf{F}(a))(x)$$

where $x \in \mathbf{F}^*Y(A)$, $a : A \to A' \in \mathbf{A}$, and $b : \mathbf{F}(A') \to B \in \mathbf{B}$. If $\chi : x \to x'$, define

$$\mathcal{G}(A)(\chi)(B)(a, b) = Y(b \circ \mathbf{F}(a))(\chi).$$

For each $A \in \mathbf{A}$, $\mathcal{G}(A)$ is a functor. The assignments $A \mapsto \mathcal{F}(A)$ and $A \mapsto \mathcal{G}(A)$ are natural in A. Consequently, $\mathcal{F}$ and $\mathcal{G}$ define natural transformations $\mathcal{HOM}(\cdot/\mathbf{F}/\cdot, Y) \Rightarrow \mathbf{F}^*Y$ and $\mathbf{F}^*Y \Rightarrow \mathcal{HOM}(\cdot/\mathbf{F}/\cdot, Y)$ respectively. Moreover, for each $A \in \mathbf{A}$, $\mathcal{G}(A)$ is left adjoint, left inverse to $\mathcal{F}(A)$. Therefore $\mathbf{F}^*Y$ is pointwise homotopy equivalent to $\mathcal{HOM}(\cdot/\mathbf{F}/\cdot, Y)$ in $(\mathbf{A}, \mathcal{CAT})$ □

5.5 Corollary **LF** *is left adjoint to* $\Sigma^{-1}\mathbf{F}^*$.

Proof. There is a pointwise weak equivalence

$$\mathbf{F}^*Y \longrightarrow \mathcal{HOM}(\cdot/\mathbf{F}/\cdot, Y)$$

for $Y \in (\mathbf{B}, \mathcal{CAT})$. Therefore, $\Sigma^{-1}\mathcal{HOM}(\cdot/\mathbf{F}/\cdot,\cdot)$ exists and is naturally isomorphic to $\Sigma^{-1}\mathbf{F}^*$. But

$$\mathbf{LF} \simeq \Sigma^{-1}L\mathbf{F}$$

is left adjoint to $\Sigma^{-1}\mathcal{HOM}(\cdot/\mathbf{F}/\cdot,\cdot)$ □

This corollary validates the designation of **LF** as the *left homotopy Kan extension of* **F**.

The remaining results show that $\otimes^{\mathbf{L}}$ behaves in many respects like the ordinary tensor product $\otimes$.

5.6 Proposition *Let Y be a diagram in $(\mathbf{B}, \mathcal{CAT})$ and $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ a functor. There is a natural isomorphism in $\Sigma^{-1}(\mathbf{A}, \mathcal{CAT})$ between $\mathbf{F}^*Y$ and $\mathcal{H}om_{\mathbf{B}}(\mathbf{F}(\cdot),\cdot) \otimes^{\mathbf{L}} Y$.*

Proof. Let $A \in \mathbf{A}$. Define $\phi(A) : (A/\mathbf{F}/\cdot)//Y \to \mathbf{F}^*(A)$ by

$$\phi(A)(a : A \to A', b : \mathbf{F}(A') \to B, y) = Y(b \circ \mathbf{F}(a))(y).$$

Extending $\phi(A)$ in the obvious way to maps defines a functor $\phi(A)$ which is natural in A and hence a natural transformation

$$\phi : (\cdot/\mathbf{F}/\cdot)//Y \Rightarrow \mathbf{F}^*Y.$$

For $A \in \mathbf{A}$ define $\theta(A) : \mathbf{F}^*(A) \to (A/\mathbf{F}/\cdot)//Y$ by

$$\theta(A)(x) = (id_A : A \to A, id_{\mathbf{F}(A)} : \mathbf{F}(A) \to \mathbf{F}(A)).$$

Again, by extending θ in the obvious way to maps, one obtains a functor $\theta(A)$ natural in A and hence a natural transformation

$$\theta : \mathbf{F}^*Y \Rightarrow (\cdot/\mathbf{F}/\cdot)//Y.$$

It is readily verified that $\theta(A)$ is left adjoint, left inverse to $\phi(A)$. It follows that ϕ is an isomorphism in $\Sigma^{-1}(\mathbf{A}, \mathcal{CAT})$. For each $A \in \mathbf{A}$, $((\mathbf{B} \int (\mathcal{H}om_{\mathbf{B}}(\mathbf{F}(A),\cdot))/\cdot)$ is a free resolution of $\mathcal{H}om_{\mathbf{B}}(\mathbf{F}(A),\cdot)$. For $B \in \mathbf{B}$ define

$$\mu(A,B)(a : A \to A', b : \mathbf{F}(A') \to B) = (\mathbf{F}(a), b).$$

Extending μ in the obvious way to maps defines a functor

$$A/\mathbf{F}/B \longrightarrow \mathbf{B} \int \mathcal{H}om_{\mathbf{B}}(\mathbf{F}(A), \cdot)/B.$$

It is easily verified that μ is natural in both variables. Define

$$\nu(B)(a : \mathbf{F}(A) \to B', b : B \to B') = (id_A, b \circ a).$$

Extending ν in the obvious way to maps defines a functor $\nu(B)$ natural in B and hence a natural transformation

$$\mathbf{B} \int \mathcal{H}om_{\mathbf{B}}(\mathbf{F}(A), \cdot) \Rightarrow (A/\mathbf{F}/\cdot).$$

There are natural transformations $\nu(B) \circ \mu(A, B) \Rightarrow 1$ and $\mu(A, B) \circ \nu(B) \Rightarrow 1$. Consequently, $\nu(A, \cdot) : A/\mathbf{F}/\cdot \Rightarrow \mathbf{B} \int \mathcal{H}om_{\mathbf{B}}(\mathbf{F}(A), \cdot)$ is a pointwise homotopy equivalence. Therefore, there are weak equivalences

$$\begin{array}{ccc} (\mathbf{B} \int \mathcal{H}om_{\mathbf{B}}(\mathbf{F}(A), \cdot))/\cdot & \longleftarrow & \mathcal{H}om_{\mathbf{B}}(\mathbf{F}(A), \cdot) \otimes^{\mathbf{L}} Y \\ \downarrow & & \\ (A/\mathbf{F}/\cdot)//Y & \longrightarrow & \mathbf{F}^*Y(A) \end{array}$$

These weak equivalences are natural in A and produce the desired natural isomorphism in $\Sigma^{-1}(\mathbf{A}, \mathcal{CAT})$,

$$\mathcal{H}om_{\mathbf{B}}(\mathbf{F}(\cdot), \cdot) \otimes^{\mathbf{L}} Y \simeq \mathbf{F}^*Y \quad \square$$

5.7 Corollary *Let* $X : \mathbf{A} \to \mathcal{CAT}$ *and* $Y : \mathbf{B}^{op} \to \mathcal{CAT}$. *There is a natural isomorphism in* $\Sigma^{-1}\mathcal{CAT}$,

$$\mathbf{LF}(X) \otimes^{\mathbf{L}} Y \simeq X \otimes^{\mathbf{L}} \mathbf{F}^*Y.$$

Proof. This follows at once from the natural isomorphisms

$$\begin{aligned} \mathbf{LF}(X) \otimes^{\mathbf{L}} Y &\simeq \left(X \otimes^{\mathbf{L}} \mathcal{H}om_{\mathbf{B}}(\mathbf{F}(\cdot), \cdot)\right) \otimes^{\mathbf{L}} Y \\ &\simeq X \otimes^{\mathbf{L}} \left(\mathcal{H}om_{\mathbf{B}}(\mathbf{F}(\cdot), \cdot) \otimes^{\mathbf{L}} Y\right) \\ &\simeq X \otimes^{\mathbf{L}} \mathbf{F}^*Y \quad \square \end{aligned}$$

5.8 Proposition *Let* $\mathbf{F}:\mathbf{A}\to\mathbf{B}$ *be a functor and let* 1 *denote the terminal diagram in* $(\mathbf{A},\mathcal{CAT})$. *There is a natural isomorphism in* $\Sigma^{-1}(\mathbf{B},\mathcal{CAT})$ *between* $\mathbf{LF}(1)$ *and* $(\mathbf{F}/\cdot)$.

Proof. For each $B\in\mathbf{B}$,

$$\begin{aligned}\mathbf{LF}(1)(B)\equiv&1\otimes^{\mathbf{L}}(\mathbf{F}(\cdot),B)\\ \simeq&(\mathbf{A}/\cdot)//(\cdot/\mathbf{F}/B).\end{aligned}$$

Define $\phi(B):(\mathbf{A}/\cdot)//(\cdot/\mathbf{F}/B)\to\mathbf{F}/B$ on objects by

$$\phi(B)(a:A\to A',a':A'\to A'',b:\mathbf{F}(A'')\to B)=(A,b\circ\mathbf{F}(a'\circ a)).$$

Extending $\phi(B)$ in the obvious way to maps defines a functor natural in B. Define $\theta(B):\mathbf{F}/B\to(\mathbf{A}/\cdot)//(\cdot/\mathbf{F}/B)$ on objects by

$$\theta(B)(A,b:\mathbf{F}(A)\to B)=(id_A,b).$$

Extending $\theta(B)$ in the obvious way to maps defines a functor natural in B. It is easily seen that $\theta(B)$ and $\phi(B)$ are homotopy inverses □

5.9 Corollary (Cofinality for Homotopy Colimits) *Let* $\mathbf{F}:\mathbf{A}\to\mathbf{B}$ *and* $\mathbf{G}:\mathbf{C}\to\mathbf{B}$ *be two functors for which there is a weak equivalence* $\mathbf{F}/\cdot\to\mathbf{G}/\cdot$ *in* $(\mathbf{B},\mathcal{CAT})$. *Then for all* $Y\in(\mathbf{B}^{op},\mathcal{CAT})$ *there is a natural isomorphism in* $\Sigma^{-1}\mathcal{CAT}$

$$1\otimes^{\mathbf{L}}\mathbf{F}^*Y\simeq1\otimes^{\mathbf{L}}\mathbf{G}^*Y.$$

Proof. There are natural isomorphisms in $\Sigma^{-1}\mathcal{CAT}$,

$$\begin{aligned}1\otimes^{\mathbf{L}}\mathbf{F}^*Y\simeq&\mathbf{LF}(1)\otimes^{\mathbf{L}}Y\\ \simeq&\mathbf{LG}(1)\otimes^{\mathbf{L}}Y\\ \simeq&1\otimes^{\mathbf{L}}\mathbf{G}^*Y\quad\square\end{aligned}$$

This result is the homotopy–theoretical analogue of the cofinality theorem for colimits proved by Paré [11].

5.10 Corollary *Assume that* $\mathbf{F}:\mathbf{A}\to\mathbf{B}$ *has "weakly contractible fibres", i.e. the unique map*

$$(\mathbf{F}/\cdot)\to1$$

is an isomorphism in $\Sigma^{-1}(\mathbf{B},\mathcal{CAT})$. *Then, for all* $Y\in(\mathbf{B}^{op},\mathcal{CAT})$,

$$1\otimes^{\mathbf{L}}\mathbf{F}^*Y\simeq1\otimes^{\mathbf{L}}Y.$$

Proof. The assumption on $\mathbf{F}$ translates into the condition that

$$\mathbf{LF}(1)\simeq1$$

in $\Sigma^{-1}(\mathbf{B},\mathcal{CAT})$ □

5.11 Proposition (Homotopy Yoneda Lemma) *For all $X \in (\mathbf{A}, \mathcal{CAT})$, there is a natural isomorphism*

$$X \otimes^{\mathbf{L}} \mathcal{H}om_{\mathbf{A}}(\cdot,\cdot) \simeq X$$

in $\Sigma^{-1}(\mathbf{A}, \mathcal{CAT})$.

Proof. By definition,

$$X \otimes^{\mathbf{L}} \mathcal{H}om_{\mathbf{A}}(\cdot,\cdot) \equiv \mathbf{L}(id_{\mathbf{A}})(X)(A).$$

Since $\mathbf{L}(id_{\mathbf{A}})$ is left adjoint to $id_{(\mathbf{A},\mathcal{CAT})}$,

$$\mathbf{L}(id_{\mathbf{A}})(X)(A) \simeq X(A) \quad \Box$$

In a sense which is made precise in the next proposition, the left derived functor of the tensor product can be expressed in terms of homotopy colimits. Let $\pi : \mathbf{A} \longrightarrow 1$ denote the unique functor to the terminal category 1. Then, since $\pi^* 1 = 1$ in $(\mathbf{A}^{op}, \mathcal{CAT})$,

$$X \otimes^{\mathbf{L}} 1 = X \otimes^{\mathbf{L}} \pi^* 1 \simeq \mathbf{L}\pi(X)$$

is the homotopy colimit of the diagram X.

5.12 Proposition *Let $X \in (\mathbf{C}, \mathcal{CAT})$ and $Y \in (\mathbf{C}^{op}, \mathcal{CAT})$. Let*

$$\mathbf{P} : \mathbf{C} \int X \to \mathbf{C}$$

be the opfibred category corresponding to X and

$$\mathbf{Q} : Y \int \mathbf{C} \to \mathbf{C}$$

the fibred category corresponding to Y. There are natural isomorphisms

$$1 \otimes^{\mathbf{L}} \mathbf{P}^* Y \simeq X \otimes^{\mathbf{L}} Y \simeq \mathbf{Q}^* X \otimes^{\mathbf{L}} 1$$

in $\Sigma^{-1}\mathcal{CAT}$.

Proof. By parity of reasoning it will be enough to establish the existence of the first isomorphism. But,

$$\begin{aligned} X \otimes^{\mathbf{L}} Y &\simeq ((\mathbf{C} \textstyle\int X)/\cdot) \otimes Y \\ &\simeq \mathbf{P}^* Y \textstyle\int \mathbf{C} \\ &\simeq 1 \otimes^{\mathbf{L}} \mathbf{P}^* Y \quad \Box \end{aligned}$$

The final result is a converse to the cofinality of homotopy colimits.

5.13 Proposition *Let* $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *and* $\mathbf{G} : \mathbf{C} \to \mathbf{B}$ *be two functors such that for all diagrams* $Y \in (\mathbf{B}^{op}, \mathcal{CAT})$, *there is a natural isomorphism in* $\Sigma^{-1}\mathcal{CAT}$

$$1 \otimes^{\mathbf{L}} \mathbf{F}^* Y \simeq 1 \otimes^{\mathbf{L}} \mathbf{G}^* Y.$$

Then there is a natural isomorphism in $\Sigma^{-1}(\mathbf{B}, \mathcal{CAT})$

$$\mathbf{F}/\cdot \simeq \mathbf{G}/\cdot\,.$$

Proof. For each $B \in \mathbf{B}$, there are isomorphisms in $\Sigma^{-1}\mathcal{CAT}$,

$$\begin{aligned} \mathbf{F}/B &\simeq (\mathbf{F}/\cdot) \otimes^{\mathbf{L}} \mathcal{H}om_{\mathbf{B}}(\cdot, B) \\ &\simeq \mathbf{LF}(1) \otimes^{\mathbf{L}} \mathcal{H}om_{\mathbf{B}}(\cdot, B) \\ &\simeq 1 \otimes \mathbf{F}^* \mathcal{H}om_{\mathbf{B}}(\cdot, B) \\ &\simeq 1 \otimes \mathbf{G}^* \mathcal{H}om_{\mathbf{B}}(\cdot, B) \\ &\simeq \mathbf{LG}(1) \otimes^{\mathbf{L}} \mathcal{H}om_{\mathbf{B}}(\cdot, B) \\ &\simeq (\mathbf{G}/\cdot) \otimes^{\mathbf{L}} \mathcal{H}om_{\mathbf{B}}(\cdot, B) \\ &\simeq \mathbf{G}/B \quad \square \end{aligned}$$

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