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ROSS STREET

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PARITY COMPLEXES

by Ross STREET

Résumé On construit un modèle pratique de la n -catégorie libre engendrée par certains polytopes orientés de manière adéquate. Pour faire la construction tout ce qui est nécessaire du polytope est l'ensemble gradué de faces et le classement par signe des $(m-1)$ -faces de chaque m -face. Le concept de complexe parité fait abstraction de cette structure.

La description de la n -catégorie libre engendrée par un complexe parité est simple mais la preuve qu'elle est une n -catégorie et qu'elle est libre entraîne de longues inductions. On considère les produits et les joints des complexes parités. Comme applications, les objets fondamentaux de la cohomologie générale non-abélienne et ceux de la théorie de descente se sont précisés pour la première fois.

Introduction

Parity complexes are a new combinatorial structure introduced here in order to provide a uniform treatment of the diagrams in all dimensions that appear in descent theory and in the theory of homotopy types.

The particular sequence of these diagrams arising from the simplexes was treated in [S1]. The modification of that work needed to deal with cubes was done in part by [AS], and a beautifully geometric treatment of cubes with a derived treatment of simplexes appeared in [A]. A different treatment of simplexes was given by [JW] with a view to an abstraction to general pasting diagrams. Such an abstraction was accomplished by [J], but the axioms there are rather complicated.

The calculations of [S1] led me to conjecture (in Montréal, 1985) that the decision to make all the odd faces part of the source and the even faces part of the target uniquely determined the cocycle conditions of higher-order non-abelian cohomology. This provided one motivation for the concept of parity complex.

Apart from the simplexes and cubes there were the diagrams arising in descent. J.W. Duskin proposed (about 1981) that there should be a descent ω -category associated with each cosimplicial ω -category and drew the diagrams in low dimensions. The first step in dealing with a class of diagrams is to find a suitable notation. For simplexes and cubes we had a good notation. On looking at the descent diagrams, I realized (early 1987) that they were simply products of simplexes with "globs"; these latter form a very easy class

of diagrams to treat. I gave a talk in the Sydney Category Seminar outlining a program to investigate these matters. The next week Michael Johnson pointed out that my description of the product in that lecture was too simplistic. I should have realized this since I discussed the associated chain complex in the same lecture, and the tensor product of the chain complexes is associated to the correct product. Johnson's lectures on pasting schemes (now appearing in his PhD thesis [J]) provided the first successful abstraction from the simplex example, and strongly influenced this work. His goal to capture the general pastable diagrams of higher-order categories is more ambitious than the goals outlined here.

These considerations led me to try to abstract the constructions of [S1] from simplexes to a structure in which "negative" and "positive" faces were given. The abstraction arose purely from the desire to deal with products of examples already understood. However, the resultant concept of parity complex is an accessible multi-dimensional generalization of loop-free graph; the free ω -category generalizes the category of paths in the graph.

The first section gives the definition of parity complex and the straight-forward proof that simplexes provide an example. A different proof will be obtained later from the fact that simplexes are iterated cones; however, presentation of this example in the first section should give readers a feeling for the axioms and convince them that the hard part of the work of [S1] is included in our general theory and not in verifying the axioms. Duality is explained. Also, the chain complex of free abelian groups associated to each parity complex is described.

The second section introduces the notion of movement which captures the idea of passage from one part of the parity complex to another by replacement of negative faces by positive faces. Composition and decomposition of moves is examined in preparation for the construction of the free ω -category on a parity complex. The details of this construction appear in the third section and its universal property is given in the fourth section.

The fifth section describes the product and join of parity complexes. These require some extra conditions in order to work well. The final section describes the descent ω -category of a cosimplicial ω -category and the special case, sought by John E. Roberts [R1], of cohomology with coefficients in an ω -category. The connection with homotopy types will be pursued elsewhere.

Added September 1991. This paper is a slight modification of a Macquarie Mathematics Report [S2]. Soon after that Report was written, John Power pointed out that the parity complexes axiomatized there were not general enough to cover all the pasting diagrams one would like to include, even for 3-categories. The pasting schemes of [J] were better in this respect. We were especially encouraged by the applications of the work foreshadowed by Kapranov-Voevodsky [KV1], [KV2]. However, in May 1990, Vaughan Pratt (through his interest in concurrency in computer science [P]) and Michael Johnson came up with a series of examples to show the real limitations of my axioms (even for the 3-cube). This meant that the Report was in error. They also had examples to show that the relation \triangleleft was not generally anti-

symmetric. I had observed that the relation \triangleleft (which contains \triangleleft) was in fact a total order for simplexes, cubes, and other examples. Since antisymmetry of \triangleleft was an indispensable tool in both [J] and [S2], I put aside my revision of [S2].

In mid 1990, Ronnie Brown (Bangor) informed me that Richard Steiner (Glasgow) had some new results in this area. I did some work on the approach of [ASn] but was not able to use it to overcome the problems we were experiencing. Also, John Power (Edinburgh) was able to extend to arbitrary dimensions his geometric approach to pasting [Pw].

Richard Steiner visited Macquarie University for the month of August. This provided the motivation for again concentrating in depth on the subject. Johnson, Steiner and I came to the conclusion that we should settle for the time being on the case of antisymmetry of the relation \triangleleft (*a fortiori* of the relation \triangleleft).

I returned to a revision and correction of [S2] and was amazed to find that I could simply eliminate the offending axioms! The present paper is therefore a minor modification of the original Report. The resultant notion of parity complex is still not as general as one would like (since it assumes \triangleleft antisymmetric), but we do obtain the cubes and the full descent ω -category.

I am grateful to Samuel Eilenberg, Michael Johnson, John Power, Vaughan Pratt, Richard Steiner for their various contributions to this work which certainly go well beyond the points mentioned above.

§1. Definitions, and the simplex example

Definition A *parity complex* is a graded set

$$C = \sum_{n=0}^{\infty} C_n$$

together with, for each element $x \in C_n$ ($n > 0$), two disjoint non-empty finite subsets $x^-, x^+ \subset C_{n-1}$ subject to Axioms 1, 2 and 3 which appear below after some simple terminology is introduced.

Elements of x^- are called *negative faces* of x , and those of x^+ are called *positive faces* of x . It is also appropriate to think of x as the name of a rule which allows the replacement of x^- by x^+ .

Symbols ε, η, ζ will be used to denote signs $-$ or $+$ when either is intended.

Each subset $S \subset C$ is graded via $S_n = S \cap C_n$. The *n-skeleton* of $S \subset C$ is defined by

$$S^{(n)} = \sum_{m=0}^n S_m.$$

Call S *n-dimensional* when it is equal to its n-skeleton. The complement of S

in C is denoted by $\neg S$. Let S^- denote the set of elements of C which occur as negative faces of some $x \in S$, and similarly for S^+ . In symbols,

$$S^\varepsilon = \bigcup_{x \in S} x^\varepsilon.$$

Also let S^\mp denote the set of negative faces of elements of S which are not positive faces of any element of S . So

$$S^\mp = S^- \cap \neg S^+;$$

also, let

$$S^\pm = S^+ \cap \neg S^-.$$

Write $S \perp T$ when $(S^- \cap T^-) \cup (S^+ \cap T^+) = \emptyset$. In such notation we identify a singleton $S = \{x\}$ with its element.

Definition A subset $S \subset C$ is called *well formed* when S_0 has at most one element, and, for all $x, y \in S_n$ ($n > 0$), if $x \neq y$ then $x \perp y$.

Write $x < y$ when $x^+ \cap y^- \neq \emptyset$. This implies $x \neq y$ since x^-, x^+ were assumed disjoint. For any $S \subset C$, let \triangleleft_S denote the preorder obtained on S as the reflexive transitive closure of the relation $<$ on S . Put $\triangleleft = \triangleleft_C$. In general, \triangleleft_S is contained in, but not equal to, the restriction of \triangleleft to S . Whenever order properties of a subset S of C are referred to in this work, it will be implicitly understood that the order \triangleleft_S is intended.

Axiom 1 $x^- \cup x^{++} = x^+ \cup x^{+-}$.

Axiom 2 x and x^+ are both well formed.

Axiom 3 (a) $x \triangleleft y \triangleleft x$ implies $x = y$.
 (b) $x \triangleleft y, x \in z^\varepsilon, y \in z^\eta$ imply $\varepsilon = \eta$.

Example A A 1-dimensional parity complex is precisely a directed graph with no circuits.

Example B The ω -glob is the parity complex G defined by

$$G_n = \{(\varepsilon, n) : \varepsilon = - \text{ or } +\}, \quad (\varepsilon, n)^- = \{(-, n)\} \quad \text{and} \quad (\varepsilon, n)^+ = \{(+, n)\}.$$

Example C The ω -simplex is the parity complex Δ described as follows. Let Δ_n denote the set of $(n+1)$ -element subsets of the set N of natural numbers $0, 1, 2, \dots$. Each $x \in \Delta_n$ is written as (x_0, x_1, \dots, x_n) where $x_0 <$

$x_1 < \dots < x_n$. Let $x\partial_i$ denote the set obtained from x by deleting x_i . Take x^- to be the set of $x\partial_i$ with i odd and x^+ to be the set of $x\partial_i$ with i even. The facial identities

$$x\partial_j\partial_i = x\partial_i\partial_{j-1} \quad \text{for } i < j$$

immediately imply the following characterizations

$$x^- = \{x\partial_j\partial_i : i < j, i \text{ odd}\}, \quad x^{++} = \{x\partial_j\partial_i : i < j, i \text{ even}\},$$

$$x^{+-} = \{x\partial_j\partial_i : i < j, j \text{ even}\}, \quad x^{-+} = \{x\partial_j\partial_i : i < j, j \text{ odd}\},$$

which make Axiom 1 clear.

Now consider Axiom 2. To see that x^- is well formed, take $y, z \in x^-$ with $y \neq z$ and $y\partial_i = z\partial_k$. By symmetry we may suppose $i \leq k$. Then $y_i = x_i$, $z_k = x_{k+1}$; so $y = x\partial_i$, $z = x\partial_{k+1}$. So $i, k+1$ are odd. So $i+k$ is odd. So $y \perp z$. A similar argument applies to x^+ .

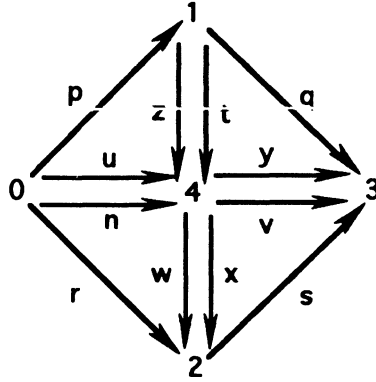
For Axiom 3 take $x \triangleleft y$ in Δ_n . Observe that $x_0 \leq y_0$. If $x_0 = y_0$ then $y\partial_0 \triangleleft x\partial_0$. Similarly, if $x_n = y_n$ then $x\partial_n \triangleleft y\partial_n$. Using the last sentence with a simple induction on n , we see that $x \triangleleft y \triangleleft x$ implies $x = y$. This gives (a). For (b), suppose $x = z\partial_i$, $y = z\partial_j$. If i is odd and j is even then $z\partial_1 \triangleleft x \triangleleft y \triangleleft z\partial_0$, so $(z_0, z_2, z_3, \dots, z_{n+1}) \triangleleft (z_1, z_2, z_3, \dots, z_{n+1})$, which implies, by repeated last-element removal, that $z_0 \triangleleft z_1$ contrary to $z_0 \neq z_1$. If i is even and j is odd then $z\partial_{n+1} \triangleleft x \triangleleft y \triangleleft z\partial_n$ for n odd, and $z\partial_n \triangleleft x \triangleleft y \triangleleft z\partial_{n+1}$ for n even; each of these implies, by repeated first-element removal, that $z_n = z_{n+1}$, a contradiction.

Example D This non-example is basically given (for other reasons) by Power [Pw], and shows that there are reasonable structures which do not satisfy our Axiom 3. Put

$$\begin{aligned} C_0 &= \{0, 1, 2, 3, 4\}, \\ C_1 &= \{p, q, r, s, t, u, v, w, x, y, z, n\}, \\ C_2 &= \{\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \iota, \kappa, \lambda, \mu\}, \\ C_3 &= \{a, b, c, d, e, f\}, \\ C_4 &= \{*\}. \end{aligned}$$

We describe the faces of elements $\chi \in C$ by writing $\chi : \chi^- \rightarrow \chi^+$. Put

$$\begin{aligned} \alpha : \{p, t\} &\rightarrow \{u\}, \quad \beta : \{q\} \rightarrow \{t, v\}, \quad \gamma : \{u, w\} \rightarrow \{r\}, \\ \delta : \{v\} &\rightarrow \{w, s\}, \quad \epsilon : \{x\} \rightarrow \{w\}, \quad \zeta : \{v\} \rightarrow \{x, s\}, \\ \eta : \{q\} &\rightarrow \{y, t\}, \quad \theta : \{y\} \rightarrow \{x, s\}, \quad \iota : \{q\} \rightarrow \{z, y\}, \\ \kappa : \{p, z\} &\rightarrow \{n\}, \quad \lambda : \{n, w\} \rightarrow \{r\}, \quad \mu : \{n, x\} \rightarrow \{r\}, \\ \nu : \{p, z\} &\rightarrow \{u\}, \quad a : \{\delta\} \rightarrow \{\zeta, \epsilon\}, \quad b : \{\beta, \zeta\} \rightarrow \{\eta, \theta\}, \\ c : \{\eta, \alpha\} &\rightarrow \{\iota, v\}, \quad d : \{v, \gamma\} \rightarrow \{\kappa, \lambda\}, \quad e : \{\lambda, \epsilon\} \rightarrow \{\mu\}, \\ f : \{\lambda, \delta\} &\rightarrow \{\mu, \zeta\}, \quad * : \{a, e\} \rightarrow \{f\}, \quad \text{and} \end{aligned}$$



Notice that $\zeta \in f^+ \cap b^-$, $\eta \in b^+ \cap c^-$, $v \in c^+ \cap d^-$, $\lambda \in d^+ \cap f^-$; hence $f < b < c < d < f$, so Axiom 3 (a) does not hold. Axiom 3 (b) does not hold either.

Proposition 1.1 $x^- \cap x^{++} = \emptyset = x^+ \cap x^{+-}$,
 $x^{\mp} = x^- \cap x^{+-} = x^{+\mp}$ and $x^{\pm} = x^+ \cap x^{++} = x^{+\pm}$.

Proof $u \in x^- \cap x^{++}$ implies there are $v \in x^-$, $w \in x^+$ such that $u \in v^-$, $u \in w^+$. So $w < v$ with $v \in x^-$, $w \in x^+$ contradicts Axiom 3(b). This gives the first equality; the second is similar. So the unions in Axiom 1 are disjoint; the other equalities now follow. **qed**

To each parity complex C there is an associated chain complex F of free abelian groups. Take F_n to be the free abelian group on the set C_n . For $x \in C_n$ we interpret $x^{\epsilon} \in F_n$ as the formal sum of its elements. Define $d : F_n \rightarrow F_{n-1}$ to be the homomorphism such that $d(x) = x^+ - x^-$. Our Axioms ensure that

$$d d(x) = d(x^+) - d(x^-) = x^{++} - x^{+-} - x^+ + x^- = 0.$$

Proposition 1.2 If $u < v$ and $v \in x^+$ then $u \cap x^+ = \emptyset$.

Proof $y \in u \cap x^+$ implies $y \in w^+$, $w \in x^-$. So $w < u < v \in x^+$, $w \in x^-$ contrary to Axiom 3(b). **qed**

This is an appropriate point to mention duality. For each subset K of N not containing 0, the K -dual CK of C consists of the same graded set as C but with x^- and x^+ interchanged when $x \in C_n$ for $n \in K$. We write xK for x as an element of CK , so that, for $x \in C_n$, xK^- is x^-K when $n \notin K$, and

xK^- is x^+K when $n \in K$. As the axioms for a parity complex are self-dual, the K -dual of a parity complex is a parity complex. Consequently, once a proposition is proved for all parity complexes, the dual propositions automatically hold.

Proposition 1.3 *If x^+ is a subset of a well-formed $S \subset C$ then x^+ is a segment of S .*

Proof Suppose $w < u \triangleleft_S v$ with $w, v \in x^+$, so that there is $y \in w^+ \cap u^-$. By Proposition 1.2, $u^- \cap x^+ = \emptyset$. So $y \notin x^+$. Now $y \in w^+ \subset x^{++}$. By Axiom 1, $y \in x^{+-}$. So $y \in z^-$ where $z \in x^+$. So $y \in u \cap z^-$. But $u, z \in S$ well formed. So $u = z \in x^+ \cdot \text{qed}$

§2. Movement

This section introduces the calculus of movement (which we might have called "replacement" or "surgery"). For these results, none of the axioms is required.

Definition Suppose S, M, P are subsets of C . Say S moves M to P when

$$P = (M \cup S^+) \cap \neg S^- \quad \text{and} \quad M = (P \cup S^-) \cap \neg S^+.$$

Denote this by $S : M \rightarrow P$.

Observe that any S moves S^{\mp} to S^{\pm} . Observe also that M and S determine P ; also S and P determine M . However, there need be no P to which a given S moves a given M .

Proposition 2.1 *For subsets S, M of C , there exists a subset P such that S moves M to P if and only if*

$$S^{\mp} \subset M \quad \text{and} \quad M \cap S^+ = \emptyset.$$

Proof If P exists then

$$M = (P \cup S^-) \cap \neg S^+ = (P \cap \neg S^+) \cup (S^- \cap \neg S^+) = (P \cap \neg S^+) \cup S^{\mp} \supset S^{\mp}$$

and

$$M \cap S^+ \subset \neg S^+ \cap S^+ = \emptyset.$$

Conversely, if the two conditions hold, define P as we are forced to, and calculate

$$\begin{aligned} (P \cup S^-) \cap \neg S^+ &= (((M \cup S^+) \cap \neg S^-) \cup S^-) \cap \neg S^+ = (M \cup S^+ \cup S^-) \cap \neg S^+ \\ &= (M \cap \neg S^+) \cup S^{\mp} = M \cup S^{\mp} = M. \text{qed} \end{aligned}$$

Proposition 2.2 *Suppose S moves A to B and $X \subset A$ has $S^{\mp} \cap X = \emptyset$. If $Y \cap S^- = Y \cap S^+ = \emptyset$ then S moves $(A \cup Y) \cap \neg X$ to $(B \cup Y) \cap \neg X$.*

Proof Using one direction of Proposition 2.1, we obtain

$$S^{\mp} \subset A \cap X \subset (A \cup Y) \cap X \quad \text{and}$$

$$(A \cup Y) \cap X \cap S^+ = ((A \cap S^-) \cup (Y \cap S^+)) \cap X = \emptyset,$$

and using the other direction yields that S moves $(A \cup Y) \cap X$ to

$$\begin{aligned} &(((A \cup Y) \cap X) \cup S^+) \cap S^- = (A \cup Y \cup S^+) \cap (\neg X \cup S^+) \cap S^- \\ &= (A \cup Y \cup S^+) \cap X \cap S^- = (((A \cup S^+) \cap S^-) \cup (Y \cap S^-)) \cap X \\ &= (B \cup Y) \cap X. \quad \text{qed} \end{aligned}$$

Proposition 2.3 (Composition of moves) *Suppose S moves M to P and T moves P to Q . If $S^- \cap T^+ = \emptyset$ then $S \cup T$ moves M to Q .*

Proof
$$\begin{aligned} Q &= (P \cup T^+) \cap T^- = (P \cup (T^+ \cap S^-)) \cap T^- \\ &= (((M \cup S^+) \cap S^-) \cup (T^+ \cap S^-)) \cap T^- = ((M \cup S^+) \cup T^+) \cap S^- \cap T^- \\ &= (M \cup (S \cup T)^+) \cap (S \cup T)^-, \end{aligned}$$

and, since the disjointness condition is self dual, the other half is dual. **qed**

Proposition 2.4 (Decomposition of moves) *Suppose $S = T \cup Z$ moves M to P with $Z^{\pm} \subset P$. If $T \perp Z$ then there exists N such that T moves M to N and Z moves N to P .*

Proof Since $Z^{\pm} \subset P$ and $P \cap Z^- \subset P \cap S^- = \emptyset$, the dual of Proposition 2.1 yields $Z : N \rightarrow P$ and $N = (P \cup Z^-) \cap Z^+$. The disjointness hypotheses imply $T^{\varepsilon} = S^{\varepsilon} \cap Z^{\varepsilon}$. Then

$$\begin{aligned} T^{\mp} &= T^- \cap T^+ = S^- \cap Z^- \cap (\neg S^+ \cup Z^+) = (S^{\mp} \cap Z^-) \cap (S^- \cap Z^{\pm}) \\ &\subset M \cup (S^- \cap P) = M. \end{aligned}$$

By Proposition 2.1, T moves M to $(M \cup T^+) \cap T^- = (M \cup (S^+ \cap Z^+)) \cap (\neg S^- \cup Z^-)$

$$\begin{aligned} &= (M \cup S^+) \cap (M \cup Z^+) \cap (\neg S^- \cup Z^-) \\ &= (((M \cup S^+) \cap S^-) \cup ((M \cup S^+) \cap Z^-)) \cap (M \cup Z^+) \\ &= (P \cup ((M \cup S^+) \cap Z^-)) \cap (M \cup Z^+) = (P \cup Z^-) \cap Z^+ = N \end{aligned}$$

where the penultimate step uses $Z^- \subset S^- = S^{\mp} \cup S^+ \subset M \cup S^+$ and $M \cap Z^+ \subset M \cap S^+ = \emptyset$. **qed**

Observe that, if S is well formed and is a disjoint union of T and Z , then we have $T \perp Z$ as required in the hypothesis of the above proposition.

§3. The ω -category of a parity complex

Definitions A *cell* of a parity complex C is a pair (M, P) of non-empty well-formed finite subsets M, P of C such that M and P both move M to P . The set of cells is denoted by $\mathcal{O}(C)$. The *n-source* and *n-target* of (M, P) are defined by

$$s_n(M, P) = (M^{(n)}, M_n \cup P^{(n-1)}), \quad t_n(M, P) = (M^{(n-1)} \cup P_n, P^{(n)}).$$

An ordered pair of cells $(M, P), (N, Q)$ is called *n-composable* when

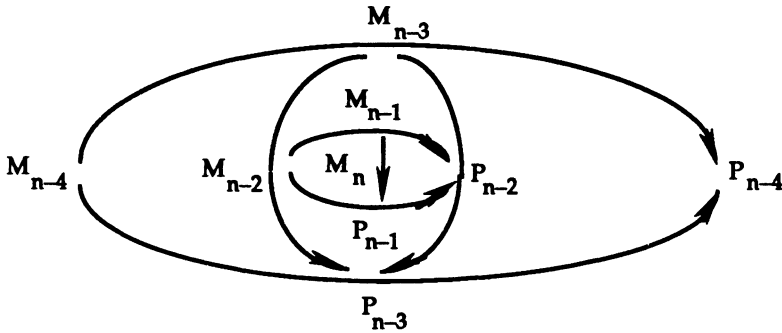
$$t_n(M, P) = s_n(N, Q),$$

in which case their *n-composite* is defined by

$$(M, P) *_n (N, Q) = (M \cup (N \cap N_n), (P \cap P_n) \cup Q).$$

The goal of this section is to show that this defines an ω -category $\mathcal{O}(C)$. It is clear that we do obtain an ω -category $\mathcal{N}(C)$ of unrestricted pairs (M, P) of finite subsets of C using the above formulas for source, target and composition; the axioms for an ω -category [S1] are immediate. That $\mathcal{O}(C)$ is closed under the source and target operations is clear. So all that remains is to show that $\mathcal{O}(C)$ is closed under the compositions.

Call $(M, P) \in \mathcal{O}(C)$ an *n-cell* when $M \cup P$ is n -dimensional. This is the same as the requirement that the cell is equal to its n -source (and/or its n -target). Then we have $M_n = P_n$.



Movement in an n -cell (M, P)

Call $S \subset C$ *receptive* when, for all $x \in C$ and $(\varepsilon, \eta) = \{-, +\}$,
if $x^{\varepsilon\eta} \cap x^{\eta\varepsilon} \subset S$ and $S \cap x^{\varepsilon\varepsilon} = \emptyset$ then $S \cap x^{\eta\eta} = \emptyset$.

A cell (M, P) is called *receptive* when both M and P are receptive. We

shall eventually show that all cells are receptive. First we shall relate receptivity to movement.

Proposition 3.1 *If x^+ moves M to P and M is receptive then x^- moves M to P .*

Proof Since x^+ moves M to something, we have $x^{+\mp} \subset M$ and $M \cap x^{++} = \emptyset$ (Proposition 2.1). By Proposition 1.1, $x^{-\mp} = x^- \cap x^{+\mp} = x^{+\mp} \subset M$. Since M is receptive, $M \cap x^{+-} = \emptyset$. Hence, by Proposition 2.1, x^- moves M to

$$\begin{aligned} (M \cup x^+) \cap x^- &= (M \cap x^-) \cup (x^+ \cap x^-) = (M \cap x^+ \cap x^-) \cup (x^+ \cap x^-) \\ &= (M \cap x^+ \cap x^{+-}) \cup (x^+ \cap x^{+-}) = (M \cup x^{++}) \cap x^{+-} = P. \text{qed} \end{aligned}$$

This leads us to the following technical lemma.

Lemma 3.2 *Suppose C is a parity complex in which all cells are receptive and suppose (M, P) is an n -cell of C . Suppose $X \subset C_{n+1}$ is finite and well formed with $X^\pm \subset M_n (= P_n)$. Put $Y = (M_n \cup X^-) \cap X^+$. Then:*

- (a) X^\pm is a segment of M_n ;
- (b) $(M^{(n-1)} \cup Y, P^{(n-1)} \cup Y)$ is a cell and $X^- \cap M_n = \emptyset$;
- (c) $(M^{(n-1)} \cup Y \cup X, P \cup X)$ is a cell.

Proof Let m be the cardinality of X . The proof involves three steps.

Step 1. (b) implies (c).

All movement conditions are covered by (b) except for “ X moves Y to M_n ”; but this follows from the definition of Y and the calculation:

$$\begin{aligned} (Y \cup X^+) \cap X^- &= (((M_n \cup X^-) \cap X^+) \cup X^+) \cap X^- \\ &= (M_n \cup X^- \cup X^+) \cap X^- \\ &= M_n \cup (X^+ \cap X^-) \text{ (using the second part of (b))} \\ &= M_n \cup X^\pm \\ &= M_n. \end{aligned}$$

Wellformedness is covered by (b) and the wellformedness of X .

Step 2. (b) for $m = 1$ implies (a) and (b) in general.

The proof is by induction on m . By hypothesis and Proposition 1.3, we can assume $m > 1$. Let x be a maximal element of X . Then $x^\pm \subset X^\pm \subset M_n$ and x^+ is well formed (Axiom 2), so (b) for $m = 1$ gives us a cell

$$(N, Q) = (M^{(n-1)} \cup ((M_n \cup x^-) \cap x^+), P^{(n-1)} \cup ((M_n \cup x^-) \cap x^+))$$

and $x \cap M_n = \emptyset$. Since X is wellformed, so is $Z = X \cap \{x\}$ and also

$$Z^\pm = (X^\pm \cup x) \cap x^\pm \subset (M_n \cup x) \cap x^\pm = N_n.$$

By induction, Z^\pm is a segment of N_n ,

$$(N_n^{(n-1)} \cup Y', Q_n^{(n-1)} \cup Y') = (M_n^{(n-1)} \cup Y', P_n^{(n-1)} \cup Y')$$

is a cell for $Y' = (N_n \cup Z^-) \cap Z^+$, and $Z^- \cap N_n = \emptyset$.

Now we prove (b). By maximality of x , $x^+ \cap Z^- = \emptyset$; so $Z^- \cap x^+$.

So

$$\begin{aligned} Y' &= (N_n \cup Z^-) \cap Z^+ = (((M_n \cup x) \cap x^+) \cup Z^-) \cap Z^+ \\ &= (M_n \cup x \cup Z^-) \cap x^+ \cap Z^+ = (M_n \cup X^-) \cap X^+ \\ &= Y. \end{aligned}$$

So $(M_n^{(n-1)} \cup Y, P_n^{(n-1)} \cup Y)$ is a cell.

We still need to show $X^- \cap M_n = \emptyset$. But

$$\begin{aligned} (N_n \cup x^+) \cap x^- &= (((M_n \cup x) \cap x^+) \cup x^+) \cap x^- \\ &= (M_n \cup x \cup x^+) \cap x^- \\ &= (M_n \cap x^-) \cup (x^+ \cap x^-) \\ &= M_n. \end{aligned}$$

Hence

$$\begin{aligned} X^- \cap M_n &= X^- \cap (N_n \cup x^+) \cap x^- \\ &= (Z^- \cup x) \cap (N_n \cup x^+) \cap x^- \\ &= ((Z^- \cap N_n) \cup (Z^- \cap x^+) \cup (x \cap N_n) \cup (x \cap x^+)) \cap x^- \\ &= Z^- \cap x^+ \cap x^- = \emptyset. \end{aligned}$$

Now we prove (a). Suppose X^\pm is *not* a segment of M_n . Then there exist $u, v \in X^\pm$ and non-empty $S \subset M_n \cap X^\mp = M_n \cap x^\mp$ (using $X^- \cap M_n = \emptyset$) such that $\{u, v\} \cup S$ is a maximal chain in M_n from u to v . Then $S \subset M_n \cap x^+ \subset N_n$. By Proposition 1.3, either u or v is not in x^+ . Suppose $v \notin x^+$. Then $v \in (X^\pm \cup x) \cap x^+ = Z^\pm$ which is a segment of N_n ; so $u \notin Z^\pm$. So $u \in x^+$. Let s be the first element of S . Then $u < s$; so

$$\emptyset \neq u^+ \cap s \subset x^+ \cap s.$$

Since M_n is well formed, $x^+ \cap s = \emptyset$. So

$$\emptyset \neq x^+ \cap s \subset (x^+ \cup x^+) \cap s = x^+ \cap s.$$

So there exists $w \in x^+$ with $w^+ \cap s \neq \emptyset$. So $w < s \leq_{S \cup \{v\}} v$ in N_n and $w, v \in Z^\pm$ contrary to the segment property of Z^\pm . Hence we must have $v \in x^+$ and $u \notin x^+$. Then let s be the last element of S . Then $s < v$; so $\emptyset \neq s^+ \cap v \subset$

$s^+ \cap x^{+-}$. Since M_n is well formed, $x^{++} \cap s^+ = \emptyset$. So

$$\emptyset \neq s^+ \cap x^{+-} \subset s^+ \cap (x^- \cup x^{++}) = s^+ \cap x^-.$$

So there exists $w \in x^-$ with $s^+ \cap w \neq \emptyset$. So $u \leftarrow_{S \cup \{w\}} s < w$ in N_n and $u, w \in Z^\pm$ contrary to the segment property of Z^\pm . So X^\pm is a segment of M_n .

Step 3. (b) holds for $m = 1$.

Let $x \in C_{n+1}$ be the element of X ; so we have $x^+ \subset M_n$ and $Y = (M_n \cup x^-) \cap x^+ = (M_n \cap x^+) \cup x^-$. What we need to prove is that $(M^{(n-1)} \cup Y, P^{(n-1)} \cup Y)$ is a cell and $x^- \cap M_n = \emptyset$. We do this by induction on n . For $n = 0$, $M \cup P = M_0$ is a singleton. So $x^+ = M_0$. By Axiom 2, $Y = x^-$ is a singleton, so that (Y, Y) is certainly a 0-cell, and we have $x^- \cap M_0 = \emptyset$ by Axiom 1.

So suppose $n > 0$. By Proposition 1.3, x^+ is a segment of M_n , so we can write M_n as a disjoint union

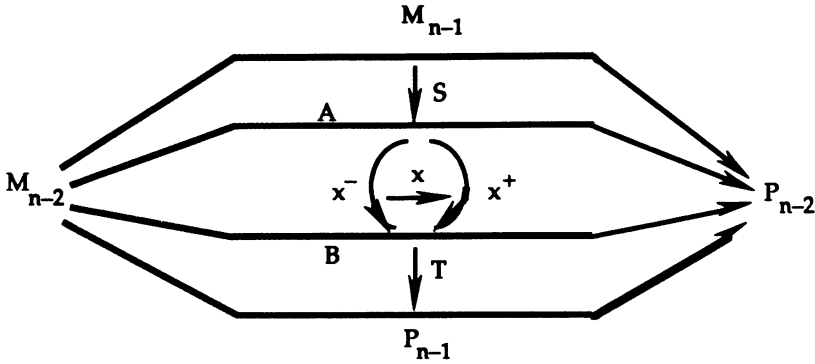
$$M_n = S \cup x^+ \cup T$$

where S is an initial and T is a final segment. By Proposition 2.4, we have $A, B \subset C_{n-1}$ such that

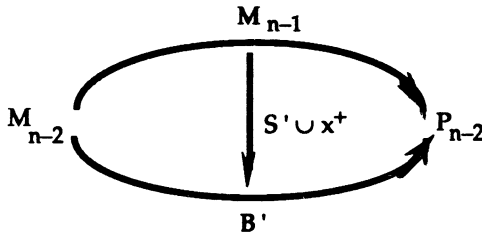
$$M_{n-1} \xrightarrow{S} A \xrightarrow{x^+} B \xrightarrow{T} P_{n-1}.$$

Since (M, P) is a cell, $t_{n-1}(M, P) = (M^{(n-2)} \cup P_{n-1}, P^{(n-1)})$ is an $(n-1)$ -cell to which, by induction, we can apply Steps 1 and 2 because T satisfies the conditions of the Lemma. This gives that $(M^{(n-2)} \cup B \cup T, P^{(n-1)} \cup T)$ is an n -cell, T^\pm is a segment of P_{n-1} , and $T^- \cap P_{n-1} = \emptyset$. Now $(M^{(n-2)} \cup B, P^{(n-2)} \cup B)$ is an $(n-1)$ -cell to which, by induction, we can apply Steps 1 and 2 because x^+ satisfies the conditions of the Lemma. This gives that $(M^{(n-2)} \cup A \cup x^+, P^{(n-2)} \cup B \cup x^+)$ is an n -cell, $x^{++} \cap x^+ = x^{+\pm}$ is a segment of B , and $x^{+-} \cap B = \emptyset$. Now $(M^{(n-2)} \cup A, P^{(n-2)} \cup A)$ is an $(n-1)$ -cell to which, by induction, we can apply Steps 1 and 2 because S satisfies the conditions of the Lemma. So $(M^{(n-1)} \cup S, P^{(n-2)} \cup A \cup S)$ is an n -cell and S^\pm is a segment of M_{n-1} .

In particular, A, B are well formed and receptive (since all cells are receptive by hypothesis). By Proposition 3.1, x^- moves A to B .



We shall prove $S^- \cap x^+ = \emptyset$ by induction on the cardinality p of S . For $p = 0$, it is clear, so suppose $p > 0$. Assume it is false; so there is $w \in S$, $v \in x^-$ with $w \cap v^+ \neq \emptyset$. Let u be maximal in S with $w \triangleleft_S u$. But $v < w \triangleleft u$ and $v \in x^-$ imply $u^+ \cap x^+ = \emptyset$ (dual of Proposition 1.2). By Proposition 2.2, x^+ moves $A' = (A \cup u) \cap u^+$ to $B' = (B \cup u) \cap u^+$. Since u is maximal in S , $u^+ \subset A$. So, by induction on n , $(M_{n-2} \cup A', P_{n-2} \cup A')$ is a cell. By hypothesis on C , we have that A' is receptive. By Proposition 3.1, x^- moves A' to B' . So $x^+ \cap u^- \subset x^+ \cap A' = \emptyset$. Hence $w \neq u$. So $w \in S \cap \{u\} = S'$ which moves M_{n-1} to A' . Since the cardinality of S' is $p-1$, induction gives $S'^- \cap x^+ = \emptyset$. But this contradicts $w \in S'$, $v \in x^-$ with $w \cap v^+ \neq \emptyset$. So $S^- \cap x^+ = \emptyset$ as claimed.



That $S \perp x^-$ follows from the calculations:

$$S^- \cap x^- = S^- \cap x^- \cap (x^+ \cup x^+) = S^- \cap x^- \cap x^+ = S^- \cap x^+ \subset S^- \cap A = \emptyset$$

$$S^+ \cap x^+ = S^+ \cap S^- \cap x^+ \subset A \cap x^+ = \emptyset.$$

Proposition 2.3 now gives $S \cup x^- : M_{n-1} \rightarrow B$. Now S is well formed as it is a subset of M_{n-1} and x^- is well formed by Axiom 2; so $S \perp x^-$ implies $S \cup x^-$ is well formed.

Dually, $T^+ \cap X^- = \emptyset$, $X \cup T$ is well formed, and $X^- \cup T : A \rightarrow P_{n-1}$.

Since S is initial and T is final in the well formed set M_n , we have $S \cup T$ well formed and $S^- \cup T^+ = \emptyset$. By Proposition 2.3 and the above, we have $S \cup X \cup T = Y = (M_n \cup X) \cap X^+$ is well formed and moves M_{n-1} to P_{n-1} . So $(M^{(n-1)} \cup Y, P^{(n-1)} \cup Y)$ is a cell as required. Also

$$X^- \cap M_n = X^- \cap (S \cup X^+ \cup T) = (X^- \cap S) \cup (X^- \cap T) = \emptyset$$

since $X^- \cap S^- = X^- \cap T^- = \emptyset$. **qed**

Proposition 3.3 *All cells of all parity complexes are receptive.*

Proof Since receptivity is a condition which applies dimension by dimension, it suffices to prove this for parity complexes of finite dimension n . We prove this by induction on n . For parity complexes of dimensions 0 and 1, receptivity is an empty condition. For complexes of dimension 2, we only need to see that singletons $\{a\}$ are receptive. Suppose $X^- \cap X^+ \subset \{a\}$ and $\{a\} \cap X^+ = \emptyset$. Now $X^- \cap X^+$ is non-empty since it contains the negative faces of minimal elements of X^- . So $\{a\} = X^- \cap X^+$; so $a \in X^+$. So $a \notin X^+$; so $\{a\} \cap X^+ = \emptyset$. So, using duality for the other condition, we have proved $\{a\}$ receptive.

Now inductively suppose that all cells are receptive in parity complexes of dimension n . Let C be a parity complex of dimension $n+1$. The n -skeleton D of C is a parity complex of dimension n and, by induction, the cells of D are receptive. Let (M, P) be an $(n-1)$ -cell of C . We must prove that $M_{n-1} = P_{n-1}$ is receptive. Suppose $X^- \cap X^+ \subset P_{n-1}$ and $P_{n-1} \cap X^+ = \emptyset$. Now (M, P) is an $(n-1)$ -cell of D and $X = X^- \cap D_n$ is finite and well formed with $X^+ = X^- \cap X^+ \subset P_{n-1}$. So the dual of Lemma 3.2 applies to give $X^+ \cap P_{n-1} = \emptyset$. That is, $X^+ \cap P_{n-1} = \emptyset$. Using duality for the other condition, we have proved P_{n-1} receptive. **qed**

Proposition 3.4 *If (M, P) is an n -cell of C and $z \in C_n$ with $z^- \subset P_{n-1}$ then*

$$M_n^- \cap z^+ = \emptyset.$$

Proof. We prove it by induction on the cardinality of M_n . If $M_n = P_n$ is empty then the result is clear. So suppose M_n is non-empty. Assume $M_n^- \cap z^+ \neq \emptyset$; so there is $w \in M_n$ with $w^- \cap z^+ \neq \emptyset$. So $z < w$. Let u be maximal in M_n with $w < u$. Then $u^+ \subset P_{n-1}$. By Proposition 2.4, $A = M_n \cap \{u\}$ moves M_{n-1} to $B = (P \cup u) \cap u^+$. By Lemma 3.2, we have a cell $(M^{(n-2)} \cup B \cup \{u\}, P^{(n-1)} \cup \{u\})$. So $(M \cap \{u\}, P^{(n-2)} \cup B \cup (M_n \cap \{u\}))$ is an n -cell. Now $z^- \subset$

$P_{n-1} \cap \gamma u^* \subset B$ (since $z \cap u^* \neq \emptyset$ implies $u < z$ which implies $u < z < w < u$, contrary to Axiom 3 (a)). By induction, $(M_n \cap \gamma(u))^- \cap z^* = \emptyset$. By Lemma 3.2 (dual), $z^- \subset P_{n-1}$ implies $z^+ \cap P_{n-1} = \emptyset$; so $z^+ \cap u^* = \emptyset$, so $z^+ \subset \gamma u^*$. Also $z^- \subset B$. So Lemma 3.2 (b) (dual) gives $z^+ \cap B = \emptyset$. So $z^+ \cap u^* = z^+ \cap \gamma u^* \cap \gamma u^* \subset z^+ \cap B = \emptyset$. So we have the contradiction

$$M_n^- \cap z^+ = ((M_n \cap \gamma(u))^- \cup u^*) \cap z^+ = ((M_n \cap \gamma(u))^- \cap z^+) \cup (u^* \cap z^+) = \emptyset. \text{qed}$$

Proposition 3.5 *Suppose $(M, P), (N, Q)$ are n -cells with*

$$M_m = N_m \text{ and } P_m = Q_m \text{ for } 0 \leq m < n-1, \text{ and } P_{n-1} = N_{n-1}.$$

*Then, $M_n^- \cap N_n^+ = \emptyset$ and the $(n-1)$ -composite $(M, P) *_{n-1} (N, Q)$ is a cell.*

Proof We prove this by induction on the cardinality q of $N_n (= Q_n)$. It is clear for $q = 0$. Suppose $q = 1$, so that $N_n = \{z\}$, say. Then $z^- \subset N_{n-1} = P_{n-1}$. By Proposition 3.4, $M_n^- \cap z^+ = \emptyset$. By Lemma 3.2 (b), $M_n^+ \cap z^+ = M_n^+ \cap \gamma M_n^- \cap z^+ \subset P_{n-1} \cap z^+ = \emptyset$. By Proposition 2.1, $M_n^- \cap z^- \subset P_{n-1} \cap M_n^- = \emptyset$. So $M_n \cup \{z\}$ is well formed. Proposition 2.3 gives $M_n \cup \{z\} : M_{n-1} \rightarrow Q_{n-1}$. So the composite $(M, P) *_{n-1} (N, Q)$ is a cell.

Now suppose $q > 1$. Let z be a minimal element of N_n . Then $z^- \subset N_{n-1} = P_{n-1}$. By Lemma 3.2 (b), putting

$$N'_m = N_m \text{ for } 0 \leq m < n-1, N'_{n-1} = (N_{n-1} \cup z^*) \cap \gamma z^-, N'_n = N_n \cap \gamma \{z\},$$

we obtain cells $(N^{(n-1)} \cup \{z\}, Q^{(n-2)} \cup N'_{n-1} \cup \{z\})$ and $(N', Q^{(n-1)} \cup N'_n)$. By the $q = 1$ case, $(M \cup \{z\}, P^{(n-2)} \cup N'_{n-1} \cup \{z\})$ is a cell and $M_n^- \cap z^+ = \emptyset$. By induction since the cardinality of N'_n is $q-1$, we obtain the cell

$$(M \cup \{z\}, P^{(n-2)} \cup N'_{n-1} \cup \{z\}) *_{n-1} (N', Q^{(n-1)} \cup N'_n) = (M, P) *_{n-1} (N, Q)$$

and $M_n^- \cap N'_n^+ \subset (M_n \cup \{z\})^- \cap N'_n^+ = \emptyset$. But $M_n^- \cap N_n = M_n^- \cap (N'_n^+ \cup z^+) = \emptyset$, so we have all that is required. **qed**

Theorem 3.6 *If C is any parity complex then $O(C)$ is an ω -category. Furthermore, if $(M, P), (N, Q)$ are n -composable cells then*

$$(M_k \cup P_k)^- \cap (N_k \cup Q_k)^+ = \emptyset \text{ for all } k > n.$$

Proof Suppose $(M, P), (N, Q)$ are both r -cells. For $r \leq n$, the result is trivial. We use induction on $r-n$ which we now assume positive. By Propositions 3.5 and 2.3, we see that $s_{n+1}(M, P) *_{n+1} s_{n+1}(N, Q)$ is an $(n+1)$ -cell, and so are the other three expressions obtained by replacing either or both s 's by t 's; moreover, $(M_k \cup P_k)^- \cap (N_k \cup Q_k)^+ = \emptyset$ for $k = n+1$. Define r -cells

$(M', P'), (N', Q')$ to agree with $(M, P), (N, Q)$, respectively, in dimensions $\leq n$ and $> n+1$, while

$M'_{n+1} = M_{n+1} \cup N_{n+1}, P'_{n+1} = P_{n+1} \cup N_{n+1} = N'_{n+1}, Q'_{n+1} = P_{n+1} \cup Q_{n+1}$;
 this uses Proposition 2.2 with $X = \emptyset$. Then $(M', P'), (N', Q')$ are $(n+1)$ -composable. By induction, $(M', P') *_{n+1} (N', Q')$ is a cell and $(M'_k \cup P'_k) \cap (N'_k \cup Q'_k)^+ = \emptyset$ for all $k > n+1$. Since

$$(M', P') *_{n+1} (N', Q') = (M, P) *_{n+1} (N, Q),$$

the result is proved. **qed**

4. Freeness of the ω -category

Definition For each $x \in C_p$, two subsets $\mu(x), \pi(x) \subset C^{(p)}$ are inductively defined as follows. Put

$$\begin{aligned} \mu(x)_p &= \pi(x)_p = \{x\} \\ \mu(x)_{n-1} &= \mu(x)_n^{\pm}, \quad \pi(x)_{n-1} = \pi(x)_n^{\mp} \quad \text{for } p \geq n > 0. \end{aligned}$$

The pair $(\mu(x), \pi(x))$ is denoted by $\langle x \rangle$.

Theorem 4.1 (Excision of extremals) *Suppose (M, P) is an n -cell and $u \in M_n (= P_n)$ is such that $(M, P) \neq \langle u \rangle$. Then there exist n -cells $(N, Q), (L, R)$ and $m < n$ such that $(N, Q), (L, R)$ are not m -cells and*

$$(M, P) = (N, Q) *_{m+1} (L, R).$$

Algorithm Since $(M, P) \neq \langle u \rangle$, there exists a largest m such that

$$(M_r, P_r) = (\mu(u)_r, \pi(u)_r) \quad \text{for } r > m+1.$$

From the definition of μ and π , and from movement properties of a cell,

$$M_{m+1} = \mu(u)_{m+1} + M_{m+1} \cap P_{m+1}, \quad P_{m+1} = \pi(u)_{m+1} + M_{m+1} \cap P_{m+1}.$$

So $M_{m+1} \cap P_{m+1}$ contains at least one element w , say. Let x be a minimal element of M_{m+1} which is less than w and let y be a maximal element of M_{m+1} which is greater than w . By Lemma 3.2 (a), either x or y is not in $\mu(u)_{m+1}$. So either x or y is in $M_{m+1} \cap P_{m+1}$. By minimality of x and maximality of y ,

$$x \subset M_m \quad \text{and} \quad y^+ \subset P_m.$$

If $x \in M_{m+1} \cap P_{m+1}$, by Lemma 3.2, we have cells $(N, Q), (L, R)$ defined as follows, and they clearly have their m -composite equal to (M, P) :

$$\begin{aligned} N &= \{x\} \cup M^{(m)}, & Q &= \{x\} \cup ((M_m \cup x^+) \cap x) \cup P^{(m-1)}, \\ L &= ((M \cap \{x\}) \cup x^+) \cap x, & R &= P \cap \{x\}. \end{aligned}$$

If $y \in M_{m+1} \cap P_{m+1}$, by Lemma 3.2, we have cells $(N, Q), (L, R)$

defined as follows, and they too clearly have their m -composite equal to (M, P) :

$$\begin{aligned} N &= M \cap \{y\}, & Q &= ((M \cap \{y\}) \cup y^-) \cap y^+, \\ L &= \{y\} \cup ((P_m \cup y^-) \cap y^+) \cup M^{(m-1)}, & R &= M^{(m)} \cup \{y\} \text{ .qed} \end{aligned}$$

Definition An element $x \in C_p$ is called *relevant* when $\langle x \rangle$ is a cell. This amounts to saying that $\mu(x)_n$ and $\pi(x)_n$ are well formed for $0 \leq n < p-1$, and

$$\mu(x)_{n-1} = \pi(x)_n^{\mp}, \quad \pi(x)_{n-1} = \mu(x)_n^{\pm} \text{ for } 0 < n < p-1.$$

Call $\langle x \rangle$ an *atom* when x is relevant.

All elements of dimension 0 and 1 are relevant. An element of dimension 2 is relevant if and only if $x^- \cap x^{*-}$ and $x^+ \cap x^{*+}$ are singletons.

Remark In many examples we find that it is easy to characterize the sets $\mu(x)$ and $\pi(x)$ and to prove the following stronger forms of Axiom 1 and 2:

- (R1) $\mu(x)^- \cup \pi(x)^+ = \mu(x)^+ \cup \pi(x)^-$ and $\mu(x)^- \cap \pi(x)^+ = \mu(x)^+ \cap \pi(x)^- = \emptyset$;
- (R2) $\mu(x)$ and $\pi(x)$ are well formed.

Then it follows that every element of C is relevant. Such an example is Δ . To see this, define a subset $u = (u_0, u_1, \dots, u_n)$ of N to be *alternating* when consecutive elements u_i, u_{i+1} have opposite parity (that is, one is even and one is odd). Call u *negative alternating* when u_0 is odd, *positive alternating* when u_0 is even. Write $x\partial_u$ for the result of deleting from $x \in \Delta$

those x_r with $r \in u$. Then it is easy to show [S1; Proposition 3.1] that

$$\begin{aligned} \mu(x) &= \{ x\partial_u : u \text{ is negative alternating} \} \text{ and} \\ \pi(x) &= \{ x\partial_u : u \text{ is positive alternating} \}, \end{aligned}$$

and that (R1) and (R2) hold. Hence, *every element of Δ is relevant*.

Recall the concept of "free ω -category" from [S1; Section 4].

Theorem 4.2 *The ω -category $\mathcal{A}(C)$ is freely generated by the atoms.*

Proof Define the *rank* of a cell (M, P) to be the cardinality of the subparity complex of C generated by (M, P) . Excision of extremals (Theorem 4.1) produces two cells of smaller rank than (M, P) . Repeated application of that algorithm must therefore terminate. It can only terminate when each cell in the decomposition has the form $\langle x \rangle$. Since the algorithm produces cells at each stage, these final $\langle x \rangle$ must be atoms. Hence the atoms do generate.

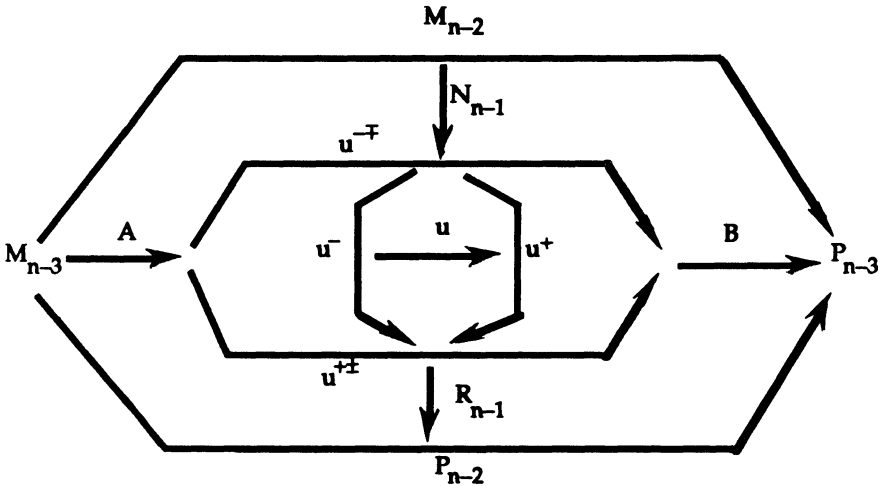
Every decomposition of an n -cell (M, P) into atoms will include all the $\langle u \rangle$ with $u \in M_n (= P_n)$, and these will be the only n -cells in the

decomposition which are not $(n-1)$ -cells.

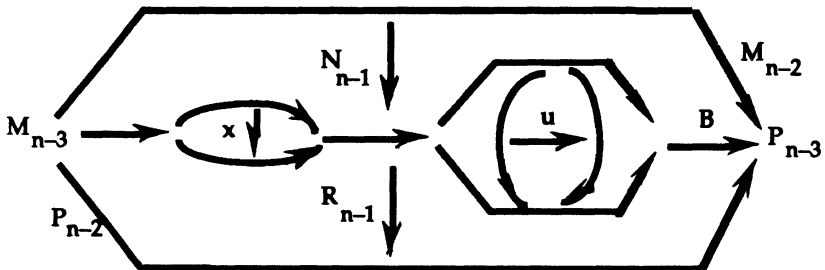
Consider the case where $M_n = \{u\}$. We shall examine the ambiguity in a decomposition

$$(M, P) = (N, Q) *_{n-2}(u) *_{n-2}(L, R).$$

This situation is depicted by the diagram below, where $Q_{n-2} = A \cup u^{\mp} \cup B$ and $L_{n-2} = A \cup u^{\pm} \cup B$ with the unions disjoint.



The second sentence of Theorem 3.6 tells us that N_{n-1} must contain all $x \in M_{n-1} \cap \uparrow$ for which there is a $y \in \uparrow$ with $x \triangleleft y$; similarly for R_{n-1} with \triangleleft reversed. These are the *only* compulsory elements of these sets since, by a sequence of processes represented by the diagram

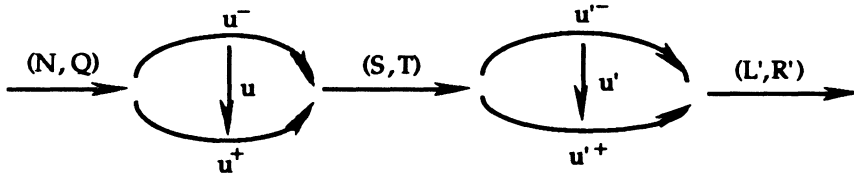


or the similar diagram with x on the B side, one can transfer all the *other* elements from N_{n-1} to R_{n-1} and vice versa. These processes only involve the middle-four-interchange law for $(n-1)$ -cells. Furthermore, there is a unique

decomposition of (M, P) , of the kind we are considering, such that the cardinality of N_{n-1} is least: namely, the one for which N_{n-1} consists of precisely its compulsory elements.

Now consider a general n -cell (M, P) with M_n of cardinality greater than 1. Then we have a decomposition of (M, P) as

$\dots *_{n-1}((N, Q) *_{n-2}(u) *_{n-2}(L, R)) *_{n-1}((N', Q') *_{n-2}(u') *_{n-2}(L', R')) *_{n-1} \dots$
 in which, if $u < u'$, the order of u and u' cannot be changed even with alteration of the "whiskers" $(N, Q), (L, R), (N', Q'), (L', R')$. However, if $u \cap u' = \emptyset$ then we either have the situation depicted in the diagram



in which

$$(N', Q') = (N, Q) *_{n-2} t_{n-1}(u) *_{n-2}(S, T) \text{ and}$$

$$(L, R) = (S, T) *_{n-2} s_{n-1}(u') *_{n-2}(L', R'),$$

or the similar situation with u, u' interchanged; so it is possible to interchange u, u' by appropriately altering the whiskers.

The only possible order changes in the decomposition are therefore consequences of the n -category axioms. Freeness follows. \square

5. Product and join

Definition The *product* $C \times D$ of two parity complexes C, D is given by

$$(C \times D)_n = \sum_{p+q=n} C_p \times D_q, \quad (x, a)^\varepsilon = x^\varepsilon \times [a] \cup [x] \times a^{\varepsilon(p)}$$

for $x \in C_p, a \in D_q, \varepsilon \in \{-, +\}$, where $\varepsilon(p) \in \{-, +\}$ is ε for p even and is not ε for p odd. The usual rebracketing associativity bijection respects dimension, and negative and positive faces. The product of parity complexes is not a parity complex in general.

For $x \in C_p, a \in D_q$, put $\theta = -(p)$ and $\phi = +(p)$. Then we have

$$(x, a)^- = x^- \times [a] \cup [x] \times a^\theta, \quad (x, a)^+ = x^+ \times [a] \cup [x] \times a^\phi,$$

$$(x, a)^{-} = x^- \times [a] \cup x^- \times a^\phi \cup x^- \times a^\theta \cup [x] \times a^{\theta\theta},$$

$$(x, a)^{+} = x^+ \times [a] \cup x^- \times a^\theta \cup x^+ \times a^\theta \cup [x] \times a^{\phi\phi},$$

$$(x, a)^{+-} = x^+ \times [a] \cup x^+ \times a^\phi \cup x^- \times a^\phi \cup [x] \times a^{\phi\theta},$$

$$\begin{aligned} (x, a)^{++} &= x^{++} \times \{a\} \cup x^+ \times a^0 \cup x^+ \times a^\dagger \cup \{x\} \times a^{\dagger\dagger}, \\ (x, a)^{-\mp} &= x^{-\mp} \times \{a\} \cup x^- \times a^\dagger \cup \{x\} \times (a^{00} \cap a^{\dagger\dagger}), \\ (x, a)^{\pm\pm} &= x^{\pm\pm} \times \{a\} \cup x^+ \times a^0 \cup \{x\} \times (a^{\dagger\dagger} \cap a^{00}). \end{aligned}$$

Proposition 5.1 *The product of parity complexes satisfies Axioms 1 and 2, and Proposition 1.1.*

Proof This is easily deduced from the above formulas and the corresponding properties of the component parity complexes. \square

The above proposition suffices for the construction of the chain complex associated with the product, and we have a canonical isomorphism of chain complexes:

$$F(C \times D) \cong FC \otimes FD,$$

where we remind readers that the tensor-product boundary formula is

$$d(x \otimes a) = dx \otimes a + (-1)^p x \otimes da \quad \text{for } x \in FC_p, a \in FD_q.$$

Remark If C, D satisfy (R1) and (R2) of the Remark of Section 4 then so does the product $C \times D$. There are explicit formulas for $\mu(x, a), \pi(x, a)$ in terms of $\mu(x), \mu(a), \pi(x), \pi(a)$. To express these, write χ^r to denote $\chi \in \{\mu, \pi\}$ when r is even and to denote the other element of $\{\mu, \pi\}$ when r is odd. Then

$$\chi(x, a)_n = \bigcup_{r+s=n} \chi(x)_r \times \chi^r(a)_s.$$

The product of two parity complexes need not satisfy Axiom 3. The problem is with the preorder \triangleleft . To study this preorder on the product, we shall introduce a larger relation \triangleleft on the factor complexes.

In a parity complex C , write $x \prec y$ when either $y \in x^+$ or $x \in y^-$. Let \triangleleft denote the reflexive transitive closure of the relation \prec .

Notice that $x \prec y$ means there exists $z \in x^+ \cap y^-$, so this implies $x \prec z \prec y$. Hence, $x \triangleleft y$ implies $x \triangleleft y$. The relation \triangleleft compares elements of the same dimension, whereas \triangleleft compares elements of all dimensions.

So clearly Axiom 3 (a) is implied by antisymmetry of \triangleleft . Also, half of Axiom 3 (b) is implied by antisymmetry of \triangleleft . For, $x \triangleleft y, x \in z^+, y \in z^-$ imply $z \prec x \triangleleft y \prec z$, which, if \triangleleft is antisymmetric, implies $z = x = y$ contrary to the difference of 1 between the dimensions of x and z . So $x \triangleleft y, x \in z^+, y \in z^\eta$ imply $+ = \eta$.

Duality was discussed at the end of Section 1. Antisymmetry of \triangleleft is a self-dual condition, whereas antisymmetry of \triangleleft is by no means self-dual.

There is a particular dual which is important here. Let od denote the set of odd natural numbers. Then Cod is called the *odd dual* of C . Notice that $\varepsilon(p)(q) = \varepsilon(p+q)$ and, for $x \in C_p$,

$$(x\text{od})^\varepsilon = x^{\varepsilon(q)}\text{od}.$$

Proposition 5.3 $(C \times D)\text{od} \cong \text{Dod} \times \text{Cod}$.

Proof Take $x \in C_p$, $a \in D_q$, $n = p+q$. Then

$$\begin{aligned} (x, a)\text{od}^\varepsilon &= (x, a)^{\varepsilon(n)}\text{od} = (x^{\varepsilon(n)} \times \{a\})\text{od} \cup (\{x\} \times a^{\varepsilon(q)})\text{od} \text{ and} \\ (\text{aod}, x\text{od})^\varepsilon &= \text{aod}^\varepsilon \times \{x\text{od}\} \cup \{\text{aod}\} \times x\text{od}^{\varepsilon(q)} \\ &= a^{\varepsilon(q)}\text{od} \times \{x\text{od}\} \cup \{\text{aod}\} \times x^{\varepsilon(n)}\text{od}. \end{aligned}$$

It follows that the bijection $(x, a)\text{od} \mapsto (\text{aod}, x\text{od})$ respects negative and positive faces. \square

Theorem 5.4 Suppose that the relation \triangleleft is antisymmetric for each of the parity complexes C, D and their odd duals. Then the product $C \times D$ and its odd dual are parity complexes for which \triangleleft is antisymmetric.

Proof By Proposition 5.1, $C \times D$ satisfies Axioms 1 and 2. Note that $(x, a) \triangleleft (y, b)$ in $C \times D$ means either

$$\begin{array}{ll} y \in x^+ \text{ and } a = b, & \text{or } x = y \text{ and } b \in a^{+(\varphi)}, \\ \text{or } x \in y^- \text{ and } a = b, & \text{or } x = y \text{ and } a \in b^{-(\varphi)}, \end{array}$$

where $x \in C_p$, $a \in D_q$. This implies $x \triangleleft y$, and, if $x = y$ then $a \triangleleft b$ for p even and $b \triangleleft a$ for p odd. Consequently, $(x, a) \triangleleft (y, b) \triangleleft (x, a)$ implies $x \triangleleft y \triangleleft x$. Since \triangleleft is antisymmetric for C , we have $x = y$. But then $a \triangleleft b \triangleleft a$; so $a = b$, since \triangleleft is antisymmetric for C . So \triangleleft is antisymmetric for $C \times D$. Hence $C \times D$ satisfies Axiom 3(a) and half of Axiom 3(b). Since \triangleleft is antisymmetric for the odd duals of C, D , Proposition 5.3 now implies that \triangleleft is antisymmetric for the odd dual of $C \times D$. So, also, $C \times D$ satisfies the other half of Axiom 3(b). So $C \times D$ is a parity complex; its odd dual is too by Proposition 5.3. \square

Definition The *join* $C \bullet D$ of two parity complexes C, D is given by

$$(C \bullet D)_n = C_n + \sum_{p+q+1=n} C_p \times D_q + D_n$$

in which the summands C and D are embedded as sub-parity complexes and the elements $(x, a) \in C_p \times D_q$ are written as xa with

$$(xa)^{\sim} = x^{\sim}a \cup xa^{\sim} \quad \text{and} \quad (xa)^{\dagger} = x^{\dagger}a \cup xa^{\dagger} \quad \text{for } p \text{ odd,}$$

$$(xa)^{\sim} = x^{\dagger}a \cup xa^{\dagger} \quad \text{and} \quad (xa)^{\dagger} = x^{\sim}a \cup xa^{\sim} \quad \text{for } p \text{ even,}$$

where, for example, $x^{\dagger}a = \{ ya : y \in x^{\dagger} \}$ is taken to mean $\{a\}$ when $p = 0$.

In particular, when D consists of a single element ∞ in dimension 0, the join $C \bullet D$ is called the *right cone* of C and denoted by C^{\triangleright} . Also $D \bullet C$ is the *left cone* of C and denoted by C^{\triangleleft} .

Another description of the join will allow us to make use of our calculations with product. Define the *suspension* ΣC of C by

$$(\Sigma C)_n = C_{n-1} \quad \text{for } n \geq 1 \quad \text{and} \quad (\Sigma C)_0 = \{*\},$$

$$\text{interchange } x^{\sim} \text{ and } x^{\dagger} \text{ for } x \in C_p, \quad p > 0,$$

$$\text{and } x^{\sim} = x^{\dagger} = \{*\} \text{ for } x \in C_0.$$

Because of the last line, ΣC is never a parity complex; but it goes close when C is. The main observation we wish to make is the identification

$$\Sigma(C \bullet D) = \Sigma C \times \Sigma D.$$

This essentially makes the following results about join consequences of the corresponding results about product.

Proposition 5.6 $(C \bullet D)_{\text{od}} \cong \text{Dod} \bullet \text{Cod}$ and $C^{\triangleright}_{\text{od}} \cong (\text{Cod})^{\triangleleft}$. *qed*

Theorem 5.7 Suppose that the relation \triangleleft is antisymmetric for each of the parity complexes C, D and their odd duals. Then the join $C \bullet D$ and its odd dual are parity complexes for which \triangleleft is antisymmetric. *qed*

The Remark of this section also applies, with appropriate changes, to join in place of product.

§6. Application to cubes and descent

The n -simplex $\Delta[n]$ is the sub-parity complex of Δ (see Section 1 Example C) consisting of subsets of $\{0, 1, 2, \dots, n\}$. Alternatively, we can obtain $\Delta[n]$ from the one-point parity complex $\Delta[0]$ by repeated right coning, calling the new point at each stage $1, 2, 3, \dots, n$, respectively, instead of ∞ . Then Δ is the union of the $\Delta[n]$, $n \in \omega$. The 1-simplex is also denoted by I (it is the parity complex version of the "interval").

The n -cube $Q[n]$ is the product

$$Q[n] = I^n = I \times I \times \dots \times I$$

of n copies of I . Just as for the simplexes, there is an ω -cube Q which is the union of the $Q[n]$, $n \in \omega$. The elements are words $x = x_1 x_2 \dots x_n$ on the three symbols $-, 0, +$. The dimension of an element is the number of 0's appearing in it. Let $x_{d_i}^-$ denote the word obtained from x by replacing the i -

th 0 by - when i is odd and by + when i is even. Similarly, $x\partial_i^+$ is defined by interchanging - and + in the last sentence. For $x \in Q_p$, define

$$x^e = \{x\partial_i^e : 1 \leq i \leq p\}.$$

Moreover, if $x = x_1 x_2 \dots x_n \in Q$ then $\mu(x)$ consists of those $y = y_1 y_2 \dots y_n \in Q$ such that:

- $x_r \neq 0$ implies $y_r = x_r$,
- $x_r = 0$ and $y_r = -$ imply $\dim(y_1 y_2 \dots y_r)$ is even, and
- $x_r = 0$ and $y_r = +$ imply $\dim(y_1 y_2 \dots y_r)$ is odd.

Also, $\pi(x)$ is given dually. It is clear from these descriptions that Q satisfies conditions (R1) and (R2) of the Remark of Section 4.

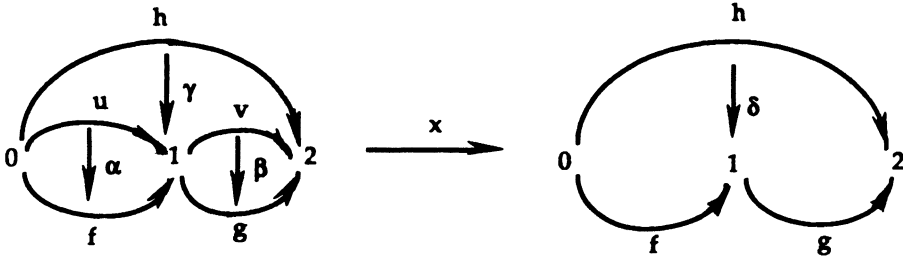
We also recall the ω -glob G from Section 1. This is a particularly trivial parity complex with which to deal. There is an obvious bijection

$$\alpha(G) \cong G.$$

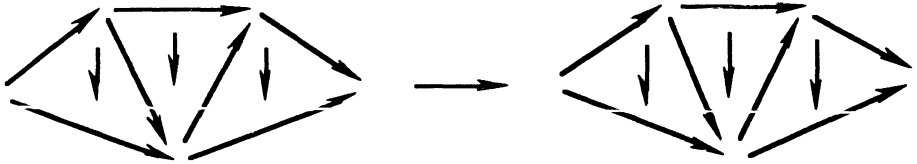
Corollary 6.1 Δ, Q, G and their odd duals are all parity complexes for which \prec is antisymmetric and all elements are relevant. *qed*

In fact, for each of the parity complexes of Corollary 6.1, \prec is a total order. We now give two reasonable examples where \prec is not antisymmetric.

Example This example, indicated by the following diagram, is due to Vaughan Pratt. Notice that $f \prec 1 \prec v \prec \beta \prec x \prec \delta \prec f$; so \prec is not antisymmetric. There is a dual of this example for which \prec is a total order.



Example This example, indicated by the following diagram, is due to Michael Johnson. Here there is no dual which has \prec antisymmetric.



Before discussing descent, we need to consider a certain countable limit construction which yields an ω -category A_∞ for each ω -category A . The elements of A_∞ are sequences $(u, v) = ((u_n, v_n) : n \in \mathbb{N})$ in the set $A \times A$ such that $s_{n-1}(u_n) = s_{n-1}(v_n) = u_{n-1}$ and $t_{n-1}(u_n) = t_{n-1}(v_n) = v_{n-1}$. Source and target functions are given by

$$s_m(u, v) = ((s_m(u_n), s_m(v_n)) : n \in \mathbb{N}),$$

$$t_m(u, v) = ((t_m(u_n), t_m(v_n)) : n \in \mathbb{N}).$$

The composition functions are given by

$$(u, v) *_m (u', v') = (h, k)$$

where

$$h_n = \begin{cases} u_n = u'_n & \text{for } n < m \\ u_n & \text{for } n = m \\ u_n *_m u'_n & \text{for } n > m \end{cases} \quad \text{and} \quad k_n = \begin{cases} v_n = v'_n & \text{for } n < m \\ v_n & \text{for } n = m \\ v_n *_m v'_n & \text{for } n > m \end{cases} .$$

There is a canonical ω -functor $A \rightarrow A_\infty$ which takes $a \in A$ to

$$((s_n(a), t_n(a)) : n \in \mathbb{N}).$$

If A is an n -category for some n then $A \rightarrow A_\infty$ is an isomorphism. If $A = \mathcal{O}(G)$ then A_∞ contains one element ∞ which is not an n -cell for any n ; and $A \rightarrow A_\infty$ is injective with ∞ the only element not in the image.

Let $\omega\text{-Cat}$ denote the category of (small) ω -categories and ω -functors. For each ω -category A , there is a canonical bijection

$$\omega\text{-Cat}(\mathcal{O}(G), A) \cong A_\infty .$$

Since A_∞ supports a functorial ω -category structure, it follows that $\mathcal{O}(G)$ is equipped with a canonical structure of co- ω -category in the category $\omega\text{-Cat}$. More generally:

Proposition 6.2 *If C is a parity complex such that \triangleleft is antisymmetric for it and its odd dual then the ω -category $\mathcal{O}(C \times G)$ admits a canonical co- ω -category structure in $\omega\text{-Cat}$.*

Let \mathbf{S} denote the category of non-empty finite ordinals

$$[n] = \{0, 1, \dots, n\}$$

and order-preserving functions. A *cosimplicial ω -category* is a functor

$$E : \mathbf{S} \longrightarrow \omega\text{-Cat}.$$

A particular example is the cosimplicial ω -category $\Delta \otimes G$ whose value on the object $[n]$ is $\mathcal{A}(\Delta[n] \times G)$. This forms the basis for the general "descent" construction which is therefore at the heart of the dichotomy between simplexes and globs for higher-order categories.

Definition For any cosimplicial ω -category E , the *descent ω -category* of E is defined to be the hom-set

$$\text{Desc}E = \omega\text{-Cat}^{\mathbf{S}}(\Delta \otimes G, E)$$

with ω -category structure induced by the co- ω -category structures (Proposition 6.2) on each $(\Delta \otimes G)[n] = \mathcal{A}(\Delta[n] \times G)$.

Suppose we have both a simplicial object $R : \mathbf{S}^{\text{op}} \longrightarrow \mathcal{E}$ in a category \mathcal{E} and an ω -category A in \mathcal{E} . Then $\mathcal{E}(R, A)$ is a cosimplicial ω -category.

Definition The *cohomology ω -category* of R with coefficients in A is defined by

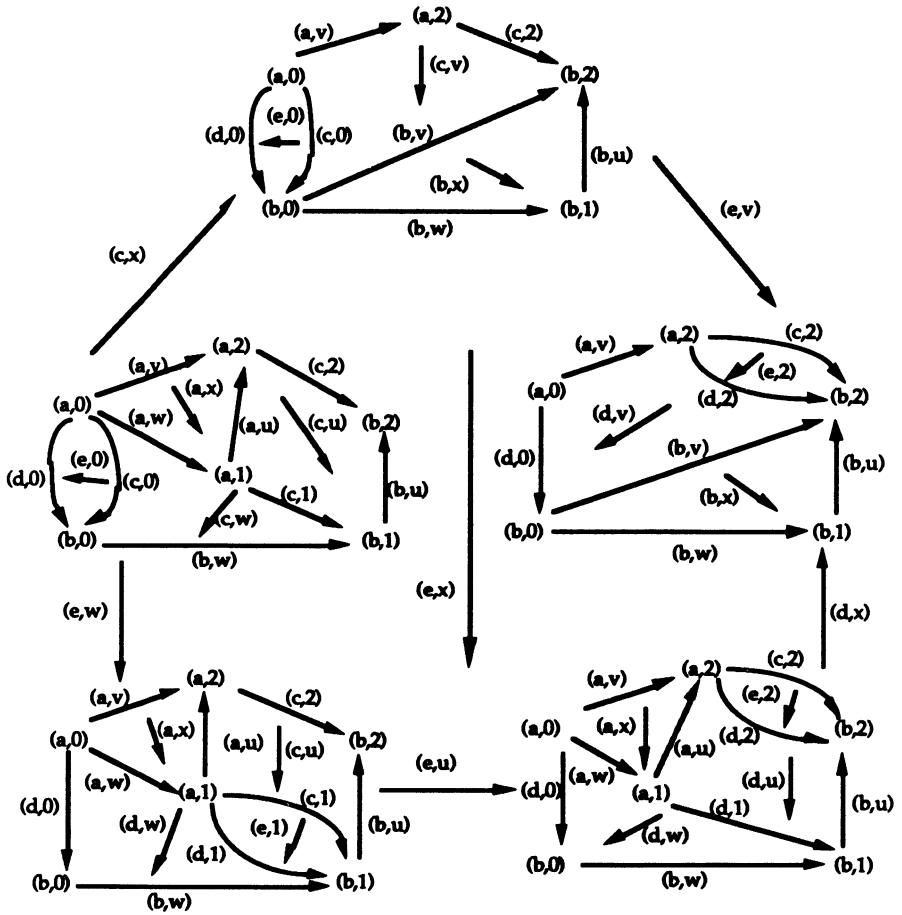
$$\mathcal{H}(R, A) = \text{Desc}\mathcal{E}(R, A).$$

The ω -categories $\mathcal{A}(\mathbf{Q}[n] \times G)$ are important in the theory of homotopy types, but it is not the purpose of this paper to pursue these concepts any further.

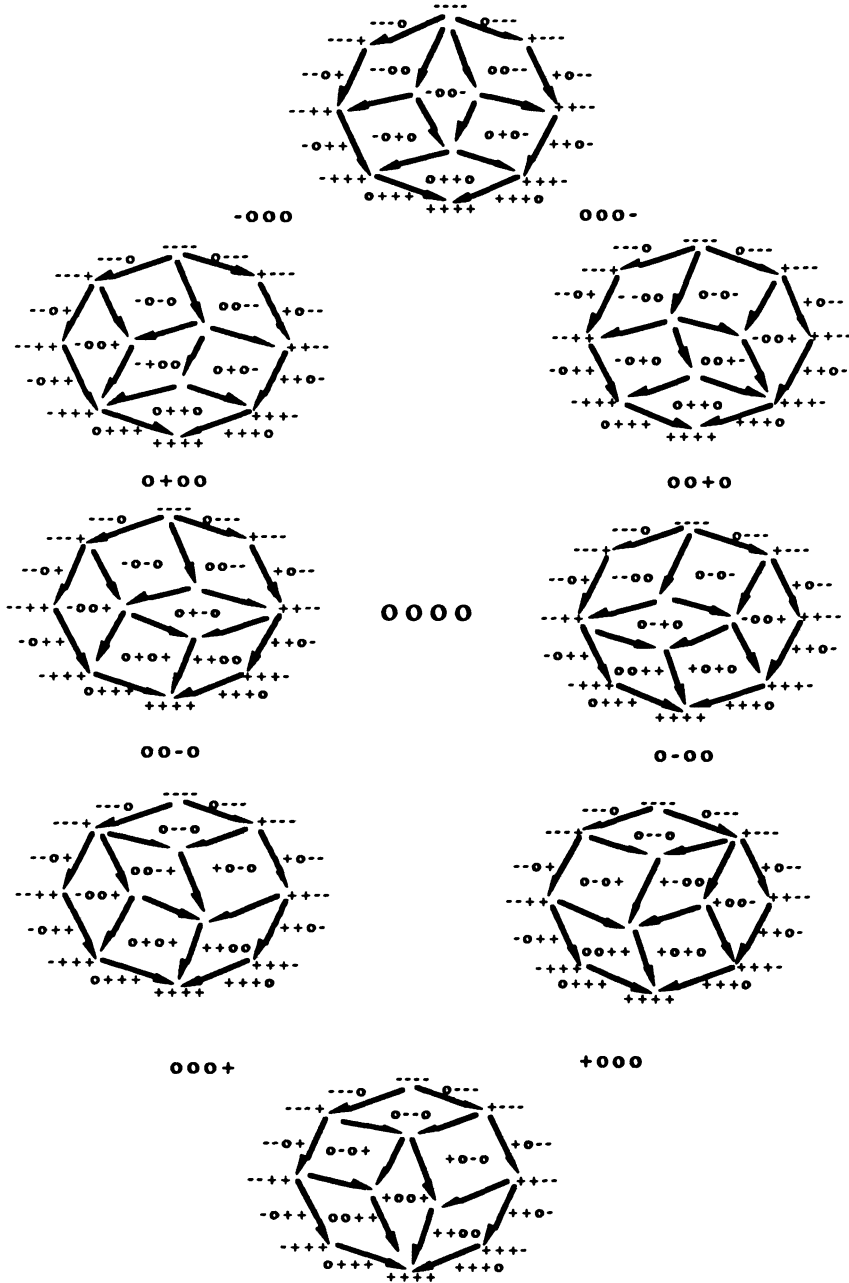
We conclude this paper with diagrams for the product of the 2-glob and the 2-simplex, and for the 4-cube.

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$$G[2] \times \Delta[2] = \begin{array}{c} \text{c} \\ \text{a} \downarrow \text{e} \text{b} \\ \text{d} \end{array} \times \begin{array}{ccc} & \text{v} & \\ 0 & \xrightarrow{\quad} & 2 \\ \text{w} & \searrow \text{x} \downarrow \nearrow & \text{u} \\ & \text{i} & \end{array}$$



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Mathematics Department
Macquarie University
New South Wales 2109
AUSTRALIA

email <street@macadam.mpce.mq.edu.au>