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κ-LINDELÖF LOCALES AND THEIR SPATIAL PARTS

by P.B. JOHNSON

Résumé: Nous définissons la classe des locales κ-Lindelöf complétement réguliers, que nous désignerons par $|\mathbf{Loc}_\kappa|$, et nous vérifions que la sous-catégorie pleine $\mathbf{Loc}_\kappa \xrightarrow{\subset} \mathbf{Locales}$ est réflective, pour chaque cardinal régulier κ indénombrable. Pour un locale quelconque A, on peut décrire les flèches d'adjonction $A \longrightarrow \lambda_\kappa A$ suivant l'optique de la théorie des treillis locaux, de façon tout à fait semblable à la construction de Banaschewski et Mulvey des flèches d'adjonction $A \longrightarrow \beta A$ pour la réflection compacte complétement régulière [1]. Plusieurs de nos théorèmes sont des généralisations directes de théorèmes de Madden et Vermeer [7]. Nous pouvons identifier les parties spatiales des locales κ-Lindelöf complétement réguliers avec les espaces κ-compacts de Herrlich [3]. Un résultat topologique de Hušek [4] est renforcé dans le sens que, pour chaque κ, il existe un locale R_κ tel qu'un locale quelconque A vérifie $A \in |\mathbf{Loc}_\kappa|$ si et seulement si A se plonge comme sous-locale fermé dans une puissance *localique* de R_κ. Selon le théorème principal de ce travail, R_κ est un *cogénérateur régulier* de $\mathbf{Loc}_\kappa$.

1 Introduction

For an introduction to the category **Frames**, of frames and frame homomorphisms, and for more detailed discussion of the topics outlined in this section, we refer the reader to [5].

1.1 For elements a and b of a frame, we say *a is well below b*, and write $a \lessdot b$, provided there exists c such that $a \wedge c = 0$ and $b \vee c = 1$. We say *a is really below b*, and write $a \overline{\overline{\lessdot}}\, b$, provided there are elements $\{c_q : q \in \mathbf{Q} \cap [0,1]\}$ satisfying: $c_0 = a$, $c_1 = b$, and $c_p \lessdot c_q$ whenever $p < q$. We say an element c of a frame A is *cozero*, and write $c \in cozA$, provided there is a frame homomorphism $f : \Omega\mathbf{R} \longrightarrow A$ satisfying $c = f\{(-\infty, 0) \cup (0, \infty)\}$. The really below relation satisfies the following *subdivisibility property*: For elements a and b of a frame A satisfying $a \overline{\overline{\lessdot}}\, b$, there exists $c \in cozA$ such that $a \overline{\overline{\lessdot}}\, c \overline{\overline{\lessdot}}\, b$.

For each element a of a frame A, the set $prin(a) = \{x \in A : x \mathrel{\overline{\overline{<}}} a\}$ is a lattice ideal.

A frame A is *completely regular* provided $a = \bigvee_A prin(a)$ for each $a \in A$, or equivalently, by the subdivisibility property, $cozA$ is a basis for A. A frame homomorphism $A \xrightarrow{f} B$ is *dense* provided $\{a \in A : fa = 0_B\} = \{0_A\}$.

The following lemma is essential; a sketch of its proof may be found in [5], page 82.

Lemma 1.2 In the following diagram of frames and frame homomorphisms:

$$C \underset{y}{\overset{x}{\rightrightarrows}} A \xrightarrow{\quad f \quad} B,$$

if f is dense and C is completely regular, then $fx = fy$ implies $x = y$.

1.3 We say a frame A is *compact* provided 1_A is a *finite element*, that is to say, every sup to 1_A admits a finite sub-sup. The full subcategory of **Frames** whose objects consist of those frames which are compact and completely regular is denoted by **K**. The finitary nature of the sup operation in the frame $Idl(A)$ of all lattice ideals in A is exemplified in the fact that all such frames are compact. In the next section, we will introduce frames of ideals for which the sup operation does *not* have this finitary nature. The following theorem of Banaschewski and Mulvey lays a foundation for our Theorem 2.3.

Theorem 1.4 Stone-Čech compactification [1]

For a fixed frame A, we define the set of lattice ideals:

$$\beta A \quad = \quad \{I \in \ Idl(A) \ / \ a \in I \quad \Longrightarrow \quad \exists b \in I : a \mathrel{\overline{\overline{<}}} b\}.$$

Then the following statements (where $\bigvee$ denotes sup in A) are true:

1. *βA is a subframe of $Idl(A)$.*
2. *$\beta A \ \in |\mathbf{K}|$.*
3. *$prin(a) \in \beta A$ for each $a \in A$.*
4. *$\beta A \xrightarrow{\subset} Idl(A) \xrightarrow{\bigvee} A$ is a dense frame homomorphism.*

5. *$\beta A \xrightarrow{\bigvee} A$ is onto if and only if A is completely regular.*

6. *$\beta A \xrightarrow{\bigvee} A$ is the coreflection of A into $\mathbf{K}$.*

Remark: It follows that β : **Frames** $\longrightarrow$ **K** is a functor. In fact β is a subfunctor of Idl; for each frame homomorphism $f : A \longrightarrow B$ and $I \in \beta A$,

$$\beta f(I) = \{b \in B \ / \ \exists a \in A : b \leq fa\}.$$

2 κ-Lindelöf Completely Regular Frames

2.1 We fix an uncountable regular cardinal κ and a frame A.

1. $a \in A$ is called a *κ-cozero element* and we write $a \in \kappa\text{-}cozA$ provided there is $X \subset cozA$ satisfying $|X| < \kappa$ and $a = \bigvee_A X$.

2. $a \in A$ is called a *κ-Lindelöf element* provided that, whenever $X \subset A$ satisfies $a = \bigvee_A X$, there is $X_0 \subset X$ such that $|X_0| < \kappa$ and $a = \bigvee_A X_0$.

3. A is a *κ-Lindelöf* frame provided 1_A is a κ-Lindelöf element.

4. $\mathbf{Lind}_\kappa \xrightarrow{\subset}$ **Frames** denotes the full subcategory whose objects consist of those frames which are both *completely regular* and *κ-Lindelöf.*

The following lemma extends to frames well-known facts about topologies, the second of which, in either context, seems to depend heavily on the assumption that κ is a regular cardinal.

Lemma 2.2

1. If A is κ-Lindelöf, then $c \in cozA \implies c \in \kappa\text{-}elA$.

2. If $A \in |\mathbf{Lind}_\kappa|$, then $c \in \kappa\text{-}cozA$ if and only if c is a κ-Lindelöf element of A.

Given a *nucleus* j on the frame A, we will denote by $j^* : A \longrightarrow A_j$ the induced regular frame epimorphism and, when there is no danger of confusion, may choose to supress the $*$. The rest of this section will be devoted to proving, via a succession of lemmas, the following:

Theorem 2.3 $\mathbf{Lind}_\kappa \xrightarrow{\subset} \mathbf{Frames}$ *is coreflective.*

We continue to consider an arbitrary frame A. It readily follows from the elementary properties of the *really below* relation that, for each $I \in \beta A$, the set

$$jI = \{x \in A \,/\, \exists C \subset I \;:\; x \,\overline{\overline{<}}\, \vee_A C \;\;\&\;\; |C| < \kappa\}$$

is an ideal $jI \in \beta A$, and that the assignment $I \mapsto jI$ defines a nucleus j on βA. We shall write $\lambda A = (\beta A)_j$.

Lemma 2.4 $\lambda A \in |\mathbf{Lind}_\kappa|$.

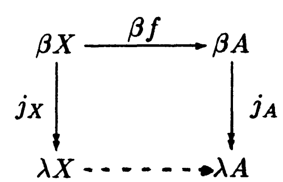
Proof : $\beta A \xrightarrow{j^*} \lambda A$ exhibits λA as a quotient of a completely regular frame and therefore λA is completely regular. To see that λA is κ-Lindelöf, observe first that, for any collection of ideals $\{I_\alpha\} \subset \lambda A$,

$$1_{\lambda A} = \bigvee_{\lambda A} I_\alpha = j^*(\bigvee_{\beta A} I_\alpha),$$

if and only if, there is $C \subset \bigvee_{\beta A} I_\alpha$ satisfying $|C| < \kappa$ and $1_A \,\overline{\overline{<}}\, \bigvee_A C$. Since sups (of collections of ideals) in βA coincide with those in $Idl(A)$, each $c \in C$ is a finite join of elements each of which lies in some I_α. As κ is a regular cardinal, the number of ideals I_α which occur in all such representations is still less than κ. Therefore, there exist $\kappa' < \kappa$, a collection of ideals, $\{I_{\alpha_i} : i \in \kappa'\}$ and $d_i \in I_{\alpha_i}$ such that $1_A = \bigvee_A\{d_i : i \in \kappa'\}$. It follows that $1_{\lambda A} = \bigvee_{\lambda A}\{I_{\alpha_i} : i \in \kappa'\}$, and λA is κ-Lindelöf.

Lemma 2.5 The object assignment $A \mapsto \lambda A$ has a unique extension to a functor $\lambda : \mathbf{Frames} \longrightarrow \mathbf{Frames}$ for which the collection of maps $\{j_A : \beta A \longrightarrow \lambda A\}$ constitute a natural transformation $\beta \longrightarrow \lambda$.

Proof: Given a frame homomorphism $f : X \longrightarrow A$, consider the diagram below:



Employing the usual criterion for factoring an algebraic homomorphism through a surjective one, $j_A \circ \beta f$ factors through j_X just in case, for each $I \in \beta A$,

$$j_A \circ \beta f(I) = j_A \circ \beta f(j_X I).$$

Now $j_A \circ \beta f$ is order preserving, so the following containment is clear:

$$j_A \circ \beta f(I) \subset j_A \circ \beta f(j_X I).$$

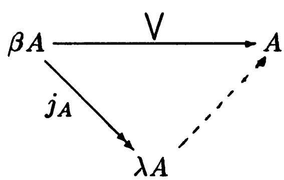
Because j_A is a nucleus, the other containment will follow, given that

$$\beta f(j_X I) \subset j_A \circ \beta f(I).$$

So let $a \in \beta f(j_X I)$. Then $a \leq fx$, for some $x \in j_X I$. It follows that there exists $C \subset I$ satisfying $|C| < \kappa$ and $x \stackrel{=}{<} \bigvee_A C$. But then $a \leq fx \stackrel{=}{<} \bigvee\{fc : c \in C\}$ and $a \in j_A \circ \beta f(I)$.

This establishes the existence of a factorization and, since j_X is an epimorphism, the factoring map, denoted by $\lambda f : \lambda X \longrightarrow \lambda A$, is unique. That the λ-data is functorial now follows immediately from this uniqueness and the fact that β is a functor.

Lemma 2.6 We can factor



and thereby produce a candidate for the universal morphism $\lambda A \longrightarrow A$.

Proof: One can easily check that if $I \in \beta A$ satisfies $\bigvee I = a$, then $I \subset jI \subset prin(a)$ and therefore $\bigvee jI = a$ also. The triangle fills in and the factoring map is denoted by $\bigvee : \lambda A \longrightarrow A$.

We mention in passing that $\{a \in A \ / \ \exists I \in \beta A \ : \ a = \bigvee I\}$ is easily seen to be the largest completely regular subframe of A, and is in fact the coreflection of A into the category of completely regular frames.

Lemma 2.7 $A \in |\mathbf{Lind}_\kappa| \implies \lambda A \xrightarrow{\bigvee} A$ is an isomorphism.

Proof: By Theorem 1.4.5, $\lambda A \xrightarrow{\bigvee} A$ is onto provided A is completely regular, so it suffices to show that $\lambda A \xrightarrow{\bigvee} A$ is injective. In fact, making no mention of a separation axiom (such as complete regularity), the following is true:

Claim: For A κ-Lindelöf, $\lambda A \xrightarrow{\bigvee} A$ is *injective.*

Note that for each $a \in A$, $prin(a) \in \lambda A$. It suffices to show, where $I \in \lambda A$ satisfies $\bigvee I = a$, that $I = prin(a)$. If $b \mathrel{\overline{\overline{<}}} a$, then $b \mathrel{\overline{\overline{<}}} c \mathrel{\overline{\overline{<}}} a$, for some $c \in cozA$. Now $c = \bigvee\{x \wedge c : x \in I\}$. Since cozero elements are κ-elements in a κ-Lindelöf frame, $c = \bigvee_A C$ for some $C \subset I$ satisfying $|C| < \kappa$. Therefore, since $b \mathrel{\overline{\overline{<}}} c$ and $I \in \lambda A$, it follows that $b \in I$, and the claim is established.

Lemma 2.8 $\lambda A \xrightarrow{\bigvee} A$ is the coreflection of A into $\mathbf{Lind}_\kappa$.

Proof: Let $X \in |\mathbf{Lind}_\kappa|$ and $X \xrightarrow{f} A$ be given and consider the diagram below:

$$\begin{array}{ccc} \beta X & \xrightarrow{\beta f} & \beta A \\ {\scriptstyle j_X}\big\downarrow & & \big\downarrow{\scriptstyle j_A} \\ \lambda X & \xrightarrow{\lambda f} & \lambda A \\ {\scriptstyle \cong}\big\downarrow & & \big\downarrow{\scriptstyle \bigvee} \\ X & \xrightarrow[f]{} & A \end{array}$$

The outer rectangle commutes, by the properties of β mentioned in Theorem 1.4. Thus the bottom square commutes, since the top one does and j_X is an epimorphism. Since X is completely regular, the exhibited factorization of the test map f through the dense frame map $\bigvee : \lambda A \longrightarrow A$ is unique, by Lemma 1.2. This completes the proof of the lemma, and hence of Theorem 2.3 as well.

3 The Localic description

For the remainder of the paper we work in the category of locales and locale maps, denoted by **Locales**, and all morphisms are written in the geometric direction of continuous maps between spaces. We adhere to the convention of treating the category of (sober) spaces, denoted **Spaces**, as a full coreflective subcategory of **Locales** with coreflection functor pt : **Locales** $\longrightarrow$ **Spaces**, and thereby deem a locale A to be *spatial* just in case $A \cong ptX$ for some locale X. It must be stressed however that while limit constructions such as product and intersection of spatial locales may be carried out in either category, **Spaces** or **Locales**, the results may quite possibly differ. The "frame of opens", by virtue of which a locale A is defined, is denoted A^*. We say a locale or locale map has a certain property, like that of being completely regular or dense, just in case that property is indicated in the corresponding frame or frame homomorphism.

A nucleus j on a locale A determines a regular subobject or *sublocale* $A_j \subset A$. The intention is that $A_j \subset A$ denote an equivalence class of regular monomorphisms with a distinguished choice of representative. The collection $Sub(A)$ of all sublocales of a given locale A, ordered by the *containment* relation $\subset$, forms a complete lattice. As in [7], we adopt a notation for the intersection, union, and forward and inverse images of sublocales that mimics the usual notation for subspaces. The open and closed sublocales of a given locale A determined by an element $c \in A^*$ are denoted $A_{c\to(\ )}$ and $A_{c\vee(\ )}$ respectively. Where $c \in cozA^*$, $A_{c\to(\ )}$ is called a *cozero* sublocale and $A_{c\vee(\ )}$ a *zero-set* sublocale of A. Where $a \in \kappa\text{-}cozA^*$, the open sublocale $A_{c\to(\ )}$ is a κ-*cozero* sublocale of A.

Again we fix an uncountable regular cardinal κ and denote the full subcategory of completely regular κ-Lindelöf locales as $\mathbf{Loc}_\kappa \xrightarrow{\subset}$ **Locales**. The results of the last section dualize as follows: $\mathbf{Loc}_\kappa \xrightarrow{\subset}$ **Locales** is a reflective subcategory. For each $A \in |\mathbf{Locales}|$ there is a reflection map $A \longrightarrow \lambda A$ where $\lambda A \in |\mathbf{Loc}_\kappa|$, and this map has the universal property *dual* to that established in Lemma 2.8. Moreover, $A \longrightarrow \lambda A$ is dense, and λA is a dense sublocale of βA.

The following theorem generalizes a result of Madden and Vermeer [7] and gives a description of λA as an intersection of open sublocales of βA.

Theorem 3.1 *Given a locale X, the following are equivalent:*

1. $X \in |\mathbf{Loc}_\kappa|$.
2. *X is completely regular and X is an intersection of κ-cozero sublocales of each completely regular locale containing X as a sublocale.*
3. *X is an intersection of open κ-Lindelöf sublocales of βX.*

Proof:

$2 \Longrightarrow 3$. Given that X is completely regular, X is a sublocale of βX. The implication then follows from the fact that (by Lemma 2.2.2) κ-cozero sublocales of βX are κ-Lindelöf.

$3 \Longrightarrow 1$. This follows immediately from the facts that $\beta X \in |\mathbf{Loc}_\kappa|$, and that $\mathbf{Loc}_\kappa$ is closed under taking intersections in **Locales**, by virtue of being a reflective subcategory.

$1 \Longrightarrow 2$. Let $X = A_j \in |\mathbf{Loc}_\kappa|$ for some nucleus j on a completely regular locale A. With reference to [5], page 51, it suffices to show that, for each $a \in A^*$ satisfying $a < ja$, that is $a \notin A_j^*$, there is $c \in \kappa\text{-}cozA^*$ satisfying $a < (c \to a)$ and $jc = 1_{A^*}$, so that $a \notin A^*_{c\to(\ )}$ and $A_j \subset A_{c\to(\ )}$. Proceed via the following steps:

(a) Using the fact that A is completely regular, find $x \mathrel{\overline{<}} ja$ satisfying $a < x \vee a$.

(b) Where $C = \{c \in cozA^* : c \leq \neg x \text{ or } c \leq a\}$, observe that $\neg x \vee a = \bigvee_{A^*} C$, whence

$$\bigvee_{A_j^*}\{jc : c \in C\} = j(\bigvee_{A^*} C) \geq j(\neg x) \vee ja = 1_{A^*}.$$

(c) Choose $C_0 \subset C$ such that $|C_0| < \kappa$ and $j(\bigvee_{A^*} C_0) = 1_{A^*}$, using the κ-Lindelöf property of A. Now let $c_0 = \bigvee_{A^*} C_0$ and note $c_0 \in \kappa\text{-}cozA^*$ and $j(c_0) = 1_{A^*}$.

Finish the argument by showing that c_0 is the κ-cozero element that works. It is easily seen, by the way C was chosen, that each $c \in C_0$ satisfies $(x \vee a) \wedge c \leq a$. It follows that $(x \vee a) \wedge c_0 \leq a$, whence

$$a < (x \vee a) \leq (c_0 \to a),$$

and therefore $a \notin A^*_{c_0 \to (\ )}$, which completes the proof.

Corollary 3.2 Where A is a completely regular locale, λA is the intersection in $Sub(\beta A)$ of all the κ-Lindelöf open sublocales of βA which contain A.

Proof: Use the universal property of the reflection map $A \longrightarrow \lambda A$.

4 The Spatial parts

We offer the following scheme for producing reflective subcategories of **Spaces** from reflective subcategories of **Locales**.

Theorem 4.1 *Given the diagram of categories and functors below,*

$$\mathbf{A} \underset{\lambda}{\overset{\subset}{\rightleftarrows}} \mathbf{Locales} \underset{\supset}{\overset{pt}{\rightleftarrows}} \mathbf{Spaces}$$

in which:

1. $\mathbf{A} \xrightarrow{\subset}$ **Locales** *is a full reflective subcategory with reflection* λ*;*
2. $|\mathbf{A}|$ *consists of completely regular locales;*
3. *The reflection maps* $\{A \longrightarrow \lambda A : A \in |\mathbf{Locales}|\}$ *are dense.*

Then $\{X \in |\mathbf{Spaces}| \ / \ \exists A \in |\mathbf{A}| : \ X \cong ptA\}$ *forms the object class of a full reflective subcategory of* **Spaces**.

For an independent treatment of the special case in which $\mathbf{A} = \mathbf{Loc}_\kappa$, the reader may skip to Definition 4.2. To sketch a proof in general, one might begin by considering the following question: When is the full isomorphism-closed image $\mathbf{X}_0$, of a *right adjoint* $F : \mathbf{A} \longrightarrow \mathbf{X}$, a reflective subcategory of $\mathbf{X}$? Certainly it suffices that the unit maps for the adjunction involving F, as morphisms in $\mathbf{X}$, are each epimorphic with respect to maps in $\mathbf{X}_0$. This sufficient condition is readily shown to hold when $X =$ **Spaces** and F is the composite (right adjoint) $\mathbf{A} \xrightarrow{\subset}$ **Locales** $\xrightarrow{pt}$ **Spaces** above, (using the fact that complete regularity of locales and density of locale maps are preserved under restriction to spatial parts, that is, under application of the functor pt).

4.2 A completely regular space X is *κ-compact* [3] provided, for each z-ultrafilter $\mathbf{p}$ on X,

$$\bigcap \mathbf{p} = \emptyset \implies \bigcap p_0 = \emptyset \text{ for some } p_0 \subset \mathbf{p} \text{ with } |p_0| < \kappa.$$

Remark: Herrlich has observed that the class of κ-compact spaces, taken as a full subcategory of **Spaces**, is reflective.

Theorem 4.3 *For $X \in$ |**Spaces**|, the following are equivalent:*

1. $X \cong ptA$ *for some* $A \in |\mathbf{Loc}_\kappa|$.
2. *X is an intersection in* **Spaces** *of open κ-Lindelöf subspaces of some completely regular space.*
3. *X is κ-compact.*
4. *X is an intersection in* **Spaces** *of open κ-Lindelöf subspaces of βX.*

Proof:

$1 \Longrightarrow 2$. As noted in Theorem 3.1, A is an intersection in **Locales** of open κ-Lindelöf sublocales of the spatial locale βA, and the functor pt : **Locales** $\longrightarrow$ **Spaces** preserves this intersection, that is, $ptA \cong X$ is the intersection in **Spaces** of these same opens.

$2 \Longrightarrow 3$. It follows immediately from the definitions that κ-Lindelöf spaces are κ-compact. The implication is then a consequence of the fact that the category of κ-compact spaces is closed under taking intersections in **Spaces**.

$3 \Longrightarrow 4$. The points of βX may be identified in the usual way with zero-set ultrafilters on X. It must be shown that each *free* zero-set ultrafilter $\mathbf{p} \in \beta X \setminus X$ is excluded from some open κ-Lindelöf subspace of βX containing X. As X is κ-compact, there are $\kappa' \in \kappa$ and $p_0 = \{F_i : i \in \kappa'\} \subset \mathbf{p}$, satisfying: $\bigcap p_0 = \emptyset$. For each F_i find a zero-set $Z_i \subset \beta X$ satisfying

$$F_i = Z_i \cap X \ \ and \ \ \mathbf{p} \in Z_i.$$

It follows that

$$X \subset \bigcup_{i \in \kappa'} \{c_i \in coz(\beta X^*) : c_i = \beta X \setminus Z_i\} \subset \beta X \setminus \{\mathbf{p}\}.$$

$4 \Longrightarrow 1$. Let X be an intersection of open κ-Lindelöf subspaces of βX. Then the description of λX afforded by Corollary 3.2 and the fact that *pt* preserves intersections combine to imply $pt(\lambda X) \cong X$.

Remark: A set X of cardinality ω^+, taken with the discrete topology, provides an example of an ω^+-compact space which fails to be an intersection of ω^+-Lindelöf open subsets in its one point compactification, underlining a sharp contrast between the statement about locales given in Theorem 3.1 and the statement about spaces given in Theorem 4.3 above.

5 A Localic version of Hušek's Theorem

Hušek has shown that for each infinite cardinal κ there is a space $\mathbf{P}_\kappa$ with the property that an arbitrary space is κ-compact just in case it embeds as a closed subspace into a power of $\mathbf{P}_\kappa$. We will prove a theorem which (at least for uncountable regular cardinals) implies this spatial result. Along the way we record two more conditions, each of which is necessary and sufficient for a locale A to satisfy $A \in |\mathbf{Loc}_\kappa|$. See Theorem 5.6.

Throughout the remainder of the paper we use the notation $\mathbf{I} = [0,1] \in |\mathbf{Spaces}|$, *cardinal* will mean uncountable regular cardinal, γ^+ will denote the cardinal successor to the cardinal γ, and the symbol $\prod$ will invariably denote product in the category of locales. We refer the reader to [5], where the construction of products in **Locales** is discussed, and it is established that products of compact completely regular locales are spatial.

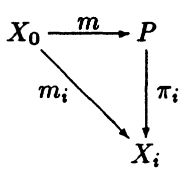
Much of the rest of this section is devoted to the development of a machinery, the full utility of which shall not be evident until Section 6. To begin with is the following definition, which is motivated by Hušek's construction [4].

5.1 We define a locale R_κ for each cardinal κ.

1. $R_\kappa = \mathbf{I}^\gamma \setminus \{\vec{0}\}$, for $\kappa = \gamma^+$.
2. $R_\kappa = \prod\{R_{\gamma^+} : \gamma^+ < \kappa\}$, for limit cardinal κ.

Lemma 5.2 Each factor X_0 in a product $P = \prod\{X_i : i \in \gamma\}$ of *pointed locales* $(X_i, *_i)$ embeds into P as a retract.

Proof: Define a locale map m as below,



where $m_0 = 1_{X_0}$, and $m_i = \{X_0 \longrightarrow 1 \xrightarrow{*_i} X_i\}$ for each $i \neq 0$.

Lemma 5.3 For each cardinal κ, the following are true:

1. $R_\kappa \in |\mathbf{Loc}_\kappa|$.

2. **I** embeds into R_κ as a retract.

Proof:

1. For each $\kappa = \gamma^+$, the frame $(R_\kappa)^*$ has a γ-sized basis (of product rectangles) and therefore $R_\kappa \in |\mathbf{Loc}_\kappa|$. For κ a limit cardinal, R_κ is the product of objects in $\mathbf{Loc}_\kappa$, and therefore, since $\mathbf{Loc}_\kappa$ is reflective in **Locales**, $R_\kappa \in |\mathbf{Loc}_\kappa|$.

2. For $\kappa = \gamma^+$, the map $m : \mathbf{I} \longrightarrow R_{\gamma^+}$, where $(mx)_0 = x$ and $(mx)_i = 1$ for $i > 0$ is clearly the embedding of a retract. Where κ is a limit cardinal, the locales R_{γ^+}, for cardinals $\gamma^+ < \kappa$, are readily pointed and Lemma 5.2 applies: Such R_{γ^+} embed into R_κ as retracts, so **I** does as well.

5.4 For convenience, we introduce the following definitions regarding contravariant set-valued functors.

1. With respect to a functor U : **Locales*** $\longrightarrow$ **Sets**, a locale map $f : A \longrightarrow B$ is *U-contractible* provided $Uf : UB \longrightarrow UA$ is a split epimorphism in **Sets**.
2. Where f is a regular monomorphism of locales and U-contractible, we say that *A is U-embedded into B* (via f).
3. $C_0 = \mathbf{Locales}(-, \mathbf{I})$.
4. $C_\kappa = \mathbf{Locales}(-, R_\kappa)$ for each cardinal κ.

Example: The reflection $A \longrightarrow \lambda A$ of a locale A into $\mathbf{Loc}_\kappa$ is C_κ-contractible.

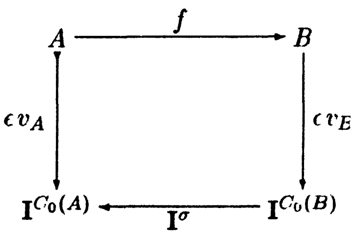
Lemma 5.5 The following are true about a locale map $A \xrightarrow{f} B$:

1. If f is C_κ-contractible, then f is C_0-contractible.
2. If f is C_0-contractible and A is completely regular, then f is the inclusion of a sublocale.

Proof:

1. This follows immediately from the fact that **I** embeds into R_κ as a retract, as established in Lemma 5.3.

2. By hypothesis there is a function $\sigma : C_0(A) \longrightarrow C_0(B)$ such that $a = \sigma a \circ f$, for each $a \in C_0(A)$. As the complete regularity of A ensures that $ev_A : A \longrightarrow \mathbf{I}^{C_0(A)}$ is a regular monomorphism of **Locales**, the following diagram exhibits f as a regular monomorphism,



where $\mathbf{I}^\sigma$ is constructed to satisfy $\pi_a \circ \mathbf{I}^\sigma = \pi_{\sigma a}$ for each $a \in C_0(A)$.

We extend the characterization of $\mathbf{Loc}_\kappa$ given in Theorem 3.1.

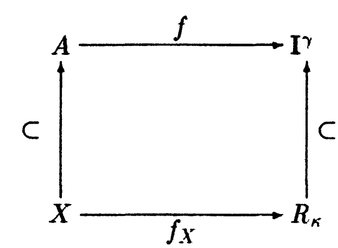
Theorem 5.6 *The following are equivalent, for a completely regular locale X:*

1. $X \in |\mathbf{Loc}_\kappa|$.
2. *If X is dense and C_κ-embedded into a completely regular locale A, then $X \cong A$.*
3. *The evaluation map $ev_\kappa X : X \longrightarrow {R_\kappa}^{C_\kappa(X)}$ is the inclusion of a closed sublocale.*

Proof:
1 $\Longrightarrow$ 2. By Theorem 3.1 it suffices to assume that X is a κ-cozero sublocale of A, so let

$$X = \bigcup_{Sub(A)} \{A_{c_i \to (\;)} : i \in \gamma\},$$

for some $\gamma < \kappa$ and $c_i \in cozA^*$.
In the case that κ is a successor cardinal, γ above may be taken to satisfy $\kappa = \gamma^+$. Then X is a pullback as depicted in the diagram of locales:



where:

(a) f satisfies $(\pi_i \circ f)^{-1}U \cong A_{c_i \to (\,)}$ for each $i \in \gamma$, where $U = (0,1]$, and

(b) $R_\kappa \cong \bigcup_{Sub(\mathbf{I}^\gamma)}\{\pi_i{}^{-1}U : i \in \gamma\}$.

Now $X \subset A$ is an C_κ-embedding, so f_X extends to A. Moreover, f must factor through this extension, that is to say $A \subset f^{-1}(R_\kappa)$, because $X \subset A$ is dense. Therefore $X \cong A$ as desired.

In the case κ is a limit cardinal, it may be assumed that γ is a successor cardinal less than κ. Argue as before, using the fact that, since R_γ is a retract in R_κ, if $X \subset A$ is an C_κ-embedding, then it is an C_γ-embedding.

$2 \Longrightarrow 3$. As $ev_\kappa X : X \longrightarrow R_\kappa{}^{C_\kappa(X)}$ is C_κ-contractible, by Lemma 5.5 it is a C_κ-embedding, and hence $X \subset \overline{X}$ (denoting the closure of X in $R_\kappa{}^{C_\kappa(X)}$) is also a C_κ-embedding. Therefore $X \cong \overline{X}$.

$3 \Longrightarrow 1$. The usual topological arguments ensure that a closed sublocale of the κ-Lindelöf locale $R_\kappa{}^{C_\kappa(X)}$ is itself κ-Lindelöf.

Remark: It follows immediately that the reflection of an arbitrary locale A into $\mathbf{Loc}_\kappa$ is the closure of the image of $ev_\kappa : A \longrightarrow R_\kappa^{C_\kappa(A)}$.

Corollary 5.7 Where $\mathbf{P}_\kappa = pt(R_\kappa)$, a space X is κ-compact if and only if X embeds as a closed subspace into a power of $\mathbf{P}_\kappa$.

Proof: Use Theorem 4.3 and the fact that pt : **Locales** $\longrightarrow$ **Spaces** preserves products.

6 R_κ is a regular cogenerator for $\mathbf{Loc}_\kappa$

6.1 The diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\ \epsilon\ } E$$

in a category $\mathbf{X}$ is called a *contractible coequalizer diagram* provided $\epsilon f = \epsilon g$ and there exist two additional maps: $X \overset{\tau}{\longleftarrow} Y \overset{\sigma}{\longleftarrow} E$, satisfying:

$$\epsilon\sigma = 1_E, \quad g\tau = 1_Y, \quad \text{and } f\tau = \sigma\epsilon.$$

Remark: It is easily seen that such data in **X** suffice to ensure that $\epsilon = coeq(f, g)$.

6.2 We say an object P *is a regular cogenerator for a category* **A**, having powers of P sufficient for the existence of evaluation maps $\{A \longrightarrow P^{\mathbf{A}(A,P)} : A \in |\mathbf{A}|\}$, provided these evaluation maps are each regular monomorphisms.

And now, the main theorem:

Theorem 6.3 *R_κ is a regular cogenerator for* $\mathbf{Loc}_\kappa$.

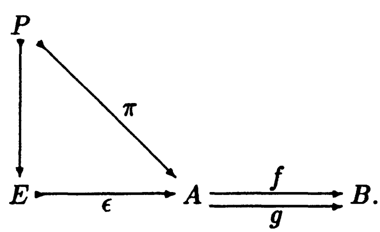
Proof: Where $\tilde{C}_\kappa$ denotes *the restriction to* $\mathbf{Loc}_\kappa{}^*$ of the functor C_κ defined previously, it suffices [2] to show that $\tilde{C}_\kappa : \mathbf{Loc}_\kappa{}^* \longrightarrow \mathbf{Sets}$ reflects coequalizers of $\tilde{C}_\kappa$-contractible pairs. Taking careful note of the contravariance of the functor C_κ, it must be demonstrated that every diagram $(*)$ in $\mathbf{Loc}_\kappa$,

$$(*) \qquad P \xrightarrow{\ \pi\ } A \underset{g}{\overset{f}{\rightrightarrows}} B$$

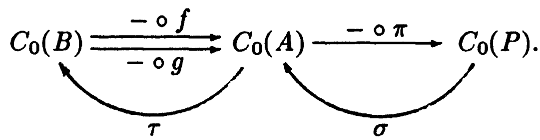
satisfying $f\pi = g\pi$, and for which $C_\kappa(*)$ is a contractible coequalizer diagram in **Sets**, is actually an equalizer diagram in $\mathbf{Loc}_\kappa$. Proceed via the following steps:

1. Observe that by hypothesis $(- \circ \pi) : C_\kappa(A) \longrightarrow C_\kappa(P)$ is a split epimorphism in **Sets**. Therefore $P \xrightarrow{\pi} A$ is C_κ-contractible, and hence (as we have argued before in Theorem 5.6) the inclusion of a *closed* sublocale.

2. $\{E \xrightarrow{\epsilon} A\} = eq(f, g)$ is also the inclusion of a closed sublocale, and by the universal property of equalizers, there is the following diagram in $\mathbf{Loc}_\kappa$:



3. As $C_\kappa(*)$ is a contractible coequalizer diagram in **Sets**, the diagram $C_0(*)$ is also. (A fairly easy diagram chase, the details of which are left to the enthusiastic reader, will verify this claim. Use the fact that **I** embeds into R_κ as a retract. Or better still, exhibit C_0 as a retract of C_κ in the appropriate functor category). Therefore, there exists the following data in **Sets**, satisfying the conditions of Definition 6.1 and exhibiting $C_0(*)$ as a contractible coequalizer diagram:



4. By assumption each $a : A \longrightarrow \mathbf{I}$ satisfies

$$\tau a \circ g = a \quad \text{and} \quad \tau a \circ f = \sigma(a \circ \pi).$$

It follows that a pair of **I**-valued maps a and a_0 agreeing on the sublocale P in fact agree on E, more precisely:

$$a_0 \circ \epsilon = a \circ \epsilon \iff a_0 \circ \pi = a \circ \pi.$$

5. Let a_0 denote the *constantly zero* locale map $A \xrightarrow{!} \{0\} \subset \mathbf{I}$. The thrust of point 4 above, then, is that a given *zero-set* sublocale $a^{-1}\{0\} \subset A$ contains E if and only if it contains P.

6. Since the locale P is closed in the completely regular locale A, it is an intersection of zero-set sublocales of A. It follows that $E \cong P$, the original diagram $(*)$ was in fact an equalizer diagram in $\mathbf{Loc}_\kappa$, and the proof is complete.

Remark: It follows immediately, from the classical theory of triples as found in [6], that $|\mathbf{Loc}_\kappa|$ is exactly that class of locales A, uniquely recoverable from the algebraic structure on the set $\mathbf{Locales}(A, R_\kappa)$.

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