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ON PRIESTLEY DUALS OF PRODUCTS

by V. KOUBEK and J. SICHLER

RÉSUMÉ. Cet article présente les espaces de Priestley représentant les produits et les produits fibrés de (0,1)-treillis distributifs et de double p-algèbres.

1. Introduction

The well-known Priestley duality [9], a contravariant equivalence of the category **P** of compact totally order disconnected spaces to the category **D** of distributive (0, 1)-lattices, has become an essential tool for structural and categorical investigations of varieties of algebras with reducts in **D**. Its applications produced a fairly extensive list of subcategories of **D** representing such varieties, and also led to catalogues of Priestley duals of numerous algebraic concepts, such as those by Priestley [11], or Davey and Duffus [5].

Since any variety **V** of algebras with reducts in **D** is closed under the formation of Cartesian products, a category $P(\mathbf{V})$ contravariantly equivalent to **V** is closed under coproducts, and a natural question of characterizing these coproducts arises.

A straightforward argument shows that the Priestley dual $P(K)$ of a product $K = \Pi\{K_i \mid i \in I\}$ contains a copy of the sum $Q = \Sigma\{P(K_i) \mid i \in I\}$ of Priestley duals $P(K_i)$ of its components as a dense ordered subspace. While always totally order disconnected, the ordered topological space Q need not be compact, in which case $P(K)$ must be a proper compactification of Q which is, up to an isomorphism, the 'maximal' compactification $M(Q)$ of Q.

Since the Priestley dual $M(Q)$ of a product K of Boolean algebras K_i is the unordered Stone-Čech compactification βQ of Q, one may be tempted to conjecture that βQ is also the underlying space of the dual $M(Q)$ of the product $K = \Pi\{K_i \mid i \in I\}$ of a set $\{K_i \mid i \in I\}$ of distributive (0, 1)-lattices. This, however, is false. In general, any transitive extension of the order of Q compatible with the topology of βQ is only a preorder on $\beta Q \setminus Q$, and hence needs to be factored out to obtain the Priestley dual $M(Q)$ of K.

An alternate approach is adopted here. We say that an object P of **P** is a Priestley compactification of an ordered topological space Q whenever Q is a dense order subspace of P, characterize these compactifications, and then investigate the special case in which Q is a sum of Priestley spaces. We show that Priestley compactifications of these sums represent weak direct products (in 3.5), characterize the Priestley dual $M(Q)$ of the full direct product, and also collections of lattices for which βQ is the underlying space of $M(Q)$. We also describe Priestley duals of ultraproducts. This and all other results are presented in Section 3.

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2. Priestley spaces

First we review the essentials of Priestley's duality for distributive $(0,1)$-lattices and distributive double p-algebras.

A triple $(X,\tau,\leq)$ is called an ordered topological space whenever (X,τ) is a topological space and $(X,\leq)$ is a poset. For any $Z\subseteq X$ define

$$(Z]=\{x\in X\mid \exists z\in Z\ x\leq z\} \text{ and } [Z)=\{x\in X\mid \exists z\in Z\ x\geq z\}.$$

A set $Z\subseteq X$ is decreasing if $(Z]=Z$, increasing if $[Z)=Z$, and clopen if it is both closed and open in (X,τ). A convex set is an intersection of a decreasing set and an increasing one.

An ordered topological space $(X,\tau,\leq)$ is totally order disconnected whenever $x\not\leq y$ in X implies the existence of a clopen decreasing set $Y\subseteq X$ such that $y\in Y$ and $x\notin Y$. A totally order disconnected ordered topological space is called a Priestley space if and only if it is compact.

To any distributive $(0,1)$-lattice, Priestley [9] assigns an ordered topological space $P(L)=(X,\tau,\leq)$ in which X is the set of all prime filters of L, $x\leq y$ if and only if $y\subseteq x\subset L$, and the topology τ has an open subbasis

$$\mathcal{S}=\{\{x\in X\mid A\in x\}\mid A\in L\}\cup\{\{x\in X\mid A\notin x\}\mid A\in L\}.$$

Every $S\in\mathcal{S}$ is clopen and, consequently, (X,τ) has an open basis formed by clopen convex sets. The space $P(L)$ is compact and totally order disconnected, [9]. For any $(0,1)$-homomorphism $f:L\longrightarrow L'$ of distributive $(0,1)$-lattices L, L', the inverse image mapping $f^{-1}:P(L')\longrightarrow P(L)$ is continuous and order preserving. Setting $P(f)=f^{-1}$ thus gives rise to a contravariant functor $P:\mathbf{D}\longrightarrow\mathbf{P}$ of the category $\mathbf{D}$ of all $(0,1)$-homomorphisms of distributive $(0,1)$-lattices into the category $\mathbf{P}$ of all continuous order preserving mappings of Priestley spaces.

Inclusion ordered clopen decreasing subsets of a Priestley space $P=(X,\tau,\leq)$ form a distributive $(0,1)$-lattice $D(P)$, and the inverse image map $g^{-1}:D(P)\longrightarrow D(P')$ of a $\mathbf{P}$-morphism $g:P'\longrightarrow P$ is a lattice $(0,1)$-homomorphism. Thus $D(g)=g^{-1}$ completes a definition of a contravariant functor $D:\mathbf{P}\longrightarrow\mathbf{D}$.

The functors P and D determine Priestley's duality as follows.

Theorem 2.1 (Priestley [9], [10]). *The composite functors $P\circ D:\mathbf{P}\longrightarrow\mathbf{P}$ and $D\circ P:\mathbf{D}\longrightarrow\mathbf{D}$ are naturally equivalent to the respective identity functors of their domains.*

A $\mathbf{D}$-morphism $f:L\longrightarrow L'$ is surjective if and only if $P(f)$ is a homeomorphism and order isomorphism of $P(L')$ onto a closed order subspace of $P(L)$, and f is one-to-one if and only if $P(f):P(L')\longrightarrow P(L)$ is surjective. ∎

For a Priestley space $(X,\tau,\leq)$, let $Max(X)$ and $Min(X)$ respectively denote the set of all elements which are maximal or minimal in $(X,\leq)$, and let $Ext(X)=Max(X)\cup Min(X)$ be the set of all extremal members of $(X,\leq)$. For any $Y\subseteq X$, set $Max(Y)=[Y)\cap Max(X)$, $Min(Y)=(Y]\cap Min(X)$ and $Ext(Y)=Max(Y)\cup$

$Min(Y)$. In any Priestley space, the sets $Max(x) = Max(\{x\})$ and $Min(x) = Min(\{x\})$ are nonvoid for every $x \in X$.

Recall that a distributive $(0,1)$-lattice L is a distributive double p-algebra provided that for every $x \in L$ it contains a largest element x^* such that $x \wedge x^* = 0$, and a smallest element x^+ satisfying $x \vee x^+ = 1$. Homomorphisms of these algebras are all **D**-morphisms preserving the two unary operations thus defined. Following is a well-known characterization of Priestley duals of distributive double p-algebras, [**11**].

THEOREM 2.2. *Let $f : L \longrightarrow L'$ be a **D**-morphism and let $g = P(f) : P(L') \longrightarrow P(L)$ be its Priestley dual. Then:*

(1) *L is a distributive double p-algebra if and only if $(Y]$ is clopen for every clopen increasing subset Y of $P(L)$ and $[Z)$ is clopen for every clopen decreasing set Z;*
(2) *a mapping f is a double p-algebra homomorphism if and only if $g(Max(x)) = Max(g(x))$ and $g(Min(x)) = Min(g(x))$ for every element x of $P(L')$.* ∎

A Priestley space satisfying 2.2(1) is called a dp-space, and a continuous order preserving mapping g for which 2.2(2) holds is a dp-map.

Elements a and b of a poset $(X, \leq)$ are connected whenever there exists a finite sequence $a = x_0, x_1, \dots, x_n = b$ such that x_{i-1} is comparable to x_i for each $i \in \{1, \dots, n\}$. Classes of the resulting equivalence are called order components of $(X, \leq)$. Since $Max(x) \neq \emptyset \neq Min(x)$ for every element x of a Priestley space $P = (X, \tau, \leq)$, a subset Y of X is a component of P if and only if $Ext(Y)$ is a component of the subposet $Ext(X)$ of X.

The proposition below summarizes some useful properties of Priestley spaces.

PROPOSITION 2.3. *Let $P = (X, \tau, \leq)$ be a Priestley space and let cT denote the τ-closure of $T \subseteq X$. Then:*

(1) *for any closed disjoint subsets Y and Z there exists a clopen $A \subseteq X$ such that $Z \subseteq A$ and $Y \subseteq X \setminus A$; if, in addition, $Y \cap (Z] = \emptyset$ then A may be chosen to be decreasing; consequently,*
(2) *the sets $(Y]$ and $[Y)$ are closed whenever $Y \subseteq X$ is closed; hence*
(3) *a union U of order components of P is closed if $Max(U)$ or $Min(U)$ is closed;*
(4) *$c(T] \subseteq (cT]$ and $c[T) \subseteq [cT)$ for any $T \subseteq X$;*
(5) *the Boolean algebra $C(P)$ of all clopen subsets of P is generated by the lattice $D(P)$ of all clopen decreasing subsets of P;*
(6) *if $D(P)$ is a double p-algebra, then $Max(X)$ and $Min(X)$ are closed sets.* ∎

According to 2.1, congruences of a distributive $(0,1)$-lattice L are in one-to-one correspondence to closed order subspaces of $P(L)$, see also [**11**]: clopen decreasing subsets A, B represent Θ-congruent elements of L exactly when $A \cap Z = B \cap Z$ for the closed subposet Z of $P(L)$ corresponding to the congruence Θ. A closed subset Z of a dp-space $P(L)$ corresponds to a congruence of a distributibe double p-algebra L if and only if $Ext(Z) \subseteq Z$, that is, when Z is a closed c-set [**4**] or [**8**].

Let $P = (X, \tau, \leq)$ be a Priestley space. By 2.3(5), every τ-clopen $A \subseteq X$ can be written in the form $A = \bigcup\{A_i \setminus B_i \mid i \in \{1, \dots, n\}\}$, where $A_i, B_i \subseteq X$ are clopen and decreasing for all $i \in \{1, \dots, n\}$; since $A_i \cap B_i$ is τ-clopen and decreasing, we may assume that $A_i \subseteq B_i$ for every $i = 1, \dots, n$. In other words, every $A \in C(P)$ is the union of finitely many clopen convex sets $C_i = A_i \setminus B_i$ with $A_i \subseteq B_i$. Let $Gen(A)$ denote the least number of clopen convex sets whose union is A. The ...al $Comp(P) = \sup\{Gen(A) \mid A \in C(P)\}$ will be called the complexity of the Priestley space P. We say that the Boolean algebra $C(P)$ of all clopen sets of P is uniformly generated whenever $Comp(P)$ is finite.

It seems clear that the complexity of P will depend on the length of chains contained in P. Let $A \in C(P)$. A chain $x_0 < x_1 < \dots < x_{2k}$ of P is a characteristic chain of A whenever

(1) $x_j \in A$ if and only if j is even, and

(2) the length of any chain of P satisfying (1) is at most $2k$.

Any $A \in C(P)$ possesses a characteristic chain, and all characteristic chains of A have equal length, say $2k$. Since no convex subset of A may contain two distinct elements of a characteristic chain, it follows that $Gen(A) \geq k + 1$.

For any $A \in C(P)$, let $T_i = T_i(A)$ consist of all $t \in X$ such that $t = x_i$ in some characteristic chain $x_0 < x_1 < \dots < x_{2k}$ of A. Clearly, the sets $T_0, \dots, T_{2k}$ are pairwise disjoint. We claim that every T_i is closed. To see this, note that $T_i \cap (T_j] \neq \emptyset$ if and only if $i \leq j$, in which case $T_i \subseteq (T_j]$ and $T_j \subseteq [T_i)$. But then 2.3(4) implies that $cT_i \subseteq c(T_j] \subseteq (cT_j]$ and $cT_j \subseteq [cT_i)$ and, because A is clopen, $cT_i \subseteq A$ for all even i, while $cT_i \subseteq X \setminus A$ when i is odd. Therefore, for every $t \in cT_i$ there is a characteristic chain $x_0 < x_1 < \dots < x_{2k}$ of A with $x_i = t$ and, consequently, each T_i is closed.

LEMMA 2.4. *If $P = (X, \tau, \leq)$ is a Priestley space and $A \in C(P)$ has a characteristic chain of length $2k$, then $Gen(A) = k + 1$.*

PROOF: We proceed by induction on k.

If $k = 0$, then $T_0(A) = A$, the clopen set A is convex, and $Gen(A) = 1$ follows trivially.

Let $k \geq 1$ and suppose that any clopen B whose characteristic chains have the length $2k - 2$ can be written in the form $B = \bigcup\{D_i \mid i \in \{0, \dots, k - 1\}\}$, where D_i is clopen convex and $T_{2i}(B) \subseteq D_i$ for all $i \in \{0, \dots, k - 1\}$. Since the sets $T_j(A)$, $[T_j(A))$, $(T_j(A)]$ are closed for $j \in \{0, \dots, 2k\}$ and because $T_j(A) \cap [T_{2k}(A)) = \emptyset$ for all $j < 2k$, by 2.3(1), we obtain a clopen increasing set I such that $\bigcup\{T_j(A) \mid j < 2k\} \subseteq X \setminus A$ and $T_{2k}(A) \subseteq I$. Characteristic chains of the clopen set $A \setminus I$ are of length $2k - 2$, and $T_{2i}(A) \subseteq T_{2i}(A \setminus I)$ for all $i \in \{0, \dots, k - 1\}$. By the induction hypothesis, there exist clopen convex sets $D_0, \dots, D_{k-1}$ such that $T_{2i}(A \setminus I) \subseteq D_i$ and $A \setminus I = \bigcup\{D_i \mid i \in \{0, \dots, k - 1\}\}$. Characteristic chains of the clopen set $A \setminus D_0$ are of length $2k - 2$, and the induction hypothesis provides clopen convex sets $C_0, \dots, C_{k-1}$ such that $T_{2i}(A \setminus D_0) \subseteq C_i$ and $A \setminus D_0 = \bigcup\{C_i \mid i \in \{0, \dots, k - 1\}\}$. The sets $E_0 = D_0$ and $E_{i+1} = C_i$ for $i \in \{0, \dots, k - 1\}$ are clopen and convex, and $A = \bigcup\{E_j \mid j \in \{0, \dots, k\}\}$. From $T_{2i+2}(A) \subseteq T_{2i}(A \setminus D_0)$ it follows that

$T_{2j}(A) \subseteq E_j$ for all $j \in \{0, \dots, k\}$. This shows that $Gen(A) \leq k + 1$. ∎

LEMMA 2.5. *Any chain of length $2k$ in a Priestley space P is a characteristic chain of some $A \in C(P)$.*

PROOF: Let $x_0 < x_1 < \dots < x_{2k}$ be a chain in P. Since P is totally order disconnected, for every $i \in \{0, \dots, 2k-1\}$ there exists a clopen decreasing set A_i such that $x_i \in A_i$ and $x_{i+1} \notin A_i$. Define $A_{-1} = \emptyset$ and $A_{2k} = P$. Then $C_i = A_{2i} \setminus A_{2i-1}$ is clopen and convex, and $x_0 < x_1 < \dots < x_{2k}$ is a characteristic chain of $A = \bigcup\{C_i \mid i \in \{0, \dots, k\}\}$. ∎

COROLLARY 2.6. *The Boolean algebra $C(P)$ of all clopen sets of a Priestley space $P = (X, \tau, \leq)$ is uniformly generated if and only if the poset $(X, \leq)$ has a finite height.* ∎

REMARK 2.7. Adams and Beazer [1] show that chains of a Priestley space $P = P(L)$ have at most n elements if and only if for any chain $a_0 \leq a_1 \leq \dots \leq a_{n-1}$ of elements of the distributive $(0,1)$-lattice L there exist $a'_0, a'_1, \dots, a'_{n-1} \in L$ such that

$$a_0 \wedge a'_0 = 0,\ a_i \vee a'_i = a_{i+1} \wedge a'_{i+1} \text{ for } 0 \leq i < n-1, \text{ and } a_{n-1} \vee a'_{n-1} = 1.$$

Hence $C(P)$ is uniformly generated if and only if the lattice $L = D(P)$ satisfies the Adams–Beazer condition for some finite n.

Let $Con(L)$ denote the congruence lattice of a distributive $(0,1)$-lattice L. A congruence Ψ of L is compact if $\Psi \leq \bigvee\{\Theta_i \mid i \in I\}$ holds in $Con(L)$ only when $\Psi \leq \bigvee\{\Theta_j \mid j \in J\}$ for some finite $J \subseteq I$. The lattice $Con(L)$ is distributive, complete, and each of its members is a join of compact congruences. The least congruence $\theta(a,b) \in Con(L)$ containing the pair $\{a,b\} \subseteq L$ of distinct elements of L is compact and, because compact elements form a join semilattice, any join $\bigvee\{\theta(a_j, b_j) \mid j = 1, \dots, n\}$ of finitely many principal congruences $\theta(a_j, b_j)$ is compact.

Let $K \subseteq P(L)$ be the closed set representing a compact congruence Ψ of L, and let $\{U_i \mid i \in I\}$ be an open covering of $P(L) \setminus K$. Then $P(L) \setminus U_i$ represents a $\Theta_i \in Con(L)$ for each $i \in I$ and $K \supseteq \bigcap\{P(L) \setminus U_i \mid i \in I\}$, that is, $\Psi \leq \bigvee\{\Theta_i \mid i \in I\}$. From the compactness of Ψ we obtain that $P(L) \setminus K \subseteq \bigcup\{U_j \mid j \in J\}$ for some finite $J \subseteq I$, so that $P(L) \setminus K$ is compact and hence closed. This shows that compact congruences of L are represented by clopen subsets of $P(L)$. Since the set $P(L) \setminus K$ is closed, it represents the complement Ψ' of Ψ in $Con(L)$. Hence all compact members of $Con(L)$ are complemented, see also Hashimoto [7].

Since $Con(L)$ is dually isomorphic to the inclusion ordered poset of all closed subsets of $P(L)$, any subset of $P(L)$ representing a complemented member of $Con(L)$ must be clopen.

For clopen decreasing sets $A \subseteq B \subseteq P(L)$, let $\Phi = \Phi(A, B) \in Con(L)$ be represented by the clopen convex set $B \setminus A$; thus $(U, V) \in \Phi$ if and only if $U \cap (B \setminus A) = V \cap (B \setminus A)$. Then $(\emptyset, A) \in \Phi$ and $(B, P(L)) \in \Phi$, so that $\theta(0, A) \vee \theta(B, 1) \leq \Phi$ in $Con(L)$. On the other hand, for any $(U, V) \in \Phi$ we obtain $(U \cup A) \cap B = (V \cup A) \cap B$, that is, $(U \cup A, V \cup A) \in \theta(B, 1)$; from $(U, U \cup A), (V, V \cup A) \in \theta(0, A)$ it then follows that $(U,V) \in \theta(0,A) \vee \theta(B,1)$. Therefore $\theta(0,A) \vee \theta(B,1) = \Phi$ and, consequently, $\theta(A,B) \vee \Phi = \theta(0,1)$ is the unit of $Con(L)$. Hence $\theta(A,B) \wedge \Phi = (\theta(A,B) \wedge \theta(0,A)) \vee (\theta(A,B) \vee \theta(B,1))$. If $(U,V) \in \theta(A,B) \wedge \theta(0,A)$ or $(U,V) \in \theta(A,B) \vee \theta(B,1)$, then $U = V$ (see Grätzer [6], p.89). This shows that $\Phi(A,B)$ is the complement $\theta(A,B)'$ of the principal congruence $\theta(A,B)$ for any $A \subsetneq B$.

Let $\Psi \in Con(L)$ be the congruence represented by a clopen set $K \subseteq P(L)$. Then the clopen set $P(L) \setminus K$ can be written as $P(L) \setminus K = \bigcup\{B_j \setminus A_j \mid j = 1, \dots, n\}$ with clopen decreasing $A_j \subseteq B_j \subseteq P(L)$. If $\Phi_j \in Con(L)$ denotes the congruence represented by the clopen convex set $B_j \setminus A_j$ for $j = 1, \dots n$, then $\Psi' = \bigwedge\{\Phi_j \mid j = 1, \dots, n\}$, so that $\Psi = \bigvee\{\Phi_j' \mid j = 1, \dots, n\} = \bigvee\{\theta(A_j, B_j) \mid j = 1, \dots, n\}$ is a join of finitely many principal congruences. This completes the proof of the claim below.

PROPOSITION 2.8. *Let $K \subseteq P(L)$ be a closed set representing $\Psi \in Con(L)$, that is, let Ψ consist of all $(U,V) \in L^2$ with $U \cap K = V \cap K$. Then the following are equivalent:*

1. *Ψ is a compact congruence;*
2. *Ψ has a complement in $Con(L)$;*
3. *Ψ is a join of finitely many principal congruences;*
4. *K is clopen.*

Moreover, a compact $\Psi \in Con(L)$ is a join of at most n principal congruences if and only if $n \geq Gen(P(L) \setminus K)$. ∎

3. PRIESTLEY COMPACTIFICATIONS AND PRIESTLEY DUALS OF PRODUCTS

DEFINITION 3.1. Let $P = (X, \tau, \leq)$ and $Q = (Y, \nu, \preceq)$ be ordered topological spaces. We say that P is a Priestley compactification of Q whenever P is a Priestley space containing Q as a dense ordered subspace, that is, whenever

1. (Y, ν) is a dense subspace of (X, τ), and
2. the partial orders $\preceq$ and $\leq$ coincide on Y.

Thus every Priestley space is its own Priestley compactification.

For any ordered topological space $Q = (Y, \nu, \preceq)$, let $C(Q)$ denote the Boolean algebra of all ν-clopen subsets of Y, and let $D(Q)$ be the $(0,1)$-sublattice of $C(Q)$ formed by all decreasing members of $C(Q)$. We say that a $(0,1)$-sublattice L of $C(Q)$ creates the order of Q provided

$$y_0 \preceq y_1 \text{ if and only if } y_1 \in A \text{ implies } y_0 \in A \text{ for all } A \in L.$$

Thus any sublattice $L \subseteq C(Q)$ creating the order of Q is, in fact, a sublattice of $D(Q)$, and the Boolean algebra $B(L) = [L]_{C(Q)}$ generated within $C(Q)$ by L is an open basis of a topology ν in which $Q = (Y, \nu, \preceq)$ is totally order disconnected.

In particular, if Q is a Priestley space, then $B(D(Q)) = C(Q)$ by 2.3(5) and, since Q is totally order disconnected, $D(Q)$ creates the order of Q.

LEMMA 3.2. *Let $P = (X, \tau, \leq)$ be a Priestley compactification of an ordered topological space $Q = (Y, \nu, \preceq)$, and let $\varphi : C(P) \longrightarrow C(Q)$ be the mapping defined by $\varphi(V) = Y \cap V$ for every $V \in C(P)$. Then φ is an embedding of the Boolean algebra $C(P)$ into $C(Q)$, and the $(0,1)$-sublattice $L = \varphi(D(P))$ of $D(Q)$ has the following properties:*

(1) *P is the Priestley space $P(L)$ of L,*
(2) *$L \subseteq D(Q)$ creates the order of Q, and*
(3) *members of $B(L) = [L]_{C(Q)}$ form an open basis of Q.*

PROOF: The mapping φ is one-to-one on $C(P)$ because Q is dense in P. It is also clear that φ preserves all Boolean operations. Since $\preceq$ coincides with the restriction of $\leq$ to Q, it follows that φ maps $D(P)$ isomorphically onto a $(0,1)$-sublattice L of $D(Q)$; thus, by 2.1, the Priestley space P is, in fact, the Priestley dual $P(L)$ of L.

If $y_0 \not\preceq y_1$ in Q, then $y_0 \not\leq y_1$ in P and, because the latter space is totally order disconnected, for some $A \in D(P)$ we have $y_1 \in A$ and $y_0 \in X \setminus A$. But then $\varphi(A) \in L \subseteq D(Q)$ is such that $y_1 \in \varphi(A)$ and $y_0 \in Y \setminus \varphi(A)$. Therefore L creates the order of Q.

To prove (3), assume that $G \subseteq Y$ is ν-closed and $y \in Y \setminus G$. Then there exists a $\tau-$closed $F \subseteq X$ such that $G = Y \cap F$. Since $\{y\}$ is τ-closed, 2.3(1) implies the existence of an $A \in C(P)$ with $F \subseteq A$ and $y \in X \setminus A$. Hence $B = \varphi(A)$ is ν-clopen, $G \subseteq B$ and $y \in Y \setminus B$. But $B \in B(L)$ because $D(P)$ generates $C(P)$ and $L = \varphi(D(P))$. Points and closed subsets of Q are thus separated by members of $B(L)$, so that $B(L)$ is an open basis of Q. ∎

Let $Q = (Y, \nu, \preceq)$ be an ordered topological space and let L be a $(0,1)$-sublattice of $C(Q)$ such that L creates the order of Q and $B(L)$ is an open basis of Q.

Next we aim to prove a converse of Lemma 3.2 by showing that the Priestley dual $P(L)$ of any such $L \subseteq C(Q)$ is a Priestley compactification of Q.

Let $e : L \longrightarrow B(L)$ denote the inclusion homomorphism of L into the Boolean algebra $B(L) \subseteq C(Q)$ generated by L, and let $F(B(L))$ be the set of all prime filters of $B(L)$. If σ is the topology on $F(B(L))$ whose open basis is formed by all sets

$$cl(B) = \{x \in F(B(L)) \mid B \in x\} \text{ with } B \in B(L),$$

then $(F(B(L)), \sigma) = P(B(L))$ is the Stone space of $B(L)$.

For every prime filter y of L there exists a unique prime filter $x \in F(B(L))$ such that $y = L \cap x$, so that the **P**-morphism $P(e) : P(B(L)) \longrightarrow P(L)$ dual to the inclusion $e : L \longrightarrow B(L)$ is a continuous bijection, and hence a homeomorphism, of $P(B(L))$ onto the (unordered) underlying compact Hausdorff space of $P(L)$. For any $x_0, x_1 \in F(B(L))$ set $x_0 \leq x_1$ exactly when $x_1 \cap L \subseteq x_0 \cap L$. Then $\leq$ is a partial order under which $(F(B(L)), \sigma, \leq)$ becomes an ordered space homeomorphic and also order isomorphic to the Priestley space $P(L)$ of L. We need only show that $Q = (Y, \nu, \preceq)$ is a dense ordered subspace of $(F(B(L)), \sigma, \leq)$.

For any $B \in B(L)$ we have $cl(F(B(L)) \setminus B) = \{x \in F(B(L)) \mid B \notin x\}$ because each $x \in F(B(L))$ is a prime filter of $B(L)$. It follows that $cl(B) \cup cl(Y \setminus B) = F(B(L))$ and $cl(B) \cap cl(Y \setminus B) = \emptyset$, so that $cl(B)$ is $\sigma-$clopen for every $B \in B(L)$.

Let $A \in L$ and $x_1 \in cl(A)$. If $x_0 \leq x_1$ then $A \in x_1 \subseteq x_0$, that is, $x_0 \in cl(A)$, so that $cl(A)$ is decreasing for every $A \in L$.

For every $y \in Y$ define $p(y) = \{B \in B(L) \mid y \in B\}$. Since $p(y)$ is a prime filter of $B(L)$, this defines a mapping $p : Y \longrightarrow F(B(L))$. Since L creates the order $\preceq$ of Q, $p(y_1) \cap L \subseteq p(y_0) \cap L$ if and only if $y_0 \preceq y_1$. This shows that p is one-to-one and that $p(y_0) \leq p(y_1)$ is equivalent to $y_0 \preceq y_1$ for all $y_0, y_1 \in Y$.

Since $p(y) \in cl(B)$ if and only if $y \in B$, it follows that $cl(B) \cap p(Y) = p(B)$ and hence also $p^{-1}(cl(B)) = B$ for all $B \in B(L)$. Thus p is continuous. In fact, since $B(L)$ is an open basis of ν, the mapping p is a homeomorphism of Q onto the ordered subspace $p(Y)$ of $(F(B(L)), \sigma, \leq)$.

Let $A \neq \emptyset$ be σ-open. Since $\{cl(B) \mid B \in B(L)\}$ is an open basis of $(F(B(L)), \sigma, \leq)$, there exists a $B \in B(L)$ for which $cl(B)$ is a nonvoid subset of A. But then $p(B) = p(Y) \cap cl(B) \subseteq p(Y) \cap A$ is nonvoid, by the definition of $cl(B)$. Thus $p(Y)$ is dense in $(F(B(L)), \sigma, \leq)$.

To conclude the proof, we identify Q with its homeomorphic and order isomorphic copy $p(Y)$ dense in $(F(B(L)), \sigma, \leq) \cong P(L)$.

Observe that if $L \subseteq D(Q)$ creates the order of Q and if $B(L)$ forms an open basis of Q, then these two properties are inherited by any $(0,1)$-sublattice K of $D(Q)$ containing L and, in particular, by the lattice $D(Q)$ itself. In conjunction with 3.2, these observations yield the result below.

THEOREM 3.3. *Let $Q = (Y, \nu, \preceq)$ be an ordered topological space. Then:*

(1) *a Priestley compactification of Q exists if and only if $D(Q)$ creates the order $\preceq$ of Q and $B(D(Q))$ is an open basis of ν;*

(2) *an ordered topological space P is a Priestley compactification of Q if and only if $P = P(L)$ for some $(0,1)$-sublattice L of $D(Q)$ such that L creates the order $\preceq$ of Q and $B(L)$ is an open basis of ν.* ▮

Thus, for example, the Stone–Čech compactification βQ is a Priestley compactification for any infinite discrete antichain Q; in fact, its Priestley compactifications are exactly its (unordered) compactifications. On the other hand, for the naturally ordered discrete set N of all positive integers, the only $(0,1)$-sublattice $L \subseteq D(N)$ creating the order of N is that consisting of $\emptyset$, N and all initial segments $\{1, 2, \ldots, n\} \subseteq N$. Hence the one-point compactification of N by a largest element is the only Priestley compactification of N.

Returning to general considerations, we now assume that $Q = (Y, \nu, \preceq)$ has a Priestley compactification $P(L)$ dual to a $(0,1)$-sublattice L of $D(Q)$. It follows that the Priestley space

$$M(Q) = P(D(Q)) = (F(D(Q)), \sigma, \leq)$$

is a Priestley compactification of Q as well. Next we show that $M(Q)$ is the 'largest' Priestley compactification of Q.

THEOREM 3.4. *Let $P(L)$ be a Priestley compactification of $Q = (Y, \nu, \preceq)$, and let $P(K)$ be the Priestley dual of a distributive $(0,1)$-lattice K. If $g : Q \longrightarrow P(K)$ is a continuous order preserving mapping, then:*

(1) *there is a unique* **P**-*morphism $P(f) = g' : M(Q) \longrightarrow P(K)$ extending g;*
(2) *g extends to a* **P**-*morphism $g'' : P(L) \longrightarrow P(K)$ if and only if $f(K) \subseteq L$, in which case $g' = g'' \circ P(e_L)$ with the inclusion $(0,1)$-homomorphism $e_L : L \longrightarrow D(Q)$.*

Moreover, $M(Q)$ is an ordered Stone-Čech compactification of Q if and only if $D(Q)$ generates $C(Q)$.

PROOF: If $g : Q \longrightarrow P(K)$ is a continuous order preserving mapping, then $g^{-1}(A) \in D(Q)$ for any clopen decreasing subset A of $P(K)$. From $D(P(K)) \cong K$ it follows that the restriction $f : K \longrightarrow D(Q)$ of g^{-1} to $D(P(K))$ is a lattice $(0,1)$-homomorphism. The **P**-morphism $P(f) : M(Q) \longrightarrow P(K)$ satisfies $P(f)^{-1}(A) \cap Y = g^{-1}(A)$ for all $A \in D(P(K))$; since $P(K)$ is totally order disconnected, this is possible only when the restriction of $P(f)$ to Y coincides with g. In other words, the **P**-morphism $g' = P(f)$ extends g. Since Q is dense in $M(Q)$, the extension g' of g is unique.

Now $g' = g'' \circ g_L$ for some continuous order preserving maps $g_L : M(Q) \longrightarrow P(L)$ and $g'' : P(L) \longrightarrow P(K)$ if and only if $f(K) \subseteq L$. If this is the case then, since Q is dense in $P(L)$, the **P**-morphism g_L is surjective and, in fact, $g_L = P(e_L)$ for the inclusion $(0,1)$-homomorphism $e_L : L \longrightarrow D(Q)$. This demonstrates (1) and (2).

The unordered reduct $Q_0 = (Y, \nu)$ of Q is a completely regular T_1-space. Its Stone-Čech compactification βQ_0 is thus one of its Priestley compactifications and, consequently, the identity mapping id_Y of Y extends to a continuous mapping $h : M(Q_0) \longrightarrow \beta Q_0$. Since $M(Q_0)$ compactifies Q_0, there is also a continuous $h' : \beta Q_0 \longrightarrow M(Q_0)$ extending id_Y and, because Y is dense in either space, h is a homeomorphism with the inverse h'. Thus $M(Q_0) = (F(C(Q)), \beta)$ with the Stone-Čech topology β.

The inclusion mapping of Q_0 into $M(Q)$ is continuous and order preserving, and Q_0 is dense in $M(Q)$. Hence there exists a unique continuous extension $k : M(Q_0) \longrightarrow M(Q)$ of id_Y. The mapping k is one-to-one if and only if $F(B(D(Q))) = F(C(Q))$; since k is surjective, it is a homeomorphism if and only if $C(Q)$ is generated by $D(Q)$. Therefore the β-compactification of Q_0 can be partially ordered to become a Priestley compactification of Q if and only if the lattice $D(Q)$ generates $C(Q)$. ∎

Next we turn our attention to Priestley duals of products.

Let $Q_i = (Y_i, \nu_i, \preceq_i)$ be nonvoid Priestley spaces for all $i \in I \neq \emptyset$, and denote $Q = \Sigma\{Q_i \mid i \in I\} = (Y, \nu, \preceq)$ their sum; that is, the partial order $\preceq$ is the union of all $\preceq_i$ and the collection of all ν_i-open sets forms an open basis of ν. Since $A \in D(Q)$ if and only if $A \cap Y_i \in D(Q_i)$ for all $i \in I$ and because each Q_i is a Priestley space, the lattice $D(Q)$ creates the order of Q and the Boolean algebra $B(D(Q))$ forms an open basis of Q. Thus the Priestley compactification $M(Q)$ of Q exists, by 3.3(1). The sum $g : Q \longrightarrow P(K)$ of any collection of **P**-morphisms $g_i : Q_i \longrightarrow P(K)$ with

$i \in I$ is a continuous order preserving mapping and hence, by 3.4(1), it extends uniquely to a **P**-morphism $g' : M(Q) \longrightarrow P(Q)$. This, of course, means that $M(Q)$ is dual to the product $\Pi\{D(Q_i) \mid i \in I\}$ of the nontrivial distributive $(0,1)$-lattices $D(Q_i)$. In particular, Q_i is a closed order subspace of $M(Q) = P(\Pi\{D(Q_i) \mid i \in I\})$ for every $i \in I$.

We show that Priestley compactifications of the sum $Q = \Sigma\{Q_i \mid i \in I\}$ are Priestley spaces of certain subdirect products of lattices $D(Q_i)$. For example, if I is infinite then a one-point compactification $Q \cup \{u\}$ of Q such that $u > y$ for all $y \in Q$ is the Priestley space of the lower weak direct product L of all $D(Q_i)$ with $i \in I$ – that is, the $(0,1)$-sublattice $L \subseteq K = \Pi\{D(Q_i) \mid i \in I\}$ consisting of the unit $1 \in K$ and of all $\kappa \in K$ for which $\{i \in I \mid \kappa(i) > 0\}$ is finite. On the other hand, if I is finite then the only Priestley compactification of the sum Q is the Priestley dual $Q = \Sigma\{Q_i \mid i \in I\}$ of the product $\Pi\{D(Q_i) \mid i \in I\}$.

Let $K_i = P(Q_i)$ be a distributive $(0,1)$-lattice with more than one element for each $i \in I \neq \emptyset$, and let L be a $(0,1)$-sublattice of the product $K = \Pi\{K_i \mid i \in I\}$. For any $J \subseteq I$, let $\pi_J \in Con(L)$ consist of all pairs $(\lambda, \lambda') \in L^2$ such that $\lambda(j) = \lambda'(j)$ for all $j \in J$. We say that L is a <u>weak direct product</u> of the set $\{K_i \mid i \in I\}$ if and only if, for every finite subset J of I, the congruence π_J is complemented and $L/\pi_J \cong \Pi\{K_j \mid j \in J\}$.

PROPOSITION 3.5. *A Priestley space $P(L)$ is a Priestley compactification of the sum $Q = \Sigma\{Q_i \mid i \in I\}$ if and only if L is a weak direct product of $\{K_i \mid i \in I\}$.*

PROOF: If $P(L)$ is a Priestley compactification of $Q = \Sigma\{Q_i \mid i \in I\}$, then there is a surjective **P**-morphism $h : M(Q) \longrightarrow P(L)$ extending the identity mapping of Q onto itself, by 3.4; therefore $Q_i = h(Q_i)$ is compact and hence closed in $P(L)$ for every $i \in I$. But then $Q_i \cup c(Q \setminus Q_i) = c(Q) = P(L)$ because $Q \subseteq P(L)$ is dense. Furthermore, since Q is a subspace of $P(L)$ and because Q_i is open in Q, it follows that $Q_i \cap c(Q \setminus Q_i) = \emptyset$ in $P(L)$. Thus $Q_i \subseteq P(L)$ is clopen for every $i \in I$, and so is $Q_J = \bigcup\{Q_j \mid j \in J\}$ for every finite $J \subseteq I$. By 2.8, the congruence π_J represented by Q_J has a complement, while $L/\pi_J \cong \Pi\{K_j \mid j \in J\}$ follows from the fact that Q_J is a closed order subspace of $P(L)$.

Conversely, let L be a $(0,1)$-sublattice of K such that $L/\pi_J \cong \Pi\{K_j \mid j \in J\}$ and $\pi_J \in Con(L)$ is complemented for every finite $J \subseteq I$. In particular, $L/\pi_{\{i\}} \cong K_i = D(Q_i)$ for each $i \in I$. By 2.1, there exists an order isomorphism and homeomorphism $g_i : Q_i \longrightarrow P(L)$ for each $i \in I$ and, consequently, a continuous order preserving joint extension $g : Q \longrightarrow P(L)$ of all $g_i : Q_i \longrightarrow P(L)$. For distinct $i, j \in I$, the hypothesis gives $L/\pi_{\{i,j\}} \cong K_i \times K_j$ which implies that g maps the sum $Q_i + Q_j$ onto its order copy in $P(L)$. Therefore g is an order isomorphism of Q into $P(L)$. Since $\pi_{\{i\}} \in Con(L)$ has a complement, the set $g(Q_i) \subseteq P(L)$ is clopen by 2.8, and hence the copy $g(Q)$ of Q is a subspace of $P(L)$. By 3.4, there exists a **P**-morphism $h : M(Q) \longrightarrow P(L)$ extending g; the mapping h is surjective because L is a $(0,1)$-sublattice of $D(M(Q)) = \Pi\{K_i \mid i \in I\}$, see 2.1. But then the copy $g(Q) \subseteq P(L)$ of Q is dense in $P(L)$ because Q is dense in $M(Q)$. Altogether, $g(Q)$ is a dense subspace of $P(L)$ that is order isomorphic to Q. ∎

REMARK 3.6. If L is a weak direct product of $\{K_i \mid i \in I\}$ then, for any finite subset J of I, the complement π'_J of $\pi_J \in Con(L)$ is the congruence $\pi_{I\setminus J}$. To see this, select a $j \in I$ and note that $\pi_j \vee \pi_i = 1 \in Con(L)$ for any $i \in I \setminus \{j\}$ because $Q_i \cap Q_j = \emptyset$ in $P(L)$. From the distributivity of $Con(L)$ it then follows that $\pi_i \geq \pi'_j$ for each $i \neq j$, and hence also $\bigwedge\{\pi_i \mid i \neq j\} \geq \pi'_j$. On the other hand, $\bigwedge\{\pi_i \mid i \neq j\} \wedge \pi_j = 0 \in Con(L)$ because L is a sublattice of $\Pi\{K_i \mid i \in I\}$, so that $\bigwedge\{\pi_i \mid i \neq j\} \leq \pi'_j$ because $Con(L)$ is distributive.

PROPOSITION 3.7. *Let $Q_i = P(K_i)$ be the Priestley space of a nontrivial distributive $(0,1)$-lattice K_i for each $i \in I \neq \emptyset$, and let $Q = \Sigma\{Q_i \mid i \in I\}$. Then the Priestley dual $M(Q)$ of the product $K = \Pi\{K_i \mid i \in I\}$ is an ordered β-compactification of Q if and only if the chain lengths of all but finitely many component spaces Q_i are uniformly bounded by a finite cardinal n.*

PROOF: Let $J \subseteq I$ be finite and let all chains of every Q_i with $i \in I \setminus J$ have length at most n. To prove that βQ is the underlying space of $M(Q)$, in view of the last claim in 3.4 we need only show that the Boolean algebra $C(Q)$ of all clopen sets is generated by $D(Q)$.

Let $C \in C(Q)$ be arbitrary. Since Q_i is a Priestley space and because $C \cap Q_i \in C(Q_i)$, for every $i \in I$ there exists an integer n_i such that $C \cap Q_i = \bigcup\{A_{k,i} \setminus B_{k,i} \mid k = 1, \dots, n_i\}$ with $A_{k,i}, B_{k,i} \in D(Q_i)$, by 2.3(5). From 2.6 it follows that there exist sets $A_{k,i}, B_{k,i} \in D(Q_i)$ such that $n_i \leq m_0$ for all $i \in I \setminus J$ and a finite m_0. If $m_1 = \max\{n_j \mid j \in J\}$ and $m \geq \max\{m_0, m_1\}$, we may write $C \cap Q_i = \bigcup\{A_{k,i} \setminus B_{k,i} \mid k = 1, \dots, m\}$. Set $A_k = \bigcup\{A_{k,i} \mid i \in I\}$ and $B_k = \bigcup\{B_{k,i} \mid i \in I\}$. Then A_k, B_k are clopen decreasing for $k = 1, \dots, m$ and $C = \bigcup\{A_k \setminus B_k \mid k = 1, \dots, m\}$. Therefore $D(Q)$ generates $C(Q)$, as claimed.

Conversely, if the order condition fails, then there exists a one-to-one countably infinite sequence $i(1), i(2), \dots$ such that $Q_{i(n)}$ contains a chain of length $2n$ for $n = 1, 2, \dots$. By 2.5 and 2.4, there exist $C_{i(n)} \in C(Q_{i(n)})$ such that $C_{i(n)} = \bigcup\{A_k \setminus B_k \mid k = 1, \dots, m\}$ for some $A_k, B_k \in D(Q_{i(n)})$ only when $m \geq n+1$. Since $Q = \Sigma\{Q_i \mid i \in I\}$, the set $C = \bigcup\{C_{i(n)} \mid n = 1, 2, \dots\}$ is clopen in Q, yet lies outside the Boolean algebra generated by $D(Q)$. ∎

REMARK 3.8. Adams and Beazer [2] show that the congruences of a distributive $(0,1)$-lattice L are $(n+1)$-permutable if and only if all chains of $P(L)$ have at most n elements. Hence 3.7 can be reformulated as follows: the Priestley dual of a product $\Pi\{K_i \mid i \in I\}$ is an ordered β-compactification of $\Sigma\{P(K_i) \mid i \in I\}$ if and only if there exists some finite n such that all but finitely many lattices K_i have $(n+1)$-permutable congruences.

REMARK 3.9. Since any product of distributive double p-algebras is a distributive double p-algebra, the Priestley compactification $M(Q)$ of the sum $Q = \Sigma\{P(K_i) \mid i \in I\}$ of dp-spaces is the dual of the double p-algebra $K = \Pi\{K_i \mid i \in I\}$, and the inclusion $Q_i \longrightarrow M(Q)$ is a dp-map for every $i \in I$. Therefore 3.7 remains valid in the category of all dp-maps between dp-spaces. According to Beazer [3], a distributive double p-algebra L has n-permutable congruences if and only if any chain in its dp-space $P(L)$ has at most $n+1$ elements, and the claim below follows immediately.

COROLLARY 3.10. *Let $Q_i = P(K_i)$ be the Priestley space of a nontrivial distributive double p-algebra K_i for each $i \in I \neq \emptyset$, and let $Q = \Sigma\{Q_i \mid i \in I\}$. Then the Priestley dual $M(Q)$ of the product $K = \Pi\{K_i \mid i \in I\}$ is an ordered β-compactification of Q if and only if the chain lengths of all but finitely many component spaces Q_i are uniformly bounded by a finite cardinal n. This is the case exactly when all but finitely many algebras K_i have n-permutable congruences.* ∎

To describe Priestley duals of ultraproducts, let $Q_i = P(K_i)$ be the Priestley dual of a nontrivial distributive $(0,1)$-lattice or a double p-algebra K_i for $i \in I \neq \emptyset$, and let $M(Q)$, where $Q = \Sigma\{Q_i \mid i \in I\}$, be the Priestley dual of $K = \Pi\{K_i \mid i \in I\}$.

Since $Q_i \subseteq Q \subseteq M(Q)$ is clopen in $M(Q)$ for every $i \in I$, the mapping $e : Q \longrightarrow \beta I$ into the unordered Stone-Čech compactification βI of the discrete space I defined by $e(q) = i$ for all $q \in Q_i$ is continuous and order preserving, and satisfies $e(Ext(p)) = Ext(e(p))$ for all $p \in M(Q)$. Since I is dense in βI, from 3.4 we obtain the existence of a continuous surjective extension $h : M(Q) \longrightarrow \beta I$ of e; the **D**-morphism $\psi = D(h)$ embeds the Boolean algebra 2^I canonically into K. Of course, $h(Ext(p)) = Ext(h(p))$ follows from the fact that βI is unordered.

For any ultrafilter u on I, let $\phi_u : K \longrightarrow K/\theta_u$ denote the canonical surjective homomorphism from K to the ultraproduct K/θ_u. Thus $\phi_u(\kappa) = \phi_u(\kappa')$ if and only if $E(\kappa,\kappa') = \{i \in I \mid \kappa(i) = \kappa'(i)\} \in u$. Let $\phi_u \circ \psi = \mu_u \circ \epsilon_u$ be a decomposition such that μ_u is one-to-one and ϵ_u is surjective. For any $\lambda \in 2^I$ exactly one of the sets $\lambda^{-1}\{0\}$, $\lambda^{-1}\{1\}$ belongs to u, so that ϵ_u maps 2^I onto the two-element Boolean algebra $2 = \{0,1\}$, and $\epsilon_u(\lambda) = 1$ if and only if $\lambda^{-1}\{1\} \in u$. Furthermore, these four morphisms form a pushout. To see this, let $\phi : K \longrightarrow L$ and $\mu : 2 \longrightarrow L$ satisfy $\phi \circ \psi = \mu \circ \epsilon_u$ and let $\kappa, \kappa' \in K$ be such that $E(\kappa,\kappa') \in u$. If $\lambda \in 2^I$ is given by $\lambda^{-1}\{1\} = E(\kappa,\kappa')$, then $\phi(\psi(\lambda)) = \mu(\epsilon_u(\lambda)) = \mu(1) = 1$ and $\kappa \wedge \psi(\lambda) = \kappa' \wedge \psi(\lambda)$ in K, so that $\phi(\kappa) = \phi(\kappa) \wedge \phi(\psi(\lambda)) = \phi(\kappa \wedge \psi(\lambda)) = \phi(\kappa' \wedge \psi(\lambda)) = \phi(\kappa')$. This shows that θ_u is contained in the kernel of ϕ. Hence $\phi = \phi' \circ \phi_u$ for some **D**-morphism ϕ'. But then $\phi' \circ \mu_u = \mu$ follows from the fact that ϵ_u is surjective, and the four **D**-morphisms in $\phi_u \circ \psi = \mu_u \circ \epsilon_u$ do, indeed, constitute a pushout. Therefore the diagram formed by their Priestley duals is a pullback in which $P(\epsilon_u) : \{1\} \longrightarrow \beta I$ is given by $P(\epsilon_u)(1) = u$. Thus the closed order subspace $h^{-1}\{u\}$ of $M(Q)$ is the Priestley dual $P(K/\theta_u)$ of the ultraproduct K/θ_u.

This concludes the proof of the claim below.

PROPOSITION 3.11. *Let $\{K_i \mid i \in I\}$ be a nonvoid set of nontrivial distributive $(0,1)$-lattices or double p-algebras, and let $h : P(K) \longrightarrow \beta I$ be the Priestley dual of the canonical embedding $e : 2^I \longrightarrow K$ of the Boolean algebra 2^I into the product $K = \Pi\{K_i \mid i \in I\}$. Then, for any ultrafilter u on I, the closed order subspace $h^{-1}\{u\} \subseteq P(K)$ is the Priestley dual $P(K/\theta_u)$ of the ultraproduct K/θ_u.* ∎

It is clear that Proposition 3.11 applies also to all varieties of distributive (0,1)-lattices with operators – such as varieties of p-algebras and of (double) Heyting algebras.

REMARK 3.12. It is easily verified that **D**-morphisms (or double p-algebra homomorphisms) $\varphi_i : L \longrightarrow K_i$ determine a subdirect product L of lattices (or double p-algebras) K_i with $i \in I$ if and and only if each $P(\varphi_i)$ is a homeomorphism and an order isomorphism (and a dp-map) onto a closed order subspace (or a closed c-set) of $P(L)$, and the union of images of all $P(\varphi_i)$ is dense in $P(L)$.

EXAMPLES AND OBSERVATIONS. While there are many minimal weak direct products of distributive lattices, there is only one minimal weak direct product in the category of distributive double p-algebras. We use Priestley compactifications to illustrate these points.

For instance, if Q is the sum of infinitely many two–element chains $Q_i = \{0_i, 1_i\}$ with $i \in I$, then its one–point compactification $R = Q \cup \{w\}$ in which $[w) \cap Q_i = \{1_i\}$ for each $i \in I$ is the dual of a minimal weak direct product of three–element chains $D(Q_i)$, by 3.3 and 3.5. Yet any singleton $\{1_i\}$ is a clopen increasing set for which $(1_i] = \{1_i, 0_i, w\}$ is not open; hence, according to 2.2(1), the Priestley space R is not the dual of any distributive double p-algebra.

For an example of another kind, consider the two–point extension $S = Q \cup \{z, u\}$ of the sum Q as above, in which $z \leq s \leq u$ for all $s \in S$, while $\{z\}$ compactifies $Min(Q)$ and $\{u\}$ compactifies $Max(Q)$. Then S is the dual of a double Stone algebra, and also a minimal Priestley compactification of Q. No insertion of Q_i into S is a dp-map, however; as a result, R is not the dual of a weak direct product of three–element double Stone algebras $D(Q_i)$.

These two examples indicate that dp-spaces of weak direct products of distributive double p-algebras must satisfy additional requirements.

Assume that $Q = \Sigma\{Q_i \mid i \in I\}$ is the sum of arbitrary nontrivial dp-spaces Q_i and that L is a weak direct product of algebras $K_i = D(Q_i)$ in the category of distributive double p-algebras. Then, as in 3.5 and 3.6, Q is dense in $P(L)$ and the order subspaces Q_i and $P(L) \setminus Q_i = c(Q \setminus Q_i)$ form a clopen decomposition of $P(L)$ for each $i \in I$. Since these sets also represent distributive double p-algebra congruences, it follows that Q_i and $P(L) \setminus Q_i$ are clopen c-sets.

If $p \in P(L)$ satisfies $p \leq q$ for some $q \in Q_i$ then there exists an $m \in Max(Q_i) \subseteq Max(P(L))$ such that $p \leq m$. But then $Min(p) \subseteq Min(m)$. Since m belongs to the c-set Q_i, we have $Min(p) \subseteq Min(Q_i)$ and then, because $P(L) \setminus Q_i$ is a c-set, we conclude that $p \in Q_i$. Together with a dual argument, this shows that Q is a union of order components of $P(L)$, and explains the findings of the two preceding examples.

Observe that the set $P(L) \setminus Q = \bigcap\{P(L) \setminus Q_i \mid i \in I\}$ is a closed union of order components of $P(L)$.

Let $Q \cup \{v\}$ be the one–point compactification of Q in which v is incomparable to any member of Q. It is easily seen that $Q \cup \{v\} = P(L_0)$ is the Priestley dual of a distributive double p-algebra $L_0 \subseteq \Pi\{K_i \mid i \in I\}$ which consists of all λ satisfying $\lambda(i) = 0$ for all but finitely many $i \in I$ or $\lambda(i) = 1$ for all but finitely many $i \in I$.

Since the mapping $h : P(L) \longrightarrow P(L_0)$ defined by $h(q) = q$ for all $q \in Q$ and $h(p) = v$ for all $p \in P(L) \setminus Q$ is, clearly, a dp-map, this shows that L_0 is the unique minimal weak direct product in the class of all distributive double p-algebras.

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