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MONADS AND COCOMPLETENESS OF CATEGORIES

by Manuela SOBRAL *

RÉSUMÉ: Soit $\mathcal{C}$ une catégorie complète et bien-potentiée. La théorie des monades est utilisée pour démontrer que, pour chaque $\mathcal{C}$-objet A, l'envel-oppe réflective $\mathcal{A}$ de (A) dans $\mathcal{C}$ est une catégorie cocomplète, lorsque A vérifie une condition d'injectivé. Ensuite, nous démontrons que ce résultat est une conséquénce d'un théoreme qui établit une rélation entre $\mathcal{A}$ et des monades induites dans $\mathcal{C}$ et $Set^{\mathbb{T}}$ par l'adjonction associée au fonteur comparaison $\Phi : \mathcal{C}^{op} \to Set^{\mathbb{T}}$, $\mathbb{T}$ est la monade induite dans Set par l'adjonction $<A^{-}, Hom(-,A)> : Set \to \mathcal{C}^{op}$.

1. INTRODUCTION.

Let $\mathcal{C}$ be a complete and well-powered category. Then, the reflective hull REF(A) of {A} in $\mathcal{C}$ is its limit-closure and A is a strong cogenerator in REF(A), for each single $\mathcal{C}$-object A (see [9] 6.1 where credit is given to Ringel [7]).

It is well-known that $\mathcal{C}$ has (epi, strong mono) - factorizations and so that extremal monomorphisms are strong. If, furthermore, $\mathcal{C}$ is co-well-powered then it has (epi, strong mono) - factorization of sources and this implies the existence of certain colimits in $\mathcal{C}$ ([5] 1.1) and, consequently, in its reflective full subcategories.

Throughout this paper $\mathcal{C}$ will denote a complete and well-powered category. We are going to prove that if a $\mathcal{C}$-object A is D-injective, D being a class of morphisms we shall define shortly, then REF(A) is a cocomplete category. For that we

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show that REF(A) is dually equivalent to a monadic category over a complete category.

Our general reference for monad theory is [1].

From now on, we consider a $\mathcal{E}$-object A and denote by $\mathbb{T}$, $\Phi : \mathcal{E}^{op} \longrightarrow \mathrm{Set}^{\mathbb{T}}$, and $\mathcal{A}$, the monad induced in Set by the adjunction $A^{-} \dashv \mathrm{Hom}(-,A) : \mathcal{E}^{op} \longrightarrow \mathrm{Set}$, the comparison functor and REF(A), respectively. We will often take $F = A^{-}$ and $H = \mathrm{Hom}(-,A)$.

2. THE $\mathcal{A}$-CLOSURE OPERATOR.

Let $\mathcal{M}$ be the full subcategory of $\mathcal{E}^2$ (where $2 = \{0 \longrightarrow 1\}$) with objects the external monomorphisms. The $\mathcal{A}$-closure operator ([2], [3]), originally defined in [8] for $\mathcal{E} = \mathrm{Top}$, is the functor

$$[\quad]_{\mathcal{A}} : \mathcal{M} \longrightarrow \mathcal{M}$$

which assigns to each extremal monomorphism m the generalized pullback of

$$\{e = \mathrm{eq}(f,g) \mid f.m = g.m \text{ and } \mathrm{cod} f = \mathrm{cod} g \in \mathcal{A}\} .$$

We consider the extension of $[\quad]_{\mathcal{A}}$ to $\mathcal{E}^2$, also denoted by $[\quad]_{\mathcal{A}}$, or simply $[\quad]$, assigning to $f = m.e$ in $\mathcal{E}$ the $\mathcal{A}$-closure $[m]_{\mathcal{A}}$ of m, $m.e.$ being the (epi, extremal mono)-factorization of f .

Let $\varepsilon_X : X \longrightarrow A^{\mathrm{Hom}(x,A)}$ be the unique $\mathcal{E}$-morphism such that $p_f.\varepsilon_X = f$ for all $f \in \mathrm{Hom}(X,A)$, (p_f) being the canonical projections. Then ε^{op} is the counit of the adjunction $F = A^{-} \dashv H = \mathrm{Hom}(-,A) : \mathcal{E}^{op} \longrightarrow \mathrm{Set}$.

The following is essentially Lemma 1 in [6], whose proof we include for convenience of the reader.

2.1 Lemma. *For each $\mathcal{E}$-object* X , $[\varepsilon_X]_{\mathcal{A}} = e_X$ *where* $e_X = \mathrm{eq}(FH\varepsilon_X, \varepsilon_{FHX})$.

Proof. The functors $\Big[\quad\Big]_{\{A\}}$ and $\Big[\quad\Big]_{\mathcal{A}}$ coincide ([4] 2.1) and so it is enough to prove that $[\varepsilon_X]_{\{A\}} = e_X$ for every $X \in \mathcal{E}$. We have that $p_h.\varepsilon_{FHX} = h$ and $p_h . FH\varepsilon_X = p_h.\varepsilon_X$ for all $\mathcal{C}$-morphisms $h : A^{\mathrm{Hom}(X,A)} \to A$. For $\mathcal{C}$-morphisms $f,g : A^{\mathrm{Hom}(X,A)} \to A$, if $f.\varepsilon_X = g.\varepsilon_X$ then $f.e_X = g.e_X$. Indeed

$$f.e_X = p_f.\varepsilon_{FHX}.e_X = p_f.FH\varepsilon_X.e_X = p_{f.\varepsilon_X}.e_X$$
$$= p_{g.\varepsilon_X}.e_X = p_g.FH\varepsilon_X.e_X = p_g.\varepsilon_{FHX}.e_X = g.e_X \ .$$

Since the converse is trivial, then $f.\varepsilon_X = g.\varepsilon_X$ if and only if $f.e_X = g.e_X$.

Furthermore, $\varepsilon_{FHX}.\varepsilon_X = FH\varepsilon_X.\varepsilon_X$. Hence

$$p_h.\varepsilon_{FHX}.[\varepsilon_X]_{\{A\}} = p_h.FH\varepsilon_X.[\varepsilon_X]_{\{A\}}$$

for all $h \in \mathrm{Hom}(A^{\mathrm{Hom}(X,A)},A)$, and so

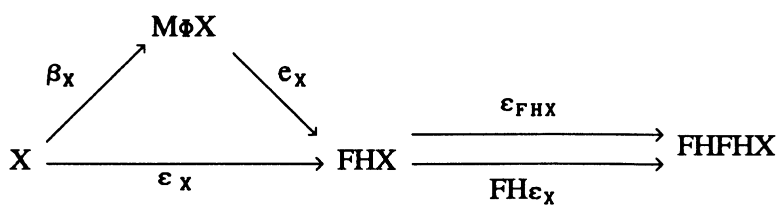
$$\varepsilon_{FHX}.[\varepsilon_X]_{\{A\}} = FH\varepsilon_X.[\varepsilon_X]_{\{A\}} \ .$$

It is now straightforward to prove that $[\varepsilon_X]_{\{A\}} = e_X$. □

Let $D = \left\{[\varepsilon_X]_{\mathcal{A}} \mid X \in \mathcal{C}\right\}$, α be the unit and β^{op} the co-unit of the adjunction $M \dashv \Phi : \mathcal{C}^{op} \to \mathrm{Set}^{\mathbb{T}}$.

2.2 Lemma. If A *is D-injective then* β_X *is an A-epi for every* $\mathcal{C}$*-object* X .

Proof. We recall that $M(Y,\theta)$ is the equalizer object of the $\mathcal{A}$-morphisms $F\theta, \varepsilon_{FY} : A^Y \to A^{\mathrm{Hom}(AY,A)}$ and so it is an $\mathcal{A}$-object, for every $\mathbb{T}$-algebra (Y,θ) . For $X \in \mathcal{C}$ we have the diagram



where the triangle commutes. Let f and g be two parallel $\mathcal{A}$-morphisms such that $f.\beta_X = g.\beta_X$. Without loss of generality, since A is a cogenerator in $\mathcal{A}$, we assume that the codomain of f and g is A . The injectivity condition of

A with respect of $[\varepsilon_X] = e_X$ implies that there exist morphisms f' and g' such that $f'.[\varepsilon_X] = f$ and $g'[\varepsilon_X] = g$. Hence $f'.\varepsilon_X = g'.\varepsilon_X$. By definition of $\mathcal{A}$-closure of ε_X $f'[\varepsilon_X] = g'[\varepsilon_X]$ and so $f = g$. □

3. COCOMPLETENESS OF A.

Let $\mathbb{S} = (S,\alpha,\delta)$ be a monad in $\mathcal{X}$ and $\mathrm{Fix}(S,\alpha)$ be the full subcategory with objects all $\mathcal{X}$-objects X for which α_X is an isomorphism.

The equivalence of the following assertions is known.

3.1 Proposition. *For a monad* $\mathbb{S} = (S,\alpha,\delta)$ *in* $\mathcal{X}$ *the following are equivalent:*

(i) δ *is an isomorphism.*

(ii) $\alpha_S = S_\alpha$.

(iii) α_S *is an isomorphism.*

(iv) $\mathcal{X}^{\mathbb{S}}$ *is concretely isomorphic to* $\mathrm{Fix}(S,\alpha)$.

A monad $\mathbb{S}$ is called **idempotent** if it satisfies one, and so all, of the conditions (i) - (iv) in 3.1.

3.2 Theorem. *If* A *is injective with respect to* D *then* $\mathcal{A}$ *is a cocomplete category.*

Proof: We consider the monad $\mathbb{S} = (\Phi M,\alpha,\Phi\beta^{op}M)$ induced in $\mathrm{Set}^{\mathbb{T}}$ by the adjunction $M \dashv \Phi : \mathcal{C}^{op} \longrightarrow \mathrm{Set}^{\mathbb{T}}$. The restriction Φ_1 of Φ to $\mathcal{A}^{op}$ is part of an adjunction which induces the same monad in $\mathrm{Set}^{\mathbb{T}}$ as the former adjunction does. Indeed, it is enough to remark that $M(Y,\Theta) \in \mathcal{A}$ for all $\mathbb{T}$-algebra (Y,Θ) (see proof of 2.2).

If $X \in \mathcal{A}$ there exists an $\mathcal{A}$-extremal monomorphism from X to some power of A and so ε_X is an extremal monomorphism in $\mathcal{A}$. Since $\varepsilon_X = [\varepsilon_X] . \beta_X$ and β_X is an epimorphism in $\mathcal{A}$ (2.2), then it is an isomorphism. Hence $\Phi\beta^{op}M$ is an isomorphism and so $\mathbb{S}$ is an idempotent monad. It is now clear that $\Phi_1 : \mathcal{A}^{op} \longrightarrow \mathrm{Fix}(\Phi M,\alpha) \cong (\mathrm{Set}^{\mathbb{T}})^{\mathbb{S}}$ is an equivalence. We have therefore concluded that $\mathcal{A}$ is dually equivalent to a monadic category $(\mathrm{Set}^{\mathbb{T}})^{\mathbb{S}}$ over a complete category $\mathrm{Set}^{\mathbb{T}}$ and so that $\mathcal{A}$ is cocomplete. □

4. MONADS INDUCED BY THE COMPARISON ADJUNCTION

The adjunction associated with the comparison functor $\Phi : \mathcal{C}^{op} \longrightarrow \mathrm{Set}^{\mathbb{T}}$ induces a monad $\mathbb{S} = (\Phi M,\alpha,\Phi\beta^{op}M)$ and a comonad in $\mathcal{C}^{op}$, i.e. a monad $S' = (M\Phi,\beta,(M\alpha\Phi)^{op})$ in $\mathcal{C}$. The monad S was the main tool for proving the cocompleteness of REF(A) = $\mathcal{A}$, whenever A is injective with respect to D. This is a consequence of the fact that the injectivity condition is equivalent to some close relations between $\mathcal{A}$ S and S'.

4.1 Theorem. *The following assertions are equivalent.*

(i) A *is D-injective.*

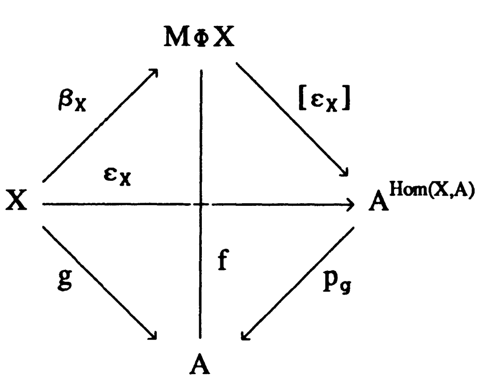
(ii) *The comparison functor* Φ *induces an equivalence* $\Phi_1 : \mathcal{A}^{op} \to (Set^{\mathbb{T}})^S$.

(iii) A *and* $\mathcal{C}^{S'}$ *are concretely isomorphic over* $\mathcal{C}$.

Proof: (i) ⇒ (ii) See proof of 3.2.

(ii) ⇒ (iii) Since Φ_1 is an equivalence, β_X is an isomorphism for every $X \in \mathcal{A}$. Then β_{M_Φ} is an isomorphism. This tells us that S' is an idempotent monad (3.1 (iii)). The functor $L : \mathcal{A} \to \mathcal{C}^{S'} \cong Fix(M\Phi,\beta)$ defined on objects by $L(X) = (X,\beta_X^{-1})$ is a concrete isomorphism, i.e. $U^{S'}.L = E$ where $E : \mathcal{A} \to \mathcal{C}$ is the embedding and $U^{S'} : \mathcal{C}^{S'} \to \mathcal{C}$ is the forgetful functor from the Eilenberg-Moore category of algebras $\mathcal{C}^{S'}$ to $\mathcal{C}$.

(iii) ⇒ (i) Let $L : \mathcal{A} \to \mathcal{C}^{S'}$ be an isomorphism such that $U^{S'}.L = E$. Then β_X is the reflection of X in $\mathcal{A}$. Indeed, $M\Phi X$ is an A-object and $\beta_X : X \to M\Phi X \cong U^{S'}F^{S'}X$ where $F^{S'}$ is the left adjoint to $U^{S'}$, is universal from X to E. Let $f : M\Phi X \to A$ be a $\mathcal{C}$-morphism. Then, by definition of ε_X, $p_g.\varepsilon_X = g$ for $g = f.\varepsilon_X$.



Since β_X is the reflection of X in $\mathcal{A}$, hence $p_g.[\varepsilon_X].\beta_X = f.\beta_X$ implies that $p_g.[\varepsilon_X] = f$ and so that A is D-injective. □

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