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A 2-categorical approach to change of base and geometric morphisms I

A 2-CATEGORICAL APPROACH TO CHANGE OF BASE
AND GEOMETRIC MORPHISMS I

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RÉSUMÉ.   Pour étudier le comportement, par rapport aux morphismes géométriques, du passage d'une catégorie régulière $\mathcal{E}$ à la 2-catégorie Rel $\mathcal{E}$ des relations dans $\mathcal{E}$, on introduit une 2-catégorie **F** dont les Rel $\mathcal{E}$ sont des objets et qui contient, pour tout foncteur F : $\mathcal{E} \rightarrow \mathcal{F}$, un morphisme Rel F : Rel $\mathcal{E} \rightarrow$ Rel $\mathcal{F}$. En fait, Rel induit une bijection entre les adjonctions G $\dashv$ F : $\mathcal{E} \rightarrow \mathcal{F}$ dans **Cat** et les adjonctions S $\dashv$ T : Rel $\mathcal{E} \rightarrow$ Rel $\mathcal{F}$ dans **F** ; et parmi ces dernières, on distingue par des propriétés simples celles qui proviennent d'un morphisme géométrique G $\dashv$ F . De plus, Rel F étant défini pour tout foncteur F , on peut étudier au niveau de **F** les morphismes géométriques *essentiels*, ou *ouverts*, ou *locaux*.

INTRODUCTION.

The general question of *change of base* is a large one. On the one hand, in enriched category theory, one wants to consider the effect on the 2-category $\mathcal{V}$-**Cat** , on the bicategory $\mathcal{V}$-**Mod** (also called $\mathcal{V}$-**Prof**), and on the various notions of limit, colimit, Kan extension, and so on, of a monoidal functor $\Phi : \mathcal{V} \rightarrow \mathcal{V}'$ in the sense of [7], or more particularly of an adjoint pair $\Psi \dashv \Phi$ in the 2-category of monoidal categories and monoidal functors; recall from [14] that the left adjoint Ψ here necessarily preserves tensor products to within isomorphism. There is particular interest, of course, in the cases where the monoidal categories $\mathcal{V}$ and $\mathcal{V}'$ are closed or biclosed, and where they have a sym-

metry or, more generally, a *braiding* in the sense of Joyal and Street [11]; while it has become clear since the work of Walters in [21] and [22], and various later developments of this, that the questions above are not seen in the proper light until the monoidal categories are replaced by suitable bicategories. Some progress has been made on these matters by Gray [9], by Jay [10] and by Betti and Power [2]; but a satisfying general account is still lacking.

In topos theory, on the other hand, the appropriate notion of a change of base is that of a geometric morphism $G \dashv F : \mathcal{E} \to \mathcal{F}$; and one wants to consider the effect of this on various fundamental constructions such as the passage from $\mathcal{E}$ to the bicategories Span $\mathcal{E}$ and Prof $\mathcal{E}$, as well as others we shall mention in a moment. It is clear, however, from the work of Walters mentioned above and that, for instance, of Rosebrugh and Wood in [19], that this latter concept of change of base is related to the former by no mere analogy but by precise mathematical connexions; one may legitimately hope for a general account that is not only satisfying but also unifying.

In the present article and its planned sequel we make a small initial step in this direction, restricting ourselves to the geometric-morphism context, and even there availing ourselves of the great simplification provided by studying only such constructions as lead to **Ord**-categories rather than to general bicategories. By an **Ord**-category we mean a category enriched over the cartesian closed category of ordered sets; it is the same things as a 2-category, or a bi⸺category, whose hom-categories are but ordered sets. At this level we do have what seems to be (so far as we have gone) a satisfactory general approach; and while we hope that this will guide the way to the study of the more general cases, we note that many constructions leading to **Ord**-categories are of great importance in themselves.

For instance, the value of considering, for a topos $\mathcal{E}$, the **Ord**-category Sl $\mathcal{E}$ of sup-lattices in $\mathcal{E}$ has been made clear by Joyal and Tierney in [12]. Intimately connected with Sl $\mathcal{E}$ are two other **Ord**-categories, namely Idl $\mathcal{E}$ given by the preorders in $\mathcal{E}$ and the ideals between these, and Rel $\mathcal{E}$ given by the objects of $\mathcal{E}$ and the relations between them; the significance of Rel $\mathcal{E}$ in the study of limits for Grothendieck toposes is evident from Pitts [18] (which draws on Carboni and Walters [6], to which we refer again below). Note that, while the definition of Sl $\mathcal{E}$ uses the full force of the topos axioms, when dealing with Idl $\mathcal{E}$ and Rel $\mathcal{E}$ we need of $\mathcal{E}$ only that it is a regular category in

the sense of Barr.

The kind of problem we wish to address then is - at least to a first approximation - that of the *functoriality* of the constructions Rel, Idl and Sl with respect to geometric morphisms. This first article really deals only with Rel, although we adumbrate the case of Idl to motivate some of our definitions. In an as yet partially-prepared sequel, we shall extend the study of Rel in connection with a "comprehension scheme", include that of Idl, and perhaps turn to Sl as well.

As to the functoriality of Rel, it is clear that a geometric morphism $G \dashv F : \mathcal{E} \to \mathcal{F}$ between regular categories induces in a natural way graph-morphisms, monotone with respect to the order on hom-sets, Rel F : Rel $\mathcal{E} \to$ Rel$\mathcal{F}$ and Rel G : Rel $\mathcal{F} \to$ Rel $\mathcal{E}$; only the latter of these, however, is a 2-functor, the former being only a *lax functor* (or what Bénabou [1] calls a *morphism of bicategories*). We think that the reader will share - or at least sympathize with - our conviction that it should be possible to exhibit Rel G and Rel F as adjoints; but no-one, so far as we know, has hitherto done so. In speaking of "adjoints" here, we are referring to the usual notion of adjoint morphisms in a bicategory - or preferably, for greater simplicity, in a 2-category - a notion replete with good and useful properties; not to the various notions of "local adjunction" given in [10] and [2], nor to any of the other weakenings of the adjunction concept given by Gray in [8]. One of our main tasks, then, is to find a 2-category **F** in which Rel G and Rel F live and are adjoint, and to demonstrate the "correctness" of our choice of **F** both by giving an elementary characterization of those adjunctions $S \dashv T :$ Rel $\mathcal{E} \to$ Rel $\mathcal{F}$ in **F** which arise from a (unique) geometric morphism $G \dashv F : \mathcal{E} \to \mathcal{F}$, and by showing that we can express in **F** such extra properties of the geometric morphism as being *local*, or *essential,* or *open.* (It was the facility for expressing these *extra* properties that we had in mind above when we used the words "to a first approximation".)

There is a seeming obstruction: it is notorious that 2-categories - even **Ord**-categories - and lax functors, with any reasonable definition of 2-cell, do not form a 2-category (or for that matter a bicategory): see [18, Section 1.3, *Warning*]. Recall, however, from [3, Proposition 5] (which generalises results of Lawvere [17] and Walters [21], [22]), that for a regular category $\mathcal{E}$, the *maps* in Rel $\mathcal{E}$ - that is, those 1-cells f which have a right adjoint f* - are just the graphs of the arrows of $\mathcal{E}$. Thus, if $F : \mathcal{E} \to \mathcal{F}$ is a

left-exact functor, the induced lax functor Rel F : Rel $\mathcal{E}$ → Rel $\mathcal{F}$ has a special property: it preserves adjoint pairs of 1-cells. This was our first idea: lax functors with this special property compose, and with a suitable definition of 2-cell (*cf.* [18, Definition 2.4 (ii)]) they form a 2-category.

The matter becomes more subtle, however, when we replace Rel by Idl; for Idl F is now a lax functor that preserves, not *all* adjoint 1-cells, but only those that arise from *functors*; this subcategory of the maps of Idl $\mathcal{E}$, related to the notions of Cauchy completion and the axiom of choice in $\mathcal{E}$, was studied by Carboni and Street in [5]. Our second idea, then, was to take as the objects of our 2-category pairs $(\mathcal{A},\mathcal{A}_*)$ consisting of an **Ord**-category $\mathcal{A}$ and a subcategory $\mathcal{A}_*$ of its maps (containing all the identities), and to take as the morphisms those lax functors which preserve the adjunctions for the maps in $\mathcal{A}_*$; again, with a suitable definition of 2-cell, this does give a 2-category, which we are now calling $F_\leq$, the $\leq$ here indicating "lax functor". Note that we are dealing in effect with "proarrow equipment" in the sense of Wood ([23] and [24]) and of Rosebrugh and Wood in [20].

We soon came to see, however, that this $F_\leq$ is still too narrow. When $\mathcal{E}$ and $\mathcal{F}$ are regular categories but the functor $F : \mathcal{E} \to \mathcal{F}$ is not left exact, we do not get a lax functor Rel F ; yet, for the proper development of the theory of geometric morphisms, we do want in our final 2-category F a morphism Rel F for *every* F , left exact or not. For instance, we want to be able to deal with an *essential* geometric morphism $G \longrightarrow F$, where G has a further left adjoint H , which will not in general be left exact. Moreover, our treatment of the "comprehension scheme", to appear in the sequel, again demands a Rel F where F is not left exact.

After a first chapter devoted to preliminaries - mostly well-known results on relations in regular categories for which we do not give full historical references, although Proposition 1.8 seems to be new - we define in Chapter 2 a 2-category F that is "Flabby" enough for the purposes above. It has sub-2-categories $F_\leq$, $F_\geq$, and $F_=$ obtained by restricting to those morphisms that are lax functors, co-lax functors, and 2-functors, respectively. For instance, when a left-exact F also preserves strong epimorphisms,

Rel F lies not only in $F_\le$ but in $F_=$. In Chapter 3 we study adjunctions in F , which turn out to have excellent properties, and give necessary and sufficient conditions for a morphism in F to admit a right adjoint. We define the Rel F : Rel $\mathcal{E}$ → Rel $\mathcal{F}$ in Chapter 4, exhibiting Rel as a colax functor from the 2-category **Reg** of regular categories and *all* functors to the 2-category F ; we determine those T : Rel $\mathcal{E}$ → Rel $\mathcal{F}$ in F of the form Rel F , as well as those F for which Rel F lies in $F_\le$, in $F_\ge$, or in $F_=$; and we give some evidence for the "correctness" of F by showing that Rel induces a bijection between adjunctions G ⊣ F : $\mathcal{E}$ → $\mathcal{F}$ in **Reg** and adjunctions S ⊣ T : Rel $\mathcal{E}$ → Rel $\mathcal{F}$ in F .

In any 2-category with products we can define a *cartesian object* $\mathcal{A}$ as one for which the diagonal $\mathcal{A}$ → $\mathcal{A} \times \mathcal{A}$ and the unique $\mathcal{A}$ → 1 admit right adjoints $\otimes$ and I , and a *cartesian morphism* between cartesian objects as one which preserves $\otimes$ and I . In Chapter 5, after pointing this out, we examine the cartesian objects of F (which turn out in fact to be cartesian in $F_\le$) along with their central properties; and discover that the cartesian objects $\mathcal{A}$ in $F_=$ (at least those where $\mathcal{A}_*$ consists of all the maps of $\mathcal{A}$) are nothing but the *cartesian bicategories* introduced *ad hoc* by Carboni and Walters in [6], which were basic to their characterizations of Rel $\mathcal{E}$ for a regular $\mathcal{E}$ and of Idl $\mathcal{E}$ for an exact $\mathcal{E}$. Since the cartesian objects of F share *all* the important properties of cartesian bicategories except the 2-functoriality of $\otimes$, this gives further evidence of the "correctness" of F . We continue on in Chapter 5 to study the cartesianness of a general morphism in F between cartesian objects, and of Rel F in particular, showing that Rel F is cartesian if and only if F is so - the latter meaning of course that F preserves finite products in the classical sense. We end the chapter and this article by characterizing in Theorem 5.9 the morphisms T : Rel $\mathcal{E}$ → Rel $\mathcal{F}$ in F of the form Rel F for a left-exact F as the cartesian morphisms in $F_\le$ satisfying one simple extra condition, and by showing in Theorem 5.10 that, in the bijection which ends the last paragraph above, G ⊣ F is a geometric morphism if and only if S (and then, automatically, T as well) is a cartesian morphism in $F_\le$.

This, we think, clinches the argument that F provides a natural setting for the study of geometric morphisms of toposes. It is indeed the case that the notions of local,

essential or open geometric morphisms find their formulations in **F** , as do various constructions such as glueing for a left-exact functor. One may accordingly be tempted to see **F** as the "algebra" of geometric morphisms; but now we must return to the beginning of this Introduction, pointing out that *not all* of the notions linked to geometric morphisms $G \longrightarrow F$ are expressible in **F** - for example, that of *locally-connected* morphism, in which G is to preserve the functors Π . To obtain a context general enough to encompass these further notions, one must study the functoriality of geometric morphisms with respect to such other constructions as Span $\mathcal{E}$ and Prof $\mathcal{E}$. This, in its turn, requires a characterization of the bicategories Span $\mathcal{E}$ and Prof $\mathcal{E}$, at least for locally-cartesian-closed $\mathcal{E}$. Such further investigations, as well as the question of change of base in enriched category theory, will, we hope, be the subject of later investigations: the main challenge here will be the generalization of **F** to an analogue whose objects are no longer **Ord**-categories but general bicategories.

1. PRELIMINARIES

1.1 A *preorder* on a set is a reflexive and transitive relation $\leq$; is an *order* when it is anti-symmetric. Ordered sets and monotone functions form a cartesian closed category **Ord** ; accordingly, by the general theory of enriched categories, we have the 2-category **Ord-Cat** of **Ord**-categories, **Ord**-functors, and **Ord**-natural transformations. An **Ord**-category $\mathcal{A}$ is the same thing as a 2-category (or, for that matter, a bicategory) in which each hom-category $\mathcal{A}(A,B)$ is but an ordered set. An **Ord**-functor $T : \mathcal{A} \to \mathcal{B}$ is the same thing as a 2-functor (which is what we usually call it); it is nothing but an ordinary functor $T : \mathcal{A}_0 \to \mathcal{B}_0$, between the underlying ordinary categories of $\mathcal{A}$ and $\mathcal{B}$, which is *locally monotone*, in the sense that $T\phi \leq T\psi :$ $TA \to TB$ whenever $\phi \leq \psi : A \to B$. An **Ord**-natural transformation $\alpha : T \to S : \mathcal{A} \to \mathcal{B}$ is simply a natural transformation $\alpha : T \to S : \mathcal{A}_0 \to \mathcal{B}_0$, the extra condition of *enriched* naturality being vacuous here. Because the closed category **Ord** is itself an **Ord**-category and hence a 2-category, the 2-category **Ord-Cat** extends to a 3-category; the 3-cells $\alpha \to \beta : T \to S : \mathcal{A} \to \mathcal{B}$, also called *modifications* from α to β , are just inequalities $\alpha \leq \beta$, this last meaning that $\alpha_A \leq \beta_A : TA \to SA$ for each A .

1.2 An arrow $f : A \to B$ in an **Ord**-category $\mathcal{A}$ is called a *map* if it has a right adjoint f^* ; that is, an arrow

$$f\star : B \to A \quad \text{with} \quad 1_A \le f\star f \quad \text{and} \quad ff\star \le B .$$

Note that $f\star$, when it exists, is unique. Clearly composites of maps are maps, as are all identities and even all isomorphisms; so that the objects of $\mathcal{A}$, the maps between them, and the inequalities between these, form a locally-full sub-**Ord**-category Map $\mathcal{A}$ of $\mathcal{A}$. Let us adopt the convention whereby Greek letters stand for general arrows of the **Ord**-category $\mathcal{A}$ and Roman letters for maps, then for $f : A \to B$, $\psi : B \to D$, $\phi : A \to C$, and $g : C \to D$ we have (as a very special case of [16, Prop. 2.1])

$$g\phi \le \psi f \quad \textit{if and only if} \quad \phi f\star \le g\star\psi . \tag{1.1}$$

The type of inequality on the left of (1.1) occurs repeatedly below; we call it an *adjoint inequality*.

1.3 We call a category $\mathcal{E}$ *regular* if (a) it is finitely complete (some authors demand only pullbacks), (b) for each $f : A \to B$ there is a smallest subobject $i : C \to B$ of B through which f factorizes as $f = ie$; then the monomorphism i , or more properly the subobject it represents, is called the *image* im f of f , while e is a strong epimorphism in the sense of [13]; and (c) every pullback of a strong epimorphism is a strong epimorphism. We call ie the *factorization* of f . Recall from [13] that strong epimorphisms are closed under composition. In a regular category every strong epimorphism is (see [5, Prop. 1]) the coequalizer of its kernel-pair, so that regular and strong epimorphisms are the same things.

1.4 By a *relation* $\phi : A \to B$ in a regular category $\mathcal{E}$ we mean a subobject of $A \times B$. If

$$<\phi_1,\phi_2> : \Phi \to A \times B$$

is one of the monomorphisms representing this subobject, we call the jointly-monic pair

$$(\phi_1 : \Phi \to A , \phi_2 : \Phi \to B)$$

a *tabulation* of the relation ϕ ; the tabulation is unique to within an isomorphism of Φ . As is well known, we get an associative composition of relations $\phi : A \to B$ and $\psi : B \to C$ by forming in $\mathcal{E}$ the diagram

.

$$\text{(1.2)}$$
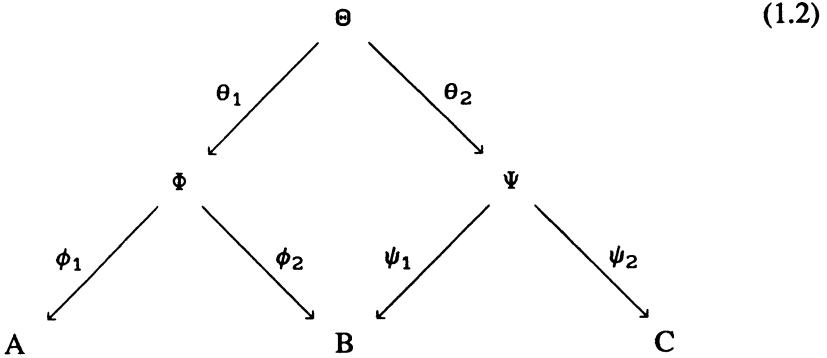
where the diamond is a pullback, and taking for $\psi\phi$ the relation given by the image of $<\phi_1\theta_1, \psi_2\theta_2> : \Theta \to A \times C$; the identity relation $A \to A$ is given by the diagonal subobject $\Delta : A \to A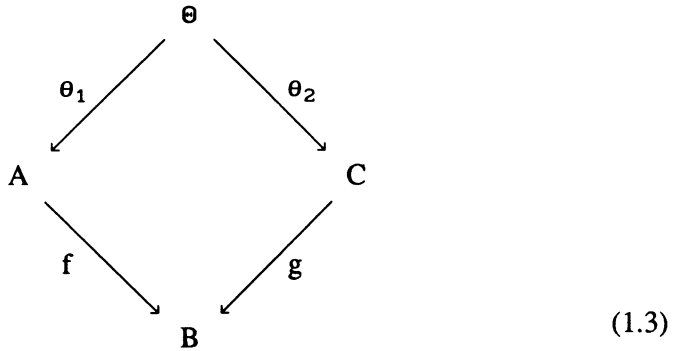 \times A$. With this composition, and with the order on relations from A to B given by inclusion of subobjects of $A \times B$, the objects of $\mathcal{E}$ and the relations between them form an **Ord**-category Rel $\mathcal{E}$. This has *local finite infima,* $\phi \wedge \chi$ for relations from A to B being their intersection as subobjects of $A \times B$ and the top element $\top_{AB}$ being the identity subobject $A \times B \to A \times B$ tabulated by the projections $p : A \times B \to A$ and $q : A \times B \to B$. There is an evident involution $(\ )^\circ : \text{Rel } \mathcal{E} \cong (\text{Rel } \mathcal{E})^{op}$ sending $\phi : A \to B$ as above to $\phi^\circ : B \to A$ given by $<\phi_2, \phi_1> : \Phi \to B \times A$. Treating the category $\mathcal{E}$ as a locally-discrete **Ord**-category, we have a 2-functor $\mathcal{E} \to \text{Rel } \mathcal{E}$ which is the identity on objects and sends $f : A \to B$ in $\mathcal{E}$ to the relation $A \to B$ given by the graph $<1_A, f> : A \to A \times B$ of f. Since $<1_A, f> \leq <1_A, g>$ only when $f = g$, we may as well identify $<1_A, f>$ with f, regarding $\mathcal{E} \to \text{Rel } \mathcal{E}$ as a locally-full inclusion. Such a relation $f : A \to B$ in $\mathcal{E}$ we call a *function*; of course $\phi : A \to B$ is a function precisely when ϕ_1 is invertible. We use Roman letters only for such relations as are functions.

1.5 If we form (1.2) when ϕ is a function f, so that $(\phi_1, \phi_2) = (1_A, f)$, the pair $(\theta_1, \psi_2\theta_2)$ is jointly monic, and is therefore itself the tabulation of ψf, there being no need to pass to an image. When, moreover, ψ is g° for a function $g : C \to B$, the composite $g^\circ f$ is just the relation θ tabulated by θ_1 and θ_2 in the pullback

$$
\begin{array}{ccc}
 & \Theta & \\
\theta_1 \swarrow & & \searrow \theta_2 \\
A & & C \\
f \searrow & & \swarrow g \\
 & B &
\end{array}
\qquad (1.3)
$$

in $\mathcal{E}$. In particular, when $C = A$ and $g = f$, the rela-
tion $f^o f$ is the equivalence relation tabulated by the kernel-
pair (θ_1, θ_2) of f in $\mathcal{E}$; it follows that $1_A \le f^o f$, with
equality precisely when f is a monomorphism.

1.6 When, on the other hand, we form (1.2) where ϕ is arb-
itrary but ψ is a function $k : B \to C$, the pullback is
trivial since θ_1 like ψ_1 is 1 and $\theta_2 = \phi_2$; yet the
pair $(\phi_1, k\phi_2)$ is not in general jointly monic, and we are
obliged to take its image. This is so even when $\phi = h^o$ for
a function $h : B \to A$; then the outer legs of (1.2) are h
and k , so that the composite relation kh^o is the sub-
object given by the *image* of $\langle h, k \rangle : B \to A \times C$. Various
consequences follow. First, *if* $\chi : A \to C$ *is a relation
given by the subobject* $\langle \chi_1, \chi_2 \rangle : X \to A \times C$, *we have some*
$t : B \to X$ *with* $\chi_1 t = h$ *and* $\chi_2 t = k$ *if and only if*
$kh^o \le \chi$. Next, since $\langle \chi_1, \chi_2 \rangle$ is a monomorphism, *for any
relation* χ *we have* $\chi = \chi_2 \chi_1^o$. Finally, combining this with
§1.5, we see that *a diagram* (1.3) *commutes and is a pullback
if and only if* θ_1 *and* θ_2 *are jointly monic with*
$\theta_2 \theta_1^o = g^o f$.

1.7 Taking $C = A$ and $k = h$ in §1.6 we observe that, A be-
ing embedded in $A \times A$ by the diagonal $\Delta : A \to A \times A$, the
relation $hh^o : A \to A$ as a subobject of $A \times A$ is im $h \subset A$
$\subset A \times A$, with tabulation (im $h \to A$, im $h \to A$) ; thus we
have $hh^o \le 1_A$, with equality precisely when h is a strong
epimorphism. For any $f : A \to B$ in $\mathcal{E}$, therefore, since
$1_A \le f^o f$ by §1.5, we have $f \dashv f^o$ in Rel $\mathcal{E}$. Thus every
function is a map in Rel $\mathcal{E}$, with $f^\star = f^o$. In fact the
only maps in Rel $\mathcal{E}$ are the functions; see [3, Prop. 5].
Accordingly, when speaking of Rel $\mathcal{E}$, we may use *function* and
map interchangeably - in the language of §1.2 we have

Map Rel $\mathcal{E}$ = $\mathcal{E}$.   This reconciles our conventions of §1.2 (reserving Roman letters for maps) and of §1.4 (reserving Roman letters for functions); *we abbreviate by using these conventions throughout*, beginning with the following on the nature in Rel $\mathcal{E}$ of the *adjoint inequalities* of §1.2.

1.8   Proposition. *For*   f : A → B , ψ : B → D , ϕ : A → C , *and*   g : C → D   *in* Rel $\mathcal{E}$, *we have the adjoint inequality* $g\phi \le \psi f$ *if and only if we have some (necessarily unique)* t : Φ → Ψ *rendering commutative*

$$
\begin{array}{ccccc}
A & \xleftarrow{\quad\phi_2\quad} & \Phi & \xrightarrow{\quad\phi_2\quad} & C \\
{\scriptstyle f}\downarrow & & {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle g} \\
B & \xleftarrow[\psi_1]{\quad\quad} & \Psi & \xrightarrow[\psi_2]{\quad\quad} & D \quad .
\end{array}
\qquad (1.4)
$$

Proof By §1.6, there is such a t if and only if $g\phi_2\phi_1^{o}f^{o} \le \psi$; since $\phi_2\phi_1^{o} = \phi$ by §1.6 again, and since   f $\dashv$ fo , we have the result by (1.1).

1.9 Consider now those relations ϕ : A → A   for which $\phi \le 1_A$.   Such relations are clearly those for which $\phi_1 = \phi_2 = i$, say; whereupon the condition that ϕ_1 and ϕ_2 be jointly monic becomes just the condition that   i   be monic.   Such relations, therefore, are in bijection with subobjects   i : Φ → A   of   A ,   the tabulation of ϕ being (i : Φ → A , i : Φ → A) .   We have $1_A = i^{o}i$ by §1.5, and $\phi = ii^{o}$ by   §1.6; so that ϕ is an idempotent in (the underlying category of) Rel $\mathcal{E}$,   split by   i   and i^{o} .   Such a ϕ may accordingly be called a *comonad* in Rel $\mathcal{E}$; necessarily a *symmetric* one, since clearly $\phi^{o} = \phi$.   For h,k : B → A ,   it is immediate that   <h,k> : B → A × A factorizes through   <i,i> : Φ → A × A   if and only if   k = h and   h   factorizes through   i .

Proposition. *Let* ϕ : A → B   *be a relation and write* p : A × B → A ,   q : A × B → B   *for the projection functions. The relation*

$$
\psi = 1 \wedge q^{o}\phi p : A \times B \to A \times B
$$

is ≤ 1 ;   *let its tabulation be*   (i,i)   *where*   i   *is a mono-*

morphism $C \to A \times B$. *Then* $pi : C \to A$ *and* $qi : C \to B$
provide a tabulation of ϕ .

Proof. Certainly pi and qi are jointly monic, for in fact
$<pi,qi> = i : C \to A \times B$; we are to show that this and
$<\phi_1,\phi_2> : \Phi \to A \times B$ are equal as subobjects. Consider any
functions $h : X \to A$ and $k : X \to B$; to say that $<h,k> : X$
$\to A \times B$ factorizes through i is, by §1.9 and §1.6, to say
that $<h,k><h,k>^o \leq \psi$. Since we always have $ff^o \leq 1$,
this is equally to say that $<h,k><h,k>^o \leq q^o\phi p$; which, as
in (1.1), is equally to say that $q<h,k><h,k>^o p^o \leq \phi$, or
$kh^o \leq \phi$; by §1.6 again, this is to say that $<h,k>$ factor-
izes through $<\phi_1,\phi_2>$.

1.11 Although the study of *ideals* in a regular category $\mathcal{E}$
must await a later article, we define them here the better to
motivate our definitions below; our nomenclature differs
slightly from that in the general reference [5]. An object
$A = (A,\pi_A)$ of the **Ord**-category Idl $\mathcal{E}$ is a *preordered
object* of $\mathcal{E}$; that is, an object A of $\mathcal{E}$ together with a
preorder relation $\pi_A : A \to A$ in Rel $\mathcal{E}$, by which (as in
§1.1) we mean a relation satisfying $1 \leq \pi_A$ and $\pi_A\pi_A \leq \pi_A$
(or equivalently a *monad structure* on A in Rel $\mathcal{E}$). An
arrow $\phi : A \to B$ of Idl $\mathcal{E}$ is an *ideal*, by which we mean a
relation ϕ satisfying $\pi_B\phi\pi_A \leq \phi$; and a 2-cell is, as in
Rel $\mathcal{E}$, an inclusion of ideals as subobjects of $A \times B$.
Composition of arrows in Idl $\mathcal{E}$ is just the composition of
relations, the composite of ideals being an ideal since
$1 \leq \pi_B = \pi_B\pi_B$; in Idl $\mathcal{E}$, however, the identity arrow of A
is $\pi_A : A \to A$. Identifying the object A of Rel $\mathcal{E}$ with
the object $(A,1_A)$ of Idl $\mathcal{E}$ exhibits Rel $\mathcal{E}$ as a full
sub-**Ord**-category of Idl $\mathcal{E}$; we may call $(A,1_A)$ a *discrete
object* of Idl $\mathcal{E}$.

For A and B in Idl $\mathcal{E}$, a *functor* $f : A \to B$ is a *mono-
tone function* - that is, an f in $\mathcal{E}$ satisfying $f\pi_A \leq \pi_B f$;
clearly the composite in $\mathcal{E}$ of functors is a functor. We
define a preorder relation on functors $A \to B$ by setting
$g \leq f$ whenever $<f,g> : A \to B \times B$ factorizes through the
subobject corresponding to π_B ; by §1.6 this is to say that
$gf^o \leq \pi_B$, which is easily seen to be equivalent to $\pi_B g$
$\leq \pi_B f$. We write PreOrd $\mathcal{E}$ for the 2-category given by the
preordered objects of $\mathcal{E}$, the functors, and the inequalities
$g \leq f$; its full sub-2-category given by the discrete ob-
jects is just $\mathcal{E}$, seen as a locally-discrete 2-category.
There is a 2-functor $( )_* : $ PreOrd $\mathcal{E} \to$ Idl $\mathcal{E}$ which is

the identity on objects and which sends the functor $f : A \to B$ to the ideal $f_\star = \pi_B f$; not only is it locally monotone, but by the above we have $g \leq f$ if and only if $g_\star \leq f_\star$; so $(\ )_\star$ fails to be locally-fully-faithful only because $g_\star = f_\star$ gives not $g = f$ but $g \cong f$, in the sense that $g \leq f$ and $f \leq g$. The restriction of $(\ )_\star$ to the discrete objects is the usual inclusion $\mathcal{E} \to \text{Rel } \mathcal{E}$ of §1.4. Following [5], we call ideals of the form $f_\star$ *principal*; by taking only these principal ideals as arrows, we get a locally-full sub-**Ord**-category PrinIdl $\mathcal{E}$ of Idl $\mathcal{E}$ with the same objects; by the above, it is isomorphic to the **Ord**-category obtained from PreOrd $\mathcal{E}$ by identifying isomorphic functors.

A principal ideal $f_\star$ is in fact a *map* in Idl $\mathcal{E}$; it has the right adjoint $(f_\star)^\star$, usually written simply as $f^\star$, given by the composite $f^\circ \pi_B$ in Rel $\mathcal{E}$. We observed in §1.7 that the only maps in Rel $\mathcal{E}$ are the functions, so that Map Rel $\mathcal{E} = \mathcal{E}$; in constrast (see [5]), maps in Idl $\mathcal{E}$ need not be principal ideals - which, by the above, are "essentially" functors - so that PrinIdl $\mathcal{E} \to$ Map Idl $\mathcal{E}$ is in general a proper inclusion. To see this, begin with a strong epimorphism $f : A \to B$ in $\mathcal{E}$. By §1.7 we have $ff^\circ = 1_B$, while $\varepsilon = f^\circ f$ is by §1.5 the equivalence relation on A given by the kernel-pair of f . Regard (A,ε) and $(B,1_B)$ as preordered objects of $\mathcal{E}$; clearly f is a functor $(A,\varepsilon) \to (B,1_B)$, the principal ideal $f_\star$ being f itself. Since $ff^\circ = 1_B$ and $f^\circ f = \varepsilon$, we see that $f^\circ = f^\star$ is in fact inverse to f in Idl $\mathcal{E}$, and so is certainly a map. Suppose that f° is a principal ideal $g_\star = \varepsilon g$ for some functor $g : (B,1_B) \to (A,\varepsilon)$. Since $ff^\circ = 1_B$ we have $f\varepsilon g = 1_B$; but $f\varepsilon = ff^\circ f = f$, giving $fg = 1_B$ and exhibiting f as a *retraction* in $\mathcal{E}$. Thus, unless $\mathcal{E}$ satisfies the axiom of choice (in the sense that all strong epimorphisms are retractions), a map in Idl $\mathcal{E}$ need not be a principal ideal, even if it is the inverse of a principal ideal.

2. THE 3-CATEGORY F AND THE 3-FUNCTOR $(\ )_\# : F \to$ Ord-Cat

2.1 While a general 3-category is a (2-**Cat**)-category, the 3-category **F** we are about to define may be seen as an (**Ord-Cat**)-category, since its only 3-cells are inequalities. An object of **F** is a pair $(\mathcal{A}, \mathcal{A}_\#)$ where $\mathcal{A}$ is an **Ord**-category and $\mathcal{A}_\#$ is a locally-full sub-**Ord**-category of $\mathcal{A}$, the objects of $\mathcal{A}_\#$ being exactly the objects of $\mathcal{A}$ and

every arrow of $\mathcal{A}_{\#}$ being a map in $\mathcal{A}$. In other words, to give $\mathcal{A}_{\#}$ is to give a subclass of the maps of $\mathcal{A}$, which we shall call the *selected* maps, and which are to contain the identities and to be closed under composition. When there is no danger of confusion, we abbreviate $(\mathcal{A},\mathcal{A}_{\#})$ to $\mathcal{A}$. The chief motivating examples are $\mathcal{A} = \text{Rel } \mathcal{E}$ (for a regular category $\mathcal{E}$) , with $(\text{Rel } \mathcal{E})_{\#}$ consisting of the functions, so that $(\text{Rel } \mathcal{E})_{\#} = \text{Map Rel } \mathcal{E} = \mathcal{E}$; and $\mathcal{A} = \text{Idl } \mathcal{E}$, with $(\text{Idl } \mathcal{E})_{\#}$ consisting of the principal ideals. In the latter example, we see from §1.11 that a map need not be selected, even if it is the inverse of a selected map. We now modify the convention of §1.2; henceforth Roman letters are reserved for *selected* maps.

2.2 The neatest description of the morphisms of **F** is in terms of the *adjoint inequalities* of (1.1), where now f and g denote *selected* maps. Given $\mathcal{A} \in \textbf{F}$, forgetting $\mathcal{A}_{\#}$, forgetting the 2-cells of $\mathcal{A}$, and forgetting the composition in $\mathcal{A}$ gives us the *underlying graph* of $\mathcal{A}$. A morphism $T : \mathcal{A} \to \mathcal{B}$ in **F** is a morphism of the underlying graphs which (a) is *normal*, in the sense that $T1_A = 1_A$, (b) *preserves adjoint inequalities*, in the sense that $g\phi \leq \psi f$ implies $Tg.T\phi \leq T\psi.Tf$, and (c) *preserves selected maps*, in the sense that Tf is in $\mathcal{B}_{\#}$ whenever f is in $\mathcal{A}_{\#}$. It is immediate, given (a), that (b) is equivalent to the conjunction of the following three conditions: (b_1) T is *locally monotone*, in the sense that $\phi \leq \psi : A \to C$ implies $T\phi \leq T\psi$; (b_2) $Tg.T\phi \leq T(g\phi)$; (b_3) $T(\psi f) \leq T\psi.Tf$. Now (a) and (b_1) - (b_3) give $Tf.Tf^{\star} \leq T(ff^{\star}) \leq T1 = 1$, and similarly $1 \leq Tf^{\star}.Tf$, so that

$$Tf \dashv Tf^{\star} \; ; \qquad\qquad (2.1)$$

accordingly (c) *is automatic whenever* $\mathcal{B}_{\#} = \text{Map } \mathcal{B}$. Since (b_2) and (b_3) give $T(gf) = Tg.Tf$, and since we have (b_1) , we see that restricting T to the selected maps provides a 2-functor $T_{\#} : \mathcal{A}_{\#} \to \mathcal{B}_{\#}$. Clearly morphisms in **F** compose, so that **F** is a category, with a functor $(\ )_{\#} : \textbf{F} \to \textbf{Ord-Cat}$.

2.3 In the definition of a morphism T in **F** we can replace the condition (b) by (d): $\phi f^{\star} \leq g^{\star}\psi$ implies $T\phi.Tf^{\star} \leq Tg^{\star}.T\psi$. The point is that, given (a), it is clear that (d) is equivalent to the conjunction of $(d_1) = (b_1)$, (d_2) $T\phi.Tf^{\star} \leq T(\phi f^{\star})$, and (d_3) $T(g^{\star}\psi) \leq Tg^{\star}.T\psi$; that (a) and the (d_1) like (a) and the (b_1) give (2.1); and

that, in the presence of (2.1), (b) and (d) are equivalent by (1.1). It is then clear that, given (a), we can go further and replace (b) or (d) by a combined form (e_1) of the conditions (b_1) and (d_1) ; namely $(e_1) = (b_1) = (d_1) =$ local monotonicity, along with the inequalities

$$(e_2) \quad Tg.T\phi.Tf^\star \leq T(g\phi f^\star) \ ,$$

$$\hspace{8cm} (2.2)$$

$$(e_3) \quad T(g^\star\psi f) \leq Tg^\star.T\psi.Tf \ .$$

There are yet further ways of describing a morphism T in **F** , in which (2.1) is not a conclusion but one of the conditions. Write $(a_\leq)$ for the weakening $1_{TA} \leq T1_A$ of (a), and $(a_\geq)$ for $T1_A \leq 1_{TA}$.

2.4 Proposition. *For $\mathcal{A}$ and $\mathcal{B}$ in* **F** *, let the morphism T of their underlying graphs be locally monotone, satisfy (2.1) and satisfy (c). Then T is a morphism in* **F** *if and only if it satisfies any one of the following conjunctions of conditions:* (i) $(a_\leq),(b_2)$, *and* (d_2); (ii) $(a_\leq)$ *and* (e_2); (iii) $(a_\geq),(b_3)$, *and* (d_3); (iv) $(a_\geq)$ *and* (e_3).

Proof. "Only if" is known. We deal with (i) and (ii), to which (iii) and (iv) are dual. Clearly (i) implies (ii), while (ii) implies (i) on taking $f = 1$ or $g = 1$. Given (ii), it suffices by §2.3 to prove (a) and (e_3). Since $1_A \dashv 1_A$, (2.1) gives $T1_A \dashv T1_A$, so that

$$T1_A.T1_A \leq 1_{TA} \ ;$$

using $(a_\leq)$ we have

$$T1_A = 1_{TA}.T1_A \leq T1_A.T1_A \leq 1_{TA} \ ,$$

giving (a). Again, $gg^\star \leq 1$ and $ff^\star \leq 1$ give $gg^\star\psi ff^\star \leq \psi$; now (e_2) and the local monotonicity of ψ give $Tg.T(g^\star\psi f).Tf^\star \leq T\psi$; since $Tg \dashv Tg^\star$ and $Tf \dashv Tf^\star$ by (2.1), we have (e_3) by (1.1).

2.5 Recall from [16] that, for **Ord**-categories $\mathcal{A}$ and $\mathcal{B}$, a *lax functor* $T : \mathcal{A} \to B$ is a locally-monotone graph morphism for which $1_{TA} \leq T1_A$ and $T\theta.T\psi \leq T(\theta\psi)$; in the language of [1] this is a *morphism of bicategories*. A *colax functor*, on the other hand, has $T1_A \leq 1_{TA}$ and $T(\theta\phi) \leq T\theta.T\phi$, while a *2-functor* is of course a T that is both a lax functor and a colax functor. Since 2-functors preserve adjunc-

tions, (2.1) is automatic for a 2-functor T . It now follows from Proposition 2.4 that *a lax functor or a colax functor* T : $\mathcal{A} \to \mathcal{B}$ *is a morphism in* **F** *if and only if it satisfies* (2.1) *and* (c); *a 2-functor* T *is a morphism in* **F** *if and only if it satisfies* (c); *and* (c) *is superfluous if* $\mathcal{B}_* = $ Map $\mathcal{B}$.

   Note that, *when* T *is a lax [resp. colax] functor, the inequality* (e_3) *[resp.* (e_2)*] of* (2.2) *is an equality.*

2.6 Given morphisms T,S : $\mathcal{A} \to \mathcal{B}$ in **F** , we define a *transformation* f : T $\to$ S to be a family $(f_A : TA \to SA)$ of *selected maps* in $\mathcal{B}$, indexed by the objects A of $\mathcal{A}$, and such that for each arrow $\phi : A \to C$ in $\mathcal{A}$ we have the adjoint inequality

$$f_C.T\phi \le S\phi.f_A . \qquad (2.3)$$

If k : S $\to$ R : $\mathcal{A} \to \mathcal{B}$ is another transformation, we have an evident composite transformation k·f : T $\to$ R given by $(k \cdot f)_A = k_A f_A$; since we have also the identity transformation $1_T : T \to T$ given by $(1_T)_A = 1_{TA}$, the morphisms $\mathcal{A} \to \mathcal{B}$ in **F** and their transformations constitute a category $\mathbf{F}(\mathcal{A},\mathcal{B})$; this becomes an **Ord**-category when, for f,g : T $\to$ S , we write f $\le$ g to mean that $f_A \le g_A$ for each A . We should note that f : T $\to$ S *is invertible in* $\mathbf{F}(\mathcal{A},\mathcal{B})$ *if and only if* (i) *each* f_A *is invertible in* $\mathcal{B}_*$, *and* (ii) *for each* ϕ *the inequality* (2.3) *is an equality.* The "if" part is clear, f having the inverse k where $k_A = f_A^{-1}$. For the converse, let f have an inverse k ; (i) is immediate since $k_A f_A$ and $f_A k_A$ are identities; and then, by (2.3) for f and for k , we have $T\phi = k_C f_C.T\phi \le k_C.S\phi.f_A \le T\phi.k_A f_A = T\phi$, giving (ii) since k_C is invertible. Two points need emphasis: first, as we observed in §2.1, the invertibility of f_A in $\mathcal{B}_*$ is strictly stronger than its invertibility in $\mathcal{B}$ in general, although not when $\mathcal{B}_* = $ Map $\mathcal{B}$; secondly, (ii) above is not a consequence of (i), even when each f_A is an identity - we shall meet below (see §4.4) pairs T,S : $\mathcal{A} \to \mathcal{B}$ of morphisms in **F** which agree on objects but have strict inequalities $T\phi \le S\phi$.

2.7 It is a central observation that, *for any transformation* f : T $\to$ S : $\mathcal{A} \to \mathcal{B}$ *in* **F** , *the inequality* (2.3) *is an equality whenever* $\phi : A \to C$ *is a selected map.* The point is that, for a selected map h : A $\to$ C , (2.3) with $\phi = $ h gives $f_C.Th \le Sh.f_A$, while (2.3) with $\phi = h^*$ gives $f_A.Th^* \le Sh^*.f_C$, equivalent by (2.1) and (1.1) to $Sh.f_A \le f_C.Th$. One consequence of this is that every transformation f as

above gives rise to a natural transformation $f_* : T_* \to S_* :$ $\mathcal{A}_* \to \mathcal{B}_*$ in **Ord-Cat** , where $(f_*)_A = f_A$; clearly we have $f_* \leq g_*$ if and only if $f \leq g$, so that $f_* = g_*$ if and only if $f = g$. We may at times abbreviate by writing f for f_* .

2.8 We now make **F** into an **(Ord-Cat)**-category. For $P : \mathcal{B} \to \mathcal{C}$ and $f : T \to S : \mathcal{A} \to \mathcal{B}$ we define a transformation $Pf : PT \to PS$ by $(Pf)_A = Pf_A$; observe that Pf_A is indeed a selected map by (c) of §2.2, and that (2.3) for Pf follows from (2.3) for f by (b) of §2.2. Since P_* is a 2-functor, it is clear that $P(k \cdot f) = Pk \cdot Pf$, that $P1_T = 1_{PT}$, and that $f \leq f'$ implies $Pf \leq Pf'$. Next, for $g : P \to Q : \mathcal{B} \to \mathcal{C}$ and $T : \mathcal{A} \to \mathcal{B}$ we define a transformation $gT : PT \to QT$ by $(gT)_A = g_{TA}$; here (2.3) for gT follows trivially from (2.3) for g , and it is clear that $(h \cdot g)T = hT \cdot gT$, that $1_P T = 1_{PT}$, and that $g \leq g'$ implies $gT \leq g'T$. Consider now $f : T \to S : \mathcal{A} \to \mathcal{B}$ and $g : P \to Q : \mathcal{B} \to \mathcal{C}$; to define the horizontal composite $gf : PT \to QS : \mathcal{A} \to \mathcal{C}$ we need the equality of $gS \cdot Pf$ and $Qf \cdot gT$; in terms of components we need $g_{SA}.Pf_A = Qf_A.g_{TA}$, which we have by applying §2.7 to (2.3) for g , since f_A is a selected map. Since we clearly have equalities of the forms

$$(nP)T = n(PT) , \quad (Vg)T = V(gT) , \quad \text{and} \quad (VQ)f = V(Qf) ,$$

it is indeed the case that **F** is a 2-category - and in fact an **(Ord-Cat)**-category by the remarks above on inequalities. The cartesian closed category **Ord-Cat** is of course itself an **(Ord-Cat)**-category and it is immediate that $\mathcal{A} \to \mathcal{A}_*$, $T \to T_*$, and $f \to f_*$ constitute an **(Ord-Cat)**-functor (a 3-functor) $( )_* : F \to \text{Ord-Cat}$. In fact this **(Ord-Cat)**-functor is *representable*; the reader will easily see that it is isomorphic to $F(1,-)$ where $1 \in F$ is the terminal object of **F** , given by the one-arrow **Ord**-category 1 with $1_* = \text{Map } 1 = 1$.

2.9 The 3-category **F** has not only the terminal object 1 above, but all products; to illustrate by binary products, the product in **F** of $\mathcal{A}$ and $\mathcal{B}$ is just the usual product $\mathcal{A} \times \mathcal{B}$ of **Ord**-categories, with $(\mathcal{A} \times \mathcal{B})_* = \mathcal{A}_* \times \mathcal{B}_*$. Clearly the projections $\pi_1 : \mathcal{A} \times \mathcal{B} \to \mathcal{A}$ and $\pi_2 : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ are 2-functors. Let us henceforth write $F_\leq, F_\geq$, and $F_=$ for the locally-full sub-3-categories of **F** obtained from **F** by keeping all the objects but restricting the morphisms to the lax functors, the colax functors and the 2-functors respect-

ively; see §2.5. It is immediate that the product $\mathcal{A} \times \mathcal{B}$ in **F** is equally the product in $\mathbf{F}_\leq$, in $\mathbf{F}_\geq$, and in $\mathbf{F}_=$.

2.10 We now consider various dualities in **F** , beginning by fixing some notation. For enriched categories $\mathcal{K}$ in general, and hence for 2-categories and 3-categories in particular, we have the opposite enriched category $\mathcal{K}^{op}$ where $\mathcal{K}^{op}(A,B) = \mathcal{K}(B,A)$; when $\mathcal{K}$ is a 2-category [or a 3-category], $\mathcal{K}^{op}$ is obtained from $\mathcal{K}$ by reversing the 1-cells but not the 2-cells [nor the 3-cells]; in these latter cases $\mathcal{K}(A,B)$ is itself a category [or a 2-category], and we get a second dual $\mathcal{K}^{co}$ given by $\mathcal{K}^{co}(A,B) = \mathcal{K}(A,B)^{op}$; in $\mathcal{K}^{co}$ the 2-cells are reversed, but not the 1-cells [nor the 3-cells]; clearly $\mathcal{K}^{co\ op} = \mathcal{K}^{op\ co}$. When $\mathcal{K}$ is a 3-category, so that each $\mathcal{K}(A,B)$ is a 2-category, we have a third dual $\mathcal{K}^{\square}$ where $\mathcal{K}^{\square}(A,B) = \mathcal{K}(A,B)^{co}$; in $\mathcal{K}^{\square}$ the 3-cells are reversed, but neither the 1-cells nor the 2-cells.

If $f : A \rightarrow B$ is a map in an **Ord**-category $\mathcal{A}$, with $f \dashv f^\star$, then $f^\star : A \rightarrow B$ is a map in $\mathcal{A}^{op}$ with $f^\star \dashv f$, and $f^\star : B \rightarrow A$ is a map in $\mathcal{A}^{co}$ with $f^\star \dashv f$. Accordingly, for $\mathcal{A} = (\mathcal{A}, \mathcal{A}_\#)$ in **F** we get objects $\mathcal{A}^{op} = (\mathcal{A}^{op}, (\mathcal{A}^{op})_\#)$ and $\mathcal{A}^{co} = (\mathcal{A}^{co}, (\mathcal{A}^{co})_\#)$ of **F** by taking the maps in $(\mathcal{A}^{op})_\#$ and in $(\mathcal{A}^{co})_\#$ to be the $f^\star$ where f is a map in $\mathcal{A}_\#$. Note that in both cases $f \longmapsto f^\star$ gives isomorphisms

$$(\mathcal{A}^{op})_\# \cong (\mathcal{A}_\#)^{co} , \qquad (\mathcal{A}^{co})_\# \cong (\mathcal{A}_\#)^{op} . \qquad (2.4)$$

Since, by (1.1), an adjoint inequality in $\mathcal{A}$ gives adjoint inequalities in $\mathcal{A}^{op}$ and in $\mathcal{A}^{co}$, it follows from §2.2 that, if $T : \mathcal{A} \rightarrow \mathcal{B}$ is a morphism in **F** , the same data constitute morphisms $T^{op} : \mathcal{A}^{op} \rightarrow \mathcal{B}^{op}$ and $T^{co} : \mathcal{A}^{co} \rightarrow \mathcal{B}^{co}$, with $(T^{op})_\# = (T_\#)^{co}$ and $(T^{co})_\# = (T_\#)^{op}$ modulo (2.4). A transformation $f : T \rightarrow S : \mathcal{A} \rightarrow \mathcal{B}$ gives transformations

$$f^{op} : T^{op} \rightarrow S^{op} : \mathcal{A}^{op} \rightarrow \mathcal{B}^{op}$$

and

$$f^{co} : S^{co} \rightarrow T^{co} : \mathcal{A}^{co} \rightarrow \mathcal{B}^{co}$$

on setting $(f^{op})_A = (f^{co})_A = f^\star_A$. The upshot is that we have isomorphisms

$$(\ )^{op} : \mathbf{F} \rightarrow \mathbf{F}^{\square} , \qquad (\ )^{co} : \mathbf{F} \rightarrow \mathbf{F}^{co} , \qquad (2.5)$$

which under the action of $(\ )_\# : \mathbf{F} \rightarrow \mathbf{Ord\text{-}Cat}$ become, modu-

lo the isomorphisms in (2.4), the isomorphisms

$$(\ )^{co} : \textbf{Ord-Cat} \rightarrow (\textbf{Ord-Cat})^{\square} \ ,$$

$$(\ )^{op} : \textbf{Ord-Cat} \rightarrow (\textbf{Ord-Cat})^{co} \ .$$

(2.6)

Moreover $\textbf{F}_{\leq}$, $\textbf{F}_{\geq}$, and $\textbf{F}_{=}$ are stable under $(\ )^{op}$, while $(\ )^{co}$ carries $\textbf{F}_{\leq}$ into $\textbf{F}_{\geq}$ and vice-versa, preserving $\textbf{F}_{=}$.

3. ADJUNCTIONS IN F

3.1 Ignoring the 3-cells of **F** and treating it just as a 2-category, we have in the usual way the notion of an *adjunction*

$$x,y : S \ \dashv \ T : \mathscr{A} \rightarrow \mathscr{B} \tag{3.1}$$

in **F** , given by morphisms $T : \mathscr{A} \rightarrow \mathscr{B}$ (the *right adjoint*) and $S : \mathscr{B} \rightarrow \mathscr{A}$ (the *left adjoint*), along with trans-formations $x : 1 \rightarrow TS$ (the *unit*) and $y : ST \rightarrow 1$ (the *counit*) satisfying the *triangular equations*

$$Ty \cdot xT = 1 \ , \qquad yS \cdot Sx = 1 \ . \tag{3.2}$$

Since 2-functors send adjunctions to adjunctions, the adjunction (3.1) gives an adjunction

$$x_*,y_* : S_* \ \dashv \ T_* : \mathscr{A}_* \rightarrow \mathscr{B}_* \tag{3.3}$$

in **Ord-Cat**. Again, from an adjunction (3.1) the isomorphisms of (2.5) give two further adjunctions (*still in* **F**)

$$x^{op},y^{op} : S^{op} \ \dashv \ T^{op} : \mathscr{A}^{op} \rightarrow \mathscr{B}^{op} \ ;$$

$$y^{co},x^{co} : T^{co} \ \dashv \ S^{co} : \mathscr{B}^{co} \rightarrow \mathscr{A}^{co} \ ;$$

(3.4)

either of these (which are "trivially" equivalent) can be used to halve our work by an appeal to "duality"; the point is that the *components* of the units and counits in (3.4) have the forms x_B^*, y_A^* .

3.2 For a morphism $T : \mathcal{A} \to \mathcal{B}$ in $\mathbf{F}$ we write $T_{AC} :$ $\mathcal{A}(A,C) \to \mathcal{B}(TA,TC)$ for the monotone function sending ϕ to $T\phi$; recall from §2.2 that Tf is in $\mathcal{B}_*$ when f is in $\mathcal{A}_*$. Given an adjunction (3.1) in $\mathbf{F}$, consider arrows $\psi : B \to D$ in $\mathcal{B}$ and $\phi : SB \to SD$ in $\mathcal{A}$; we have

$$S\psi \leq \phi \quad \textit{if and only if} \quad x_D\psi \leq T\phi.x_B \ . \tag{3.5}$$

For applying T to the left side of (3.5) and using (2.3) for x gives $x_D\psi \leq TS\psi.x_B \leq T\phi.x_B$; while applying S to the adjoint inequality on the right side of (3.5) - which it preserves by (b) of §2.2 - and using (2.3) for y gives

$$y_{SD}.Sx_D.S\psi \leq y_{SD}.ST\phi.Sx_B \leq \phi y_{SB}.Sx_B \ ,$$

whence $S\psi \leq \phi$ by the second equation of (3.2). By a dual argument, for arrows $\phi : A \to C$ in $\mathcal{A}$ and $\psi : TA \to TC$ in $\mathcal{B}$ we have

$$\psi \leq T\phi \quad \textit{if and only if} \quad y_C.S\psi \leq \phi y_A \ . \tag{3.6}$$

Now define monotone functions $\check{S}_{BD}:\mathcal{A}(SB,SD) \to \mathcal{B}(B,D)$ and $\hat{T}_{AC}:\mathcal{B}(TA,TC) \to \mathcal{A}(A,C)$ by

$$\check{S}_{BD}\phi = x_D^\star.T\phi.x_B \ , \quad \hat{T}_{AC}\psi = y_C.S\psi.y_A^\star \ ; \tag{3.7}$$

using (1.1) we can rewrite (3.5) and (3.6) as

$$S\psi \leq \phi \quad \textit{if and only if} \quad \psi \leq \check{S}\phi \ ,$$
$$\psi \leq T\phi \quad \textit{if and only if} \quad \hat{T}\psi \leq \phi \ ; \tag{3.8}$$

thus *for any adjunction* $S \dashv T$ *in* $\mathbf{F}$, *each* $S_{BD} : \mathcal{B}(B,D)$ $\to \mathcal{A}(SB,SD)$ *has the right adjoint* $\check{S}_{BD}$ *in* **Ord**, *while each* $T_{AC} : \mathcal{A}(\dot{A},C) \to \mathcal{B}(TA,TC)$ *has the left adjoint* $\hat{T}_{AC}$ *in* **Ord**.

3.3 Still supposing (3.1) to be an adjunction in $\mathbf{F}$, consider composable arrows $\psi : B \to D$, $\chi : D \to F$ in $\mathcal{B}$. Two applications of (2.3) for x give $x_F\chi\psi \leq TS\chi.x_D\psi \leq TS\chi.TS\psi.x_B$. It follows from (3.5) that

$$TS\chi.TS\psi \leq T(S\chi.S\psi) \quad \textit{implies} \quad S(\chi\psi) \leq S\chi.S\psi \ . \tag{3.9}$$

The same argument works for three composable arrows, giving

$$TS\varepsilon.TS\chi.TS\psi \leq T(S\varepsilon.S\chi.S\psi) \quad \textit{implies} \quad S(\varepsilon\chi\psi) \leq S\varepsilon.S\chi.S\psi \; .$$
$$(3.10)$$

Dually, for $\phi : A \rightarrow C$ and $\theta : C \rightarrow E$, we have

$$S(T\theta.T\phi) \leq ST\theta.ST\phi \quad \textit{implies} \quad T\theta.T\phi \leq T(\theta\phi) \; , \qquad (3.11)$$

as well as the three-arrow dual of (3.10). We conclude from (3.9) and (3.11) that *for any adjunction* $S \dashv T$ *in* $\mathbf{F}$, T *is a lax functor precisely when* S *is a colax functor. In particular, if* $S \dashv T$ *is an adjunction in* $\mathbf{F}_{\leq}$, *so that both* T *and* S *are lax functors, then* S *is in fact a 2-functor.* We see no reason to believe that every right adjoint in $\mathbf{F}$ is a lax functor; yet it is so in all the practical examples we have so far studied, and we have no counter-example; it is certainly so, as we shall see in §4.5, when $\mathscr{A}$ and $\mathscr{B}$ are of the form $\mathrm{Rel}\,\mathscr{E}$ and $\mathrm{Rel}\,\mathscr{F}$. At any rate, a right adjoint T has more laxness and a left adjoint S more colaxness than general morphisms in $\mathbf{F}$; for when $S \dashv T$ we can (*cf.* the last sentence of §2.5) replace the inequalities of (2.2) by the equalities

$$Tg^{*}.T\phi.Tf = T(g^{*}\phi f) \; , \quad Sk.S\psi.Sh^{*} = S(k\psi h^{*}) \; . \qquad (3.12)$$

It suffices by duality to prove the second of these. Since Sk and Sh are again selected maps, with $Sh \dashv Sh^{*}$, the inequality (e_{2}) of (2.2) gives $TSk.TS\psi.TSh^{*} \leq T(Sk.S\psi.Sh^{*})$; and now (3.10) gives $S(k\psi h^{*}) \leq Sk.S\psi.Sh^{*}$, while we have the reverse inequality by (e_{2}) again. The second equality of (3.12) may be seen as asserting commutativity for the first diagram below; replacing each arrow by its second adjoint in **Ord** then gives commutativity for the right diagram:

$$
\begin{array}{ccc}
\mathscr{B}(D,F) & \xrightarrow{\;S_{DF}\;} & \mathscr{A}(SD,SF) \\[2mm]
{\scriptstyle\mathscr{B}(h^{*},k)}\Big\downarrow & & \Big\downarrow{\scriptstyle\mathscr{A}(Sh^{*},Sk)} \\[2mm]
\mathscr{B}(B,H) & \xrightarrow[\;S_{BH}\;]{} & \mathscr{A}(SB,SH)
\end{array}
\qquad ,
$$

$$(3.13a)$$

$$\mathcal{B}(D,F) \xleftarrow{\quad \check{S}_{DF} \quad} \mathcal{A}(SD,SF)$$

$$\mathcal{B}(h,k^\star) \uparrow \qquad\qquad\qquad \uparrow \mathcal{A}(Sh,Sk^\star)$$

$$\mathcal{B}(B,H) \xleftarrow{\quad \check{S}_{BH} \quad} \mathcal{A}(SB,SH) \qquad ;$$

$$(3.13b)$$

with the dual results for T and $\hat{T}$. We end this section with the following formulae which "invert" (3.7), in that they express T and S in terms of $\check{S}$ and $\hat{T}$ respectively; for $\phi : A \to C$ in $\mathcal{A}$ and $\psi : B \to D$ in $\mathcal{B}$ we have

$$T\phi = \check{S}_{TA,TC}(y_C^\star \phi y_A) \ , \qquad S\psi = \hat{T}_{SB,SD}(x_D \psi x_B^\star) \ . \qquad (3.14)$$

We prove the first of these dual assertions by direct calculation; by (3.7) its right side is $x^\star_{TC} T(y_C^\star \phi y_A).x_{TA}$; by (3.12) this is $x_{TC}^\star.Ty_C^\star T\phi.Ty_A.x_{TA}$, which is $T\phi$ by two applications of the first equation of (3.2).

3.4 For an adjunction (3.1) in **F**, define monotone functions

$$\tau_{BA} : \mathcal{A}(SB,A) \to \mathcal{B}(B,TA) \ , \quad \sigma_{BA} : \mathcal{B}(B,TA) \to \mathcal{A}(SB,A) \ ,$$

$$\tau'_{AB} : \mathcal{A}(A,SB) \to \mathcal{B}(TA,B) \ , \quad \sigma'_{AB} : \mathcal{B}(TA,B) \to \mathcal{A}(A,SB)$$

by

$$\tau(\phi) = T\phi.x_B \ , \qquad \sigma(\psi) = y_A.S\psi \ ,$$

$$\tau'(\theta) = x_B^\star.T\theta \ , \qquad \sigma'(\chi) = S\chi.y_A^\star \ ; \qquad (3.15)$$

note that we could equally give element-free descriptions such as

$$\tau_{BA} = \mathcal{B}(x_B,TA).T_{SB,A} \ , \quad \sigma_{BA} = \mathcal{A}(SB,y_A).S_{B,TA} \ . \qquad (3.16)$$

It is classical that, for an adjunction (3.1) not in **F** but in **Cat**, τ and σ are mutually inverse isomorphisms; the

same is true (see [15, (1.51)]) for an adjunction in $\mathcal{V}$-**Cat** , where now τ and σ must be defined by (3.16); it is true in particular for adjunctions in **Ord-Cat**, so that by (3.3) the restrictions of the τ and σ of (3.15) to selected maps are indeed inverses; but it is false in general for adjunctions in **F** . For a simple example, let $\mathcal{A} = 1$, let $\mathcal{B} = $ Rel **Set** , and let $S : \mathcal{B} \to 1$ be the unique morphism in **F** ; S is easily seen to have in **F** the right adjoint $T : 1 \to \mathcal{B}$ naming the object $1 \in$ **Set** $\subset \mathcal{B}$; yet here $\mathcal{A}(SB,A) = 1$ for each B and the only A , while $\mathcal{B}(B,TA)$ is in bijection with the subsets of $B \in$ **Set** . What we *can* assert for an adjunction (3.1) in **F** is that we have in **Ord** (not isomorphisms but) *adjunctions*

$$\sigma_{BA} \dashv \tau_{BA} : \mathcal{A}(SB,A) \to \mathcal{B}(B,TA) ,$$

$$\sigma'_{AB} \dashv \tau'_{AB} : \mathcal{A}(A,SB) \to \mathcal{B}(TA,B) . \tag{3.17}$$

It suffices to prove the first of these dual statements; since $T_{SB,A}$ has by §3.2 the left adjoint $\hat{T}_{SB,A}$, the τ_{BA} of (3.16) has the left adjoint $\hat{T}_{SB,A}.\mathcal{B}(x_B^\star,TA)$; the value of this at $\psi : B \to TA$ is $\hat{T}(\psi,x_B^\star)$, which by (3.7), (3.12) and the second equation of (3.2) is

$$y_A.S(\psi x_B^\star).y_{SB}^\star = y_A.S\psi.Sx_B^\star.y_{SB}^\star = y_A.S\psi = \sigma_{BA}(\psi) .$$

Leaving it to the reader to formulate the "naturality" properties of τ,σ,τ' and σ' that follow from (3.12) and (3.13), we pass on to the observation - perhaps surprising - that the expression [15, (1.53)] of T_{AC} in terms of τ for adjunctions in $\mathcal{V}$-**Cat** continues to hold for adjunctions in **F** ; that is to say, T_{AC} *is the composite*

$$\mathcal{A}(A,C) \xrightarrow[\mathcal{A}(y_A,C)]{} \mathcal{A}(STA,C) \xrightarrow[\tau_{TA,C}]{} \mathcal{B}(TA,TC) . \tag{3.18}$$

For the value at ϕ of this composite, which by (3.15) is $T(\phi y_A).x_{TA}$, is $T\phi.Ty_A.x_{TA}$ by (3.12), and hence $T\phi$ by (3.2).

3.5 Theorem. *A morphism* $S : \mathcal{B} \to \mathcal{A}$ *in* **F** *admits a right adjoint if and only if it satisfies*

(i) $\qquad S(k\psi h^\star) \leq Sk.S\psi.Sh^\star$; $\tag{3.19}$

(ii) $S_\# : \mathcal{B}_\# \to \mathcal{A}_\#$ *admits a right adjoint in* **Ord-Cat**, *given say by* $x,y : S_\# \dashv P : \mathcal{A}_\# \to \mathcal{B}_\#$;

(iii) *for each* B *and* D , *the monotone function*

$$S_{BD} : \mathcal{B}(B,D) \to \mathcal{A}(SB,SD)$$

admits a right adjoint $\check{S}_{BD}$ *in* **Ord**;

(iv) *for each* A *in* $\mathcal{A}$ *we have the inequality*

$$\check{S}_{PS,PS}(y_S^\star y_S) \le 1_{PS} . \qquad (3.20)$$

In fact, given (i)-(iv), there is a unique morphism $T : \mathcal{A} \to \mathcal{B}$ *in* **F** *such that* $T_\# = P$ *and such that* x *and* y *are the unit and the counit for an adjunction* $x,y : S \dashv T : \mathcal{A} \to \mathcal{B}$ *in* **F** ; *here* TA = PA *and* $T\phi$ *is given by* (3.14).

We make some remarks before giving the proof. First, we are for brevity confusing x,y with $x_\#,y_\#$, which by §2.7 have the same components. Secondly, to require the inequality (3.19) is in fact to require equality as in (3.12), since by (2.2) we always have the opposite inequality; otherwise put, (i) gives commutativity in (3.13(a)), and hence - on taking right adjoints - in (3.13(b)), once we have (iii). Again, since $S1_{PA} = 1_{SPA} \le y_A^\star y_A$, the inequality opposite to (3.20) is automatic, so that to require (3.20) is in fact to require equality. Finally, given (i), (ii), and (iii), if (3.20) holds for one choice of the adjunction in (ii), it holds for any other choice $\bar{x},\bar{y} : S_\# \dashv P$; for by the general theory of adjunctions (see [16]) there is an invertible $n : P \to P : \mathcal{A}_\# \to \mathcal{B}_\#$ with $\bar{y} = y \cdot Sn^{-1}$, and it follows easily from (3.13) and (3.20) that $\check{S}(\bar{y}_A^\star \bar{y}_A) = 1$.

Proof. As for "only if", we have (i) by (3.12), (ii) by (3.3) and (iii) by §3.2; while (iv) (with equality) follows at once from (3.7), (3.12) and (3.2). Suppose then that we have (i)-(iv). Wanting $T_\# = P$ and $x,y : S \dashv T$, we are forced - as stated in the theorem - to set TA = PA and to define $T\phi$ by (3.14). For $f : A \to C$ in $\mathcal{A}_\#$ we have, since $y : S_\# P \to 1$ is a natural transformation,

$$fy_A = y_C.SPf ; \qquad (3.21)$$

now (3.14) gives $Tf = \check{S}(y_C^\star fy_A) = \check{S}(y_C^\star y_C.SPf)$, which by

(3.13(b)) is $\check{S}(y_C y_C^\star).Pf$; it follows from (3.20) with equa-

lity that $Tf = Pf$. We henceforth replace PA and Pf by TA and Tf , observing that we shall indeed have $T_\# = P$ once T has been shown to be a morphism in $\mathbf{F}$.

The result $Tf = Pf$ gives (a) and (c) of §2.2, so that to exhibit T as a morphism in $\mathbf{F}$ we must establish (b) of §2.2. Suppose that $f : A \to B$, $\psi : B \to D$, $\phi : A \to C$, and $g : C \to D$ satisfy $g\phi \leq \psi f$, which by (1.1) is equivalent to $\phi \leq g^\star\psi f$. This gives $y_C^\star\phi y_A \leq y_C^\star g^\star\psi f y_A$; the right side here, by two applications of (3.21), is $S(Tg)^\star.y_D^\star\psi y_B.STf$; applying the monotone function $\check{S}_{TA,TC}$

and using (3.13(b)) gives $\check{S}(y_C^\star\phi y_A) \leq (Tg)^\star.\check{S}(y_D^\star\psi y_B).Tf$, which by (3.14) may be written as $T\phi \leq (Tg)^\star.T\psi.Tf$; now (1.1) gives the desired $Tg.T\phi \leq T\psi.Tf$.

It remains to show that the $y_A : STA \to A$ and the $x_B : B \to TSB$ are the components of transformations in $\mathbf{F}$ satisfying (3.2), or equivalently

$$Ty_A.x_{TA} = 1 , \qquad y_{SB}.Sx_B = 1 ; \qquad (3.22)$$

in fact (3.22) is automatic, since these are equally the triangular equations for the adjunction $x, y : S_\# \dashv P$. Because $\check{S}_{TA,TC}$ is the right adjoint of $S_{TA,TC}$, (3.14) gives $ST\phi \leq y_C^\star\phi y_A$, equivalent by (1.1) to $y_C.ST\phi \leq \phi y_A$, which is the condition (2.3) for y to be a transformation. Again, (1.1) applied to the second equation of (3.22) gives $Sx_B \leq y_{SB}^\star$; so for $\psi : B \to D$ we have, using (3.19), that $S(x_D\psi) = Sx_D.S\psi \leq y_{SD}^\star.S\psi$. Using the second equation of (3.22) again gives $S(x_D\psi) \leq y_{SD}^\star.S\psi.y_{SB}.Sx_B$, or equivalently $x_D\psi \leq \check{S}(y_{SD}^\star.S\psi.y_{SB}.Sx_B)$; by (3.13) and (3.14), the right side here is $\check{S}(y_{SD}^\star.S\psi.y_{SB}).x_B = TS\psi.x_B$; so that we have $x_D\psi \leq TS\psi.x_B$, which is the condition (2.3) for x to be a transformation.

3.6 Condition (iv) in Theorem 3.5, connecting the "local"

right adjoint $\check{S}$ with the counit y of the "global" adjunction in (ii), cannot be omitted, even when S is a 2-functor. Consider the case where S is the unique morphism $\mathcal{B} \to 1$ in $\mathbf{F}$. Then (i) is trivially satisfied, S being a 2- functor; (ii) asserts that the **Ord**-category $\mathcal{B}_*$ has a terminal object I, which is equally to say that $\mathcal{B}_*$, as a mere category, has this terminal object; (iii) asserts that each $\mathcal{B}(B,D)$ has a top element, and (iv) asserts that the top element of $\mathcal{B}(I,I)$ is 1_I. To see that (iv) does not follow from (i)-(iii), let $\mathcal{B}$ have a single object I, two arrows 1_I and $\psi : I \to I$ with $\psi^2 = \psi$, and the inequality $1_I < \psi$; the only map 1_I is of course selected.

For other independence results among Conditions (i)-(iv), see [4, Remarks 2.17]; we omit them here for lack of space, (i)-(iii) seeming more natural than (iv) and being so clearly necessary. Because Condition (iv) is less limpid, we give an alternative formulation of it. First recall from the remarks preceding the proof of Theorem 3.5 that the right assertion of (3.13) follows from (i) and (iii) alone; accordingly, for any selected map $h : B \to PA$, (iv) implies $\check{S}_{B,PA}(y_A^* y_A.Sh) \leq h$, of which (iv) itself is the special case $h = 1_{PA}$. Since $\check{S}$ is the "local" right adjoint as in (iii), we can by (1.1) express this generalized Condition (iv) as follows, where $\psi : NB \to PA$ is an arbitrary arrow:

$$y_A.S\psi \leq y_A.Sh \quad implies \quad \psi \leq h ; \qquad (3.23)$$

the analogue for the original (iv) has $\psi : PA \to PA$, and reads

$$y_A.S\psi \leq y_A \quad implies \quad \psi \leq 1_{PA} . \qquad (3.24)$$

4. THE COLAX FUNCTOR Rel : **Reg** $\to$ **F**

4.1 We write **Reg** for the 2-category of regular categories, *all* functors $F : \mathcal{E} \to \mathcal{F}$, and all natural transformations, and use $J : \mathbf{Reg} \to \mathbf{Ord\text{-}Cat}$ for the fully-faithful inclusion sending $\mathcal{E}$ to itself seen as a locally-discrete **Ord**-category. It is to be understood that all the categories $\mathcal{E}, \mathcal{F}, \mathcal{G}, \dots$ that we introduce are regular. Recall from §2.1 that Rel $\mathcal{E} \in \mathbf{F}$, with (Rel $\mathcal{E})_* = $ Map Rel $\mathcal{E} = \mathcal{E}$, and from §1.7 that $f^* = f^\circ$ for a map f. Given any $F : \mathcal{E} \to \mathcal{F}$ we define a graph-morphism Rel $F :$ Rel $\mathcal{E} \to$ Rel $\mathcal{F}$ by

$$(\text{Rel } F)A = FA , \qquad (\text{Rel } F)\phi = (F\phi_2)(F\phi_1)^\circ , \qquad (4.1)$$

where (ϕ_1, ϕ_2) is a tabulation of ϕ . Equivalently, by §1.6, $(\text{Rel } F)\phi = \psi$ where

$$<F\phi_1, F\phi_2> \textit{ has the factorization } F\Phi \xrightarrow[\Gamma]{} \Psi \xrightarrow[<\psi_1, \psi_2>]{} FA \times FC \ .$$

(4.2)

Clearly $(\text{Rel } F)(\phi o) = ((\text{Rel } F)\phi)^o$, allowing us to write $(\text{Rel } F)\phi^o$ unambiguously. Note that, when ϕ is a *map* $f : A \to C$, so that $\phi_1 = 1$ and $\phi_2 = f$, (4.1) gives

$$(\text{Rel } F)f = Ff \ .$$

(4.3)

In fact $\text{Rel } F : \text{Rel } \mathcal{E} \to \text{Rel } \mathcal{F}$ *is a morphism in* $\mathbf{F}$ *with* $(\text{Rel } F)_* = F : \mathcal{E} \to \mathcal{F}$. Conditions (a) and (c) of §2.2 follow from (4.3); as for condition (b), an adjoint inequality $g\phi \leq \psi f$ in $\text{Rel } \mathcal{E}$ gives by Proposition 1.8 the commutative (1.4), applying F to which gives $Fg.F\phi_2 = F\psi_2.Ft$ and $Ff.F\phi_1 = F\psi_1.Ft$. Using (1.1) on the second equation, we get $F\phi_1.(Ft)^o \leq (Ff)^o.F\psi_2$, from which the involution $( )^o$ gives $Ft.(F\phi_1)^o \leq (F\psi_1)^o.Ff$; combining this with the first equation above we have $Fg.F\phi_2.(F\phi_1)^o = F\psi_2.Ft.(F\phi_1)^o \leq F\psi_2.(F\psi_1)^o.Ff$, which by (4.1) and (4.3) is the desired $(\text{Rel } F)g.(\text{Rel } F)\phi \leq (\text{Rel } F)\psi.(\text{Rel } F)f$.
  Consider now a natural transformation $\alpha : F \to G :$ $\mathcal{E} \to \mathcal{F}$. In fact, *the components* α_A *constitute a transformation* $\text{Rel } \alpha : \text{Rel } F \to \text{Rel } G : \text{Rel } \mathcal{E} \to \text{Rel } \mathcal{F}$ *in* $\mathbf{F}$, *with* $(\text{Rel } \alpha)_* = \alpha : F \to G : \mathcal{E} \to \mathcal{F}$. The α_A being maps, we need only the inequality (2.3) for α , where $\phi : A \to C$ is given by $<\phi_1, \phi_2> : \Phi \to A \times C$. The naturality of α giving $\alpha_C.F\phi_2 = G\phi_2.\alpha_\Phi$ and $\alpha_A.F\phi_1 = G\phi_1.\alpha_\Phi$, and the latter of these giving $\alpha_\Phi.(F\phi_1)^o \leq (G\phi_1)^o.\alpha_A$ by (1.1), we have $\alpha_C.F\phi_2.(F\phi_1)^o = G\phi_2.\alpha_\Phi.(F\phi_1)^o \leq G\phi_2.(G\phi_1)^o.\alpha_A$, as desired. Clearly, if we also have $\beta : G \to H : \mathcal{E} \to \mathcal{F}$ then $\text{Rel}(\beta \cdot \alpha) = (\text{Rel } \beta) \cdot (\text{Rel } \alpha)$, while $\text{Rel } 1_F = 1$; thus we have a functor $\text{Rel} = \text{Rel}_{\mathcal{E},\mathcal{F}} : \mathbf{Reg}(\mathcal{E},\mathcal{F}) \to \mathbf{F}(\text{Rel } \mathcal{E}, \text{Rel } \mathcal{F})$.

4.2 We now consider the image of this functor. If $T : \text{Rel } \mathcal{E} \to \text{Rel } \mathcal{F}$ is a morphism in $\mathbf{F}$, the 2-functor $( )_* : \mathbf{F} \to \mathbf{Ord\text{-}Cat}$ sends it to a morphism $T_* : \mathcal{E} \to \mathcal{F}$ in $\mathbf{Ord\text{-}Cat}$ which, the inclusion J of §4.1 being fully faithful, is the same thing as a functor $T_* : \mathcal{E} \to \mathcal{F}$ in $\mathbf{Reg}$. Similarly, for a transformation $f : T \to S : \text{Rel } \mathcal{E} \to \text{Rel } \mathcal{F}$ in $\mathbf{F}$, we can see f_* as a natural transformation $f_* : T_* \to S_* : \mathcal{E} \to \mathcal{F}$ in $\mathbf{Reg}$. Thus we have a functor $( )_* :$ $\mathbf{F}(\text{Rel } \mathcal{E} , \text{Rel } \mathcal{F}) \to \mathbf{Reg}(\mathcal{E} , \mathcal{F})$, which by §4.1 satisfies

$$(\;)_*\mathrm{Rel}_{\mathcal{E},\mathcal{F}} = 1 : \mathbf{Reg}(\mathcal{E} \; , \; \mathcal{F}) \to \mathbf{Reg}(\mathcal{E} \; , \; \mathcal{F}) \; . \qquad (4.4)$$

Suppose now that we start with $T : \mathrm{Rel} \; \mathcal{E} \to \mathrm{Rel} \; \mathcal{F}$ in $\mathbf{F}$. Since $T_*A = TA$ and $T_*f = Tf$ for a map f , while by §1.7 and (2.1) we have $(Tf)^\circ = (Tf)^* = Tf^* = Tf^\circ$, here (4.1) gives $(\mathrm{Rel}(T_*))A = TA$ and $(\mathrm{Rel}(T_*))\phi = T\phi_2.T\phi_1^\circ$. Because $\phi = \phi_2\phi_1^\circ$ by §1.6, while $T\phi_2.T\phi_1^\circ \leq T(\phi_2\phi_1^\circ)$ by (2.2), we have $(\mathrm{Rel}(T_*))\phi \leq T\phi$. Thus *there is a transformation*

$$n_T : \mathrm{Rel}(T_*) \to T \qquad (4.5)$$

in $\mathbf{F}$ *whose A-component is the identity of* TA . Moreover, *given a transformation* $f : T \to S : \mathrm{Rel} \; \mathcal{E} \to \mathrm{Rel} \; \mathcal{F}$ *in* $\mathbf{F}$, *we have*

$$fn_T = n_S.\mathrm{Rel}(f_*) \; ; \qquad (4.6)$$

for a transformation in $\mathbf{F}$ is fully determined by its components, and $(\mathrm{Rel}(f_*))_A = (f_*)_A = f_A$ by §4.1 and §2.7. So the n_T constitute a natural transformation

$$n : \mathrm{Rel}_{\mathcal{E},\mathcal{F}}(\;)_* \to 1 : \mathbf{F}(\mathrm{Rel} \; \mathcal{E} \; , \; \mathrm{Rel} \; \mathcal{F}) \to \mathbf{F}(\mathrm{Rel} \; \mathcal{E} \; , \; \mathrm{Rel} \; \mathcal{F}) \; .$$
$$(4.7)$$

Because the components of n_T are identities, n_T *is invertible precisely when it is an equality*; that is to say, when we have

$$T\phi = T\phi_2.T\phi_1^\circ \qquad (4.8)$$

for any relation ϕ , tabulated by ϕ_1 and ϕ_2 : such a T is said to be *tabulation-defined*. It follows from §4.1 and the above (using (4.6) in particular) that *a morphism* $T : \mathrm{Rel} \; \mathcal{E} \to \mathrm{Rel} \; \mathcal{F}$ *in* $\mathbf{F}$ *is of the form* $\mathrm{Rel} \; F$ *for some* $F : \mathcal{E} \to \mathcal{F}$ *if and only if it is tabulation-defined, and then* F *is necessarily* T_* . *More generally, if there is some* $F : \mathcal{E} \to \mathcal{F}$ *and an isomorphism* $f : T \cong \mathrm{Rel} \; F$, *we have* $f_* : T_* \cong F$, *and* $T = \mathrm{Rel}(T_*)$ *while* $f = \mathrm{Rel}(f_*)$. *The functor* $\mathrm{Rel}_{\mathcal{E},\mathcal{F}}$ *is fully faithful, every transformation* $f : \mathrm{Rel} \; F \to \mathrm{Rel} \; G : \mathrm{Rel} \; \mathcal{E} \to \mathrm{Rel} \; \mathcal{F}$ *being* $\mathrm{Rel} \; \alpha$ *for a unique* $\alpha : F \to G : \mathcal{E} \to \mathcal{F}$, *namely* $\alpha = f_*$. *The identity in (4.4) and the* n *of (4.7) are the unit and the counit of an adjunction* $\mathrm{Rel}_{\mathcal{E},\mathcal{F}} \dashv (\;)_*$.

Not every $T : \mathrm{Rel} \; \mathcal{E} \to \mathrm{Rel} \; \mathcal{F}$ in $\mathbf{F}$ is tabulation-defined; see §4.4 below. On the other hand, let us say that

T *preserves tabulations* if, whenever ϕ is a relation tabulated by ϕ_1 and ϕ_2, the maps $T\phi_1$ and $T\phi_2$ form a jointly-monic pair which tabulates $T\phi$. By §1.6, T preserves tabulations if and only if it is tabulation-defined and such pairs $(T\phi_1, T\phi_2)$ are jointly monic; thus T : Rel $\mathcal{E}$ → Rel $\mathcal{F}$ in **F** *is* Rel F *for an* F *that preserves jointly-monic pairs if and only if* T *preserves tabulations*. Since a general F in **Reg** need not preserve jointly-monic pairs, preserving tabulations is strictly stronger than being tabulation-defined.

4.3 We now determine those F : $\mathcal{E}$ → $\mathcal{F}$ in **Reg** for which the morphism Rel F : Rel $\mathcal{E}$ → Rel $\mathcal{F}$ of **F** is a lax functor, a colax functor, or a 2-functor. Consider first the effect of F on a pullback diagram:

(4.9)
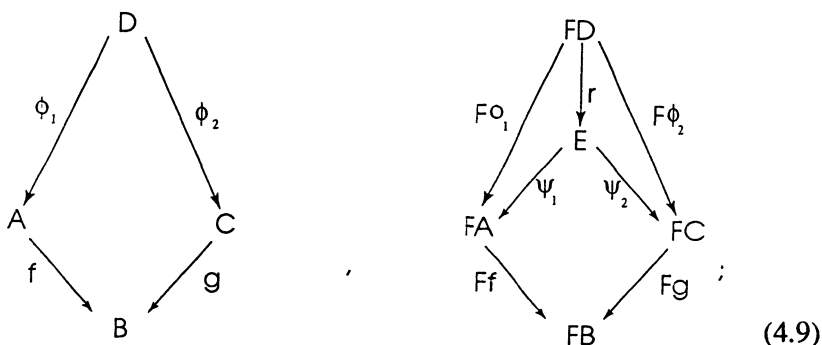
here the left diagram is an arbitrary pullback in $\mathcal{E}$, the interior diamond in the right diagram is a pullback in $\mathcal{F}$, and r is the unique map making that diagram commute; we say that F *nearly preserves pullbacks* if r is always a strong epimorphism. Of course, if F *preserves pullbacks*, in the sense that r is always invertible, it nearly preserves pullbacks; on the other hand, *if* F *preserves jointly-monic pairs and nearly preserves pullbacks, it preserves pullbacks*; for then r is monomorphic since $\langle F\phi_1, F\phi_2 \rangle$ is so, whence r is invertible if strongly epimorphic. {Note that an F that nearly preserves pullbacks preserves monomorphisms; when A = C and f = g is monomorphic, we can take $\phi_1 = \phi_2 = 1$, so that again r is monomorphic and hence invertible if strongly epimorphic.} It is immediate from §1.5, (4.2) and (4.3) that F *nearly preserves pullbacks if and only if* T = Rel F *satisfies*

$$T(g^{o}f) = Tg^{o}.Tf \ . \tag{4.10}$$

We conclude that Rel F *is a lax functor if and only if* F *nearly preserves pullbacks.* For (4.10) holds by §2.5 if T = Rel F is a lax functor; while for the converse, if $\theta\phi$ is a composite in Rel $\mathcal{E}$, (4.8), (4.10), (2.2) and §1.6 give

$$T\theta.T\phi = T\theta_2.T\theta_1^o.T\phi_2.T\phi_1^o = T\theta_2.T(\theta_1^o\phi_2).T\phi_1^o \le T(\theta_2\theta_1^o\phi_2\phi_1^o)$$
$$= T(\theta\phi) \ .$$

By §4.2, therefore, an arbitrary T : Rel $\mathcal{E}$ → Rel $\mathcal{F}$ *in* **F** *is* Rel F *for an* F *that nearly preserves pullbacks if and only if* T *is a tabulation-defined lax functor* (or equally, T is tabulation-defined and satisfies (4.10)). The last paragraph of §4.2 and the first paragraph of this section give a corollary: T : Rel $\mathcal{E}$ → Rel $\mathcal{F}$ *in* **F** *is* Rel F *for an* F *that preserves pullbacks and jointly-monic pairs if and only if* T *is a lax functor that preserves tabulations.* Going a little further, write I and I for the terminal objects of $\mathcal{E}$ and $\mathcal{F}$; then T : Rel $\mathcal{E}$ → Rel $\mathcal{F}$ *in* **F** *is* Rel F *for a left-exact* F *if and only if* T *is a lax functor, preserving tabulations, for which the unique map* TI → I *in* $\mathcal{F}$ *is invertible.* (We return in §5.9 to the study of left-exact F , giving an alternative characterization of the corresponding Rel F which, while built on that above, is more 2-categorical in spirit.)

To approach the case where Rel F is a *colax* functor, we begin by observing that F *preserves strong epimorphisms if and only if* T = Rel F *satisfies*

$$T(kh^{o}) = Tk.Th^{o} \tag{4.11}$$

for all h : D → A *and* k : D → C *in* $\mathcal{E}$. One direction is immediate from (4.3) and §1.7: if h is a strong epimorphism we have $hh^{o} = 1$, whence (4.11) gives $Th.Th^{o} = 1$, so that Fh = Th is a strong epimorphism. For the other direction, recall from §1.6 that $kh^{o} = \psi$ where $\langle h,k \rangle = \langle \psi_1,\psi_2 \rangle r$ for a strong epimorphism r ; this gives $\langle Fh,Fk \rangle = \langle F\psi_1,F\psi_2 \rangle.Fr$, so that if F preserves strong epimorphisms §1.6 gives $Fk.(Fh)^{o} = F\psi_2.(F\psi_1)^{o}$, which by (4.1) and (4.3) is (4.11).

We can now show that Rel F *is a colax functor if and only if* F *preserves strong epimorphisms.* Writing T for Rel F , we recall from §2.5 that (4.11) holds if T is a

colax functor; it remains to prove the converse. First, for a map g, we conclude from §1.6, (4.11) and (4.8) that $T(g\phi) = T(g\phi_2\phi_1^o) = T(g\phi_2).T\phi_1^o = Tg.T\phi_2.T\phi_1^o = Tg.T\phi$. Applying the involution $(\ )^o$ gives, by §4.1, $T(\psi f^o) = T\psi.Tf^o$; so that T satisfies $T(g\phi f^o) = Tg.T\phi.Tf^o$. Now, for composable θ and ϕ in Rel $\mathcal{E}$, this last, along with (2.2) and (4.8), gives

$$T(\theta\phi) = T(\theta_2\theta_1^o\phi_2\phi_1^o) = T\theta_2.T(\theta_1^o\phi_2).T\phi_1^o \leq T\theta_2.T\theta_1^o.T\phi_2.T\phi_1^o$$
$$= T\theta.T\phi ,$$

as desired. Since (4.11) contains (4.8) as a special case and holds by §2.5 for any colax functor, it follows from §4.2 that, for any morphism $T : $ Rel $\mathcal{E} \to$ Rel $\mathcal{F}$ in $\mathbf{F}$, the following are equivalent: T *is* Rel F *for an* F *that preserves strong epimorphisms;* T *is a colax functor;* T *satisfies* (4.11).
    Putting the results above together, we see that Rel F *is a 2-functor precisely when* F *preserves strong epimorphisms and nearly preserves pullbacks; and that every 2-functor* $T : $ Rel $\mathcal{E} \to$ Rel $\mathcal{F}$ *is* Rel F *for such an* F. Note that Rel F *is certainly a 2-functor if* F *preserves strong epimorphisms and pullbacks, and in particular if* F *preserves strong epimorphisms and is left exact.*

4.4 We now compare Rel(GF) with (Rel G)(Rel F) , where $F : \mathcal{E} \to \mathcal{F}$ and $G : \mathcal{F} \to \mathcal{G}$ in **Reg**. On objects, (4.1) gives

$$(Rel(GF))A = (Rel\ G)(Rel\ F)A = GFA .$$

For $\phi : A \to C$ in Rel $\mathcal{E}$, let $(Rel\ F)\phi = \psi$, let

$$(Rel\ G)(Rel\ F)\phi = (Rel\ G)\psi = \theta ,$$

and let $(Rel(GF))\phi = \chi$. Then by (4.2) we have

$$\langle F\phi_1, F\phi_2 \rangle = \langle \psi_1, \psi_2 \rangle r ,$$
$$\langle G\psi_1, G\psi_2 \rangle = \langle \theta_1, \theta_2 \rangle s ,$$

and

$$\langle GF\phi_1, GF\phi_2 \rangle = \langle \chi_1, \chi_2 \rangle t ,$$

where r , s , t are strong epimorphisms. Since the first two of these equations give $\langle GF\phi_1, GF\phi_2 \rangle = \langle \theta_1, \theta_2 \rangle s.Gr$, it follows that $(Rel(GF))\phi \leq (Rel\ G)(Rel\ F)\phi$, *with equality if and only if* $s.Gr$ *is a strong epimorphism.*

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We do not in general have equality here. Take $\mathcal{E} = \mathcal{F} = \mathcal{G} = \mathbf{Ab}$, the category of abelian groups; writing $\mathbf{Z}/n$ for the finite cyclic group, take $F = \mathbf{Z}/4 \otimes -$ and $G = \mathrm{Hom}(\mathbf{Z}/2, -)$; and let $\phi_1 = 2 : \mathbf{Z} \to \mathbf{Z}$ while $\phi_2 : \mathbf{Z} \to 0$. We find that s.Gr $= 0 : \mathbf{Z}/2 \to \mathbf{Z}/2$, which is not epimorphic.

By the second-last paragraph, *there is a transformation* $k_{G,F} : \mathrm{Rel}(GF) \to (\mathrm{Rel}\ G)(\mathrm{Rel}\ F) : \mathrm{Rel}\ \mathcal{E} \to \mathrm{Rel}\ \mathcal{G}$ *in* $\mathbf{F}$ *whose A-component is the identity of* GFA ; of course $(k_{G,F})_{\#} = 1$, *and* $k_{G,F}$ *is invertible precisely when it is an equality.* For nullary composites, we clearly have the equality $\mathrm{Rel}(1_{\mathcal{E}}) = 1_{\mathrm{Rel}\ \mathcal{E}}$, sometimes called the *normality* condition. When $\mathcal{K}$ and $\mathcal{L}$ are 2-categories, what Bénabou [1] calls a *morphism of bicategories* $\mathcal{K} \to \mathcal{L}$ is often called a *lax functor;* when $\mathcal{K}$ and $\mathcal{L}$ are merely **Ord**-categories, this reduces to the notion so named above. A *colax functor* has the sense of the comparison transformations reversed, to be that of our $k_{G,F}$. For a colax functor, these transformations have to be natural in F and G , and to satisfy three coherence conditions: all of which is trivial here because the $k_{G,F}$ have identities as components. We conclude that *the functors* $\mathrm{Rel}_{\mathcal{E},\mathcal{F}}$ *of §4.1, along with the transformations* $k_{G,F}$, *constitute a normal colax functor* $\mathrm{Rel} : \mathbf{Reg} \to \mathbf{F}$, *whose composite with the 2-functor* $(\ )_{\#} : \mathbf{F} \to \mathbf{Ord\text{-}Cat}$ *is the fully-faithful inclusion* $J : \mathbf{Reg} \to \mathbf{Ord\text{-}Cat}$ *of §4.1.*

There is a simple connection between the $k_{G,F}$ and the n_T of (4.5); for morphisms $T : \mathrm{Rel}\ \mathcal{E} \to \mathrm{Rel}\ \mathcal{F}$ and $S : \mathrm{Rel}\ \mathcal{F} \to \mathrm{Rel}\ \mathcal{G}$ in $\mathbf{F}$, we see that *the composite*

$$\mathrm{Rel}(S_{\#}T_{\#}) \xrightarrow[k_{S_{\#},T_{\#}}]{} \mathrm{Rel}(S_{\#})\mathrm{Rel}(T_{\#}) \xrightarrow[n_S n_T]{} ST \quad is \quad n_{ST} ,$$

because every component of each of these transformations is an identity. If we now take our example above of a non-invertible $k_{G,F}$ and set $S = \mathrm{Rel}\ G$ and $T = \mathrm{Rel}\ F$, so that $S_{\#} = G$ and $T_{\#} = F$ by §4.1 while n_S and n_T are equalities by §4.2, we see that n_{ST} is not invertible, so that ST provides an example, promised in §4.2, of a morphism $\mathrm{Rel}\ \mathcal{E} \to \mathrm{el}\ \mathcal{G}$ in $\mathbf{F}$ that is not tabulation-defined; it further provides an example promised at the end of §2.6.

Let us say of a functor $F : \mathcal{E} \to \mathcal{F}$ that it *almost preserves jointly-monic pairs* if, for any jointly-monic pair $(\phi_1 : \Phi \to A , \phi_2 : \Phi \to C)$, the strong epimorphism r in the factorization (4.2) is a retraction; of course F almost preserves jointly-monic pairs if it preserves them, but the

former condition is strictly weaker, being automatic when $\mathcal{F} = $ **Set** ; both conditions are *a fortiori* satisfied when F is left exact. It is clear from the last sentence of the first paragraph of this section that $k_{G,F}$ *is an equality* $Rel(GF) = (Rel\ G)(Rel\ F)$ *if either* F *almost preserves jointly-monic pairs or* G *preserves strong epimorphisms.*

Write **Reg$_{1ex}$**[resp. **Reg$_{se}$**] for the locally-full sub-2-categories of **Reg** given by restricting to those $F : \mathcal{E} \to \mathcal{F}$ that are left exact [resp. those that preserve strong epimorphisms], and **Reg$_{reg}$** for **Reg$_{1ex}$** $\cap$ **Reg$_{se}$**, where F now preserves the whole regular-category structure. We conclude from the last remark and the results of §4.3 that *the restrictions of* $Rel :$ **Reg** $\to$ F *to* **Reg$_{1ex}$** , *to* **Reg$_{se}$** , *and to* **Reg$_{reg}$** *are 2-functors, which take their values respectively in* $F_{\le}$, *in* $F_{\ge}$, *and in* $F_{=}$.

4.5 A general colax functor, unlike a 2-functor, does not produce an adjunction from an adjunction; yet the special properties of the colax functor Rel enable us to exhibit *a bijection between adjunctions*

$$x,y : S \dashv T : Rel\ \mathcal{E} \to Rel\ \mathcal{F} \qquad (4.13)$$

in F *and adjunctions*

$$\eta,\varepsilon : G \dashv F : \mathcal{E} \to \mathcal{F} \qquad (4.14)$$

in **Reg** . In one direction, the 2-functor $(\ )_{\#}$ applied to (4.13) gives as in (3.3) an adjunction

$$x_{\#},y_{\#} : S_{\#} \dashv T_{\#} : \mathcal{E} \to \mathcal{F} \qquad (4.15)$$

in **Ord-Cat**, which we may equally see as an adjunction (4.14) in **Reg** . For the other direction we start with (4.14); because the right adjoint F preserves jointly-monic pairs, while the left adjoint G preserves strong epimorphisms, the last section gives two distinct reasons for the conclusion $Rel(GF) = (Rel\ G)(Rel\ F)$. This allows us to define a transformation $v : (Rel\ G)(Rel\ F) \to 1$ by

$$v \text{ is } (Rel\ G)(Rel\ F) = Rel(GF) \xrightarrow[Rel\ \varepsilon]{} 1 , \qquad (4.16)$$

while we define a transformation $u : 1 \to (Rel\ F)(Rel\ G)$ by

u *is the composite* $1 \xrightarrow[\text{Rel } \eta]{} \text{Rel}(FG) \xrightarrow[k_{F,G}]{} (\text{Rel } F)(\text{Rel } G)$.

$$(4.17)$$

Since $(\text{Rel } \alpha)_\# = \alpha$ by §4.1 and $(k_{F,G})_\# = 1$ by §4.4, we have

$$u_\# = \eta \quad and \quad v_\# = \varepsilon . \qquad (4.18)$$

Because the triangular equations for u and v , when written in terms of components, are by §2.7 and (4.18) just those for η and ε , we do indeed have an adjunction

$$u,v : \text{Rel } G \dashv \text{Rel } F : \text{Rel } \mathcal{E} \to \text{Rel } \mathcal{F} \qquad (4.19)$$

in F . Clearly, if we now apply $(\ )_\#$ to (4.19), we regain (4.14). It remains to show that, if we begin with (4.13), and take for (4.14) the adjunction (4.15), so that $G = S_\#$, $F = T_\#$, $\eta = x_\#$, and $\varepsilon = y_\#$, the (4.19) we obtain is (4.13) itself.

By (3.12), however, the left adjoint S satisfies (4.11) in the form $S(kh^\circ) = Sk.Sh^\circ$, and *a fortiori* satisfies (4.8). By §4.2 therefore, $S = \text{Rel } G$. Note that (3.12) does not give (4.8) for T - it gives rather (4.10) - so we reason indirectly as follows. Since $S \dashv T$ by (4.13) and $S \dashv$ Rel F by (4.19), we have $T \cong \text{Rel } F$; by §4.2, this suffices for the conclusion $T = \text{Rel } F$. Now (4.18) gives $u_\# = x_\#$ and $v_\# = y_\#$, so that u = x and v = y by §2.7; this completes the proof.

Since, as we have just remarked, S satisfies (4.11) and T satisfies (4.10), it follows from §4.3 that, *for any adjunction* (4.13) *in* F , S *is a colax functor and* T *is a lax functor*; compare this with the remarks in §3.3, where the Rel $\mathcal{E}$ and Rel $\mathcal{F}$ of (4.13) are replaced by general objects $\mathcal{A}$ and $\mathcal{B}$ of F . As there, S is a 2-functor when it is lax as well as colax, which is to say that (4.13) is in fact an adjunction in $F_{\leq}$; by §4.3, this is so precisely when G nearly preserves pullbacks.

We call the adjunction (4.14) in **Reg** a *geometric morphism* when G is left exact. (In contrast to some authors, for whom the geometric morphism is F , or G , or the pair (F,G) , we understand by it the whole adjunction.) In other words a geometric morphism is an adjunction (4.14) in **Reg**$_{lex}$. By §4.4, the left exactness of G gives $k_{F,G} = 1$, so that *here* (4.17) *simplifies to* $u = \text{Rel } \eta$.

By §4.3, *the geometric morphisms* (4.14) *are in bijection with those adjunctions* (4.13) *in* $\mathbf{F}_\leq$ *for which* S *preserves tabulations and the unique map* SI → I *in* $\mathcal{E}$ *is invertible.* We return to geometric morphisms in §5.10, replacing the tabulation-preserving condition on S by one more 2-categorical in spirit; see the corresponding remark in §4.3 above.

4.6 We end with some comments on *mates* under a pair of adjunctions, in the sense of [16, Section 2]. Suppose that alongside (4.13) we have an adjunction x′,y′ : S′ ⊣ T′ : Rel $\mathcal{E}'$ → Rel $\mathcal{F}'$, and morphisms P : Rel $\mathcal{E}$ → Rel $\mathcal{E}'$, Q : Rel $\mathcal{F}$ → Rel $\mathcal{F}'$ in **F** ; then the formulas of [16] give a bijection, natural in the sense described there, between transformations f : QT → T′P and transformations g : S′Q → PS . Since mates are preserved by a 2-functor, the natural transformations

$$f_* : Q_*F \to F'P_* \quad \text{and} \quad g_* : G'Q_* \to P_*G$$

in **Reg** are again mates. Now suppose that, instead, we begin with the adjunction (4.14) and another such η′,ε′ : G′ ⊣ F′ : $\mathcal{E}'$ → $\mathcal{F}'$, with functors H : $\mathcal{E}$ → $\mathcal{E}'$ and K : $\mathcal{F}$ → $\mathcal{F}'$, and that we have a pair of mates λ : KF → F′H , μ : G′K → HG . The transformations $k_{H , F}$ and $k_{G', K}$ being identities by §4.4, we can define transformations

$$f : (Rel\ K)(Rel\ F) \to (Rel\ F')(Rel\ H)$$

and

$$g : (Rel\ G')(Rel\ K) \to (Rel\ H)(Rel\ G)$$

along the lines of (4.17) by

$$f = k_{F', H}.Rel\ \lambda , \quad g = k_{H , G}.Rel\ \mu . \tag{4.20}$$

The calculations leading from (4.17) to (4.18) here give $f_* = \lambda$ and $g_* = \mu$, so that f_* and g_* are mates. It follows that *in fact* f *and* g *are mates*; for if the mate of f is h , that of f_* is h_* , giving $g_* = h_*$, whence g = h by §2.7.

5. CARTESIAN OBJECTS IN F AND VARIOUS MORPHISMS BETWEEN THEM

5.1 Let $\mathcal{K}$ be any 2-category with finite products - in the 2-categorical sense, of course: the projections

$$\pi_1 : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \quad \text{and} \quad \pi_2 : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$$

induce for each $\mathcal{C}$ an *isomorphism of categories*

$$K(\mathcal{C}, \mathcal{A} \times \mathcal{B}) \cong K(\mathcal{C}, \mathcal{A}) \times K(\mathcal{C}, \mathcal{B}) ,$$

while for the terminal object 1 each $K(\mathcal{C}, 1)$ is a one-arrow category. For each object $\mathcal{A}$ of K we have the diagonal morphism $\delta : \mathcal{A} \to \mathcal{A} \times \mathcal{A}$ and the unique morphism $! : \mathcal{A} \to 1$; we call $\mathcal{A}$ a *cartesian object* of K when δ has a right adjoint $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and $!$ has a right adjoint $I : 1 \to \mathcal{A}$. Although there is a strong pressure of analogy to use $\times$ for $\otimes$ and $\ulcorner 1 \urcorner$ for I , it is probably less confusing in our context *not* to do so; when $K = \mathbf{F}$ and $\mathcal{A} = \text{Rel } \mathcal{C}$, for instance, $\otimes$ is not the product in Rel $\mathcal{C}$, nor I the terminal object, in the usual "external" sense; moreover our present notation agrees with that of the seminal article [6].

We denote the unit of the adjunction $\delta \dashv \otimes$ by $\Delta : 1 \to \otimes\delta : \mathcal{A} \to \mathcal{A}$, and the counit by

$$<p,q> : \delta\otimes \to 1 : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \times \mathcal{A} ;$$

the latter has the components

$$p : \otimes \to \pi_1 : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \quad \text{and} \quad q : \otimes \to \pi_2 : \mathcal{A} \times \mathcal{A} \to \mathcal{A} .$$

We leave it to the reader to write out the triangular equations in this generality; one of them splits into two component equations. For the unit of the adjunction $! \dashv I$ we use $t : 1 \to I! : \mathcal{A} \to \mathcal{A}$; here the counit is trivially the identity, and the triangular equations reduce to $tI = 1 : I \to I$.

Write

$$\alpha : (\mathcal{A} \times \mathcal{B}) \times \mathcal{C} \to \mathcal{A} \times (\mathcal{B} \times \mathcal{C}) , \quad \gamma : \mathcal{A} \times \mathcal{B} \to \mathcal{B} \times \mathcal{A} ,$$

and $\rho : \mathcal{A} \times 1 \to \mathcal{A}$ for the canonical associativity, commutativity, and unit isomorphisms for the product in K . Since we trivially have

$$(\delta \times \mathcal{A})\delta = \alpha^{-1}(\mathcal{A} \times \delta)\delta : \mathcal{A} \to (\mathcal{A} \times \mathcal{A}) \times \mathcal{A} ,$$

we have for any cartesian $\mathcal{A}$ an isomorphism

$$a : \otimes(\otimes \times \mathcal{A}) \to \otimes(\mathcal{A} \times \otimes)\alpha : (\mathcal{A} \times \mathcal{A}) \times \mathcal{A} \to \mathcal{A} \qquad (5.1)$$

in $\varkappa$ between the corresponding right adjoints. Similarly we have isomorphisms in $\varkappa$

$$c : \otimes \to \otimes\gamma : \mathcal{A} \times \mathcal{A} \to \mathcal{A} , \ r : \otimes(\mathcal{A} \times I) \to \rho : \mathcal{A} \times 1 \to \mathcal{A} .$$

$$(5.2)$$

When the 2-category $\varkappa$ is of the form $\mathcal{V}$-**Cat** for some symmetric monoidal closed $\mathcal{V}$, a cartesian object $\mathcal{A}$ is just a $\mathcal{V}$-category that admits finite products in the usual sense; the only such cases that occur below are those where $\varkappa = $ **Cat** or **Ord-Cat** . Here, of course, we *do* write $\times$ for $\otimes$, although we retain I for the terminal object of $\mathcal{A}$; now the units and counits are given by their *components*

$$\Delta_A : A \to A \times A , \ p_{AB} : A \times B \to A ,$$

$$(5.3)$$

$$q_{AB} : A \times B \to B , \ t_A : A \to I ,$$

as are a , c , and r .

Let $\varkappa'$ be a second 2-category with finite products; we still use $\times$ and 1 for the products in $\varkappa'$, but we use δ' , $\otimes '$, a' for the $\varkappa'$- analogues of δ , $\otimes$, a , and so on. Suppose now that $\Phi : \varkappa \to \varkappa'$ is a 2-functor which *preserves finite products*, in the sense that the usual comparison morphisms $\mu : \Phi(\mathcal{A} \times \mathcal{B}) \to \Phi\mathcal{A} \times \Phi\mathcal{B}$ and $\nu : \Phi 1 \to 1$ are invertible. For our applications it suffices to suppose that μ and ν are identities; if they are not, the formulae that follow need slight but trivial modifications, without any change to the essential conclusions. For any $\mathcal{A}$ in $\varkappa$, the diagonal $\delta' : \Phi\mathcal{A} \to \Phi\mathcal{A} \times \Phi\mathcal{A}$ is $\Phi\mathcal{A}$ and the unique $\Phi\mathcal{A} \to 1$ is $\Phi!$; so that *if $\mathcal{A}$ is cartesian so is $\Phi\mathcal{A}$* , with $\otimes' = \Phi\otimes$ and $I' = I$, with p' $= \Phi$p , q' $= \Phi$q , $\Delta' = \Phi\Delta$, t' $= \Phi$t , *and moreover* (since Φ preserves mates) *with* a' $= \Phi$a , c' $= \Phi$c , *and* r' $= \Phi$r .

In particular, taking Φ to be $\varkappa(\mathcal{C},-) : \varkappa \to$ **Cat** , we see that a' $= \varkappa(\mathcal{C},a)$ is the usual associativity isomorphism in **Cat** . The *coherence* of a , c , r for any cartesian $\mathcal{A}$ in $\varkappa$ now follows from the classical coherence of a' , c' , r' in **Cat** , since 2-cells $\alpha,\beta : P \to Q : \mathcal{A} \to \mathcal{B}$ coincide if $\varkappa(\mathcal{C},\alpha) = (\varkappa(\mathcal{C},\beta)$ for all $\mathcal{C}$.

5.2 Consider now a morphism $T : \mathcal{A} \to \bar{\mathcal{A}}$ in $\varkappa$ between cartesian objects, where $\bar{\mathcal{A}}$ has $\bar\delta,\bar\otimes,I,\bar a$ and so on. We have equalities $\bar\delta T = (T \times T)\delta : \mathcal{A} \to \bar{\mathcal{A}} \times \bar{\mathcal{A}}$ and $\bar! T = ! : A \to 1$; these equalities, regarded as identity 2-cells, have *mates*

$$\tilde{T} : T\otimes \to \bar{\otimes}(T \times T) , \quad T^\circ : TI \to I . \qquad (5.4)$$

By the results of [16], $\tilde{T}$ is the unique 2-cell with

$$\bar{p}(T \times T).\tilde{T} = Tp \quad \text{and} \quad \bar{q}(T \times T).\tilde{T} = Tq , \qquad (5.5)$$

while T° is simply the unique 2-cell. In the case $\kappa = V\text{-}\mathbf{Cat}$, $\tilde{T}$ is given by its components

$$\tilde{T}_{AB} : T(A \times B) \to TA \times TB ,$$

and is the usual comparison morphism, the components of (5.5) then being

$$\bar{p}_{TA,TB}\tilde{T}_{AB} = Tp_{AB} , \quad \bar{q}_{TA,TB}\tilde{T}_{AB} = Tq_{AB} . \qquad (5.6)$$

When $\Phi : \kappa \to \kappa'$ is a finite-product-preserving 2-functor as above, we can compare $(\Phi T)^{\sim}$ and $\Phi(\tilde{T})$, these having the same domain and codomain since $\Phi(T \times T) = \Phi T \times \Phi T$; similarly for $(\Phi T)^\circ$ and $\Phi(T^\circ)$. In fact it is immediate from (5.5) and the uniqueness of T° that

$$(\Phi T)^{\sim} = \Phi(\tilde{T}) , \quad (\Phi T)^\circ = \Phi(T^\circ) . \qquad (5.7)$$

We call $T : \mathcal{A} \to \bar{\mathcal{A}}$ a *cartesian morphism* if $\tilde{T}$ and T° are invertible 2-cells of κ ; that is to say, if " T preserves finite products" in the internal sense - which in the case of $V\text{-}\mathbf{Cat}$ is equally the usual external sense. (Although aware that this nomenclature conflicts with the use of "cartesian arrow" in the theory of fibrations, we think it unambiguous in the context.) It follows from (5.7) that: *when* Φ *is finite-product-preserving,* ΦT *is a cartesian morphism whenever* T *is so.* Finally, we observe that *any right adjoint* $T : \mathcal{A} \to \bar{\mathcal{A}}$ *in* κ *between cartesian objects is a cartesian morphism.* For, this being classically so when $\kappa = \mathbf{Cat}$, we conclude that each $\kappa(\mathcal{C},T)$ is cartesian, and hence by (5.7) that $\kappa(\mathcal{C},\tilde{T})$ and $\kappa(\mathcal{C},T^\circ)$ are invertible for each $\mathcal{C}$, whence $\tilde{T}$ and T° are invertible 2-cells in κ .

5.3 We now seek conditions for an object $\mathcal{A}$ of $\mathbf{F}$ to be cartesian. First, $\delta : \mathcal{A} \to \mathcal{A} \times \mathcal{A}$ admits a right adjoint $\otimes$ in $\mathbf{F}$ if and only if it satisfies Conditions (i)-(iv) of Theorem 3.5. Since δ is clearly a 2-functor, Condition (i) is automatic. Because $\delta_* : \mathcal{A}_* \to (\mathcal{A} \times \mathcal{A})_* = \mathcal{A}_* \times \mathcal{A}_*$ is the usual diagonal in $\mathbf{Ord\text{-}Cat}$, Condition (ii) is the requirement that the $\mathbf{Ord}$-category $\mathcal{A}_*$ admit binary products; we denote these

by $A \times B$, with the units and counits as in (5.3). Next,

$$\delta_{AB} : \mathcal{A}(A,B) \longrightarrow (\mathcal{A} \times \mathcal{A})((A,A) , (B,B)) = \mathcal{A}(A,B) \times \mathcal{A}(A,B)$$

being the usual diagonal in **Ord,** Condition (iii) is the requirement that each ordered hom-set $\mathcal{A}(A,B)$ admit binary infima $\phi \wedge \psi$. Now Condition (iv) takes the form

$$p_{AB}^* p_{AB} \wedge q_{AB}^* q_{AB} \leq 1 : A \times B \longrightarrow A \times B \tag{5.8}$$

which - see the remarks preceding the proof of Theorem 3.5 - is an equality when it holds. Similarly, for $! : \mathcal{A} \rightarrow 1$ to have in **F** a right adjoint $I : 1 \rightarrow \mathcal{A}$ (which we identify of course with an object I of $\mathcal{A}$) , Condition (i) is again automatic; Condition (ii) is the requirement that the **Ord**-category $\mathcal{A}_\#$ have a terminal object I , whose unit we denote as in (5.3) by $t_A : A \rightarrow I$; Condition (iii) is the requirement that each ordered hom-set $\mathcal{A}(A,B)$ admit a top element τ_{AB} ; and Condition (iv) - trivially an equality when it holds - takes the form

$$\tau_{II} \leq 1_I : I \rightarrow I . \tag{5.9}$$

In summary, *the object $\mathcal{A}$ of* **F** *is cartesian if and only if* (i) *the* **Ord**-*category $\mathcal{A}_\#$ admits finite products,* (ii) *each ordered set $\mathcal{A}(A,B)$ admits finite infima, and* (iii) *we have* (5.8) *and* (5.9).

The counter-example in §3.6 shows precisely that (5.9) cannot be omitted; one presumes that the same is true of (5.8). The alternative formulation of (5.8) corresponding to (3.23) here reads

$$p_{AB}\phi \leq f \quad and \quad q_{AB}\phi \leq g \quad imply \quad \phi \leq \langle f,g \rangle , \tag{5.10}$$

where $\phi : C \rightarrow A \times B$ is any arrow in $\mathcal{A}$ while $f : C \rightarrow A$ and $g : C \rightarrow B$ are selected maps; the more special but equivalent formulation (3.24) contemplates $\phi : A \times B \rightarrow A \times B$ and reads

$$p_{AB}\phi \leq p_{AB} \quad and \quad q_{AB}\phi \leq q_{AB} \quad imply \quad \phi \leq 1_{A \times B} . \tag{5.11}$$

Similarly the formulation (3.23) of (5.9) becomes

$$\tau_{AI} = t_A ; \tag{5.12}$$

while (3.24), as the special case $A = I$ of this, is just

(5.9) again.

Suppose now that $\mathscr{A}$ is a cartesian object of $\mathbf{F}$, that the finite products in $\mathscr{A}_*$ (along with their units and co-units) are given, and that $\otimes$ is then constructed as in the proof of Theorem 3.5; this ensures that we have an actual equality $\otimes_* = \times$, and not merely an isomorphism. So $\otimes$ is given on objects by $A \otimes B = A \times B$ (and it is often appro-priate to replace $\times$ by $\otimes$ when writing the maps (5.3)), while by (3.14) the value of $\otimes$ on arrows $\phi : A \to C$ and $\psi : B \to D$ is

$$\phi \otimes \psi = (p_{CD}^{\star}\phi p_{AB}) \wedge (q_{CD}^{\star}\psi q_{AB}) ; \qquad (5.13)$$

of course - recall our convention whereby Roman letters de-note selected maps - we have $f \otimes g = f \times g$. Because δ is a 2-functor it follows from §3.3 that $\otimes : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is a (normal) lax functor; that is

$$1_A \otimes 1_B = 1_{A \otimes B} , \quad (\phi' \otimes \psi')(\phi \otimes \psi) \leq \phi'\phi \otimes \psi'\psi . \qquad (5.14)$$

However we have equality in some cases: whether by §2.5 ($\otimes$ being a lax functor) or by (3.12) ($\otimes$ being a right adjoint), we have

$$(h^{\star} \otimes k^{\star})(\phi \otimes \psi)(f \otimes g) = (h^{\star}\phi f) \otimes (k^{\star}\psi g) . \qquad (5.15)$$

Recalling from §2.9 that the product in $\mathbf{F}$ is also the pro-duct in $\mathbf{F}_{\leq}$, in $\mathbf{F}_{\geq}$, and in $\mathbf{F}_{=}$, we conclude (cf. §3.3) that $\mathscr{A}$ is a cartesian object not only in $\mathbf{F}$ but also in $\mathbf{F}_{\leq}$; the case where (5.14) is always an equality, so that $\otimes$ is a 2-functor, is of course that in which $\mathscr{A}$ is a cartesian object of $\mathbf{F}_{=}$.

Since, by Theorem 3.5, <p,q>, Δ, and t are transfor-mations in $\mathbf{F}$, (2.3) gives for $\phi : A \to C$ and $\psi : B \to D$ the inequalities

$$p_{CD}(\phi \otimes \psi) \leq \phi p_{AB} , \quad q_{CD}(\phi \otimes \psi) \leq \psi q_{AB} ,$$

$$\Delta_C\phi \leq (\phi \otimes \phi)\Delta_A , \quad t_C\phi \leq t_A . \qquad (5.16)$$

Because, in the notation of §3.2, $\check{\delta}$ here is $\wedge$ and $\check{!}$ is $\top$, the first formula of (3.7) gives

$$\phi \wedge \psi = \Delta_B^\star(\phi \otimes \psi)\Delta_A \ , \qquad \tau_{AB} = t_B^\star t_A \ , \qquad (5.17)$$

for ϕ , ψ : $A \to B$; since $t_I = 1_I$, (5.12) is a special case of the second of these equations.

Consider finally the isomorphisms a , c , r of (5.1) and (5.2) for $\mathcal{A}$. By §5.1, $a_\#$, $c_\#$, $r_\#$ are the usual canonical isomorphisms for the cartesian product in $\mathcal{A}_\#$; since a and $a_\#$, for instance, have by §2.7 the same components, we use a for $a_\#$ when no confusion is likely. The a of (5.1), however, is an isomorphism in F ; consequently, by §2.6, we have commutativity in

$$
\begin{array}{ccc}
(A \otimes B) \otimes C & \xrightarrow{\ a_{ABC}\ } & A \otimes (B \otimes C) \\
{\scriptstyle (\phi \otimes \psi)\, \otimes\, \theta}\Big\downarrow & & \Big\downarrow{\scriptstyle \phi\, \otimes\, (\psi \otimes \theta)} \\
(D \otimes E) \otimes F & \xrightarrow[\ a_{DEF}\ ]{} & D \otimes (E \otimes F)
\end{array}
$$

$$(5.18)$$

for arbitrary arrows ϕ, ψ, θ in $\mathcal{A}$; and similarly for c and r .

5.4 The bicategories considered by Carboni and Walters in [6] are in fact **Ord**-categories. They define in [6, Definition 1.2] the notion of a *cartesian bicategory* - but not as a cartesian object in some 2-category, no 2-category apt for this purpose being known at the time of their writing. In hindsight, comparing our conditions for cartesianness in §5.3 with [6, Theorem 1.6] shows that an **Ord**-category $\mathcal{A}$ is a cartesian bicategory in their sense precisely when, with $\mathcal{A}_\#$ consisting of all maps, $\mathcal{A}$ is a cartesian object in $F_=$. Note that our (5.8) - or rather its generalization (5.13) - as well as (5.9) occur in the conditions of their Theorem 1.6; the last two inequalities of our (5.16) occur in their Definition 1.2; our (5.17) occurs in the proof of their Theorem 1.6; and our (5.18) is part of their Definition 1.1.

5.5 We turn to some examples of cartesian objects in F , each of which is in fact cartesian in $F_=$. Recalling the nomenclature in the last paragraph of §4.4, observe that, for regular categories $\mathcal{E}$ and $\mathcal{F}$, the usual product $\mathcal{E} \times \mathcal{F}$ in **Cat** is a regular category; that its projections onto $\mathcal{E}$ and

$\mathcal{F}$ lie in $\mathbf{Reg_{reg}}$; and that it constitutes the product of $\mathcal{E}$ and $\mathcal{F}$ not only in **Cat** (or in **Ord-Cat**, for that matter) but also in each of $\mathbf{Reg}$, $\mathbf{Reg_{lex}}$, $\mathbf{Reg_{se}}$, and $\mathbf{Reg_{reg}}$. Similarly, the one-arrow category 1 is terminal in each of these. In fact, every regular category $\mathcal{E}$ is a cartesian object in $\mathbf{Reg_{reg}}$, and so *a fortiori* in **Reg** and in the intermediate 2-categories above. For $\delta : \mathcal{E} \to \mathcal{E} \times \mathcal{E}$ and $! : \mathcal{E} \to 1$, with their usual meanings in **Cat**, lie in $\mathbf{Reg_{reg}}$, while the same is true of their right adjoints $\times : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ and $I : 1 \to \mathcal{E}$ in **Cat** : these preserve finite limits since they are right adjoints, and preserve strong epimorphisms since $f \times g$ is a strong epimorphism - as the composite of the pullbacks $f \times 1$ and $1 \times g$ - when f and g are such.

For our first example, which is the central one for the present article, consider the 2-functor $\mathrm{Rel} : \mathbf{Reg_{reg}} \to \mathbf{F_=}$ of §4.4; when we recall from §2.9 the description of the product in $\mathbf{F}$, which is also that in $\mathbf{F_\leq}$, in $\mathbf{F_\geq}$, and in $\mathbf{F_=}$, it is clear from §4.1 that Rel preserves products. Accordingly, by §5.1, Rel sends the cartesian object $\mathcal{E}$ of $\mathbf{Reg_{reg}}$ to an object $\mathrm{Rel}\,\mathcal{E}$ that is cartesian in $\mathbf{F_=}$ and *a fortiori* in $\mathbf{F_\leq}$ and in $\mathbf{F}$. By §5.1 again, the I of $\mathrm{Rel}\,\mathcal{E}$ is just the terminal object I of $\mathcal{E}$, while the $\otimes$ of $\mathrm{Rel}\,\mathcal{E}$ is $\mathrm{Rel}(\times)$ - so that $A \otimes B = A \times B$ and, for $\phi : A \to C$ and $\psi : B \to D$ in $\mathrm{Rel}\,\mathcal{E}$, *the relation* $\phi \otimes \psi : A \otimes B \to C \otimes D$ *is that tabulated by*

$$(\phi_1 \times \psi_1 : \Phi \times \Psi \to A \times B , \quad \phi_2 \times \psi_2 : \Phi \times \Psi \to C \times D) .$$
$$(5.19)$$

Since, by §4.1, the components of $\mathrm{Rel}\,p$ are those of p , and so on, we may as well by §5.1 denote the units and co-units for $\mathrm{Rel}\,\mathcal{E}$ by the same letters p , q , Δ , t used for $\mathcal{E}$; which agrees not only with our standard notation for a general cartesian object, but also with the notation of §1.4 for $\mathrm{Rel}\,\mathcal{E}$. Of course the 2-functor

$$(\ )_* : \mathbf{F} \to \mathbf{Ord\text{-}Cat}$$

sends $\mathrm{Rel}\,\mathcal{E}$ back to $\mathcal{E}$, seen now as a cartesian object of **Ord-Cat**.

For our second example we refer back to §1.11, observing that in a later article we shall exhibit $\mathcal{E} \mapsto \mathrm{Idl}\,\mathcal{E}$ as part

of a product-preserving 2-functor $Idl : \mathbf{Reg}_{reg} \to \mathbf{F}_=$; thus $Idl\ \mathcal{E}$ too is a cartesian object of $\mathbf{F}_=$. The $\otimes$ of $Idl\ \mathcal{E}$ is again given by (5.19), the $\phi \otimes \psi$ there being an ideal when ϕ and ψ are ideals.

Our final example is of quite a different flavour, and we establish its cartesianness by a direct argument. We remarked in §1.1 that $\mathbf{Ord}$ is itself an $\mathbf{Ord}$-category; in fact $\phi \le \psi : A \to B$ in $\mathbf{Ord}$ precisely when $\phi a \le \psi a$ for all $a \in A$. Now write $\mathcal{A} = \mathbf{Ord}_{f i}$ for the full sub-$\mathbf{Ord}$-category of $\mathbf{Ord}$ determined by those ordered sets that admit finite infima, and make it into an object of $\mathbf{F}$ by selecting *all* of its maps. These, of course, are those monotone functions $f : A \to B$ that have right adjoints f^* in $\mathbf{Ord}$; note that an arrow in $\mathcal{A}$ - even a map - is not required to preserve the finite infima. The $\mathbf{Ord}$-category $\mathcal{A}$ has a terminal object 1 and binary products $A \times B$, the latter being the usual product of sets with the pointwise order; thus $\delta : \mathcal{A} \to \mathcal{A} \times \mathcal{A}$ and $! : \mathcal{A} \to 1$ have in $\mathbf{Ord\text{-}Cat}$ right adjoints $\times : \mathcal{A} \to \mathcal{A}$ and $I : 1 \to \mathcal{A}$. Since $\mathcal{A}_{\#} = \mathrm{Map}\ \mathcal{A}$, these 2-functors $\times$ and I are by §2.5 morphisms in $\mathbf{F}_=$. Moreover the components (5.3) of the units and the counits are in fact maps: $\Delta^*_A : A \times A \to A$ sends (a,a') to $a \wedge a'$, $p^*_{AB} : A \to A \times B$ sends a to (a,τ_B) , $q^*_{AB} : B \to A \times B$ sends b to (τ_A,b) , and $t^*_A : I \to A$ sends the unique element of I to τ_A . Since the inequalities (5.16) are valid here, being in fact equalities, the units and counits above are transformations in $\mathbf{F}$; accordingly the adjunctions $\delta \dashv \times$ and $! \dashv I$ lie in $\mathbf{F}_=$, so that $\mathcal{A} = \mathbf{Ord}_{fi}$ is a cartesian object of $\mathbf{F}_=$.

The cartesian objects $\mathcal{A}$ of $\mathbf{F}_=$ which are, to within biequivalence, of the form $\mathrm{Rel}\ \mathcal{E}$ for a regular $\mathcal{E}$ are characterized (although not in the language of $\mathbf{F}$ - see §5.4 above) by Carboni and Walters in [6, Theorem 3.5]; they are those for which $\mathcal{A}_{\#} = \mathrm{Map}\ \mathcal{A}$, for which every object is *discrete* in the sense of [6, Definition 2.1], and which are *functionally complete* in the sense of [6, Definition 3.1].

5.6 Consider now a morphism $T : \mathcal{A} \to \bar{\mathcal{A}}$ in $\mathbf{F}$ where $\mathcal{A}$ and $\bar{\mathcal{A}}$ are cartesian objects of $\mathbf{F}$. The transformation $\tilde{T}$ of (5.4) in $\mathbf{F}$ has components $\tilde{T}_{AB} : T(A \otimes B) \to TA \bar{\otimes} TB$ in $\bar{\mathcal{A}}_{\#}$. Since

$$(\tilde{T})_{\#} = (T_{\#})^{\sim} \quad and \quad (T^{o})_{\#} = (T_{\#})^{o} \tag{5.20}$$

by (5.7), and since a transformation f in $\mathbf{F}$ has the same components as $f_{\#}$, it follows from (5.5) that $\tilde{T}_{AB}$ is *the unique selected map (that is, the unique arrow in $\overline{\mathscr{A}}_{\#}$) satisfying (5.6), while $T : TI \to I$ is simply the unique arrow in $\overline{\mathscr{A}}_{\#}$*.

As we saw more generally in §5.2, it follows from (5.20) that $T_{\#}$ *is cartesian whenever* T *is so*; recall from the same §5.2 that, for $T_{\#} : \mathscr{A}_{\#} \to \overline{\mathscr{A}}_{\#}$, to be cartesian is just to preserve finite products - that is, to have T^{o} and each $\tilde{T}_{AB}$ invertible. Note that, by §2.6, the cartesianness of T requires, *besides* this invertibility of T^{o} and the $\tilde{T}_{AB}$, *also* the commutativity for all arrows ϕ and ψ of the diagram

$$
\begin{array}{ccc}
T(A \otimes B) & \xrightarrow{\tilde{T}_{AB}} & TA \;\overline{\otimes}\; TB \\
{\scriptstyle T(\phi \otimes \psi)}\Big\downarrow & & \Big\downarrow{\scriptstyle T\phi \;\overline{\otimes}\; T\psi} \\
T(C \otimes D) & \xrightarrow[\tilde{T}_{CD}]{} & TC \;\overline{\otimes}\; TD \;.
\end{array}
\tag{5.21}
$$

Note that, since by §5.3 the cartesian objects of $\mathbf{F}$ are in fact cartesian objects of $\mathbf{F}_{\leq}$, to say of $T : \mathscr{A} \to \overline{\mathscr{A}}$ that it is a cartesian morphism in $\mathbf{F}$ is, when T happens to be a lax functor, equally to say that T is a cartesian morphism in $\mathbf{F}_{\leq}$. Similarly, when $\mathscr{A}$ and $\overline{\mathscr{A}}$ like Rel $\mathscr{C}$ and Rel $\mathscr{D}$ are cartesian objects of $\mathbf{F}_{=}$, to say of a 2-functor T that it is a cartesian morphism in $\mathbf{F}$ is equally to say that it is a cartesian morphism in $\mathbf{F}_{=}$.

We make use below of the following observation: *for cartesian objects $\mathscr{A}$ and $\overline{\mathscr{A}}$ of $\mathbf{F}$ and a cartesian morphism $T : \mathscr{A} \to \overline{\mathscr{A}}$ in $\mathbf{F}_{\leq}$, each $T_{AB} : \mathscr{A}(A,B) \to \overline{\mathscr{A}}(TA,TB)$ preserves finite infima*. To see this, note that by (5.6) the composite of $T\Delta_{A} : TA \to T(A \otimes A)$ and $\tilde{T}_{AA} : T(A \otimes A) \to TA \;\overline{\otimes}\; TA$ is $\overline{\Delta}_{TA}$; whence, replacing A by B, taking right adjoints, using (2.1), and observing that the right adjoint of the invertible $\tilde{T}_{BB}$ is its inverse, we have $\overline{\Delta}_{TB}^{\star}.\tilde{T}_{BB} = T\Delta_{B}^{\star}$. Combining these equations with (5.21) for the cartesian T gives

us, for $\phi, \psi : A \rightarrow B$, the equation

$$\bar{\Delta}^{\star}_{TB}(T\phi \; \bar{\otimes} \; T\psi)\bar{\Delta}_{TA} = T\Delta^{\star}_{B}.T(\phi \otimes \psi).T\Delta_{A} \; ;$$

by §2.5. T being a lax functor, the right side here is $T(\Delta^{\star}_{B}(\phi \otimes \psi)\Delta_{A})$; so that by (5.17) our equation reads $T\phi \wedge T\psi = T(\phi \wedge \psi)$. The argument for nullary infima is similar but easier, using the trivial $T^{o}.Tt_{A} = \bar{t}_{TA} : TA \rightarrow I$, the right adjoint $Tt^{\star}_{B} = \bar{t}^{\star}_{TB}.T^{o}$ of this, and the second equation of (5.17) .

5.7 Since, as we have seen, the product in $\mathbf{Reg}_{reg}$ is equally that in $\mathbf{Reg}$ and the product in $\mathbf{F}_{=}$ is equally that in $\mathbf{F}$, the expressions

$$F \times F' : \mathcal{E} \times \mathcal{E}' \rightarrow \mathcal{F} \times \mathcal{F}' \text{ and } T \times T' : \mathcal{A} \times \mathcal{A}' \rightarrow \mathcal{B} \times \mathcal{B}'$$

have unambiguous meanings for arbitrary morphisms F,F' in $\mathbf{Reg}$ and T,T' in $\mathbf{F}$. We remarked in §5.5 that the 2-functor Rel : $\mathbf{Reg}_{reg} \rightarrow \mathbf{F}_{=}$ preserves products; but it is equally clear from §4.1 that more is true; not only do we have $\mathrm{Rel}(\mathcal{E} \times \mathcal{E}') = \mathrm{Rel} \; \mathcal{E} \times \mathrm{Rel} \; \mathcal{E}'$, but for *any* $F : \mathcal{E} \rightarrow \mathcal{F}$ and $F' : E' \rightarrow \mathcal{F}'$ in $\mathbf{Reg}$ we have

$$\mathrm{Rel}(F \times F') = (\mathrm{Rel} \; F) \times (\mathrm{Rel} \; F') \; .$$

In particular we have $\mathrm{Rel}(F \times F) = (\mathrm{Rel} \; F) \times (\mathrm{Rel} \; F)$.
Consider now the effect of Rel on the mates

$$\tilde{F} : F\times \rightarrow \bar{\times}(F \times F) \text{ and } F^{o} : FI \rightarrow I$$

of the appropriate

identity 2-cells; we are using here $\bar{\times}$ and I for the products in $\mathcal{F}$, and shall use $\otimes = \mathrm{Rel}(\times)$ and $\bar{\otimes} = \mathrm{Rel}(\bar{\times})$ - see §5.5 - for the tensor products in Rel $\mathcal{E}$ and Rel $\mathcal{F}$. While the result (5.22) below follows quite easily from §4.6, some may prefer the following elementary argument. Since, as we remarked in §5.5, $\times$ and $\bar{\times}$ lie in $\mathbf{Reg}_{reg}$, §4.4 gives $\mathrm{Rel}(F\times) = (\mathrm{Rel} \; F)\otimes$ and

$$\mathrm{Rel}(\bar{\times}(F \times F)) = \bar{\otimes} \; \mathrm{Rel}(F \times F) = \bar{\otimes}((\mathrm{Rel} \; F) \times (\mathrm{Rel} \; F)) \; ;$$

accordingly (Rel F) and Rel($\tilde{\text{F}}$) have the same domain and codomain. Because (Rel F)I = F I , the same is trivially the case for (Rel F)$^{\text{o}}$ and Rel(F$^{\text{o}}$) . In fact we have

$$(\text{Rel F})^{\sim} = \text{Rel}(\tilde{\text{F}}) \ , \qquad (\text{Rel F})^{\text{o}} = \text{Rel}(\text{F}^{\text{o}}) \ . \qquad (5.22)$$

The second of these is immediate: each is the unique map FI → I . As for the first, the components of $\tilde{\text{F}}$ satisfy the analogue of (5.6); but the components of Rel $\tilde{\text{F}}$ and Rel p being those of $\tilde{\text{F}}$ and of p by §4.1, and (Rel F)p$_{\text{AB}}$ being Fp$_{\text{AB}}$ by (4.3), the (Rel $\tilde{\text{F}}$)$_{\text{AB}}$ also satisfy (5.6); the desired result now follows from the first paragraph of §5.6.

We conclude from (5.20) and (5.22), along with (Rel α)$_{\text{*}}$ = α from §4.1 and the results of §4.2, that *a functor* F : $\mathcal{E}$ → $\mathcal{F}$ *in* **Reg** *is cartesian (that is, finite-product-preserving) if and only if* Rel F : Rel $\mathcal{E}$ → Rel $\mathcal{F}$ *is a cartesian morphism in* **F** *; while a morphism* T : Rel $\mathcal{E}$ → Rel $\mathcal{F}$ *in* **F** *is of the form* Rel F *for a cartesian* F *if and only if* T *is tabulation-defined and cartesian.*

We remark that Rel F for a cartesian F need be neither a lax functor nor a colax one. Let **Grp** be the category of groups and F : **Grp** → **Ab** the abelianization functor sending A to A/[A,A] ; although F preserves finite products, it does not preserve monomorphisms - as we see by considering the inclusion into a non-abelian simple group of one of its cyclic subgroups; by the first paragraph of §4.3, therefore, F does not nearly preserve pullbacks; thus, by §4.3 once more, Rel F is not a lax functor. Again, F = Hom(Z/2, -) : **Ab** → **Ab** , although cartesian, does not preserve the strong epimorphism Z → Z/2 ; by §4.3, therefore, Rel F is not a colax functor.

5.8 The characterizations in §4.3 of those T : Rel $\mathcal{E}$ → Rel $\mathcal{F}$ in **F** of the form Rel F for a left-exact F , and in §4.6 of those adjunctions S ⊣ T : Rel $\mathcal{E}$ → Rel $\mathcal{F}$ in **F** arising from applying Rel to a geometric morphism G ⊣ F : $\mathcal{E}$ → $\mathcal{F}$ in **Reg**, have the disadvantage that the "preserving tabulations" condition they contain refers to the *internal* structure of Rel $\mathcal{E}$; in the next two sections we replace this condition by others which can be stated directly in terms of **F** . One of these is cartesianness; but we need another as well, for which we now prepare the ground.

For a map i : A → B in an **Ord**-*category* $\mathcal{A}$, *the following are equivalent*: (i) *is a coretraction in* $\mathcal{A}$, *in that we*

*have   θi = 1   for some arrow   θ : B → A ;   (ii)   i   is
"representably fully faithful", in the sense that, for any
arrows   φ,ψ : C → A ,   the inequality   iφ ≤ iψ   implies
φ ≤ ψ ;   (iii)   we have   i*i = 1ₐ .*   It is in fact trivial
that (i) implies (ii) and that (iii) implies (i); since the
adjunction inequalities   ii* ≤ 1   and   1 ≤ i*i   together give
ii*i = i ,   it follows that (ii) implies (iii).

We call such a map   i   in   ⍲   *intrinsically monomorphic.*
Recalling from §1.7 that   f* = f°   for a map   f   in   Rel 𝓔 ,
we see from §1.4 that the intrinsically monomorphic maps in
Rel 𝓔   are precisely the monomorphisms in   𝓔 .

5.9   Theorem. *For a morphism   T : Rel 𝓔 → Rel 𝓕   in   F   to
be of the form   Rel F   for a left-exact   F : 𝓔 → 𝓕 ,   it is
necessary and sufficient that   T   be a cartesian morphism in
F≤   satisfying*

$$T(ii\star) = Ti.Ti\star \qquad (5.23)$$

*for every intrinsically monomorphic map   i : A → B   in   Rel 𝓔
- that is, for every monomorphism   i   in   𝓔 .   For   T   to be
of the form   Rel F   for an   F   in   **Reg**ᵣₑ𝓰 ,   it is necessary
and sufficient that   T   be a cartesian morphism in   F₌ .*

Remark.   The latter assertion is essentially [6, Corollary
3.6].

Proof.   To facilitate reference to results above, let us
write   f°   rather than   f*   for a map   f   in   Rel 𝓔 .   As for
the necessity of the conditions, we saw in §4.3 that   T   must
be a lax functor; it must by §5.7 be cartesian, since any
left-exact   F   is cartesian in   **Reg**;   and it must satisfy
(5.23), by (4.8) applied to the relation   ψ = ii° : B → B
tabulated by   (i : A → B ,   i : A → B) .   Turning to the
sufficiency, it suffices by the result of §4.3 - since the
invertibility of the unique map   TI → I   is part of cartesi-
anness - to show that   T   preserves tabulations.

We first show that   T   preserves the tabulations of such
relations as   ψ = ii°   with   i   monomorphic in   𝓔 .   By §2.5,
the lax functor   T   applied to the equality   i°i = 1   gives
Ti°.Ti = 1 ,   so that   Ti   is by §1.7 a monomorphism in   𝓕 ;
since   Tψ = Ti.Ti°   by (5.23),   Tψ   is indeed the relation
tabulated by (Ti,Ti) .

We now turn to a general relation   φ : A → B ,   using

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Proposition 1.10 to describe its tabulation (ϕ_1, ϕ_2). We write $\bar{p}$ and $\bar{q}$ for $\bar{p}_{TA,TB}$ and $\bar{q}_{TA,TB}$, and $\tilde{T}$ for the invertible $\tilde{T}_{AB}$. With $\psi = 1 \wedge q^{\circ}\phi p$ as in Proposition 1.10, the final observation of §5.6 gives us $T\psi = 1 \wedge T(q^{\circ}\phi p)$, which is $1 \wedge Tq^{\circ}.T\phi.Tp$ by §2.5 since T is a lax functor. Now (5.6) gives, since the right adjoint of $\tilde{T}$ is its inverse,

$$T\psi = 1 \wedge \tilde{T}^{-1}\bar{q}^{\circ}.T\phi.\bar{p}\tilde{T} = \tilde{T}^{-1}(1 \wedge \bar{q}^{\circ}.T\phi.\bar{p})\tilde{T} .$$

Since, as we have just seen, the tabulation of $T\psi$ is (Ti, Ti), it follows from Proposition 1.10 that that of $T\phi$ is $(\bar{p}\tilde{T}.Ti, \bar{q}\tilde{T}.Ti)$, which by (5.6) again is (Tp.Ti, Tq.Ti), or $(T\phi_1, T\phi_2)$, as desired. The latter part of the theorem now follows from the final paragraph of §4.3, since (5.23) is automatically satisfied by a 2-functor T.

5.10 Theorem. *In the bijection of §4.5 between adjunctions $\eta, \varepsilon : G \dashv F : \mathcal{E} \to \mathcal{F}$ in* **Reg** *and adjunctions $x, y : S \dashv T :$ Rel $\mathcal{E} \to$ Rel $\mathcal{F}$ in* **F** *, the former is a geometric morphism if and only if S is a cartesian morphism in* $\mathbf{F}_{\leq}$ *. Then S is in fact a 2-functor, T too is a cartesian morphism in* $\mathbf{F}_{\leq}$ *, and S $\dashv$ T may be seen as an adjunction in* $\mathbf{F}_{\leq}$ *.*

Proof. Since, by (3.12), the left adjoint S automatically satisfies the S-version $S(ii^{\star}) = Si.Si^{\star}$ of (5.23), the first assertion follows at once from Theorem 5.9. Then, by Theorem 5.9 again, the right adjoint $F = T_{\#}$ being left exact, T too is a cartesian morphism in $\mathbf{F}_{\leq}$. Because S is necessarily a colax functor by §4.5, it is in fact a 2-functor.

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