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∞-GROUPOIDS AND HOMOTOPY TYPES

by M.M. KAPRANOV and V.A. VOEVODSKY

RÉSUMÉ. Nous présentons une description de la categorie homotopique des CW-complexes en termes des ∞-groupoïdes. La possibilité d'une telle description a été suggérée par A. Grothendieck dans son mémoire "A la poursuite des champs".

It is well-known [GZ] that CW-complexes X such that $\pi_i(X,x) = 0$ for all $i \geq 2$, $x \in X$, are described, at the homotopy level, by groupoids. A. Grothendieck suggested, in his unpublished memoir [Gr], that this connection should have a higher-dimensional generalisation involving polycategories, viz. polycategorical analogues of groupoids. It is the purpose of this paper to establish such a generalisation.

To do this we deal with the globular version of the notion of a polycategory [S1], [MS], [BH2], where a p-morphism has the "shape" of a p-globe i.e. of a p-ball whose boundary is subdivided into two (p-1)-balls, whose common boundary is subdivided into two (p-2)-balls etc. This notion is to be contrasted with the (more general) cubical notion studied in [BH1], [L].

Usually one defines groupoids (among all categories) by requiring either invertibility of all morphisms, or (equivalently) solvability of binary equations of the form $ax = b$, $ya = b$ in situations when these equations make sense. It is the second approach that we take up for the definition of n-groupoids among n-categories. But we require only *weak* solvability of such equations. This means that if b is a p-morphism, $p<n$, then ax and b are required to be not necessarily equal, but merely connected by a (p+1)-morphism (see Definition 1.1). Also, there are various types of binary equations, since ax can be understood in the sense of one of the compositions $\underset{i}{*}$, $i = 0,\dots,n-1$. Therefore we require more than just (weak) invertibility of each p-

morphism with respect to $*_{p-1}$. In fact, our axioms can be viewed as certain coherence conditions for weak inverses.

For a weak n-groupoid, $n \leq \infty$, we define its homotopy groups in a purely algebraic way similar to the definition of homology groups of a chain complex. We call a weak equivalence of n-groupoids (the analog of a quasi-isomorphism of chain complexes) a morphism which induces isomorphisms of all the homotopy groups and sets π_0. Our main result (corollary 3.8 of theorem 3.7) states that the localisation of the category of n-groupoids with respect to weak equivalences is equivalent to the category of homotopy n-types.

There are several versions of a notion of a polygroupoid in the literature, but they do not lead to a description of homotopy n-types. The first is the cubical version due the J.-L. Loday [L] who called them n-categorical groups. He proved that every homotopy n-type can be represented as the nerve of some (n-1)-categorical group, but in many essentially different ways. In fact, n-categorical groups naturally describe n-cubes of spaces and not spaces themselves [L]. The second version due to R. Brown and P.J. Higgins [BH2] is based on strict $*_{p-1}$-invertibility of all p-morphisms in a globular n-category. However, it turns out that for $n \geq 3$ such strict n-groupoids do not represent all homotopy n-types [BH2]. Finally a notion of a weak n-groupoid was mentioned by R. Street [S1] but his conditions amount only to weak invertibility of each p-morphism with respect to $*_{p-1}$. This does not suffice, for example, to ensure the Kan condition for the nerve of such a category.

Our proof of theorem 3.6 is based on a construction of Poincaré (weak) ∞-groupoid $\Pi\mathrm{sing}(T)$ of a CW-space T. Intuitively it is clear that objects of $\Pi\mathrm{sing}(T)$ should correspond to points of T, 1-morphisms to paths, 2-morphisms to homotopies between paths etc. This is precisely the approach proposed by Grothendieck. There is, however, a serious obstacle to realisation of this idea. Namely, what are 1-morphisms? We cannot consider only homotopy classes of paths, since a notion of homotopies between them, i.e. 2-morphisms, then loses its sense. Therefore, we need paths themselves. If we call paths just maps $\gamma : [0,1] \longrightarrow T$, then we face the problem of defining a composition $\gamma_1 \circ \gamma_2$ of such maps. Usually this is done by dividing $[0,1]$ to $[0,1/2]$ and $[1/2,1]$, dilating each of them to $[0,1]$ and then apply γ_i. But this composition is not strictly associative, but only associative up to homotopy. This has led Grothendieck to the idea of considering "weak ∞-categories" where all the identities (associativities and so on) hold only up to higher morphisms. Such a theory would be, of course, highly natural, but it has never been fully developed.

There is a well-known way to overcome the non-associativity of the usual composition of paths, namely considering Moore paths, i.e., maps $[0,n] \longrightarrow T$, where the "length" $n \in \mathbb{Z}_+$ may be arbitrary. The composition of a path of length m and a path of length n is a path of length $m+n$, and this composition is strictly associative. Having done this first step we have the problem of defining 2-morphisms. Clearly we have to connect Moore paths of different lengths. Therefore, for each $m,n > 0$ we are led to introduce elementary 2-morphisms as maps of a polygon with $m+n$ edges oriented as in fig 1A.

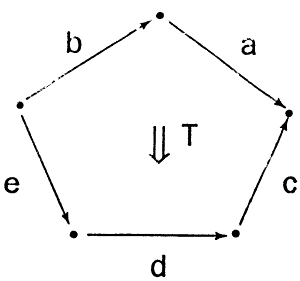

Fig. 1A

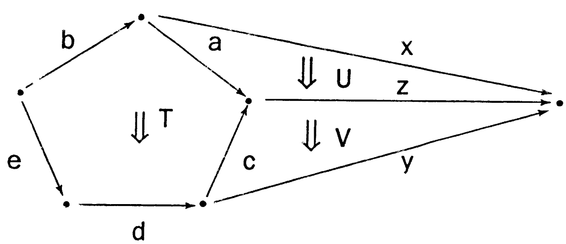

Fig. 1B

As before, there is no way to associatively compose such 2-morphisms, except formally. This means that we have to consider Moore homotopies of the form depicted in Fig. 1B, i.e. roughly speaking, maps $P \longrightarrow T$, where P is an oriented polygon subdivided, in a compatible way, into several other oriented polygons. This gives a good 2-category. If we want 3-morphisms between two such "subdivided" polygons P,Q with common boundary then we are forced to introduce elementary 3-morphisms as maps $\Sigma \longrightarrow T$, where Σ is something like a 3-dimensional polytope with boundary $P \cup Q$. Then we have to consider formal composition of these, and so on and so on.

The right language to accomplish this task is not that of polytopes but of composable pasting schemes introduced by M. Johnson [J]. They are n-categorical analogues of algebraic expressions. In fact the category J of these schemes (see §.2) is a natural candidate for what A. Grothendieck [Gr] called a *coherator,* i.e. category incorporating all possible coherence relations. We introduce the category J^0Set of diagrammatic sets generalising simplicial sets and show that it is a model category for the homotopy theory. By using these techniques we then prove our theorem.

Finally we would like to thank V.V. Schechtman for useful discussions.

1. WEAK N-GROUPOIDS AND THEIR HOMOTOPY CATEGORY.

We use the globular version of the notion of an n-category, see [BH2], [MS], [S1]. An n-category C is given by the following data:

1) Sets $C_0, \ldots, C_n$, whose elements are called objects (0-morphisms), 1-morphisms, 2-morphisms, ..., n-morphisms. We denote also $C_i = Mor_i C$, $C_0 = ObC$.

2) Maps

$$\begin{tikzcd}[column sep=large]
C_0 \arrow[r, bend right=60, "Id_1"'] & C_1 \arrow[l, shift right=1, "s_0"'] \arrow[l, shift left=1, "t_0"] \arrow[r, bend right=60, "Id_2"'] & \cdots \arrow[l, shift right=1, "s_1"'] \arrow[l, shift left=1, "t_1"] \arrow[r, bend right=60, "Id_n"'] & C_n \arrow[l, shift right=1, "s_{n-1}"'] \arrow[l, shift left=1, "t_{n-1}"]
\end{tikzcd}$$

satisfying the relations:

2a) $s_{i-1}s_i = s_{i-1}t_i$, $t_{i-1}s_i = t_{i-1}t_i$.

2b) $s_i Id_i = t_i Id_i = 1_{A_i}$.

We shall denote (somewhat abusively) by

$$t_k, s_k : C_p \longrightarrow C_k \ , \ Id_p : C_k \longrightarrow C_p \ , \quad p \geq k \ ,$$

the composite maps $t_k \ldots t_{p-1}$, $s_k \ldots s_{p-1}$, and $Id_p \ldots Id_{k+1}$.

3) Partial compositions $\underset{i}{*}$ on C_p for $0 \leq i \leq p-1$ such that $a \underset{i}{*} b$ is defined whenever $s_i a = t_i b$.

These multiplications should satisfy the following axioms [MS]:

(A0) Let $f, g \in C_p$, $q < p$, $s_q f = t_q g$. Then
$s_p(f \underset{q}{*} g) = s_p(f) \underset{q}{*} s_p(g)$, $t_p(f \underset{q}{*} g) = t_p(f) \underset{q}{*} t_p(g)$
for $q < p-1$,
$s_p(f \underset{q}{*} g) = s_p(g)$, $t_p(f \underset{q}{*} g) = t_p(f)$ for $q = p-1$.

(Ass1) $(f \underset{q}{*} g) \underset{q}{*} h = f \underset{q}{*} (g \underset{q}{*} h)$

(Ass2) Let $p < q$, $f, f', g, g' \in C_r$ and $t_q(f) = s_q(f')$, $t_q(g) = s_q(g')$, $t_p(f) = s_p(g)$, $t_p(f') = s_p(g')$.

Then $(f \underset{q}{*} f') \underset{p}{*} (g \underset{q}{*} g') = (f \underset{p}{*} g) \underset{q}{*} (f' \underset{p}{*} g')$.

(Id) $f \underset{q}{*} Id_{q-1}(s_{q-1}(f)) = Id_{q-1}(t_{q-1}(f)) \underset{q}{*} f = f$.

An *∞-category* is defined similarly, but sets $C_i = Mor_i C$ should be given for all $i \geq 0$.

The maps s_i and t_i are called the source and target maps of dimension i . For a p-morphism $f \in C_p$ we shall sometimes use the notation $f : a \longrightarrow b$ which will mean that $s_{p-1}(f) = a$, $t_{p-1}(f) = b$.

An n-functor (or simply a functor, when no confusion can occur) between two n-categories C, D is a family of maps $C_i \longrightarrow D_i$ compatible with all the structures described. We denote by Cat_n the (1-)-category of all small n-categories and their functors. Here $n = 0,1,...,\infty$. The category Cat_n has obvious direct products, and an (n+1)-category can be seen as a category enriched in the cartesian closed category Cat_n , see [S1]. In particular, for any two objects x,y of an n-category C we have an (n-1)-category $Hom_C(x,y)$.

Each n-category C can be regarded as an m-category for any $m \geq n$, if we set $C_i = C_n$ for $i \geq n$.

For an n-category C and $k \leq n$ there are two natural k-categories: $\sigma_{\leq k}C$ and $\tau_{\leq k}C$. By definition,

$$(\sigma_{\leq k}C)_p = C_p \quad \text{for } p \leq k ,$$
$$(\tau_{\leq k}C)_p = C_p \quad \text{for } p < k ,$$
$$(\tau_{\leq k}C)_k = \mathrm{Coker}(s_k,t_k:C_{k+1} \to C_k).$$

We shall call them the *stupid* and *canonical truncations* of C , see [MS].

Now we turn to the notion of a (weak) n-groupoid.

Definition 1.1. An n-category C $(n \leq \infty)$ is called an n-*groupoid* if for all $i < k \leq n$ the following conditions $(GR'_{i,k})$, $(GR''_{i,k})$ hold:

$(GR'_{i,k}, i < k-1)$ For each $a \in C_{i+1}$, $b \in C_k$, $u, v \in C_{k-1}$ such that

$$s_i(a) = t_i(u) = t_i(v) , \; a \underset{i}{*} u = s_{k-1}(b) , \; a \underset{i}{*} v = t_{k-1}(b)$$

there exist $x \in C_k$, $\varphi \in C_{k+1}$ such that

$$s_k(\varphi) = a \underset{i}{*} x , \; t_k(\varphi) = b , \; s_{k-1}(x) = u , \; t_{k-1}(x) = v .$$

$(GR'_{k-1,k})$ For each $a \in C_k$, $b \in C_k$ such that $t_{k-1}(a) = t_{k-1}(b)$ there exist $x \in C_k$, $\varphi \in C_{k+1}$ such that

$$s_k(\varphi) = a \underset{k-1}{*} x \ , \ t_k(\varphi) = b \ .$$

($GR''_{i,k}$, $i < k-1$) For each $a \in C_{i+1}$, $b \in C_k$, $u,v \in C_{k-1}$ such that

$$t_i(a) = s_i(u) = s_i(v) \ , \ u \underset{i}{*} a = s_{k-1}(b) \ , \ v \underset{i}{*} a = t_{k-1}(b)$$

there exist $x \in C_k$, $\varphi \in C_{k+1}$ such that

$$s_k(\varphi) = x \underset{i}{*} a \ , \ t_k(\varphi) = b \ , \ s_{k-1}(x) = u \ , \ t_{k-1}(x) = v \ .$$

($GR''_{k-1,k}$) For each $a \in C_k$, $b \in C_k$ such that $s_{k-1}(a) = s_{k-1}(b)$ there exist $x \in C_k$, $\varphi \in C_{k+1}$ such that

$$s_k(\varphi) = x \underset{k-1}{*} a \ , \ t_k(\varphi) = b \ .$$

Remarks.

1) Note that the conditions of Street's definition [S1] are equivalent to our ($GR''_{k-1,k}$) and ($GR'_{k-1,k}$) (and therefore, our definition is more restrictive). Namely, Street assumes the existence of a two-sided $\underset{p-1}{*}$-quasi-inverse for each $a \in C_p$. Our conditions ($GR''_{k-1,k}$) and ($GR'_{k-1,k}$) imply the existence of left and right quasi-inverses. But it is easy to see (similarly to the well-known lemma for groups) that each left quasi-inverse is also a right quasi-inverse and vice versa.

2) Our additional requirements on an ∞-groupoid imply the existence of a "coherent" system of quasi-inverses. Let us explain this in more detail. Let a be an i-morphism in an n-category C , and a^{-1} its (two-sided) $\underset{i-1}{*}$-quasi-inverse. Then, there are (i+1)-morphisms

$$\rho_a(a \underset{i-1}{*} a^{-1} \longrightarrow \mathrm{Id}(t_{i-1}(a)) \ , \ \lambda_a : a^{-1} \underset{i-1}{*} a \longrightarrow \mathrm{Id}(s_{i-1}(a)) \ .$$

From them, we can construct two (i+1)-morphisms

$$a \underset{i-1}{*} \rho_a \ , \ \lambda_a \underset{i-1}{*} a : a \underset{i-1}{*} a^{-1} \underset{i-1}{*} a \longrightarrow a$$

and it is natural to ask whether there is a homotopy connecting them, i.e. an (i+2)-morphism $\rho_{a,2} : \lambda_a \underset{i-1}{*} a \longrightarrow a \underset{i-1}{*} \rho_a$. Similarly, there are two (i+1)-morphisms

$$a^{-1} \underset{i-1}{*} \lambda_a \ , \ \rho_a \underset{i-1}{*} a^{-1} : a^{-1} \underset{i-1}{*} a \underset{i-1}{*} a^{-1} \longrightarrow a^{-1}$$

and we can look for an (i+2)-morphism

$$\lambda_{a,2} : a^{-1} \underset{i-1}{*} \lambda_a \longrightarrow \rho_a \underset{i-1}{*} a^{-1}$$

connecting them. Assuming their existence, we can construct two (i+2)- morphisms $\lambda_a \underset{i}{*} (\rho_{a,2} \underset{i-1}{*} a^{-1})$, $\lambda_a \underset{i}{*} (a \underset{i-1}{*} \lambda_{a,2})$ from

$$\lambda_a \underset{i}{*} (\lambda_a \underset{i-1}{*} a \underset{i-1}{*} a^{-1}) = \lambda_a \underset{i}{*} (a \underset{i-1}{*} a^{-1} \underset{i-1}{*} \lambda_a)$$

to $\lambda_a \underset{i}{*} (a \underset{i-1}{*} \rho_a \underset{i-1}{*} a^{-1})$, where the latter (i+1)-morphisms act from $a \underset{i-1}{*} a^{-1} \underset{i-1}{*} a \underset{i-1}{*} a^{-1}$ to $Id(t_{i-1}(a))$. It is natural to require the existence of a connecting (i+3)-morphism $\lambda_{a,3}$ and so on. Clearly there is no way to construct such a system of homotopies from Street's definition. But having our conditions we can construct them recursively: λ_a is arbitrary, ρ_a is a weak solution of the "equation" $a^{-1} \underset{i-1}{*} \lambda_a \approx \rho_a \underset{i-1}{*} a^{-1}$, $\lambda_{a,2}$ is the corresponding connecting homotopy, $\rho_{a,2}$ is a weak solution of the "equation"

$$a \underset{i}{*} (\rho_{a,2} \underset{i-1}{*} a^{-1}) \approx \lambda_a \underset{i}{*} (a \underset{i-1}{*} \lambda_{a,2})$$

and so on.

It is easy to show that for 2-categories being a 2-groupoid in our sense is equivalent to the existence of such a coherent system of quasi-inverses for any morphism. It would be interesting to investigate this connection in a general case.

We denote by $Grp_n \subset Cat_n$ the full subcategory of all n-groupoids. Clearly if G is an n-groupoid then $\tau_{\leq k}(G)$ (but not in general $\sigma_{\leq k}(G)$) is a k-groupoid for any $k \leq n$. Also, if x,y are to objects of an n-groupoid G then the (n-1)-category $Hom_G(x,y)$ is a (n-1)-groupoid.

Let G be an n-groupoid, and $i \leq n$, $x \in Ob\, G$. Consider the set

$$Hom^i_G(x,x) : = \{f \in G_i : s_{i-1}(f) = t_{i-1}(f) = Id_{i-1}(x)\} .$$

Introduce on it a relation $\approx$("homotopy") setting $f \approx g$ if there is $u \in G_{i+1}$ such that $s_i(u) = f$, $t_i(u) = g$.

Proposition 1.2. *If* G *is an* n-*groupoid then the relation* $\underset{i}{\approx}$ *on each* $\mathrm{Hom}^i_G(x,x)$ *is an equivalence relation.*

Proof. The transitivity is obvious. Let us prove the symmetry. Suppose $f \approx g$ and consider $u \in G_{i+1}$ such that $s_i(u) = f$, $t_i(u) = g$. Let $v \in G_{i+1}$ be the $\underset{i}{*}$-quasi-inverse for u. The existence of v implies that $g \approx f$.

Introduce also the relation $\approx$ on G_0 setting $x \approx y$ if there is a morphism from x to y. Clearly it is also an equivalence relation.

Definition 1.3. For an n-groupoid G, $x \in \mathrm{Ob}\, G$, $0 < i \le n$, the i-*th homotopy set* $\pi_i(G,x)$ of G with base object x is defined as the quotient $\mathrm{Hom}^i_G(x,x)/\approx$ by the homotopy relation. We also set $\pi_0(G) = (\mathrm{Ob}\, G)/\approx$.

Proposition 1.4. *For* $i \ge 1$ *the operation* $\underset{i-1}{*}$ *endows* $\pi_i(G,x)$ *with a structure of a group, Abelian for* $i \ge 2$.

Proof. Almost obvious. The commutativity for $i \ge 2$ follows from two-dimensional associativity (Ass2).

Clearly each functor $f : G \longrightarrow G'$ of n-groupoids induces a map $\pi_0(f) : \pi_0(G) \longrightarrow \pi_0(G')$ and group homomorphisms $\pi_i(f) : \pi_i(G,x) \longrightarrow \pi_i(G',f(x))$ for $i > 0$, $x \in \mathrm{Ob}\, G$.

Call a morphism (i.e. a functor) $f : G \longrightarrow G'$ of n-groupoids a weak equivalence if it induces bijections $\pi_0(G) \longrightarrow \pi_0(G')$ and $\pi_i(G,x) \longrightarrow \pi_i(G',f(x))$ for $i > 0$, $x \in \mathrm{Ob}\, G$. The class of all weak equivalences will be denoted $W_n \subset \mathrm{Mor}\,(\mathrm{Grp}_n)$. Denote $\mathrm{HoGr}_n = \mathrm{Grp}_n[W_n^{-1}]$ the localisation of Grp_n with respect to the family W_n.

Proposition 1.5. *The full subcategory in* $\mathrm{Grp}_n[W_n^{-1}]$ *whose objects are k-groupoids* $(k \le n)$, *is equivalent to* $\mathrm{Grp}_k[W_k^{-1}]$.

Proof. Use the truncation functor $\tau_{\le k}$.

Proposition 1.6. *An* n-*category* C *is an* n-*groupoid if and only if for each* $x,y \in \mathrm{Ob}\, C$ *the* (n-1)-*category* $\mathrm{Hom}_C(x,y)$ *is an* (n-1)-*groupoid and for any* $x, y, z, t \in \mathrm{Ob}\, G$, *any* 1-*morphism* $f : y \longrightarrow z$ *the functors*

$$\mathrm{Hom}_C(x,y) \longrightarrow \mathrm{Hom}_C(x,z)\ ,\quad \mathrm{Hom}_C(z,t) \longrightarrow \mathrm{Hom}_C(y,t)$$

given by composition with f *, are weak equivalences.*

Proof. Left to the reader.

2. PASTING SCHEMES AND DIAGRAMMATIC SETS

A general theory of what an algebraic expression in an n-category should be was developed by M. Johnson [J], see also [S2], [KV], [P]. To this end he introduced combinatorial objects called *composable (loop free* and *well-formed) pasting schemes.* The definitions of Johnson (especially that of the loop-free property) are very cumbersome. Moreover, these subtleties will not play any essential role in our considerations. Therefore we shall not give detailed definitions, relying heavily on [J] and on our paper [KV].

A pasting scheme A is a collection of finite sets A_i , $i = 0,1,2,\ldots$, $A_i = \emptyset$ for $i >> 0$ and binary relations $E_i, B_i \subset A_i \times A_{i+1}$ called "end" and "beginning". Elements of A_k should be thought of as k-cells, and any such cell a should be thought of as an "indeterminate" k-morphism (in our algebraic expression) from

$$\prod_{b:(b,a)\in B_{k-1}} b \quad \text{to} \quad \prod_{b:(b,a)\in E_{k-1}} b\ ,$$

where the products are to be understood as pasting. For example, on Fig. 1B the cell T is a 2-morphism from ab to cde , U-from x to za , V-from zc to y , and the result of pasting (the "product" of T,U and V) is a 2-morphism from xb to yde .

Remark. In fact, Johnson includes in the data for a pasting scheme also the relations E_i^j, $B_i^j \subset A_i \times A_j$, $i \le j$ of beginnings and ends for higher codimensions. But for composable pasting schemes (see below) these relations can be recovered from $E_i = E_i^{i+1}$, $B_i = B_i^{i+1}$. See [KV]. §1 for explicit formulas.

The data (A_i,E_i,B_i) are subject to three groups of axioms (being a pasting scheme, well-formedness and loop-freeness) which we do not recall here, referring to [J]. We shall call the system $A = (A_i,E_i,B_i)$ satisfying these axioms a *composable pasting scheme* (CPS) and denote dimA (the dimension of A) the maxim[illegible]

For a CPS A of dimension n an n-category $\mathrm{Cat}(A)$ was constructed in [J]. Its polymorphisms are composable pasting subschemes in A, the compositions $\underset{i}{*}$ are given (when defined) by the union. For the description of the operators s_i, t_i in $\mathrm{Cat}(A)$ see [J], [KV].

Simplices Δ^n and cubes $\square^n$ are canonically endowed with structures of composable pasting schemes [S1-2], [A], [KV]. The n-category $\mathrm{Cat}(\Delta^n)$ is the n-th oriental of Street [S1]. Another important (and, in a sense, simplest) example of a CPS is the n-dimensional globe G^n [S2]. Explicitly, G^n has one n-dimensional cell c_n and, for each $i \in \{0,\ldots,n-1\}$, two i-dimensional cells e_i, b_i. The structure is as follows:
for $i \leq n-1$ we have

$$E_{i-1}(e_i) = E_{i-1}(b_i) = e_{i-1}\ ,\ B_{i-1}(e_i) = B_{i-1}(b_i) = e_{i-1}\ ,$$

and

$$E_{n-1}(c_n) = e_{n-1}\ ,\ B_{n-1}(c_n) = b_{n-1}\ .$$

The category $\mathrm{Cat}(G^n)$ was denoted by Street in [S1] by 2_n.

If A is a CPS and $a \in A_n$ is a cell, then we denote, following [J], by $R(a)$ the smallest (composable) pasting subscheme in A containing a.

Definition 2.1. Let A, A' be composable pasting schemes. A *morphism* $f : A \longrightarrow A'$ is, by definition, a functor $\mathrm{Cat}(A) \longrightarrow \mathrm{Cat}(A')$ sending each polymorphism of the form $R(a)$, $a \in A_k$, to a polymorphism of the form $R(a')$, $a' \in A'_l$, $l \leq k$.

The category of all CPS's and morphisms just defined will be denoted J.

Remark. There are natural functors $\Delta \longrightarrow J$, $\square \longrightarrow J$, where Δ and $\square$ are the standard simplicial and cubical categories respectively. The first of them is a full embedding.

Definition 2.2 A *diagrammatic* object in a category C is a contravariant functor $J \longrightarrow C$. The category of such objects will be denoted J^0C.

For example, to any CPS A we can associate a diagrammatic set $[A]$ (the representable functor).

All the homotopy theory of simplicial (or cubical) sets [GZ] can be carried almost verbatim to the case of diagrammatic sets. Let us outline this generalisation, omitting tedious repetitions.

For a CPS $A \in \mathrm{Ob}\, J$ its geometric realisation $|A|$ is defined [KV] and is a contractible CW-complex. This defines a functor $|?| : J \longrightarrow \mathrm{Top}$. It generates, in the usual way, a pair of adjoint functors

$$|?| : J^0\mathrm{Set} \longrightarrow \mathrm{Top}\ ,\ \mathrm{Sing} : \mathrm{Top} \longrightarrow J^0\mathrm{Set}\ ,$$

called the *geometric realisation* and the *singular diagrammatic* set functors. Explicitly,

$$|X| = \varinjlim_{A \in J} X(A) \times |A| \quad \text{for} \quad X \in J^0\mathrm{Set}\ ,$$

$\mathrm{Sing}(T)(A) = \mathrm{Hom}_{\mathrm{Top}}(|A|,T)$ for $T \in \mathrm{Ob\,Top}$, $A \in \mathrm{Ob}\, J$. It will be useful for us to consider them as a particular case of a more general construction of Kan extensions (or coends) [M], [GZ], which we shall now recall.

Let $\mathcal{A}$, $\mathcal{C}$ be categories, $\mathcal{C}$ has direct limits and $F : \mathcal{A} \longrightarrow \mathcal{C}$ is a covariant functor. Define a functor $S_F : \mathcal{C} \longrightarrow \mathrm{Funct}(\mathcal{A}^0,\mathrm{Set})$ setting for $T \in \mathrm{Ob}\,\mathcal{C}$, $A \in \mathrm{Ob}\,\mathcal{A}$, $S_F(T)(A) = \mathrm{Hom}_{\mathcal{C}}(F(A),T)$.

Proposition 2.3. [GZ]. *If* $\mathcal{C}$ *has direct limits, then the functor* S_F *has a left adjoint* L_F .

Explicitly, for a contravariant functor $\varphi : \mathcal{A} \longrightarrow \mathrm{Set}$ the object $L_F(\varphi)$ is $\varinjlim_{A \in \mathcal{A}} F(A) \times \varphi(A)$. Here $F(A) \times \varphi(A)$ is the direct sum of $\varphi(A)$ copies of $F(A)$.

Now we define the Kan property for diagrammatic sets. Call a pasting scheme A *simple* if there is $a \in A_n$ such that $A = R(a)$, i.e. A is the closure of some cell. Clearly in this case $n = \dim A$.

Definition 2.4. A *filler* is an embedding $j : H \longrightarrow [A]$ of diagrammatic sets such that:

(i) $A = R(a)$ is a simple CPS of dimension n ,

(ii) H is the union of $R(b')$ for all $b' \in A_{n-1}$ except precisely one.

Remark. Results of [KV] show that the geometric realisation of the pair (H,A) is homeomorphic to the pair (S_+^{n-1},B^n) , where B^n is the n-ball and S_+^{n-1} is a hemisphere in its boundary.

Definition 2.5. A diagrammatic set X is said to be *Kan*, if for any filler $j : H \to [A]$ and any morphism $f : H \to X$ there is a morphism $g : [A] \to X$ such that $f = gj$.

The full subcategory in J^0Set formed by Kan sets will be denoted $\mathcal{J}\mathcal{K}an$.

Proposition 2.6. *For any* CW-*complex* T *its singular diagrammatic set* Sing(T) *is Kan.*

Proof. Obvious.

Let X be a Kan diagrammatic set. Consider, on the set $X(pt)$ of points of X , the relation $\approx$ setting $x \approx y$ if there is an interval from $X(\Delta^1)$ joining them. Clearly this relation is an equivalence relation and we define a set $\pi_0(X)$ as the quotient $X(pt)/\approx$.

Let X be a Kan diagrammatic set and $x \in X(pt)$ some point of X . Consider the set $Sph_n(X,x)$ consisting of globes $g \in X(G^n)$ such that $s_{n-1}(g)$ and $t_{n-1}(g)$ are globes degenerated to the point x . On this set we introduce the relation $\approx$, setting $g_1 \approx g_2$ if there is an $(n+1)$-globe $h \in X(G^{n+1})$ such that $g_1 = s_n(h)$, $g_2 = t_n(h)$. It is clear from the Kan property that $\approx$ is an equivalence relation. We shall denote the quotient $Sph_n(X,x)/\approx$ by $\pi_n(X,x)$. Clearly there is a map $\pi_n(X,x) \to \pi_n(|X|,x)$ where on the right we have the usual homotopy groups of the realisations.

Proposition 2.7. *If* X *is a Kan diagrammatic set then for each* $x \in X(pt)$, $n > 0$ *the map* $\pi_n(X,x) \to \pi_n(|X|,x)$ *is a bijection, as well as the map* $\pi_0(X) \to \pi_0(|X|)$.

The proof is the same as for the simplicial or cubical case.

Definition 2.8. A morphism $f : X \to Y$ of Kan diagrammatic sets is called a *weak equivalence* if it induces a bijection $\pi_0(X) \to \pi_0(Y)$ as well as bijections $\pi_i(X,x) \to \pi_i(Y,f(x))$ for all $i > 0$, $x \in X(pt)$.

Denote the class of weak equivalences by

$$W \subset Mor(\mathcal{J}\mathcal{K}an) .$$

Theorem 2.9. *The localisation* $\mathcal{J}\mathcal{K}an\,[W^{-1}]$ *of the category of Kan diagrammatic sets with respect to the class of weak equivalences is equivalent to the usual homotopy category* Hot *of* CW-*complexes.* ∎

3. MAIN THEOREM.

Let us define a pair of adjoint functors

$$\mathrm{Nerv} : \mathrm{Cat}_\infty \to J^0\mathrm{Set}\ ,\ \Pi : J^0\mathrm{Set} \to \mathrm{Cat}_\infty\ ,$$

which we shall call the J-nerve and Poincaré functors. They are defined as Kan extensions (see proposition 2.3) of the natural functor $J \to \mathrm{Cat}_\infty$ taking A to $\mathrm{Cat}(A)$. Explicitly, for an ∞-category C and a CPS $A \in \mathrm{Ob}\, J$ the set $\mathrm{Nerv}(C)(A)$ is the set of all realisations of A in C, i.e., all algebraic expressions in C of type A. For a diagrammatic set X the category $\Pi(X)$ is $\varinjlim_{A\in J} X(A) \times \mathrm{Cat}(A)$. (Note that the category Cat_∞ has direct limits, as does any category of universal algebras of a given type, see [GU]. In particular, for a representable diagrammatic set $[A]$ associated to a CPS A we have $\Pi([A]) = \mathrm{Cat}(A)$.

Theorem 3.1. *An* ∞-*category* C *is a (weak)* ∞-*groupoid if and only if* $\mathrm{Nerv}\, C$ *is a Kan diagrammatic set.*

Proof. "Only if". Suppose C is a ∞-groupoid. Let $j : H \to [A]$ be a filler of dimension n and $x \in A_{n-1}$ is the unique (n-1)-cell of A not lying in H. Then $\Pi(H)$ is a sub-∞-category in $\Pi(A) = \mathrm{Cat}(A)$. A morphism $H \to \mathrm{Nerv}\, C$ is just a functor $f : \Pi(H) \to C$. Our task is to extend it to a functor $\mathrm{Cat}(A) \to C$. Let $\varphi \in A_n$ be the unique n-cell. Then in $\mathrm{Cat}(A)$ we have the equality

$$s_{n-1}(\varphi) = a_1 \underset{n-2}{*} (a_2 \underset{n-3}{*} \cdots \underset{1}{*} (a_{n-1} \underset{0}{*} x \underset{0}{*} b_{n-1}) \underset{1}{*} \cdots$$
$$\cdots \underset{n-3}{*} b_2) \underset{n-2}{*} b_1 \quad (!)$$

where a_i and b_i are some (n-i)-morphisms of $\Pi(H)$. To obtain $f(x) = y$ and $f(\varphi)$ (and therefore to extend f to whole $\mathrm{Cat}(A)$) it is enough to solve (weakly) the equation

$$f(s_{n-1}(\varphi)) = f(a_1) \underset{n-2}{*} (f(a_2) \underset{n-3}{*} \cdots$$
$$\underset{1}{*} (f(a_{n-1}) \underset{0}{*} y \underset{0}{*} f(b_{n-1})) \underset{1}{*} \cdots \underset{n-3}{*} f(b_2)) \underset{n-2}{*} f(b_1)$$

in C. This can be done inductively by applying the properties (GR'_{in}) and (GR''_{in}) of definition 1.1.

"If". Suppose $\mathrm{Nerv}\ C$ is a Kan diagrammatic set. Let us verify, say, the condition (GR'_{in}). To do this, i.e., to construct a weak solution of an equation of the form $a \underset{i}{*} x = b$ it suffices to construct a filler $j : H \longrightarrow [A]$ and a functor $f : \Pi(H) \longrightarrow C$ such that:

1) A is a cube with its standard structure of a pasting scheme,
2) $f(t_{n-1}(\varphi)) = b$, and all $f(b_j)$, $j \in \{1,\ldots,n-1\}$,
   $f(a_j)$, $j \in \{1,\ldots,n-1\} - \{n-1-i\}$ in the formula (!), are degenerate (n-j)-morphisms, and $f(a_{n-1-i}) = a$.

We leave this construction to the reader.

Remark. This theorem remains true if one replaces diagrammatic sets by simplicial (resp. cubical) sets and the diagrammatic nerve by simplicial (resp. cubical) nerve.

Theorem 3.2. *If* X *is a Kan diagrammatic set then the* ∞-*category* $\Pi(X)$ *is an* ∞-*groupoid.*

Proof. We shall show that $\mathrm{Nerv}(\Pi(X))$ is a Kan diagrammatic set and apply theorem 3.1. We shall need one lemma, which describes the structure of the category $\Pi(X)$.

Definition 3.3. Let X be a diagrammatic set, and $\varphi \in \mathrm{Mor}_k(\Pi(X))$, a k-morphism. A *materialisation of* φ is a pair (A,γ), where A is a CPS of dimension k and $\gamma \in X(A)$ is an element such that φ is the image of $A \in \mathrm{Mor}(\mathrm{Cat}(A))$ under the functor $\Pi(\gamma) : \mathrm{Cat}(A) \longrightarrow \Pi(X)$.

Lemma 3.4. *Let* X,φ *be as above and* (S,γ_S), (T,γ_T) *be materialisations of* $s_{k-1}(\varphi)$, $t_{k-1}(\varphi)$ *such that there are identifications*

$$s_{k-2}(S) \cong s_{k-2}(T), t_{k-2}(S) \cong t_{k-2}(T)$$

compatible with γ_S,γ_T. *Then there exists a materialisation* (A,γ) *of* φ *and identifications* $s_{k-1}(A) \cong S$, $t_{k-1}(A) \cong (T)$ *compatible with* γ_S,γ_T,γ. *In particular, each morphism of* $\Pi(X)$ *has a materialisation.*

Proof. By construction, $Mor_k \Pi(X)$ is generated by polymorphisms of the form $f_x(A)$, where A is a CPS of dimension $\leq k$, $x \in X(A)$, $f_x : Cat(A) = \Pi([A]) \longrightarrow \Pi(X)$ is the functor corresponding to x, and A is considered as a polymorphism of $Cat(A)$. Using this fact and the induction, we can reduce the problem to the situation when φ is a degenerated k-morphism and the statement is supposed to be true in dimensions $k' < k$. The fact that φ is degenerated means that (S,γ_S) and (T,γ_T) represent the same (k-1)-morphism in $\Pi(X)$. Therefore there exists a sequence of "elementary" transformations (rebracketings, introducing and collapsing of degenerate cells) starting from (S,γ_S) and ending in (T,γ_T). Each such transformation can be represented as a CPS. Pasting all the intermediate CPS together, we obtain the required materialisation of φ. ∎

Now suppose $j : H \longrightarrow [A]$ is a filler of dimension n. For each cell b of H we have a CPS $R(b)$ and a functor $Cat(R(b)) \longrightarrow \Pi(X)$. Let $\langle b \rangle$ be the image of $R(b)$ under this functor. Using lemma 3.4 and the induction of skeletons we can construct a compatible system of materialisations of all $\langle b \rangle$. This system of materialisations can be included in a CPS A'. By definition, cells of A' are cells of all the materialisations together with one (n-1)-dimensional cell (corresponding to the deleted (n-1)-cell of A) and one n-dimensional (corresponding to the maximal cell of A). This defines a filler $j' : H' \dashrightarrow [A']$ in X and we have a commutative diagram of functors

$$\begin{tikzcd}
\Pi(H) \arrow[rr] \arrow[dd] \arrow[dr, "f"] & & \Pi(A) \arrow[dd] \arrow[dl, dashed] \\
& X & \\
\Pi(H') \arrow[rr] \arrow[ur] & & \Pi(A') \arrow[ul, "g"']
\end{tikzcd}$$

The arrow g is obtained by applying the Kan condition to the filler j'. Therefore the dotted arrow, defined as the composition, is a filling of j. ∎

Remarks.

a) For Kan simplicial sets the analogue of theorem 3.2 is not true since, for example, all 2-morphisms act from Moore

paths of smaller length to Moore paths of greater length.
b) For Kan cubical sets it is very plausible that the analogue of theorem 3.2 is true. In particular, for a Kan cubical set X, it is not difficult to prove that each k-morphism of the ∞-category $\Pi_{\square}(X)$ is ${}_k *_{k-1}$-quasi-invertible.

Theorems 3.1 and 3.2 show that Π and Nerv define a pair of adjoint functors between Grp_∞ and $\mathcal{J}\mathcal{K}an$. Let us show that these functors descend to localisations of these categories with respect to weak equivalences.

Proposition 3.5.
a) *For each ∞-groupoid* G *and* $x \in \mathrm{Ob}\, G$ *there is a natural isomorphism*

$$\pi_i(G,x) \cong \pi_i(\mathrm{Nerv}(G),x) .$$

In particular, if G *is an n-groupoid, then* $\pi_i(\mathrm{Nerv}(G),x) = 0$ *for all* $i > n$, $x \in \mathrm{Ob}(G)$.
b) *For each Kan diagrammatic set* X *and any* $x \in X(pt)$ *there is a natural isomorphism* $\pi_i(X,x) \cong \pi_i(\Pi(X),x)$.

Proof.
a) Since $\mathrm{Nerv}(G)$ is Kan, we have $\pi_i(\mathrm{Nerv}(G),x)$ = set of homotopy classes of i-globes with boundary contracted to x. This is exactly $\pi_i(G,x)$.
b) We have a natural map $h : \pi_i(X,x) \longrightarrow \pi_i(\Pi(X),x)$.

Let us verify its injectivity. Suppose γ is an i-globe in X with boundary in x, and $h(\gamma) = 0$. By lemma 3.4 we have a CPS A which is a subdivision of an (i+1)-globe such that $s_i(A)$, $t_i(A)$ are globes and a map $A \longrightarrow X$ such that $s_i(A)$ maps as γ and $t_i(A)$ maps to x. Therefore the element of $\pi_i(|X|,x)$ defined by γ, is trivial. Since for Kan sets combinatorial π_i coincide with the topological, we obtain that γ is trivial.

Surjectivity of h is proved similarly. ∎

Proposition 3.6.
a) *For each ∞-groupoid* G *the natural functor* $\alpha : \Pi(\mathrm{Nerv}(G)) \longrightarrow G$ *is a weak equivalence in* Grp_∞.
b) *For each Kan diagrammatic set* X *the natural morphism* $\beta : X \longrightarrow \mathrm{Nerv}(\Pi(X))$ *is a weak equivalence in* $\mathcal{J}\mathcal{K}an$.

The proof is similar to the previous proposition.
Now we obtain the following theorem.

Theorem 3.7. *The homotopy category* $\mathrm{Grp}_\infty[W^{-1}]$ *of weak* ∞-*groupoids is equivalent to the usual homotopy category of* CW-*complexes.*

Corollary 3.8. *Let* $n \geq 0$. *The functors* Nerv *and* $\tau_{\leq n}\Pi$ *establish mutually (quasi-)inverse equivalences between the categories* $\mathrm{Grp}_n[W_n^{-1}]$ *and* $\mathrm{Hot}_{\leq n}$ *of* n-*groupoids and homotopy* n-*types.*

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