CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

PAUL CHERENACK

Convenient affine algebraic varieties

Cahiers de topologie et géométrie différentielle catégoriques, tome 31, n° 4 (1990), p. 315-346

<http://www.numdam.org/item?id=CTGDC_1990__31_4_315_0>

© Andrée C. Ehresmann et les auteurs, 1990, tous droits réservés.

L'accès aux archives de la revue « Cahiers de topologie et géométrie différentielle catégoriques » implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

CAHIERS DE TOPOLOGIE
ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

VOL. XXXI-4 (1990)

CONVENIENT AFFINE ALGEBRAIC VARIETIES

by Paul CHERENACK

RÉSUMÉ. On étend la catégorie des variétés algébriques sur un corps k en une catégorie AF qui est cartésienne fermée. Pour cela on construit d'abord la catégorie cartésienne fermée SP des applications polynomiales entre espaces vectoriels structurés. Les zéros de morphismes dans AF vers k déterminent des variétés algébriques (de dimension infinie) et les morphismes de AF entre variétés algébriques (comme dans le cas de dimension finie) sont juste des restrictions de morphismes dans SP .

0. INTRODUCTION.

Frölicher [5] has identified a Cartesian closed category, the category of convenient vector spaces, in which a meaningful study of differential calculus can be pursued. Nel [11] has treated functional analysis in a similar way and his work motivates some of the directions taken here. Here a category AF becomes a convenient category for affine algebraic geometry. The construction of AF although complicated, assumes little specialized knowledge on the part of the reader in contrast to analogous results found in our study [1] of ind-affine schemes. However, where ind-affine schemes are ringed spaces, objects in AF are not. The construction proceeds in stages since each stage has its own difficulties. Thus hom-sets may not have their canonical associated basis and have only a canonical linearly independent subset; or the hom-functor in the contravariant variable may depend on infinitely many variables.

What possible applications can be made of convenient affine algebraic varieties? Because AF is Cartesian closed and should in important cases have suitable quotients (see [1]), as in ordinary homotopy theory, one should be able to define appropriate suspension and loop functors. See [3] where suspension and loop functors have been defined for the restricted category of ind-affine schemes. The infinite dimensional varieties here can also be useful in the study of affine moduli problems. For more details see [4]. We intend to prove in a later paper that the structure that we place on hom-sets here corresponds to that of the Hilbert scheme construction [7]. Finally, one can use the structure placed on $\mathrm{Hom}_{AF}(V_1,V_2)$ to provide it with a topology that we call the Zariski topology. In analogy to the situation for C^{∞}-differentiable maps [6] one would like to determine whether certain classes of transversal maps in $\mathrm{Hom}_{AF}(V_1,V_2)$ contain an open subset. Some work in this connection can be found in [2] for ind-affine schemes.

All vector spaces will be defined over an uncountable infinite field k (so that in particular there is a bijective correspondence between polynomials and polynomial maps). The symbol I_X will always denote the identity map on a set X. Let V be a vector space and V' a vector subspace of V. Let $\iota: V' \to V$ be the inclusion map. A map $\pi: V \to V'$ is called a projection if it is linear and satisfies $\pi \circ \iota = I_{V'}$. Suppose that L is a linearly independent subset of V, U is a subspace of V generated by the subset $\{u_1,\dots,u_n\}$ and W is a subspace of U generated by $\{u_1,\dots,u_m\}$. Then, letting $v = a_1u_1 + \cdots + a_nu_n$ the projection $\pi: U \to W$ associated to L sends v to $a_1u_1 + \cdots + a_mu_m$.

DEFINITION 0.1. A *polyspace* is a triple $(V,L,\leq)$ where V is a vector space, L is a basis of V and $\leq$ is a well ordering on L. A *partial polyspace* is a quadruple $(V,L,P,\leq)$ where L is a linearly independent subset of the vector space V, $\leq$ is a well ordering of L and P consists of one projection $\pi_W: V \to W$ for each finite dimensional subspace W of V generated by elements of L subject to the condition:

If $W \subset U$ where U is also a vector subspace of V generated by a finite number of elements of L and $\pi: U \to W$ is the projection associated to L, then $\pi \circ \pi_U = \pi_W$.

To every polyspace $(V,L,\leq)$ one can associate (Section 1) in a natural way a partial polyspace $(V,L.P,\leq)$. We will usually write V instead of $(V,L,\leq)$ (resp. $(V,L,P,\leq)$) and then when necessary write L_V for the basis associated to the polyspace V (resp. L_V and P_V for the linearly independent set and the collection of projections associated to the partial polyspace $(V,L,P,\leq)$). In Section 1 we define for polyspaces some useful constructs and provide some useful motivating examples. To avoid multiple subscripting we sometimes write X*i* instead of X_i.

DEFINITION 0.2. Let $L_1 = L_{V1}$ and $L_2 = L_{V2}$ be linearly independent sets defining the polyspaces V_1 and V_2. Define a map $\alpha : V_1 \to k^{L1}$ (and similarly a map $\beta : V_2 \to k^{L2}$) by sending $v \in V_1$ to $(a_j) \in k^{L1}$ where if $a_{i1}, \dots, a_{im}$ are the coordinates of k^{L1} corresponding to the elements $v_{i1}, \dots, v_{im}$ of L, V'_1 is the vector space generated by $v_{i1}, \dots, v_{im}$ and $\pi : V_1 \to V'_1$ is the unique projection in $P_1 = P_{V1}$, then $(a_{i1}, \dots, a_{im})$ is the coordinate representation of $\pi(v)$ with respect to $v_{i1}, \dots, v_{im}$. The condition of Definition 0.1 guarantees that α is well defined. A map $f : V_1 \to V_2$ is a *polymap* of partial polyspaces if there is a commutative diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{\quad f \quad} & V_2 \\ \alpha \big\downarrow & & \big\downarrow \beta \\ k^{L1} & \xrightarrow{\quad f' \quad} & k^{L2} \end{array}$$

such that, for finitely many (or just one) coordinate $y_1, \dots, y_r$ in k^{L2}, there are finitely many coordinates $x_1, \dots, x_m$ in k^{L1} such that $y_p = (f')_p$ $(p = 1, \dots, r)$ is a polynomial in the $x_1, \dots, x_m$.

Thus, for a polymap each relevant coordinate in V_2 is a polynomial in the relevant coordinates of V_1. From Definition 0.2 it is clear that the collection of polymaps between polyspaces (resp. partial polyspaces) forms a category P = *Poly* (resp. PP = *PPoly*). For the purpose of viewing f in Definition 0.2 as a ringed space map it would have been useful to assume that f sent linear subspaces of V_1 generated by finitely many elements of L_1 into linear subspaces of V_2 generated by finitely many elements of L_2. The difficulty here is that if $g : V_1 \times V_2 \to V_3$ is a polymap of polyspaces, the induced map $g_0 : V_1 \to \mathrm{Hom}_P(V_2, V_3)$ need not send finite dimensional subspaces to finite dimensional subspaces even if g does. In Section 1 examples of polymaps are given and an alternate coordinate free description of polymaps is provided.

We will ignore the well ordering on the linearly independent set L_V of a partial polyspace V unless it is essential in our considerations. Note that if L_V has two different well orderings, then the identity map is an isomorphism between the two different partial polyspaces.

We prove in Section 1 that *Poly* has products. The set $H = \mathrm{Hom}_P(V_1, V_2)$ will be given the structure of a partial polyspace. The subspace of H generated by L_H contains functions which roughly speaking determine the other functions in H. A major difficulty forcing us to extend *PPoly* is the following. If $V_1 \to V_2$

is in *PPoly*, then the induced map from $Hom_P(V_2,V_3)$ to $Hom_P(V_1,V_3)$ need not be. See Example 1.4. Hence we make the following definition.

DEFINITION 0.3. Using the conventions of Definition 0.2, a map $f: V_1 \to V_2$ of partial polyspaces is a *basic stratawise polymap* if, for every finite dimensional subspace V of V_1 generated by the elements of L_1 and the inclusion $\iota_V: V \to V_1$, the map $f \circ \iota_V$ is a polymap. Unfortunately the collection of basic stratawise polymaps does not form a category. Hence we define $f: V_1 \to V_2$ to be a *stratawise polymap* if $f = f_1 \circ f_2 \circ \cdots \circ f_n$ where the f_i (for $i = 1, \ldots, n$) are basic stratawise polymaps.

Clearly the collection of stratawise polymaps whose objects are the partial polyspaces forms a category that we denote *SP* or *SPoly*. We can now define a bifunctor (Section 2)

$$F: P^{op} \times P \to PP \text{ where } F(V_1,V_2) = (Hom_P(V_1,V_2), L_H, P_H, \leq).$$

We wish to identify the maps $V_1 \to Hom_P(V_2,V_3)$ in *PPoly* (with V_1, V_2 and V_3 objects in *P*) corresponding to maps $V_1 \times V_2 \to V_3$ in *P*. The following definition leads to an answer.

DEFINITION 0.4. A map of polyspaces $f: V_1 \to V_2$ *maps into strata* if $f(V_1)$ is contained in a finite dimensional subspace of V_2.

The collection of polymaps mapping into strata need not form a category since for instance the identity map need not map into strata. We note the inclusions

$$Poly \subset PPoly \subset SPoly$$

where (Section 1) we identify every polyspace with its associated partial polyspace. We will eventually show that *SPoly* is Cartesian closed by working up the ladder of inclusions.

Let $\Gamma(V_1, Hom_P(V_2,V_3))$ be the set of all elements f in $Hom_{PP}(V_1, Hom_P(V_2,V_3))$ such that, for every V'_3 of V_3 generated by a single element of L_3 and the projection map $\pi: V_3 \to V'_3$, the composition

$$V_1 \xrightarrow{f} Hom_P(V_2,V_3) \xrightarrow{\pi^*} Hom_P(V_2,V'_3)$$

is a polymap mapping into strata. We show in Section 3:

THEOREM 0.5. *There is a natural equivalence*

$$Hom_P(V_1 \times V_2, V_3) \approx \Gamma(V_1, Hom_P(V_2,V_3))$$

arising from that in sets where the right and left hand sides of the natural equivalence are trifunctors

$$P^{op} \times P^{op} \times P \to Sets.$$

For finite dimensional V_1, V_2 and V_3 one can readily show that there is a natural equivalence arising from that in *Sets*:

$$\mathrm{Hom}_P(V_1 \times V_2, V_3) \approx \mathrm{Hom}_{PP}(V_1, \mathrm{Hom}_P(V_2, V_3)).$$

See [1]. Theorem 0.5 shows how one can extend this equivalence to polyspaces where the "internal hom-functor" is a partial polyspace.

In Section 4 we describe a construction of Nel [11] which enables one to extend separately the category of affine algebraic varieties and projective algebraic varieties over the complex numbers to a Cartesian closed category. This construction may be too general for many purposes. However it motivated looking at *SPoly*.

DEFINITION 0.6. Let C be a concrete category (see MacLane [10]) which is generated by a collection T of maps in C which contains all the identities of C. Thus every map f in C has the form $f = f_1 \circ \cdots \circ f_n$ with $f_1, \ldots, f_n$ in T. If a and b are objects of C, the T-*product* $a \times b$ of a and b is an object of C with projections p_1, p_2 in T (whose underlying set maps are set projections) such that, for every pair of maps f: $c \to a$ and g: $c \to b$ in T, there is a unique morphism h: $c \to a \times b$ in T such that $p_1 \circ h = f$ and $p_2 \circ h = g$. T-*limits* and T-*colimits* are defined in the same way. Let $\mathrm{Hom}_T(b,c)$ denote the collection of elements in $\mathrm{Hom}_C(b,c)$ which lie in T. Let D be a full subcategory of C and a, b, c be objects in D. We assume that the T-products $a \times b$ define a functor $D \times D \to C$ and that the sets $\mathrm{Hom}_T(b,c)$ plus additional structure define a functor $D \to C$ with respect to the covariant variable. Then D is called *Cartesian closed relative to C and* T-*generated* if there is, for fixed a and b in D, a natural equivalence

$$\mathrm{Hom}_T(a \times b, c) \approx \mathrm{Hom}_T(a, \mathrm{Hom}_T(b,c))$$

in C between functors $D \to Sets$ arising from the natural equivalence in *Sets* .

Using these notions we obtain (Section 5):

THEOREM 0.7. *The category Poly is Cartesian closed relative to SPoly and basic stratawise polymaps generated.*

We chose k to be uncountable in order to prove Lemma 5.2 and hence Theorem 0.7. A stronger result could have been proven by introducing the notion of partial bifunctor to Definition 0.6.

Let W be a partial polyspace and f: W→ k a stratawise polymap. Then $V(f) = \{p \in W \mid f(p) = 0\}$ is called a *hypersurface* in W. The set A[W] = $\mathrm{Hom}_{SP}(W,k)$ can be given the structure of a ring using pointwise addition and multiplication. A[W] will be called the *affine ring of* W. Let I be an ideal in A[W]. Then

$$V(I) = \{p \in W \mid f(p) = 0 \text{ for all } f \in I\}$$

will be called an affine variety and also an affine subvariety of W. Note that W = V({0}). The V(I) define the closed sets for a topology on W which will be called the *Zariski topology*. Let V_1 (resp. V_2) be a subvariety of W_1 (resp. W_2). Then a map f: $V_1 \to V_2$ is a basic map of affine varieties if there is a commutative diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{\ f\ } & V_2 \\ \downarrow & & \downarrow \\ W_1 & \xrightarrow{\ f'\ } & W_2 \end{array}$$

where the vertical arrows are inclusions and f ' is a basic stratawise polymap. The collection of basic maps B of affine varieties generates a category that we denote *AF*. A morphism f in *AF* is always continuous for the Zariski topologies. On restricting to finite dimensional vector spaces, the above definitions become the usual ones.

Let *AF*^ denote the full subcategory of *AF* consisting of those affine varieties which are affine subvarieties of a polyspace and where the morphisms between the associated polyspaces are polymaps. Using Theorem 0.7, we obtain:

THEOREM 0.8. *The category AF^ is Cartesian closed relative to AF and basic maps of affine varieties generated.*

Finally, in Section 7, using heavily our earlier results, we prove our main result.

THEOREM 0.9. *1. SPoly is Cartesian closed.*
2. AF is Cartesian closed.

Initially we defined the notion of polymap by looking at coordinates. Using this approach it became, for instance, immediate that *PPoly* was a category. The construction of internal hom-functors in Section 2 on the other hand used except where necessary a coordinate free approach. The coordinate free approach allowed one to see more clearly the steps needed to form the convenient category *AF*. Finally, instead of proving one gigantic theorem, we have decided to prove simpler results whose proof generated by analogy in an obvious way the proof of more general results (proof bootstrapping).

1. POLYSPACES AND POLYMAPS.

We will establish some concepts which will be used later in placing structure on the hom-sets of *Poly*. First, for a partial polyspace V, let F_V denote the collection of finite dimensional vector subspaces of V generated by a finite number of vectors in L_V. When V is a polyspace, one can form the collection P_V of projections $\pi_{V'}: V \to V'$ $(V' \in F_V)$ with respect to the basis L. Since these projections satisfy the conditions stated in Definition 0.1 every polyspace is a partial polyspace in a natural way.

Let V be a partial polyspace. For $\pi_1, \pi_2 \in P_V$ we write $\pi_1 \geq \pi_2$ if there is a commutative triangle

$$\begin{array}{ccc} V & & \\ {\scriptstyle \pi_1}\downarrow & \searrow^{\pi_2} & \\ V_1 & \xrightarrow{\alpha} & V_2 \end{array}$$

with α a projection. Clearly α is unique. Note that $\geq$ is a partial ordering. A set X with a partial ordering $\geq$ is *up-directed* if, for $a, b \in X$, there is a $c \in X$ such that $c \geq a, b$.

LEMMA 1.1. *Let* V *be a partial polyspace. Then, as* F_V *is up-directed by inclusion,* P_V *is up-directed by* $\geq$.

The proof is clear.

EXAMPLE 1.2. a) Let

$$A^n = \{(a_i) \in k^N \mid a_i = 0 \text{ for } i > n\}.$$

Since $\cup A^n$ $(n \in N)$ can be used to represent polynomial maps from A^1 to A^1, we sometimes write

$$H^* = Hom_P(A^1, A^1) = \cup A^n \ (n \in N).$$

A natural selection of basis for H^* making it into a polyspace is

$$E = \{e_i \mid e_i = (\varepsilon_{ji})_{j\in N}\}_{i\in N} \text{ where } \varepsilon_{ji} = 0 \text{ if } i \neq j \text{ and } \varepsilon_{ii} = 1.$$

Here P_{H^*} consists of projections onto various finite dimensional coordinate planes. Note that H^* is the direct sum of copies of k indexed by N.

b) The vector space k^N is not viewed as a polyspace in its own right but as a subspace of k^B where B is a basis of k^N and

$$\mathrm{card}(B) = \dim_k(k^N) > \mathrm{card}(N).$$

For k^N, when referrred to later, we assume that $B \supset E$ where E is defined in (a). Note that k^N is the direct sum of copies of k indexed by B.

Let V, W be finite dimensional vector spaces. Choosing a basis for V and W, a map $f: V \to W$ is represented by a map $f^0: k^n \to k^m$ where $n = \dim_k V$ and $m = \dim_k W$. If $f^0 = (f_i^0)_{i=1,\dots,m}$, f_i^0 is a polynomial map $f_i^0: k^n \to k$, then f is called a *primitive polymap.* Note that this definition is independent of bases chosen for V and W.

One can now provide a coordinate free interpretation of polymaps.

LEMMA 1.3. *A map $f: V_1 \to V_2$ of partial polyspaces is a polymap if, for all $\pi \in P_2$, there is a $\psi \in P_1$ sufficiently large and a primitive polymap f' such that the diagram*

$$\begin{array}{ccc} V_1 & \xrightarrow{\quad f \quad} & V_2 \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\pi} \\ V'_1 & \xrightarrow{\quad f' \quad} & V'_2 \end{array}$$

commutes.

Note that the term "sufficiently large" is used with respect to $\geq$ on P_1. The proof is straighforward.

EXAMPLE 1.4. a) Let $f: A^1 \to A^1$ be a polynomial map and

$$f(x) = \sum_{i=0}^{n-1} a_i x^i.$$

Then the map

$$f^*: \mathrm{Hom}_P(A^1, A^1) \to \mathrm{Hom}_P(A^1, A^1),$$

induced by f in the contravariant variable, embeds in a diagram (with ι_m the inclusion):

$$\begin{array}{ccc} H^* & \xleftarrow{\quad \iota_m \quad} & A^m \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f'} \\ H^* & \xleftarrow{\quad \pi_{mn} \quad} & A^{mn} \end{array}$$

where $(c_i) \in A^m$ is sent by f' to $(b_p) \in A^{mn}$ with

$$b_p = \sum_{j=0}^{m-1} c_j \Big(\sum_{m[0]+\ldots+m[n]=j} (a_0)^{m[0]} (a_1)^{m[1]} \ldots (a_{m-1})^{m[n]} \Big)$$

where

$m[1], m[2], \ldots, m[n] \in N$ and $m[1]+2\,m[2]+\ldots+n\,m[n] = p$.

Similarly, the map f_* induced by f in the covariant variable embeds in a diagram

$$\begin{array}{ccc} H^* & \xrightarrow{\quad \pi_m \quad} & A^m \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f'} \\ H^* & \xrightarrow{\quad \pi_{mn} \quad} & A^{mn} \end{array}$$

where $(c_i) \in A^m$ is sent by f' to $(b_p) \in A^{mn}$ with

$$b_p = \sum_{j=0}^{m-1} a_j \Big(\sum_{m[0]+\ldots+m[n]=j} (c_0)^{m[0]} (c_1)^{m[1]} \ldots (c_n)^{m[n]} \Big)$$

where

$m[1], m[2], \ldots, m[n] \in N$ and $m[1]+2\,m[2]+\ldots+n\,m[n] = p$.

Because this last expression for b_p is independent of c_q for q large, f_* is a polymap but clearly f^* is only a stratawise polymap.

b) Let B be the basis defined in Example 1.2 for k^N. Then if f: $k^N \rightarrow k^N$ is a polymap, it is not difficult to see that each coordinate of f is a polynomial in a finite number of variables. Not every map f: $k^N \rightarrow k^N$ whose coordinates are polynomials need be a polymap. For instance, f defined by setting

$$f(x_j)_{j\in N} = (y_i)_{i\in N} \text{ where } y_i = x_1 x_2 \cdots x_i$$

is not a polymap if $\underline{1} = (1,1,\ldots,1,\ldots) \in B$. To see this note that the coordinate corresponding to $\underline{1}$ depends on $x_1, x_2, \ldots, x_n, \ldots$ which form an infinite set of variables.

Let V_1 and V_2 be polyspaces with bases L_1 and L_2, respectively. Suppose that the vector space product $V_1 \times V_2$ has the basis

$$L_{12} = \{(b_1,0) \mid b_1 \in L_1\} \cup \{(0,b_2) \mid b_2 \in L_2\}.$$

One well orders L_{12} via the orders on L_1 and L_2 and by setting

$$(b_1,0) \leq (0,b_2) \text{ for } b_1 \in L_1 \text{ and } b_2 \in L_2.$$

Then one sees quickly:

LEMMA 1.5. *The projection* $p_1: V_1 \times V_2 \to V_1$ *is a polymap.*

Next we have:

PROPOSITION 1.6. *The polyspace* $V_1 \times V_2$ *with the usual projections* $p_1: V_1 \times V_2 \to V_1$ *and* $p_2: V_1 \times V_2 \to V_2$ *is the product of* V_1 *and* V_2 *in Poly.*

PROOF. Let $f: V_3 \to V_1$ and $g: V_3 \to V_1$ be polymaps and define $h: V_3 \to V_1 \times V_2$ by setting $h(x) = (f(x), g(x))$. Clearly, h satisfies the uniqueness required. Let $(b,0)$ be a basis element of L_{12} with corresponding coordinate x_i. Then $(h(x))_i = (f(x))_i$ is a polynomial in only a finite number of variables corresponding to a basis element of L_{V3}. A similar statement holds for $(0,b) \in L_{12}$.

The projections and injections which come automatically with a partial polyspace are polymaps. We state this in the following readily proven proposition.

PROPOSITION 1.7. *Let* V *be a partial polyspace and* $V' \in F_V$. *Then the projection* $\pi: V \to V'$ *in* P_V *and the inclusion* $\iota: V' \to V$ *are polymaps.*

2. HOM-SETS AS PARTIAL POLYSPACES.

We wish to put a partial polyspace structure on $\mathrm{Hom}_P(V_1,V_2)$. We write L_m for L_U and $P_m = P_U$ where $U = V_m$ $(m \in N)$.

LEMMA 2.1. *Let* $f,g \in \mathrm{Hom}_P(V_1,V_2)$ *and* $c \in k$. *Let* $f+g$ *and* cf *be formed by pointwise addition and scalar multiplication. Then* $f+g$, $cf \in \mathrm{Hom}_P(V_1,V_2)$. *Thus,* $\mathrm{Hom}_P(V_1,V_2)$ *is a vector space under pointwise operations.*

PROOF. Let x_i be a coordinate corresponding to an element of L_2. Then as f_i and g_i involve only a finite number of variables corresponding to elements of L_1, the same is true for $(f+g)_i = f_i + g_i$. A similar argument works for cf.

REMARK. The notation introduced below is for a fixed hom-set. If we change our hom-set, we change our notation by adding a particular superscript or subscript depending on context.

We wish to determine a suitable linearly independent subset of $Hom_P(V_1,V_2)$ and suitable projections which make $Hom_P(V_1,V_2)$ into a partial polyspace. Let $\pi_1\varphi \in P_1$ and $\pi_2\delta \in P_2$. There are commutative diagrams in *Poly*:

(1)
$$\begin{array}{ccc} V_1\varphi & \xrightarrow{\quad I \quad} & V_1\varphi \\ {\scriptstyle \iota_1\varphi}\downarrow & \nearrow{\scriptstyle \pi_1\varphi} & \\ V_1 & & \end{array}$$

(2)
$$\begin{array}{ccc} V_2\delta & \xrightarrow{\quad I \quad} & V_2\delta \\ {\scriptstyle \iota_2\delta}\downarrow & \nearrow{\scriptstyle \pi_2\delta} & \\ V_2 & & \end{array}$$

with I the appropriate identity and $\iota_1\varphi$, $\iota_2\delta$ inclusions. This follows from Proposition 1.7. Applying the functor $Hom_P(-,V_2)$ to triangle (1), one obtains a commutative triangle

$$\begin{array}{ccc} Hom_P(V_1\varphi,V_2) & \xrightarrow{\quad I^* \quad} & Hom_P(V_1\varphi,V_2) \\ {\scriptstyle (\pi_1\varphi)^*}\downarrow & \nearrow{\scriptstyle (\iota_1\varphi)^*} & \\ Hom_P(V_1,V_2) & & \end{array}$$

with $(\pi_1\varphi)^*$ an injection and

$$\pi\varphi = (\pi_1\varphi)^* \circ (\iota_1\varphi)^* : Hom_P(V_1,V_2) \to (\pi_1\varphi)^*(Hom_P(V_1\varphi,V_2))$$

a projection. For simplicity we will now write H instead of $Hom_P(V_1,V_2)$.

Similarly, applying $Hom_P(V_1\varphi,-)$ to triangle (2), one obtains a commutative triangle

$$\begin{array}{ccc} \mathrm{Hom}_P(V_1\varphi, V_2\delta) & \xrightarrow{\quad I_* \quad} & \mathrm{Hom}_P(V_1\varphi, V_2\delta) \\ {\scriptstyle (\iota_2\delta)_*}\downarrow & & (\pi_2\delta)_* \\ \mathrm{Hom}_P(V_1\varphi, V_2) & & \end{array}$$

with $(\iota_2\delta)_*$ an injection. Set $H_\varphi = \mathrm{Hom}_P(V_1\varphi, V_2)$. The map

$$\pi_2\varphi\delta = (\iota_2\delta)_* \circ (\pi_2\delta)_* : H_\varphi \to (\iota_2\delta)_*(\mathrm{Hom}_P(V_1\varphi, V_2\delta))$$

is a projection. Let

$$H_{\varphi\delta} = \mu_{\delta\varphi}(\mathrm{Hom}_P(V_1\varphi, V_2\delta) \text{ where } \mu_{\delta\varphi} = (\pi_1\varphi)^* \circ (\iota_2\delta)_*.$$

The next results show how the $H_{\varphi\delta}$ fit into H.

LEMMA 2.2. *The map*

$$\pi_{\varphi\delta} = (\pi_1\varphi)^* \circ (\pi_2\varphi\delta) \circ (\iota_1\varphi)^* : H \to H_{\varphi\delta}$$

is a projection.

PROOF. Let $f \in H_{\varphi\delta}$. Then

$$f = \mu_{\delta\varphi}(f') = \iota_2\delta \circ f' \circ \pi_1\varphi \text{ with } f' \in \mathrm{Hom}_P(V_1\varphi, V_2\delta).$$

Applying the definitions we have made,

$$\pi_{\varphi\delta}(f) = (\pi_1\varphi)^* \circ (\pi_2\varphi\delta) \circ (\iota_1\varphi)^* (\iota_2\delta \circ f' \circ \pi_1\varphi) =$$
$$= (\pi_1\varphi)^* \circ (\pi_2\varphi\delta)(\iota_2\delta \circ f') = (\pi_1\varphi)^* \circ (\iota_2\delta)_* \circ (\pi_2\delta)_* (\iota_2\delta \circ f') = f.$$

Thus the $\pi_{\varphi\delta}$ which act by restricting variables in domain and image are projections. To determine a suitable linearly independent subset of $\mathrm{Hom}_P(V_1, V_2)$ one must choose bases on the various $H_{\varphi\delta}$ and make sure that these bases fit together properly. To satisfy, in part, this last point we will show that the elements of

$$P = \{\pi_{\varphi\delta} \mid \pi_1\varphi \in P_1, \ \pi_2\delta \in P_2\}$$

fit together compatibly.

Suppose that $\pi_1\varphi' \geq \pi_1\varphi$ and $\pi_2\delta' \geq \pi_2\delta$. There are commutative triangles

$$\begin{array}{ccc} V_1\varphi' & \xrightarrow{\quad \alpha_1 \quad} & V_1\varphi \\ {\scriptstyle \pi_1\varphi'}\downarrow & \swarrow {\scriptstyle \pi_1\varphi} & \\ V_1 & & \end{array} \qquad \begin{array}{ccc} V_1\delta' & \xrightarrow{\quad \alpha_2 \quad} & V_1\delta \\ {\scriptstyle \pi_2\delta'}\downarrow & \swarrow {\scriptstyle \pi_2\delta} & \\ V_2 & & \end{array}$$

with α_1 and α_2 projections. Let j_1 (resp. j_2) be the inclusion of $V_1\varphi$ into $V_1\varphi'$ (resp. of $V_2\delta$ into $V_2\delta'$). Applying the hom-functors, as above, one obtains maps j_1^* and $(\alpha_2)_*$ where

$$\mathrm{Hom}_P(V_1\varphi', V_2\delta') \xrightarrow{j_1^*} \mathrm{Hom}_P(V_1\varphi, V_2\delta) \xrightarrow{(\alpha_2)_*} \mathrm{Hom}_P(V_1, V_2)$$

and $\upsilon(\varphi,\delta,\varphi',\delta') = (\alpha_2)_* \circ j_1^*$ has a right inverse $\alpha_1^* \circ (j_2)_*$. Pulling back to $\mathrm{Hom}_P(V_1,V_2)$ one obtains a map

$$\pi(\varphi,\delta,\varphi',\delta') = \mu_{\delta\varphi} \circ \upsilon(\varphi,\delta,\varphi',\delta') \circ \mu_{\delta'\varphi'}^{-1} : H_{\varphi'\delta'} \to H_{\varphi\delta}.$$

LEMMA 2.3. *One has an inclusion* $H_{\varphi\delta} \subset H_{\varphi'\delta'}$.

PROOF. Let $f \in H_{\varphi\delta}$. Then

$$f = (\pi_1\varphi)^* \circ (\iota_2\delta)_*(g) = \iota_2\delta \circ g \circ \pi_1\varphi \text{ with } g \in \mathrm{Hom}_P(V_1\varphi, V_2\delta).$$

Thus

$$f = \iota_2\delta' \circ (j_2 \circ g \circ \alpha_1) \circ \pi_1\varphi' \text{ with } j_2 \circ g \circ \alpha_1 \in \mathrm{Hom}_P(V_1\varphi', V_2\delta').$$

LEMMA 2.4. *The map* $\pi(\varphi,\delta,\varphi',\delta') : H_{\varphi'\delta'} \to H_{\varphi\delta}$ *is a projection.*

PROOF. Let $f \in H_{\varphi\delta}$ and $f = \iota_2\delta' \circ (j_2 \circ g \circ \alpha_1) \circ \pi_1\varphi'$ as in the previous lemma. Then

$$(\mu_{\delta'\varphi'})^{-1} f = j_2 \circ g \circ \alpha_1,\ \upsilon(\varphi,\delta,\varphi',\delta')(j_2 \circ g \circ \alpha_1) = \alpha_2 \circ j_2 \circ g \circ \alpha_1 \circ j_1 = g$$

and

$$\mu_{\delta\varphi} = f.$$

In the next lemma, we see how H can be viewed as a limit cone over the $H_{\varphi\delta}$.

LEMMA 2.5. *Suppose that* $\pi_1\varphi' \geq \pi_1\varphi$ *and* $\pi_2\delta' \geq \pi_2\delta$. *Then* $\pi_{\varphi'\delta'} \geq \pi_{\varphi\delta}$. *Thus, since* P_1 *and* P_2 *are up-directed (Lemma* 1.1*),* P *is up-directed.*

PROOF. One has

$$\pi_{\varphi'\delta'} = (\pi_1\varphi')^* \circ (\iota_2\delta')_* \circ (\pi_2\delta')_* \circ (\iota_1\varphi')^* = \mu_{\delta'\varphi'} \circ (\pi_2\delta')_* \circ (\iota_1\varphi')^*$$

and $\rho(\varphi.\delta.\varphi',\delta') = (\pi_1\varphi)^* \circ (\iota_2\delta)_* \circ (\alpha_2)_* \circ j_1^* \circ (\mu_{\delta'\varphi'})^{-1}$. Hence

$$\begin{aligned}\rho(\varphi.\delta.\varphi',\delta') \circ \pi_{\varphi'\delta'} &= (\pi_1\varphi)^* \circ (\iota_2\delta)_* \circ (\alpha_2)_* \circ j_1^* \circ (\pi_2\delta')_* \circ (\iota_1\varphi')^* \\ &= (\pi_1\varphi)^* \circ (\iota_2\delta)_* \circ (\alpha_2 \circ \pi_2\delta')_* \circ (\iota_1\varphi' \circ j_1)^* \\ &= (\pi_1\varphi)^* \circ (\iota_2\delta)_* \circ (\pi_2\delta)_* \circ (\iota_1\varphi)^* = \pi_{\varphi\delta}.\end{aligned}$$

As a corollary of Lemma 2.5, taking the cone determined by the $\pi\varphi\delta$, one has an embedding $H \to \varprojlim H_{\varphi\delta}$, the inverse limit being taken in the category of vector spaces.

Next we determine a suitable linearly independent subset

L_H of $\mathrm{Hom}_P(V_1,V_2)$ making H into a partial polyspace. We first show that the $H_{\varphi\delta}$ have well ordered bases $L_{\varphi\delta}$ whose order is determined by the orders on L_1 and L_2. Taking the union of the $L_{\varphi\delta}$ we obtain the well ordered linearly independent subset L_H of H.

From the basis L_1 (resp. L_2), we obtain a basis $L_1\varphi = L_1 \cap V_1\varphi$ (resp. $L_2\delta = L_1 \cap V_2\delta$) consisting of $n_\varphi = \dim_k V_1\varphi$ elements for $V_1\varphi$ (resp. consisting of $m_\delta = \dim_k V_2\delta$ elements for $V_2\delta$). In this way one obtains a unique isomorphism

$$\Psi_2\colon \mathrm{Hom}_P(V_1\varphi, V_2\delta) \to \mathrm{Hom}_P(k^{n\varphi}, k^{m\delta})$$

of vector spaces. Let

$$\Psi_1 = (\mu_{\delta\varphi})^{-1}\colon H_{\varphi\delta} \to \mathrm{Hom}_P(V_1\varphi, V_2\delta).$$

The map Ψ_1 is a vector space isomorphism.

An element $f \in \mathrm{Hom}_P(k^{n\varphi}, k^{m\delta})$ has coordinates polynomials f_j in the x_i ($i \in L_1\varphi$, $j \in L_2\delta$). Let $\underline{M}$ denote the collection of monomonials in $k[x_i]_{i\in L_1}$, the polynomial ring in the x_i ($i \in L_1$). Let

$$m_1 = (x_{i[1]})^{p[1]} \cdots (x_{i[r]})^{p[r]} \text{ and } m_2 = (x_{i[1]})^{q[1]} \cdots (x_{i[r]})^{q[r]}$$

belong to $\underline{M}$ and $i[1] \geq i[2] \geq \cdots \geq i[r]$. Let

$$u = \min\{i[j] \mid p[j] = q[j],\ j = 1,\dots,r\}-1.$$

Let $m_2 \geq m_1$ if $\deg m_2 > \deg m_1$ or $q_u > p_u$ or $m_2 = m_1$. Then $\geq$ is a well ordering (reverse lexicographic) on $\underline{M}$ and also $\underline{M}_\varphi = \underline{M} \cap k[x_i]_{i\in \underline{L}}$ where $\underline{L} = L_1\varphi$. Now f_j, for $j \in L_2\delta$, can be written $f_j = \sum_i (a_{ij})_i$ ($i \in \underline{M}_\varphi$) where $a_{ij} \in k$. Let

$$T_{\varphi\delta} = \{(a_{ij}) \mid i \in \underline{M}_\varphi,\ j \in L_2\delta \text{ and } a_{ij} \in k\}$$

and $Q_{\varphi\delta}$ be the set

$\{(a_{ij}) \mid (a_{ij}) \in T_{\varphi\delta}$ and, for fixed j, $a_{ij} \neq 0$ for only finitely many $i\}$.

There is then a canonical vector space isomorphism

$$\Psi_3\colon \mathrm{Hom}_P(k^{n\varphi}, k^{m\delta}) \to Q_{\varphi\delta}$$

sending f to (a_{ij}). The set

$$N_{\varphi\delta} = \{(a_{ij}) \in Q_{\varphi\delta} \mid a_{ij} = 0 \text{ or } 1,\ a_{ij} = 1 \text{ for precisely one pair } (i,j)\}$$

is clearly a basis for $Q_{\varphi\delta}$. Let $\Psi_{\varphi\delta} = \Psi_3 \circ \Psi_2 \circ \Psi_1$. Then $L_{\varphi\delta} = \Psi_{\varphi\delta}^{-1}(N_{\varphi\delta})$ is a basis for $H_{\varphi\delta}$. The set $L_{\varphi\delta}$ is clearly well ordered first by $j \in L_2\delta$ and then according of $i \in \underline{M}_\varphi$. Next to complete our construction we show:

LEMMA 2.6. *The set* $L_H = \cup L_{\varphi\delta}$ ($\pi_1\varphi \in P_1$, $\pi_2\delta \in P_2$) *consists of linearly independent vectors in* $\mathrm{Hom}_P(V_1,V_2)$. *Furthermore, the* *orders on* L_1 *and* L_2 *induce in a natural way a well ordering on* L_H.

PROOF. One needs to show that if $\pi_1\varphi \geq \pi_1\varphi'$ and $\pi_2\delta \geq \pi_2\delta'$, then $L_{\varphi'\delta'} \subset L_{\varphi\delta}$ since, using Lemma 2.5, the $L_{\varphi\delta}$ are up-directed. There is a commutative diagram

$$\begin{array}{ccccccc}
H_{\varphi\delta} & \xrightarrow{\Psi_1} & \mathrm{Hom}_P(V_1\varphi, V_2\delta) & \xrightarrow{\Psi_2} & \mathrm{Hom}_P(k^{n\varphi}, k^{m\delta}) & \xrightarrow{\Psi_3} & Q_{\varphi\delta} \\
\downarrow \rho & & \downarrow \upsilon & & \downarrow \zeta & & \downarrow \chi \\
H_{\varphi'\delta'} & \xrightarrow{\Psi'_1} & \mathrm{Hom}_P(V_1\varphi', V_2\delta') & \xrightarrow{\Psi'_2} & \mathrm{Hom}_P(k^{n\varphi'}, k^{m\delta'}) & \xrightarrow{\Psi'_3} & Q_{\varphi'\delta'}
\end{array}$$

where $\rho = \rho(\varphi,\delta,\varphi',\delta')$, $\upsilon = \upsilon(\varphi,\delta,\varphi',\delta')$, where the various Ψ' are defined in the same way as the various Ψ and ζ, χ are the unique maps defined by commutativity (as the various Ψ and Ψ' are isomorphisms). Using the notation preceding Lemma 2.3 we see that υ sends $f \in \mathrm{Hom}_P(V_1\varphi, V_2\delta)$ to $\alpha_2 \circ f \circ j_1$. Using coordinates and the definitions of α_2 and j_1 we see that ζ sends an element $(f_i(x_j))$ to $(g_p(x_q))$ where $g_p(x_q) = f'_i(x_j)$ for $p \in L_2\delta'$ and $p = i$ and furthermore $f'_i(x_j)$ is obtained from $f_i(x_j)$ by setting $x_j = 0$ in $f_i(x_j)$ for $j \in L_1\varphi - L_1\varphi'$. But then $\chi(a_{ij}) = (b_{qp}) \in Q_{\varphi'\delta'}$ where $b_{qp} = a_{ij}$ if $p = j$, $q = i$ and $i \in \underline{M}_{\varphi'}$, $p \in L_2\delta'$. From this description it is clear that χ has a right inverse ϑ_1 satisfying

$$\Psi_{\varphi\delta}^{-1} \circ \vartheta_1 = \vartheta_2 \circ \Psi_{\varphi'\delta'}^{-1}$$

where $\vartheta_2 \colon H_{\varphi\delta} \to H_{\varphi'\delta'}$ is the inclusion. But then

$$L_{\varphi'\delta'} = \Psi_{\varphi'\delta'}^{-1}(N_{\varphi'\delta'}) = \vartheta_2 \circ \Psi_{\varphi'\delta'}^{-1}(N_{\varphi'\delta'}) = \Psi_{\varphi\delta}^{-1} \circ \vartheta_1(N_{\varphi'\delta'}).$$

By construction $\vartheta_1(N_{\varphi'\delta'}) \subset N_{\varphi\delta}$. Hence $L_{\varphi'\delta'} \subset L_{\varphi\delta}$.

Since each $v \in L_H$ is non-zero in only one coordinate corresponding to a vector of L_2, it is clear that L_H can be well ordered first according to $j \in L_2$ and then according to $i \in \underline{M}$. The proof is complete.

We cover $Q_{\varphi\delta}$ by the sets

$$(Q_{\varphi\delta})_{\underline{l}} = \{(a_{ij}) \in Q_{\varphi\delta} \mid a_{ij} = 0 \text{ if } i > l_j\}$$

where $\underline{l} = (l_j)_{j \in L'} \in (\underline{M}_\varphi)_{L'}$ and $L' = L_2\delta$. Let $(H_{\varphi\delta})_{\underline{l}} = \Psi_3^{-1}((Q_{\varphi\delta})_{\underline{l}})$ and $(\tau_{\varphi\delta})_{\underline{l}} \colon H_{\varphi\delta} \to (H_{\varphi\delta})_{\underline{l}}$ be the projection induced by the evident projection $(\nu_{\varphi\delta})_{\underline{l}} \colon Q_{\varphi\delta} \to (Q_{\varphi\delta})_{\underline{l}}$. Recall that the basis on $Q_{\varphi\delta}$ chosen induces one on $H_{\varphi\delta}$. It is clear that since the $(\nu_{\varphi\delta})_{\underline{l}}$ $(\pi_1\varphi \in P_1,\ \pi_2\delta \in P_2)$ satisfy the condition of Definition 0.1, so do the $(\tau_{\varphi\delta})_{\underline{l}}$. Hence, using Lemma 2.5, we obtain:

PROPOSITION 2.7. *The triple* $(\mathrm{Hom}_P(V_1,V_2),L_H,\leq)$ *together with*

$$P_H = \{(\tau_{\varphi\delta})_{\underline{l}}\circ\pi_{\varphi\delta} \mid \pi_1\varphi \in P_1,\ \pi_2\delta \in P_2,\ \underline{l}\in(\underline{M}_\varphi)^{L^*},\ L^* = L_2\delta\}$$

forms a partial polyspace.

We must show that the assignment of

$$F(V_1,V_2) = (\mathrm{Hom}_P(V_1,V_2),L_H,P_H,\leq)$$

to two polyspaces V_1 and V_2 determines a bifunctor

$$F:\ P\times P \to SP.$$

LEMMA 2.8. *The assignment* $W\to\mathrm{Hom}_P(W,V_3)$ *determines a contravariant functor* $P\to SP$.

PROOF. Let $f: V_1\to V_2$ be a map of polyspaces. There is an induced map

$f^*: H_2\to H_1$ where $H_2=\mathrm{Hom}_P(V_2,V_3)$ and $H_1=\mathrm{Hom}_P(V_1,V_3)$.

First we show that, provided π_1 is sufficiently large, the diagram

$$\begin{array}{ccc} H_2 & \xrightarrow{\ f^*\ } & H_1 \\ {\scriptstyle\pi_2}\downarrow & & \downarrow{\scriptstyle\pi_1} \\ H_{2\mu\delta} & \xrightarrow{\ f^\circ\ } & H_{1\varphi\delta} \end{array}$$

commutes where

$$\pi_2 = \pi_2\mu\delta \in P_{H2},\quad \pi_1 = \pi_1\varphi\delta \in P_{H1}$$

and, if $i_2: H_{2\mu\delta}\to H_2$ is the inclusion, then $f^\circ=\pi_1\circ f^*\circ i_2$. Let a in H_2. The single superscripted inclusions and projections are those for V_1, V_2 and V_3 . We have

$$f^\circ\circ\pi_2(a) = \pi_1\circ f^*\circ i_2(\iota_3\delta\circ a\circ\iota_2\mu\circ\pi_2\mu) = \pi_1\circ\iota_3\delta\circ\pi_3\delta\circ a\circ\iota_2\mu\circ\pi_2\mu\circ f.$$

Since the image of $\iota_3\delta$ has only finitely many coordinates, for $\pi_2\mu$ and hence π large enough, this last expression equals

$$\pi_1\circ\iota_3\delta\circ\pi_3\delta\circ a\circ f = \iota_3\delta\circ\pi_3\delta\circ\iota_3\delta\circ\pi_3\delta\circ a\circ\iota_1\varphi\circ\pi_1\varphi = \pi_1\circ f^*(a).$$

Thus, to show that f is a stratawise polymap, it suffices to show that f° is a stratawise polymap. Note that $H_{2\mu\delta}$ (resp. $H_{1\varphi\delta}$) acquires a polyspace structure from the partial polyspace structure of H_2 (resp. H_1). Let $\iota_{2\underline{l}}: (H_{2\mu\delta})_{\underline{l}}\to H_{2\mu\delta}$ be the inclusion corresponding to a projection in P_L $(L = H_{2\mu\delta})$. We need to show that $f\circ i_{2\underline{l}}$ is a partial polymap. If $\rho_{1\underline{m}}: H_{1\varphi\delta}\to(H_{1\varphi\delta})_{\underline{m}}$ is a projection in P_M $(M = H_{1\varphi\delta})$, we must show that $\rho_{1\underline{m}}\circ f\circ i_{2\underline{l}}$ is a primitive polymap. Taking coordinates one must show that the composition Υ of the maps in the chain

$$(Q_{2\mu\delta})_{\underline{l}} \to Q_{2\mu\delta} \to Q_{1\varphi\delta} \to (Q_{1\varphi\delta})_{\underline{m}},$$

corresponding to the composition $\rho_{1\underline{m}} \circ f \circ i_{2\underline{l}}$ is a polynomial map. Let $\pi_{3\varphi\mu}: \mathrm{Hom}_P(W_1,W_2) \to H_{3\varphi\mu}$ be a projection in P_{H3} for the partial polyspace $H_3 = \mathrm{Hom}_P(W_1,W_2)$. Let $(a_{ij}) \in (Q_{2\mu\delta})_{\underline{l}}$ correspond to the associated map $(\sum a_{ij} i)$. Then

$$(\sum a_i ij) \circ \pi_{3\varphi\mu}(f) = (\sum b_{qp} q) \text{ with } (b_{qp}) \in Q_{1\underline{m}\varphi\delta}$$

provided $\underline{m}$ is sufficiently large. Since the b_{qp} are polynomials in the a_{ij}, Υ is a polynomial map for $\underline{m}$ large enough. However, since projection is a polynomial map, Υ is always a polynomial map.

The proof for the covariant variable is almost identical and omitted.

LEMMA 2.9. *The assignment* $W \to \mathrm{Hom}_P(V,W)$ *extends to a covariant functor* $P \to PP$.

From Lemmas 2.8 and 2.9 and, since the result holds on the set level, we obtain:

THEOREM 2.10. *The assignment*

$$(V_1,V_2) \to (\mathrm{Hom}_P(V_1,V_2), L_H, P_H, \le)$$

extends to a bifunctor $Poly^{op} \times Poly \to SPoly$.

We note that, if V_2 is finite dimensional, then $\mathrm{Hom}_P(V_1,V_2)$ is a polyspace. Furthermore one has the readily proven result:

PROPOSITION 2.11. *Let* $(V,L,P,\le)$ *and* $(V,L',P',\le')$ *be two partial polyspaces such that the vector space generated by* L *is the same as the vector space generated by* L'. *Then the identity* $i_V: V \to V$ *is an isomorphism in SPoly.*

3. WEAK CARTESIAN CLOSEDNESS OF POLYSPACES.

We prove Theorem 0.5.

Let V_1, V_2 and V_3 be vector spaces. There is a set isomorphism

$$\mu: \mathrm{Hom}_{Sets}(V_1 \times V_2, V_3) \to \mathrm{Hom}_{Sets}(V_1, \mathrm{Hom}_{Sets}(V_2,V_3))$$

where

$$\mu(f) = f_{(\,)} \text{ and } f_{(x)}(v) = f(x,v).$$

Let now (V_1,L_1), (V_2,L_2) and (V_3,L_3) be polyspaces. Suppose that $f\colon V_1\times V_2 \to V_3$ is a polymap. We want first to show that $f_{(x)}\colon V_2 \to V_3$ is a polymap for each $x \in V_1$. Let $\pi_2 \in P_2$, $\pi_3 \in P_3$ and choose some $\pi_1 \in P_1$ so that $\pi_1\colon V_1 \to V_1^\circ$ and $x \in V_1^\circ$. Note that the union of elements in F_{V_1} is $V1 = V_1$, so that the last possibility exists. If π_1 and π_2 are large enough, there is a primitive polynomial f' such that

$$\begin{array}{ccc} V_1\times V_2 & \xrightarrow{\quad f \quad} & V_3 \\ {\scriptstyle \pi_1\times\pi_2}\downarrow & & \downarrow{\scriptstyle \pi_3} \\ V_1^\circ\times V_2^\circ & \xrightarrow{\quad f' \quad} & V_3^\circ \end{array}$$

commutes. Restricting one has a commutative diagram

$$\begin{array}{ccc} V_2 & \xrightarrow{\quad f_{(x)} \quad} & V_3 \\ \downarrow & & \downarrow \\ V_2^\circ & \xrightarrow{\quad f'_{(x)} \quad} & V_3^\circ \end{array}$$

where $f'_{(x)}$ is a primitive polymap. We thus have:

LEMMA 3.1. *The map μ restricts to a map*

$$\mu'\colon \operatorname{Hom}_P(V_1\times V_2,V_3) \to \operatorname{Hom}_{Sets}(V_1,\operatorname{Hom}_P(V_2,V_3))$$

which is necessarily injective.

We will first show that μ' maps onto $\Gamma(V_1,\operatorname{Hom}_P(V_2,V_3))$ via the following lemma.

LEMMA 3.2. *Let $f_{(\,)}$ be a polymap mapping into to strata. Then f is a polymap.*

PROOF. In order to show that each coordinate of the map $f\colon V_1\times V_2 \to V_3$ involves only a finite number of variables, it suffices to assume that V_3 is one-dimensional. Then $f_{(\,)}$ maps into the strata. Hence, for $V^* = V_1$, $f_{(V^*)} \subset (H_{\varphi\delta})_{\underline{I}}$ for some $\varphi, \delta, \underline{I}$. Thus $f_{(x)} = (a_i 1(x))$ $(i \in \underline{M}_\varphi)$ and, if y_1 is the coordinate on V_3, then $y_1 = \sum a_i 1(x) i$ $(i \in \underline{M}_\varphi)$ involves only a finitely many variables of the product $V_1\times V_2$.

LEMMA 3.3. *Suppose that f is a polymap. Then $f_{(\,)}$ is a poly-*

map.

PROOF. Let $\pi_1\mu \in P_1$, $\pi_2\varphi \in P_2$ and $\pi_3\delta \in P_3$. Choose $\pi_1\mu$, $\pi_2\varphi$ large enough so that the diagram

$$\begin{array}{ccc} V_1 \times V_2 & \xrightarrow{\quad f \quad} & V_3 \\ {\scriptstyle \pi_1\mu\times\pi_2\varphi}\downarrow & & \downarrow{\scriptstyle \pi_3\delta} \\ V_1\mu \times V_2\varphi & \xrightarrow{\quad f' \quad} & V_3\delta \end{array}$$

commutes with f' a primitive polymap. Let $H = \mathrm{Hom}_P(V_2, V_3)$. Define $g' : V_1\mu \to H_{\varphi\delta}$ by sending x to $g'(x)$ where:

1. $g'(x) = \iota_3\delta \circ f'_{(x)} \circ \pi_2\varphi$ with $\iota_3\delta$: $V_3\delta \to V_3$ the inclusion.
2. $f'_{(x)}(y) = f'(x,y)$.

We show that, in fact, $g'(x) \in H_{\varphi\delta}$. For this first check that, for $x \in V_1\mu$, the diagram

$$\begin{array}{ccc} V_2 & \xrightarrow{\quad g'(x) \quad} & V_3 \\ {\scriptstyle \pi_2\varphi}\downarrow & & \downarrow{\scriptstyle \pi_3\delta} \\ V_2\varphi & \xrightarrow{\quad f'_{(x)} \quad} & V_3\delta \end{array}$$

commutes. But

$$\pi_3\delta \circ g'(x) = \pi_3\delta \circ \iota_3\delta \circ f'_{(x)} \circ \pi_2\varphi = f'_{(x)} \circ \pi_1\varphi.$$

Since f' is a primitive polymap, clearly $f'_{(x)}$ is also. Hence $g'(x) \in H_{\varphi\delta}$ for $x \in V_1\mu$.

Next we verify that

$$\begin{array}{ccc} V & \xrightarrow{\quad f_{(x)} \quad} & H \\ {\scriptstyle \pi_1\mu}\downarrow & & \downarrow{\scriptstyle \pi_{\varphi\delta}} \\ V_1\mu & \xrightarrow{\quad g'(x) \quad} & H_{\varphi\delta} \end{array}$$

commutes. Let $x \in V_1$. Then, referring to Lemma 2.5, one finds

$$\begin{aligned} \pi_{\varphi\delta} \circ f_{(x)}(y) &= \iota_3\delta \circ \pi_3\delta \circ f_{(x)} \circ \iota_2\varphi \circ \pi_2\varphi(y) = \iota_3\delta \circ \pi_3\delta \circ f(x, \pi_2\varphi(y)) \\ &= f'(\pi_1\mu(x), \pi_2 f(y)) = f'(\pi_1\mu(x))(\pi_2\varphi(y)) = \pi_3\delta(g'(\pi_1\mu(x))(y)) \\ &= g'(\pi_1\mu(x))(y) \qquad \text{since } \pi_1\mu(x) \in V_1\mu. \end{aligned}$$

Finally we show that for $\underline{I} \in \underline{M}'' = (\underline{M}_\varphi)_{L'}$ ($L' = L_3\delta$) with $\underline{I}$ sufficiently large and hence for all $\underline{I} \in \underline{M}''$, the composite

$$\alpha \circ g' : V_1\mu \to H_{\varphi\delta} \to (H_{\varphi\delta})_{\underline{I}} \text{ where } \alpha \in P_H$$

is a primitive polymap. Taking coordinates f' becomes a map f'':

$k^{l\mu}\times k^{n\varphi}\to k^{m\delta}$ which can be written, since f' is a primitive polymap, $f'' = (\sum Q_{ij} i)$ $(i \in \underline{M}_\varphi)$ with $j \in L_3\delta$ and with the Q_{ij} polynomials in the variables of $k^{l\mu}$. But then $f'_{(x)}$ and thus also g' corresponds to the map $f''_{(x)}$: $k^{l\mu}\to \mathrm{Hom}_P(k^{n\varphi},k^{m\delta})$ defined by sending x to $(Q_{ij}(x))$ where $i \in \underline{M}_\varphi$ and $j \in L_3\delta$. Since, for $j \in L_3\delta$, $Q_{ij}(x) \neq 0$ for only finitely many i, g' maps $V_1\mu$ into $(H_{\varphi\delta})$ for some $\underline{l} \in \underline{M}''$ sufficiently large. Since $f'_{(x)}$ is a primitive polymap, so is g'.

The final lemma that we need for Theorem 0.5 is:

LEMMA 3.4. *If f: $V_1\times V_2 \to V_3$ is a polymap, then*

$$f_{(\ )}: V_1 \to \mathrm{Hom}_P(V_2,V_3)$$

is a polymap mapping into the strata.

PROOF. We need only show that $f_{(\ )}$ maps into the strata. Let V'_3 be a one-dimensional subspace of V_3 generated by an element of L_3. There is a factorization

$$\begin{array}{ccc} V_1 & \xrightarrow{f_{(\ )}} & \mathrm{Hom}_P(V_2,V_3) \\ & \searrow^{g} & \downarrow \pi_* \\ & & \mathrm{Hom}_P(V_2,V'_3) \end{array}$$

with π_* arising from the projection π: $V_3\to V'_3$. If g did not map into the strata, then $\pi\circ f$: $V_1\times V_2\to V_3\to V'_3$ would not be a polymap as one sees from the argument in the proof of Lemma 3.2. But since π is a polymap, this is a contradiction.

Thus the proof of Lemma 3.4 and hence Theorem 0.5 is complete.

4. NEL'S TOPOLOGICAL UNIVERSE COMPLETION.

Let k be an algebraically closed field of characteristic 0. Let *Proj* be the category whose objects are normal projective varieties of finite type over k and whose morphisms are morphisms of schemes over k. See Hartshorne [8]. Let X be an object in *Proj*. A covering of X is a singleton set containing a surjective map f: Y→X in *Proj*. We show that the pair (*Proj*,C_P) where C_P consists of all coverings of objects in *Proj* is a pre-universe. The requirement for *Proj* to be a pre-universe are introduced as we go along. See Nel [11] for more information.

First one must show that a covering $\{f\}$ with f: X→Y is final. This means that if $g: Y \to Z$ is a set map and $g \circ f$ is a map in *Proj*, then g is also a map of varieties. For this look at the graphs $\Gamma(g)$, $\Gamma(g \circ f)$ and $\Gamma(f)$ of g, $g \circ f$ and f, respectively. Then

$$\Gamma(g) = \Gamma(g \circ f) \circ \Gamma(f^{-1})$$

is readily seen to be a closed subset of $Y \times Z$. The projection of $\Gamma(g)$ onto Y is one-one and onto. Hence, by Zariski's Main Theorem [8], this projection has an inverse $h: Y \to \Gamma(g)$ which is a map of varieties. But then $g = p_2 \circ h$, where p_2 is a projection onto the second coordinate, is a map of varieties.

Clearly the identities are coverings. Suppose that one has two maps $g, f \in$ *Proj* with g a covering. One must show that there is a commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & Z \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

with h a covering. But one can take

$$W = X \times_Y Z = \{(x,z) \in X \times Z \mid f(x) = g(z)\}.$$

Then W has the structure of a projective variety. Let h be the projection to X. Since g is surjective, so is h. Thus *Proj* is stable under "push-backs". Finally, the composition of two coverings is clearly a covering. Thus (*Proj*,C_P) is a pre-universe.

We note that a similar construction unfortunately does not work for affine varieties since the covering families need not be final. However here one can replace *Proj* by the category *AF* of affine schemes of finite type over k and let C_A be the collection of singleton sets whose only morphism is an isomorphism in *AF*. Then, in this case (*AF*,C_A) is readily seen to be a pre-universe.

In the following, we use the word *"imprint"* rather than the word "native" used in [11] since it seems to better represent the construction looked at here. A topological universe is much more than a Cartesian closed category. For instance it has arbitrary limits and colimits.

To *Proj* we associate a topological universe $\mathcal{U}_P$ where the objects of $\mathcal{U}_P$ are the pairs (X,ξ) such that X is an arbitrary set and ξ is a collection of maps $B \to X$ called imprints with B an object of *Proj* subject to the conditions:

(1) Every constant map $B \to X$ with B an object of *Proj* is an imprint.

(2) If $\alpha: B \to X$ is an imprint, and $\beta \in Proj$, then $\alpha \circ \beta$ is an imprint.

(3) Given a map $q: B \to X$ such that, for some covering $p: B' \to B$, the composite $q \circ p$ is an imprint, then q is an imprint.

A morphism in the topological universe U_P is a map $f: (X,\xi) \to (Y,\xi')$ such that if $\alpha: B \to X$ is in ξ, then $f \circ \alpha$ is in ξ'. This definition has its origin in differential geometry where structure on a differentiable manifold is often defined in terms of curves in it. The definition also suggests that stratawise polymaps are more likely to provide a Cartesian closed category than polymaps. An example in U_P that one might want to study is (X,ξ) where X is a complete variety and ξ is the non-empty set (Chow's Lemma [8]) of maps $f: X^* \to X$ where X^* is projective and f is an onto map of varieties.

For (AF, C_A) the corresponding topological universe U_A contains, for instance, algebraic spaces (see Knutson [9]) if, for an algebraic space X, we identify X with (X,ξ) where ξ is the set of all maps $f: Y \to X$ where Y is an object of AF and f is an étale map.

As further motivation we mention the U structure on

$$\underline{H} = \mathrm{Hom}_U((X,\xi_X),(Y,\xi_Y))$$

which makes U into a Cartesian closed category. This is simply defined. A map $s: A \to \underline{H}$ is an imprint if and only if, for the evaluation map ev: $X \times H \to Y$ and every imprint $t: B \to X$, the composite $ev \circ (s \times t): A \times B \to Y$ is in U.

EXAMPLE 4.1. Consider $\underline{H} = \mathrm{Hom}_U((A^1,\xi),(A^1,\xi))$ where $U = U_A$ and ξ consists of all maps $f: X \to A^1$ in AF. If $A^1 \to \mathrm{Hom}_U(A^1,A^1)$ is an mprint, then for all imprints $t: A^1 \to A^1$, the composite $h: A^1 \times A^1 \to A^1$ defined by sending (a,b) to $s(a)(t(b))$ is a map in U_A. But then h must be a polynomial map.

5. RELATIVE CARTESIAN CLOSEDNESS OF PARTIAL POLYSPACES.

Let T denote the collection of stratawise polymaps. Before proving Theorem 0.7, we show:

LEMMA 5.1. *The category SPoly has* T*-products.*

PROOF. A little diagram chasing below shows that products exist in *PPoly*. Let V_1 and V_2 be two partial polyspaces. As the projection maps $p_1: V_1 \times V_2 \to V_1$ and $p_2: V_1 \times V_2 \to V_3$ are in *PPoly*, they are also in *SPoly*. Let $f: V \to V_1$ and $g: V \to V_2$ be two maps in *SPoly* and $h: W \to V_1 \times V_2$ be the unique map such that $p_1 \circ h = f$ and $p_2 \circ h = g$. Let $W' \in F_W$ and $\iota: W' \to W$ the inclusion. Then $f \circ \iota$ and $g \circ \iota$ are polymaps. Since $V_1 \times V_2$ is the product of V_1 and V_2 in *PPoly*, $h \circ \iota$ is in *PPoly*. Thus h is in *SPoly* and the proof is complete.

It is clear that the product $\times$ defines a bifunctor $P \times P \to P$. For a polyspace (V_3, L_3) and finite dimensional vector spaces V_1 and V_2, since T-maps from a finite dimensional vector space are the same as *P* maps, we conclude from Theorem 0.5 that there is a natural equivalence

(3) $\qquad \mathrm{Hom}_T(V_1 \times V_2, V_3) \approx \mathrm{Hom}_T(V_1, \mathrm{Hom}_T(V_2, V_3))$

where the right and left hand sides are functors in V_3 from *P* to *Sets* with the following proviso: Let W be a one-dimensional subspace of V_3 generated by one element of L_3. Let β be a T-map (here a polymap) which is the composite of a T-map from V_1 to $\mathrm{Hom}_T(V_2, V_3)$ followed by a projection to $\mathrm{Hom}_T(V_2, W)$. Then, for (3), one must show that β maps into the strata.

To show that β maps into the strata, one must show that $\beta(V_1)$ is contained in a finite dimensional subspace of $\mathrm{Hom}_T(V_2, W)$. Since $\mathrm{Hom}_T(V_2, W)$ is of infinite countable dimension, by looking at the coordinates of β, we see that it suffices to prove:

LEMMA 5.2. *A finite dimensional vector space* V *over an infinite uncountable field cannot be covered by countably many hypersurfaces.*

PROOF. If V is one-dimensional, since a hypersurface then consists of finitely many points, we are done. Otherwise, it is easy to see that there are uncountably many hyperplanes. Suppose that V is covered by countably many hypersurfaces. Then there is a hyperplane L which is not contained in one of the countably many hypersurfaces covering V. The hyperplane L is in turn covered by countably many hypersurfaces. Applying induction we obtain a contradiction.

Clearly, if the field in Lemma 5.2 were countable, then V could be covered by countably many hypersurfaces.

Let now V_1 be finite dimensional and (V_2, L_2), (V_3, L_3) be arbitrary polyspaces. One readily shows that $\mathrm{Hom}_T(V_2, V_3)$ is a vector space under pointwise addition and scalar multiplication.

Clearly

$$\mathrm{Hom}_P(V_2,V_3) \subset \mathrm{Hom}_T(V_2,V_3).$$

We wish to make $\mathrm{Hom}_T(V_2,V_3)$ into a partial polyspace. For this note the following readily proven result.

LEMMA 5.3. 1. *Let* V *be a partial polyspace and* ι *be the inclusion right inverse to some* $\pi \in P_V$. *Suppose that* f *is in* T. *Then, whenever* $f \circ \iota$ *or* $f \circ \pi$ *is well defined, it lies in* T.

2. *Let* g *be a morphism in PP and* f *belong to* T. *If* $g \circ f$ *is defined then* $g \circ f$ *is in* T.

In particular, in Lemma 5.3 (2), g could be $\pi \in P_V$ or its right inverse ι.

Repeating the construction of §2 almost word for word, one shows that $\mathrm{Hom}_T(V_2,V_3)$ is a partial polyspace whose linearly independent set is the same as the linearly independent set of $\mathrm{Hom}_P(V_2,V_3)$ and whose projections are extensions of those belonging to $\mathrm{Hom}_P(V_2,V_3)$. Furthermore, as in §2, using Lemma 5.3 (2), the assignment $V_3 \to \mathrm{Hom}_T(V_2,V_3)$ extends to a functor from *P* to *PP*.

Let $V'_2 \in F_2 = F_{V[2]}$, $f \in \mathrm{Hom}_T(V_1, \mathrm{Hom}_T(V_2,V_3))$ and

$$\rho: \mathrm{Hom}_T(V_2,V_3) \to \mathrm{Hom}_T(V'_2, V'_3)$$

be the restriction map. Using (3) $\rho \circ f$ determines a map $\mu(f)$: $V_1 \times V'_2 \to V_3$ in *Poly*. The $\mu(f)$, for $V'_2 \in F_2$, extend to a map $g: V_1 \times V_2 \to V_3$ such that $g(x,y) = f_{(x)}(y)$. The map g is in T since elements of F_W, where $W = V_1 \times V_2$, are of the form $V_1 \times V'_2$ $(V'_2 \in F_2)$.

Conversely let $g: V_1 \times V_2 \to V_3$ be a T-map and $g \circ \Phi$ the restriction of g to $V_1 \times V'_2$. Then $g \circ \Phi$ determines a map

$$\delta(g): V_1 \to \mathrm{Hom}_T(V'_2,V_3) = \mathrm{Hom}_P(V'_2,V_3)$$

inducing a map $f: V_1 \to \mathrm{Hom}_T(V_2,V_3)$ such that $f_{(x)}(y) = g(x,y)$. Since the projections of $\mathrm{Hom}_T(V_2,V_3)$ are just the extensions of the projections of $\mathrm{Hom}_P(V_2,V_3)$, clearly f is a polymap.

In order to complete the proof of Theorem 0.7 let (V_1,L_1) be an arbitrary polyspace. We write T-dirlimF to denote the T-direct limit of a subcategory *F* of *LP* whose arrows are contained in T. One easily shows:

LEMMA 5.6. 1. *The set* $F_1 = F_{V_1}$ *can be viewed as a subcategory of LP with arrows inclusions. Then* T-dirlim$F_1 = V_1$.

2. *Let* V_2 *be a polyspace and* G *the subcategory of LP whose objects are of the form* $V'_1 \times V_2$ $(V'_1 \in F_1)$ *and whose ar-*

rows are inclusions. Then

$$\text{T-lim}G = V_1 \times V_2 = \text{T-dirlim}(V'_1 \times V_2) \quad (V'_1 \in F_1).$$

One then has $(V'_1 \in F_1)$:

$$\text{Hom}_T(V_1 \times V_2, V_3) = \text{Hom}_T(\text{T-dirlim}(V'_1 \times V_2, V_3))$$
$$\approx \lim(\text{Hom}_T(V'_1 \times V_2, V_3)) \approx \lim(\text{Hom}_T(V'_1, \text{Hom}_T(V_2, V_3)))$$
(using our previous result)
$$\approx \text{Hom}_T(V_1, \text{Hom}_T(V_2, V_3)).$$

Thus Theorem 0.7 is proven.

6. WEAK CARTESIAN CLOSEDNESS IN AFFINE VARIETIES.

We prove Theorem 0.8 by combining various lemmas. Let B again denote the collection of basic maps between affine varieties.

LEMMA 6.1. *The category AF has* B-*products.*

PROOF. Let V_1 (resp. V_2) be a subvariety of W_1 (resp. W_2). Then it is easy to see that $V_1 \times V_2 = V(J)$, where if $V_1 = V(I_1)$ and $V_2 = V(I_2)$, then

$$J = I_1 A[W_1 \times W_2] + I_2 A[W_1 \times W_2].$$

Clearly the projections p_1 and p_2 from $V_1 \times V_2$ to V_1 and V_2 are in B. Suppose that there are B-maps $f: V_3 \to V_1$, $g: V_3 \to V_2$ with V_3 a subvariety of the partial polyspace W_3. Since f, g are induced by stratawise polymaps $f^*: W_3 \to V_1$, $g^*: W_3 \to W_2$, the unique morphism $h: V_3 \to V_1 \times V_2$ such that $p_1 \circ h = f$ and $p_2 \circ h = g$ is induced from a stratawise polymap $h^*: W_3 \to W_1 \times W_2$ such that $p_1 \circ h^* = f^*$ and $p_2 \circ h^* = g^*$ (p_1 and p_2 extended to $W_1 \times W_2$). The proof is complete.

Suppose that V_2 is an affine subvariety of a polyspace W_2 and that W_3 is a polyspace. Then $\text{Hom}_B(V_2, W_3)$ can be given the structure of a vector space via pointwise operations. We suppose first that W_3 is one-dimensional. Since the linearly independent set associated with W_2 has a well ordering $\leq$, from Lemma 2.6, the linearly independent set L_H of

$$H = \text{Hom}_B(V_2, W_3) = \text{Hom}_T(V_2, W_3)$$

is also well ordered. Note that, as we saw in §5, $\text{Hom}_T(V_2, W_3)$ and $\text{Hom}_P(V_2, V_3)$ have, as partial polyspaces, the same linearly independent set. There is a restriction map

$$\rho: \mathrm{Hom}_B(W_2, W_3) \to \mathrm{Hom}_B(V_2, W_3).$$

Applying transfinite induction one can choose a linearly independent subset L' of L_H such that

1. $\rho(L')$ is a maximal linearly independent subset of $\rho(L_H)$.
2. If L'' is a subset of L and $\rho(L'')$ is a maximal linearly independent subset of $\rho(L_H)$, then $L' \leq L''$. The condition $L' \leq L''$ means the following: If $l'' \in L''$, then $\rho(l'')$ is a linear combination of $\rho(l'_i)$ ($l'_i \in L'$) where $l'_i \leq l''$ for each i.

REMARK. The above procedure provides a method for assigning to A[W], where W is an usual affine variety of finite type over k, a canonical basis. As an object in *PPoly* the structure on $\mathrm{Hom}_B(V_2, W_3)$ depends on the well ordering on W_2.

We let $\rho(L')$ be the maximal linearly independent subset of $\rho(L_H)$ making it into a polyspace. Taking products one can extend this construction to place a polyspace structure on $\mathrm{Hom}_B(V_2, W_3)$ for W_3 a finite dimensional vector space with a well ordered basis. Finally, if W_3 is an arbitrary polyspace, as in §2, using the projections of W_3, one can define a partial polyspace structure on $\mathrm{Hom}_B(V_2, W_3)$. In fact it is easier here since one need only worry about the range variable. The well ordering is given first according to the coordinates determined by L_3 and then in each coordinate using the case where W_3 is one-dimensional.

We need to show that the assignment $W_3 \to \mathrm{Hom}_B(V_2, W_3)$ extends to a functor $P \to SP$. Let $f: V' \to V$ be in P,

$$Q = \mathrm{Hom}_B(V_2, V),\quad Q' = \mathrm{Hom}_B(V_2, V'),\quad K \in F_Q \text{ and } K' \in F_{Q'}.$$

We need to show that the map $h = \pi_K \circ f^* \circ i$, where $i: K' \to Q'$ is the inclusion, $\pi_K \in P_Q$ and f^* is induced from f, is a primitive polymap. But, from our choice of basis, one can rewrite h as the composite

$$K' \xrightarrow{e_1} \mathrm{Hom}_B(W_2, V') \xrightarrow{f_*} \mathrm{Hom}_B(W_2, V) \xrightarrow{e_2} K,$$

where e_1 is a vector space isomorphism of K' with an element of $F_{N'}$ ($N' = \mathrm{Hom}_B(V_2, V')$) and e_2 equals some π ($\pi \in P_N$, $N = \mathrm{Hom}_B(V_2, V)$) followed by a vector space isomorphism with K. Since f_* in the diagram has been shown to be in *SPoly*, we obtain:

LEMMA 6.2. *Let* V_2 *be an object of* $AF^\wedge$. *Then the assignment*

$$W_3 \to \mathrm{Hom}_B(V_2, W_3)$$

extends to a functor $P \to SP$.

Next, let V_3 be a subvariety of a polyspace W_3. Then, in the evident way, one can view $Hom_B(V_2,V_3)$ as a subset of $Hom_B(V_2,W_3)$. Now, if $Q = Hom_B(V_2,W_3)$ and $J \in F_Q$, then J can be identified with $J' \in F_R$ $(R = Hom_B(W_2,W_3))$. But J', using the notation of §2, is contained in some $(H_{\varphi\delta})_{\underline{I}}$.

Before continuing with our main argument we show:

LEMMA 6.3. *Let* $x \in W_3$. *The evaluation map*

$$ev_x : Hom_B(W_2,W_3) \to W_3$$

sending f *to* $f(x)$ *is a basic stratawise polymap.*

PROOF. For $\pi_2\varphi \in P_{W_2}$, $\pi_3\delta \in P_{W_3}$ the composition

$$(H_{\varphi\delta})_{\underline{I}} \xrightarrow{j} Hom_B(W_2,W_3) \xrightarrow{ev_x} W_3 \xrightarrow{\pi_3\delta} W_{3\delta}$$

is also evaluation at x and a primitive polynomial map.

As a consequence one obtains:

COROLLARY 6.4. *The evaluation map*

$$ev_x : Hom_B(V_2,W_3) \to W_3$$

sending f *to* $f(x)$ *is a stratawise polymap* $(x \in V_2)$.

Let $V_3 = V(I)$, $a \in I$ and $ev_x : Hom_B(V_2,W_3) \to W_3$ be the evaluation map. Since $a \in A[W_3]$, $a : W_3 \to k$ is a stratawise polymap. As ev_x sends $(H_{\varphi\delta})_{\underline{I}}$ to $W_{3\delta}$, clearly $a \circ ev_x$ is again a basic stratawise polymap. Let

$$J = \{a \circ ev_x \mid a \in I,\ x \in V_2\}.$$

Then J is an ideal in $A[Hom_B(V_2,W_3)]$ and clearly $Hom_B(V_2,V_3) = V(J)$. Thus $Hom_B(V_2,V_3)$ is an affine subvariety of $Hom_B(V_2,W_3)$.

Using the definition of morphisms in $AF^\wedge$ and Lemma 6.2, one readily obtains:

PROPOSITION 6.5. *Let* V_2 *be a fixed object of* $AF^\wedge$. *The assignment of* $Hom_B(V_2,W_3)$ *to an object* W_3 *of* $AF^\wedge$ *extends to a functor* $AF^\wedge \to AF$.

Next we show that, for V_1, V_2 in $AF^\wedge$ and W_3 in *Poly*, one has the bijection

$$(4) \qquad \tau : Hom_B(V_1 \times V_2, W_3) \to Hom_B(V_1, Hom_B(V_2,W_3))$$

required to prove Theorem 0.8. Let f belong to the left side of (4). There is a commutative diagram

$$\begin{array}{ccc} V_1\times V_2 & \xrightarrow{\quad f \quad} & W_3 \\ {\scriptstyle i}\big\downarrow & \nearrow {\scriptstyle g} & \\ W_1\times W_2 & & \end{array}$$

with i the inclusion of $V_1\times V_2$ as a subvariety of the polyspace $W_1\times W_2$ and g a basic stratawise polymap. From Theorem 0.7 there is an induced map $g'\colon W_1\to \mathrm{Hom}_B(W_2,W_3)$ in T. There is also a sequence of maps

$$V_1 \xrightarrow{\ i'\ } W_1 \xrightarrow{\ g'\ } \mathrm{Hom}_B(W_2,W_3) \xrightarrow{\ \rho'\ } \mathrm{Hom}_B(V_2,W_3)$$

whose composite equals $\tau(f)$ where i' is the inclusion and ρ' the restriction map. Since the linearly independent set defining $\mathrm{Hom}_B(V_2,W_3)$ can be identified with a subset of the one defining $\mathrm{Hom}_B(W_2,W_3)$ and since g' is a stratawise polymap, $\rho'\circ g'$ is a stratawise polymap. Hence $\tau(f)$ is in B.

Conversely, let $d\colon V_1\to \mathrm{Hom}_B(V_2,W_3)$ be in B. Then there is a commutative diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{\quad d \quad} & \mathrm{Hom}_B(V_2,W_3) \\ {\scriptstyle j}\big\downarrow & \nearrow {\scriptstyle d'} & \\ W_1 & & \end{array}$$

with d' a stratawise polymap. We want a commutative diagram

$$\begin{array}{ccc} W_1 & \xrightarrow{\quad d' \quad} & \mathrm{Hom}_B(V_2,W_3) \\ {\scriptstyle d''}\big\downarrow & \nearrow {\scriptstyle \rho} & \\ \mathrm{Hom}_B(V_2,W_3) & & \end{array}$$

where d'' is a stratawise polymap. In the common coordinate of d' and d'' we let d' and d'' agree. Otherwise we set the coordinates of d'' to zero. One sees immediately that d'' is then a stratawise polymap. Hence, by Theorem 0.7, there is an induced map $m\colon W_1\times W_2\to W_3$ in *SPoly* and then a commutative diagram

$$\begin{array}{ccc} V_1\times V_2 & \xrightarrow{\quad \tau^{-1}(d) \quad} & W_3 \\ \big\downarrow & & \big\downarrow \\ W_1\times W_2 & \xrightarrow{\quad m \quad} & W_3 \end{array}$$

with the left hand arrow an inclusion. Hence $\tau^{-1}(d)$ is in B.

Let now V_3 be an arbitrary object of $AF^\wedge$. Since it is essentially just a matter of mapping into V_3 on the left hand side of (4) and correspondingly into $Hom_B(V_2, W_3)$ on the right hand side of (4), it is clear that (4) restricts to the natural isomorphism required to prove Theorem 0.8.

7. CARTESIAN CLOSEDNESS OF *SPoly* AND *AF*.

Let T again be the collection of basic stratawise polymaps. First we show:

LEMMA 7.1. *The category SPoly of stratawise polymaps has products.*

PROOF. Let V_1 and V_2 be partial polyspaces. Since the projections $p_1: V_1 \times V_2 \to V_1$ and $p_2: V_1 \times V_2 \to V_2$ are polymaps; they are stratawise polymaps. Let g: $V \to V_1$ and h: $V \to V_2$ be two stratawise polymaps. Then $h = h_n \circ \cdots \circ h_1$ and $g = g_m \circ \cdots \circ g_1$ with each function in the compositions a T-map. Let g: $V \to V'_1$ and $h: V \to V'_2$. Since T-products exist (Lemma 5.1), there is a T-map $l_1: V \to V'_1 \times V_2$ which when followed by the projections p'_1 and p'_2 to V'_1 and V'_2 yields g_1 and h_1, respectively. We can replace V by $V'_1 \times V'_2$, g_1 by p'_1, and g'_2 by p'_2. But since $g_2 \circ p'_1$ and $h_2 \circ p'_2$ are in T, applying induction, we obtain a clearly unique map $l = l_1 \circ \cdots \circ l_j$ in *SPoly* such that $p_1 \circ l = g$ and $p_2 \circ l = h$, thus completing the proof.

Thus we have a bifunctor $\times$: *SPoly* $\times$ *SPoly* $\to$ *SPoly* . Using the construction in §2, aside from Lemma 7.2 below, one obtains a partial polyspace structure on $Hom_{SP}(V_1, V_2)$ for partial polyspaces V_1 and V_2.

LEMMA 7.2. *The pointwise addition of two elements h, g in* $Hom_{SP}(V_1, V_2)$ *belongs to SPoly.*

PROOF. Let

$$h = h_n \circ \cdots \circ h_1 \text{ and } g = g_m \circ \cdots \circ g_1$$

with each function in the compositions a T-map. Using the argument of Lemma 7.1 one has

$$h = h_n \circ \cdots \circ (h_2 \circ p'_2) \circ l_1 \text{ and } g = g_m \circ \cdots \circ (g_2 \circ p'_1) \circ l_1$$

with l_1, $h_2 \circ p'_2$ and $g_2 \circ p'_1$ in T (Lemma 5.3 (2)). But then one need only show that the sum of

$$h_n \circ \cdots \circ (h_2 \circ p'_2) \text{ and } g_m \circ \cdots \circ (g_2 \circ p'_1)$$

is in *SPoly* which is possible, using induction, since we saw earlier that the sum of two T-maps between the same polyspaces is again a T-map. The proof is complete.

To show that the assignment $(W,V) \to Hom_{SP}(W,V)$ determines a bifunctor $SP^{op} \times SP \to SP$ it suffices to prove the next two lemmas. We use the notation of §2.

LEMMA 7.3. *The assignment* $W \to Hom_{SP}(W,V)$ *determines a contravariant functor* $SP \to SP$.

PROOF. Let $f: W_1 \to W_2$ be in T. There is an induced map

$$f^*: Hom_{SP}(W_2,V) \to Hom_{SP}(W_1,V).$$

The composite map $f \circ \iota_1\mu: W_1\mu \to W_1 \to W_2$, by definition, is a polymap for each $\pi_1\mu: W_1 \to W_1\mu$ in P_{W_1} with $\iota_1\mu$ the inclusion right inverse to $\pi_1\mu$. Hence, by Lemma 2.8, the composite

$$Hom_{SP}(W_2,V) \xrightarrow{f^*} Hom_{SP}(W_1,V) \xrightarrow{(\iota_1\mu)^*} Hom_{SP}(W_1\mu,V)$$

is in T. Using the notation of §2, let $\iota_{\underline{I}}: (H_{\varphi\delta})_{\underline{I}} \to Hom_{SP}(W_2,V)$ be an inclusion map. Using the construction of §2, the linearly independent set of $Hom_{SP}(W_1,V)$ is the union of the images of the linearly independent sets of the $Hom_{SP}(W_1\mu,V)$ under $(\pi_1\mu)^*$ $(\pi_1\mu \in P_{W_1})$. It is then clear that, since the composite $\iota_1\mu^* \circ f^* \circ \iota_{\underline{I}}$ is a polymap, that f^* is in T. Since every element f in *SP* is a composite of elements in T, one can complete the proof in an evident fashion.

Using the same type of argument one has:

LEMMA 7.4. *The assignment* $W \to Hom_{SP}(V,W)$ *induces a functor* $SP \to SP$.

Finally we obtain the main result of the paper:

THEOREM 7.5. 1. *The category SPoly of stratawise polymaps is Cartesian closed.*

2. *The category AF of affine varieties is Cartesian closed.*

PROOF. Using the results and methods of §6, one can prove the analogs of Lemmas 7.1-7.4. Then, using Theorem 0.8 instead of Theorem 0.7, the proof of part 2 will be similar to the proof of part 1 given below.

Let $f \in Hom_{SP}(V_1 \times V_2, V_3)$. One has a factorization

$$V_1 \times V_2 \xrightarrow{f_{()}} V_3^1 \xrightarrow{f_1} V_3^2 \longrightarrow \cdots \xrightarrow{f_n} V_3^{n+1} = V_3$$

with the $f_i \in$ T. Using Theorem 0.7, corresponding to $f_{()}$ one obtains a map g_0: $V_1 \to \mathrm{Hom}_{SP}(V_2, V'_3)$ in T. Corresponding to the f_i $(i = 1, \ldots, n)$ one obtains maps

$$g_i = (f_i)_*: \mathrm{Hom}_{SP}(V_2, V_3^i) \to \mathrm{Hom}_{SP}(V_2, V_3^{i+1})$$

in T $(i = 1, \ldots, n)$. Hence $g = g_n \circ g_{n-1} \circ \cdots \circ g$ is in *SP* and evidently g: $V_1 \to \mathrm{Hom}_{SP}(V_2, V_3)$ satisfies $g(x)(y) = f(x,y)$.

Let now $f \in \mathrm{Hom}_{SP}(V_1, \mathrm{Hom}_{SP}(V_2, V_3))$. There is a factorization

$$V_1 \xrightarrow{f_0} V_1^1 \xrightarrow{f_1} V_1^2 \longrightarrow \cdots \longrightarrow V_1^n \xrightarrow{f_n} \mathrm{Hom}_{SP}(V_2, V_3)$$

with the f_i in T. Corresponding to f_n, using Theorem 0.7, one obtains a map g_n: $V_1^n \times V_2 \to V_3$ in T. There are natural maps $f_n \times$ I: $V_1^i \times V_2 \to V_1^{i+1} \times V_2$ in T $(i = 0, \ldots, n)$. Hence the map g: $V_1 \times V_2 \to V_3$ corresponding to f $(g(x,y) = f_{(x)}(y))$ can be rewritten

$$g = g_n \circ (f_{n-1} \times \mathrm{I}) \circ \cdots \circ (f_0 \times \mathrm{I})$$

with factors in T. Again note that the natural equivalence arises from that on the sets level.

BIBLIOGRAPHY.

1. P. CHERENACK, A Cartesian closed extension of a category of affine schemes, *Cahiers Top. et Géom. Diff.* XXIII-3 (1982), 298-318.
2. P. CHERENACK, Algebraic transversality in *Global Analysis - Analysis on Manifolds* (dedicated to M. Morse; ed. T.Rassias) Teubner Texte Band 57, Teubner 1983.
3. P. CHERENACK, Cones and comparisons, in ind-affine homotopy theory, *Quaestiones Math.* 6 (1983), 49-66.
4. P. CHERENACK, Extending schemes to a Cartesian closed category, *Quaestiones Math.* 9 (1986), 95-133.
5. A. FRÖLICHER, Smooth structures in category theory, *Lecture Notes in Math.* 962, Springer (1982).
6. M. GOLUBITSKY & GUILLEMIN, *Stable mappings and their singularities*, Springer 1973.
7. A. GROTHENDIECK, Techniques de construction et théorèmes d'existence en Géométrie Algébrique, IV, in *Séminaire Bourbaki 1960/61*, Benjamin, 1966.
8. R. HARTSHORNE, *Algebraic Geometry*, Springer, 1977.
9. D. KNUTSON, *Algebraic spaces*, Springer, 1971.
10. S. MAC LANE, *Categories for the working mathematician*, Springer, 1971.
11. L. D. NEL, Topological universes and smooth Gelfand-Naimark duality, *Contemporary Math.* 30 (1984), 244-276.

Financial support from the South African Council for Scientific and Industrial Research via the Topology Research Group at the University of Cape Town is acknowledged.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CAPE TOWN
7700 RONDEBOSCH
SOUTH AFRICA