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THE CATEGORY OF UNIFORM FRAMES

by J. L. FRITH

RÉSUMÉ. Cet article étudie la catégorie des cadres uniformes et ses relations à d'autres catégories. En particulier, les relations à la catégorie des espaces uniformes décrites à l'aide des foncteurs ouvert et spectre sont examinées. Les cadres proximaux de Banaschewski lui sont aussi reliés de façon appropriée.

0. INTRODUCTION.

Frames (and locales, the dual category) are presently structures of considerable interest (see e.g. Johnstone [3]). This paper arises from a desire to consider frames with various uniform-type structures, the topic of the author's thesis. Frame-analogues of the open set, open cover structure of (covering) uniform spaces are considered. These have also been investigated by Pultr [5] where it is established that the uniformizable frames are precisely the completely regular ones.

In this paper, a functor from the category of completely regular frames to uniform frames leaving the underlying frame unchanged is constructed, and it is shown to be the coarsest possible such functor; the result of Pultr is a consequence. Open- and spectrum-type functors are constructed which lift the well-known situation for frames and topologies.

An application includes a demonstration that the proximal structures considered by Banaschewski [1] are just the totally bounded uniform frames, with a particularly simple proof which seems to be new in the spatial setting. In forthcoming papers, non-symmetric analogues of these uniform structures are considered.

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1. BACKGROUND.

A *frame* is a complete lattice satisfying the (infinite) distributive law:

$$a \wedge \bigvee x_\alpha = \bigvee (a \wedge x_\alpha).$$

Frame maps preserve top and bottom elements (denoted by 1 and 0 resp.), finite meets, arbitrary joins. The resulting category is *FRM*.

The topology of a topological space is a frame, functorially so (but contravariantly: continuous functions are mapped to the associated inverse image lattice map). This functor *O* (the *open* functor) from *TOP* to *FRM* is adjoint on the right to the (contravariant) *spectrum* functor $\Sigma: FRM \to TOP$ which may be defined as follows: the points of ΣL are the "points" of L, viz. hom(L,2), where 2 is the 2-element chain: the family

$$T_{\Sigma L} = \{\Sigma_a \mid a \in L\}, \quad \text{where } \Sigma_a = \{p \in \Sigma L \mid p(a) = 1\},$$

is the *spectral* topology. This dual adjunction of *O* and Σ restricts to a dual equivalence on the subcategories of sober spaces (join irreducible closed sets are unique singleton closures) and spatial frames (topologies). (See [3] for the details.)

Two binary relations we shall need are:

(i) $a \overline{\lessdot} b$ (a is *rather below* b) if there is an element s satisfying

$$a \wedge s = 0, \ s \vee b = 1.$$

(ii) $a \overline{\lessdot\lessdot}\, b$ (a is *completely below* b) if there is a family of elements, $\{x_\alpha \mid \alpha \in \mathbb{Q} \cap [0,1]\}$, satisfying

(i) $x_0 = a, \ x_1 = b,$

(ii) $\alpha < \beta \Rightarrow x_\alpha \overline{\lessdot} x_\beta$.

The completely below relationship interpolates: relabel $x_{1/2}$ as c; it is clear that $a \overline{\lessdot\lessdot}\, c \overline{\lessdot\lessdot}\, b$. A frame L is *regular* if, for each b in L, $b = \bigvee\{a \mid a \overline{\lessdot} b\}$; it is *completely regular* if, for each $b \in$ L, $b = \bigvee\{a \mid a \overline{\lessdot\lessdot}\, b\}$.

2. COVERS AND UNIFORM FRAMES.

$C \subset L$ is a *cover* of L if $\bigvee C = 1$. For C, D covers of L, we say that $C \leq D$ (C *refines* D) if, for each $c \in C$, there is $d \in D$ with $c \leq d$; $C \wedge D$ denotes the cover $\{c \wedge d \mid c \in C, \ d \in D\}$; for $a \in L$,

$$\mathrm{st}(a,C) = \bigvee\{c \in C \mid c \wedge a \neq 0\}$$

and C^* denotes the cover $\{\mathrm{st}(c,C) \mid c \in C\}$; we say that C *star refines* D if $C^* \leq D$.

DEFINITION 1. (a) Let L be a frame, q a non-empty family of covers of L. (L, q) is a *uniform frame* if:

(i) q is a filter with respect to $\wedge$, $\leq$.
(ii) $D \in q \Rightarrow$ there is $C \in q$ with $C^* \leq D$.
(iii) For each $b \in L$, $b = \bigvee\{a \in L \mid st(a,C) \leq b$ for some $C \in q\}$.

(b) Let (L,q), (M,p) be uniform frames; $f\colon L \to M$ is a *uniform map* if f is a frame map and

$$C \in q \Rightarrow f[C] = \{f(c) \mid c \in C\} \in p.$$

The resulting category of uniform spaces we denote by *UNIFRM*.

We say in (a) above that q is a *compatible uniform structure on* L. Not all frames have compatible uniform structures (e.g. the 3-chain, 3). Any complete boolean algebra B does admit a canonical uniform structure; let q have as filter sub-base the family of covers $\{C_b \mid b \in B\}$ where $C_b = \{b, \neg b\}$ ($\neg b$ denotes the complement of b). It is already known (see Pultr [5]) that if (L,q) is a uniform frame, then L as a frame is regular and even completely regular (given the axiom of countable dependent choice). Conversely:

PROPOSITION 2. *Let* L *be a completely regular frame,* L *has a compatible uniform structure.*

PROOF. For $a \,\overline{\lessdot\lessdot}\, b$ in L define $C_a^b = \{a^*, b\}$. (Here as elsewhere

$$a^* = \bigvee\{t \in L \mid t \wedge a = 0\}.)$$

C_a^b is certainly a cover of L. Select $c', c'' \in L$ with $a \,\overline{\lessdot\lessdot}\, c' \,\overline{\lessdot\lessdot}\, c'' \,\overline{\lessdot\lessdot}\, b$. Denote by C the cover $C_a^{c'} \wedge C_{c'}^{c''} \wedge C_{c''}^{b}$. One checks readily that $C^* \leq C_a^b$. Now let $q_C(L)$ be the family of covers of L having as sub-base all covers of the form C_a^b for $a \,\overline{\lessdot\lessdot}\, b$; since $st(a, C_a^b) = b$, and L is completely regular, we see that $q_C(L)$ is indeed a compatible uniform structure. ∎

PROPOSITION 3. *Let $f\colon L \to M$ be a frame map between completely regular frames. Then $f\colon (L, q_C(L)) \to (M, q_C(M))$ is a uniform map.*

PROOF. To prove this, one only need observe that

$$a \,\overline{\lessdot}\, c \Rightarrow f(c)^* \,\overline{\lessdot}\, f(a)^*$$

and use the interpolation property of $\overline{\lessdot\lessdot}$. ∎

We thus have a functor from the completely regular frames to the uniform frames leaving the underlying frame unchanged. We also see that (with the countable dependent choice principle) the uniformizable frames are the completely regular ones, a result established by Pultr.

PROPOSITION 4. *Every compact, completely regular frame has unique compatible uniform structure (the set of **all** covers).*

PROOF. Let L be compact, completely regular; L has at least one compatible uniform structure, q. Suppose C is a cover of L, for $c \in C$ we have

$$c = \bigvee\{t \mid st(t,D) \leq c \text{ for some } D \in q\}.$$

So

$$\{t \mid st(t,D) \leq c \text{ for } c \in C,\ D \in q\}$$

is a cover of L; select a finite subcover $\{t_i \mid i = 1,2,\ldots,n\}$ of this cover. For each such i, there is $D_i \in q$, $c_i \in C$ with $st(t_i,D_i) \leq c_i$. Set $D_0 = \wedge D_i$; it is easy to see that D_0 refines C, so C is a uniform cover. ∎

Johnstone [3] shows that if $a \,\overline{\lessdot\lessdot}\, b \in L$, then there is a frame map f from the frame of open sets of the unit interval, $O[0,1]$, to L satisfying $f\,[0,1) \leq a^*$, $f(0,1] \leq b$. As an immediate consequence, if $T: L \to (L, q_T(L))$ is *functorial* from the category of completely regular frames to the category of uniform frames, then every pair $\{a^*,b\}$, where $a \,\overline{\lessdot\lessdot}\, b$, is indeed a member of $q_T(L)$. This means that $q_C(L) \subset q_T(L)$, so q_C is the coarsest functorial way of constructing compatible uniform structures. It is appropriate then to call $q_C(L)$ the Cech uniformity on L. We turn now to the relationship of *UNIFRM* to other categories.

3. UNIFORM SPACES.

Isbell [2] is a good reference for uniform spaces via covers. Given (X,μ) a uniform space, we denote the canonical topology generated by μ by $T(\mu)$; the $T(\mu)$-open covers of μ form a base for μ. (We require *no* separation conditions on (X,μ).)

LEMMA 5. *Let* (X,μ) *be a uniform space,* $U \in T(\mu)$. *Then*

$$U = \bigcup\{V \in T(\mu) \mid st(V,C) \subset U \textit{ for some } C \textit{ (open) in } \mu\}.$$

PROOF. Suppose $x \in U$; for some $C \in \mu$, $st(\{x\},C) \subset U$. Select a $T(\mu)$-open member D of μ with D a star-refinement of C. Of course $st(\{x\},D)$ is open, but it is also easy to show that we have $st(\, st(\{x\},D),D) \subset U$, giving the required result. ∎

With the help of the above lemma, it is easy to see that the $T(\mu)$-open subsets of a uniform space (X,μ) together with the (T,μ)-open covers give us a uniform frame; denote this uniform frame by $(OX, O\mu)$. In fact, it is almost obvious that we have a functor, O, (contravariant) from the category of uniform spaces and uniformly continuous functions, *UNIF*, to *UNIFRM*. We provide also a spectrum-type functor from *UNIFRM* to *UNIF*.

DEFINITION 6. Let (L, q) be a uniform frame, ΣL the set of "points" of L. For $C \in q$, set

$$\Sigma C = \{\Sigma_c \mid c \in C\};\quad \cup \Sigma C = \cup\{\Sigma_c \mid c \in C\} = \Sigma_{\vee C} = \Sigma_1 = \Sigma L$$

so ΣC is indeed a cover of ΣL. Denote by Σq the family of all covers of ΣL refined by a cover of the form ΣC.

PROPOSITION 7. *Let* (L, q) *be a uniform frame; then* $(\Sigma L, \Sigma q)$ *is a uniform space (possible empty).*

PROOF. We show only that if $C^* \leq D$ $(C, D \in q)$, then ΣC star-refines ΣD. To see this, suppose $c \in C$;

$$\mathrm{st}(\Sigma_c, \Sigma C) = \bigcup\{\Sigma_t \mid t \in C,\ \Sigma_t \cap \Sigma_c \neq \emptyset\} = \bigcup\{\Sigma_t \mid t \in C,\ \Sigma_{t\wedge c} \neq \emptyset\}$$
$$\subset \bigcup\{\Sigma_t \mid t \in C,\ t \wedge c \neq 0\} = \Sigma_{\mathrm{st}(c,C)} \subset \Sigma_d$$

for some $d \in D$ since $C^* \leq D$, so ΣC star-refines ΣD. ■

In fact, we can provide the claimed spectrum functor Σ: *UNIFRM*→*UNIF* as follows: suppose $f: (L, q) \to (M, q')$ is a uniform map. Then define

$$\Sigma f: (\Sigma M, \Sigma q') \to (\Sigma L, \Sigma q) \text{ by } \Sigma f(p) = p \circ f.$$

It is straightforward to see that Σf is uniformly continuous.

PROPOSITION 8. *Let* (L, q) *be a uniform frame. The topologies* $T(\Sigma q)$ *and* $T_{\Sigma L}$ *on* ΣL *coincide.*

PROOF. Suppose $U \in T(\Sigma q)$; then for each $p \in U$, there is $C \in q$ with $\mathrm{st}(p, \Sigma C) \subset U$. But members of ΣC are $T_{\Sigma L}$-open, so $U \in T_{\Sigma L}$. Conversely, suppose $U \in T_{\Sigma L}$: then $U = \Sigma_a$ for some $a \in L$; but

$$a = \bigvee\{b \in L \mid \mathrm{st}(b, C) \leq a \text{ for some } C \in q\}$$

and $p \in \Sigma_a$ implies $p(a) = 1$, so for some $b \in L$ with $\mathrm{st}(b, C) \leq a$, we must have $p(b) = 1$ (p is a frame map). Now

$$\mathrm{st}(\{p\}, \Sigma C) = \bigcup\{\Sigma_c \mid c \in C,\ p(c) = 1\};$$

but $p(c) = 1$ and

$$p(b) = 1 \Rightarrow p(c \wedge b) = 1 \Rightarrow c \wedge b \neq 0 \Rightarrow c \leq \mathrm{st}(b, C) \leq a \Rightarrow \Sigma_c \subset \Sigma_a .$$

This yields Σ_a as a $T(\Sigma q)$-open set. (The case where L is a degenerate frame is no problem.) ■

THEOREM 9. *The two contravariant functors O and Σ are adjoint on the right.*

PROOF. One simply verifies that the two (well-known at the "topological" level) maps

$$\eta_L : L \to O\Sigma L \ (\eta_L(a) = \Sigma_a),$$
$$\varepsilon_X : X \to \Sigma OX \ (\varepsilon_X(x) = p_x \text{ [where } p_x(U) = 1 \Leftrightarrow x \in U])$$

are uniform maps (Proposition 8 says that we are lifting the topological level to the uniform level). Then η_L and ε_X are the required adjunction maps. ■

We naturally call a uniform frame (L,q) *spatial* if it satisfies $(L,q) \approx (OX, O\mu)$ for some uniform space (X,μ). It is an easy matter to see that the separated uniform spaces and the spatial uniform frames are the fixed objects of the adjunction in Theorem 9; in particular this adjunction induces a dual equivalence on these two subcategories. One may also notice that $(\Sigma O, \Sigma O\mu)$ is the separated reflection of (X,μ) for any uniform space (X,μ). Note that not all uniform frames are spatial; complete atomless Boolean algebras with the canonical uniform structure are not spatial.

4. PROXIMITY.

Let (L,q) be a uniform frame, as before. For $a, b \in L$, we set $a \ll_q b$ if for some $C \in q$, $st(a,C) \leq b$. It is an easy matter to prove:

P1. $1 \ll_q 1$, $0 \ll_q 0$.

P2. $a \ll_q b \Rightarrow a \prec b$.

P3. $a \leq b \ll_q c \leq d \Rightarrow a \ll_q d$.

P4. $a_1, a_2 \ll_q b \Rightarrow a_1 \vee a_2 \ll_q b$; $a \ll_q b_1, b_2 \Rightarrow a \ll_q b_1 \wedge b_2$.

P5. $a \ll_q b \Rightarrow$ there is $c \in L$ with $a \ll_q c \ll_q b$.

P6. $a \ll_q b \Rightarrow b^* \ll_q a^*$.

P7. $a = \bigvee\{ b \mid b \ll_q a \}$ for any $a \in L$.

The reader familiar with proximity spaces (see Naimpally & Warrack [4]) will realize that $\ll_q$ exhibits properties reminiscent of the so-called inclusion relation on a proximity space. We define a *proximal frame* to be a pair $(L,\ll)$ where L is a frame and $\ll$ a binary relation on L which satisfies P1 to P7. These are considered by Banaschewski [1]. What we see is that we have constructed a functor from *UNIFRM* to the category of proximal frames and proximal maps, *PROXFRM*. (A map $f:(L,\ll)\to(L',\ll')$ is proximal if it is a frame map and $a \ll b \Rightarrow f(a) \ll' f(b)$.) We call $\ll_q$ the proximal relation (on L) induced by q.

We consider the converse problem of endowing a proximal frame with a (functorial) compatible uniform structure (in the sense that the induced proximal relation is the original proximal relation).

THEOREM 10. *Let* $(L,\ll)$ *be a proximal frame, a compatible uniform structure* $q_{\ll}(L)$ *exist such that* $q_{\ll}(L)$ *induces* $\ll$.

PROOF. Suppose $a \ll b \in L$. Set $C_a^b = \{a^*, b\}$; the fact that $a \ll b \Rightarrow a \,\overline{\lessdot}\, b$ ensures that C_a^b is a cover of L; it is straightforward (mimicking Proposition 3) to see that we may use all such covers to generate a compatible uniform structure $q_{\ll}(L)$ on L. It remains to show that $q_{\ll}(L)$ induces $\ll$. Suppose $a \ll b$, then $\mathrm{st}(a, C_a^b) \le b$ and so $a \ll_{q_{\ll}(L)} b$. Conversely suppose, for some $C \in q_{\ll}(L)$, that $\mathrm{st}(a,C) \le b$. Our aim is to show that $a \ll b$. We may as well assume that

$$C = C_{a_1}^{b_1} \wedge \cdots \wedge C_{a_n}^{b_n} \quad \text{where} \quad a_i \ll b_i \quad (i=1,\dots,n).$$

Using P5 repeatedly, select

$$c_i, d_i \in L \ (i=1,\dots,n) \text{ with } a_i \ll c_i \ll d_i \ll b_i$$

for each i. Note that then $c_i^* \ll a_i^*$. Let t be a typical element of C; then

$$t = \bigwedge_{i\in I} a_i^* \wedge \bigwedge_{j\in J} b_j \text{ where } \{I,J\} \text{ is a partition of } \{1,2,\dots,n\}.$$

Set $t^r = \bigwedge_{i\in I} c_i^* \wedge \bigwedge_{j\in J} d_j$; note that

$$t^r \in C_{c_1}^{d_1} \wedge \cdots \wedge C_{c_n}^{d_n} \text{ and } t^r \ll t.$$

Now

$$a \le \bigwedge\{s^* \mid s \in C,\ s\wedge a = 0\} = \left(\bigvee\{s \mid s \in C,\ s\wedge a = 0\}\right)^*$$
$$\ll \left(\bigvee \{s^r \mid s \in C,\ s\wedge a = 0\}\right)^* \le \bigvee\{s^r \mid s \in C,\ s\wedge a \ne 0\}$$
$$\ll \bigvee\{s \mid s \in C,\ s\wedge a \ne 0\} = \mathrm{st}(a,C), \text{ so } a \ll b. \quad \blacksquare$$

In fact, if $f:(L,\ll)\to(L',\ll')$ is proximal, then

$$f:(L,q_{\ll}(L))\to(L',q_{\ll'}(L'))$$

is a uniform map, so we have a functor $U_{\ll}$ from *PROXFRM* to *UNIFRM* which is right inverse to the "inducing" functor mentioned at the beginning of §6.

We now establish that the category of totally bounded uniform frames is isomorphic to the category of proximal frames.

PROPOSITION 11. *Suppose* $(L, \ll)$ *is a proximal frame,* q *a compatible uniform structure on* L *which furthermore induces* $\ll$. *Then* $q_{\ll}(L) \subset q$.

PROOF. Omitted.

PROPOSITION 12. *Let* $(L, \ll)$ *be a proximal frame. Suppose* q *is a totally bounded compatible uniform structure which induces* $\ll$, *then* $q \subset q_{\ll}(L)$.

PROOF. Let $C \in q$; select $D \in q$ with $D^* \leq C$. Total boundedness means that there is a finite subset E of D which still covers L (E need not be a member of q). Suppose $E = \{d_1, d_2, \ldots, d_n\}$. For each such d_i $(i = 1, \ldots, n)$, $st(d_i, D) \leq c_i$ (some $c_i \in C$), so in fact $d_i \leq c_i$. Now $C^{c_1}_{d_1} \wedge \cdots \wedge C^{c_n}_{d_n}$ refines C (since $d_1^* \wedge \cdots \wedge d_n^* = 0$!) showing that $C \in q_{\ll}(L)$. ∎

COROLLARY 13. *For a proximal frame* $(L, \ll)$ *is the unique compatible totally bounded uniform structure which induces* $\ll$.

The isomorphism of the two categories mentioned above is now clear. Open and spectrum type functors (adjoint) for proximal frames and proximity spaces exist which again lift the well-known adjunction for frames and spaces.

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