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CONVERGENCE IN SOME RINGED TOPOSES

by A. Barbara VEIT *

Résumé. A l'aide de la sémantique des faisceaux, nous interprétons un certain nombre de notions de convergence pour les suites de nombres réels (toutes classiquement équivalentes entr'elles) dans différents topos annelés. L'anneau de base étant généralement un faisceau de fonctions à valeurs réelles, nous sommes amenés à traduire les suites de nombres réels par un faisceau de suites de fonctions réelles. Dans chacun des topos considérés, on obtient alors plusieurs faisceaux de suites de fonctions, chacun capturant un aspect différent de la convergence.

Dans la première partie nous travaillons dans le (gros) topos sur le site des espaces topologiques. Les notions de convergence dégagées s'appliquent donc aux suites de fonctions réelles continues, et nous les confrontons d'une part entr'elles, et d'autre part avec la notion classique de convergence uniforme sur les compacts. La deuxième partie est consacrée aux 'topos lisses', tels qu'ils ont été introduits en Géométrie Différentielle Synthétique. Nous travaillons donc généralement avec des faisceaux de fonctions lisses, et nous montrons que la notion (externe) de convergence de Whitney peut se caractériser internement par la notion de 'convergence avec les sous-suites', par exemple dans le 'Cahiers topos'.

Introduction

This paper deals with the interpretation of various notions of convergence in a ringed topos. While those notions are equivalent in the topos of sets with the real numbers as base ring, they provide interesting tools for analyzing and differentiating phenomena of convergence in more specific situations.

A preliminary section gives a short introduction to sheaf semantics, the main technique employed in this paper.

In part I we work in the topos of sheaves over the site of topological spaces and open covers, the base ring being the object of Dedekind reals. Internal Cauchy sequences turn out to admit various alternative descriptions in terms of the internal language of this ringed topos, and their external counterpart provides a notion of convergence of sequences of real-valued continuous functions, which seems to be an efficient substitute for uniform convergence on compact sets. We also examine a notion of 'convergence with subsequences' which goes back to Kuratowski, and we

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show up some of the pathologies this notion suffers when applied to spaces that fail to be locally connected. Still — and this will be essential in part II — convergence with subsequences is internally equivalent with Cauchy convergence in a wide class of spaces (cf. proposition 4.8 of part I).

The toposes we deal with in part II arise from differential geometry: they have objects sheaves over 'loci', i.e. duals of C^∞–rings, and the arrows in the ground–site are equivalence–classes of smooth maps. These toposes have been studied mainly as models for Synthetic Differential Geometry (cf. [9]). I would like to emphasize the fact that the definition of smooth topos given in this paper is fairly 'ad hoc': it simply isolates the features of those models, where our analysis of convergence does apply. Extremely interesting toposes are thus left out, for example the 'smooth Zariski topos' of [8], where the existence of 'smooth integers' should presumably give special insight into convergence.

In the context of smooth toposes, the various notions of convergence each interpret as a different sheaf. Surprisingly enough, it is the notion of convergence with subsequences that appears to be the most efficient one, in so far as it captures (external) Whitney convergence, for example in the Cahiers topos.

Finally, I wish to express my deep gratitude to Gonzalo E. Reyes whom I owe most of what I learnt in categorical logic, all of the stimulation for writing the present paper, and many helpful conversations. It was he who attired my attention to this approach to convergence, who taught me the notion of convergence with subsequences, and who conjectured that it might characterize Whitney convergence internally. The definition of convergence with N^* is due to Jacques Penon who suggested it during some conversations we had in Montreal.

A Brief Outline of Sheaf Semantics

Sheaf semantics relies on the possibility of viewing Grothendieck toposes as semantical universes; in more technical terms, it exploits the fact that any interpretation of the non–logical constants of a (higher order, multisorted) language $\mathcal{L}$ in a Grothendieck topos $\mathcal{E}$ extends canonically to an interpretation of all the formulae of $\mathcal{L}$ in $\mathcal{E}$. This technique generalizes Kripke semantics, and the corresponding notion of truth is intuitionistic, in the sense that any inference which is valid in intuitionistic logic is valid in the internal logic of a Grothendieck topos. Since sheaf semantics is the main tool used in this paper, we shall describe it quickly below. This account is neither exhaustive nor self–contained; for a more detailed exposition we refer the interested reader to [9], [11], [6] or [15].

In ordinary semantics one associates with a formula, say $\exists\, n\; x^n = 0$, its 'extension': in the example, this would be the set of nilpotent elements of the ring referred to by x; more precisely, if we interpret the sort of the variable n by the set $\mathbb{N}$ and the sort of x by a set R equipped with a ring structure, then our formula interprets as the set of nilpotent elements of R.

Similarly in sheaf semantics: for a language $\mathcal{L}$, to give an interpretation of its non–logical constants in a Grothendieck topos $\mathcal{E}$ means to choose a sheaf in $\mathcal{E}$ for each of the sorts, a subobject for each of the relational symbols, and a morphism in

$\mathcal{E}$ for each of the functional symbols of $\mathcal{L}$, in a way that respects the sorts specified for those symbols. Then for any formula φ of $\mathcal{L}$ having its free variables among $x_1, \ldots, x_n$, if the sorts of the x_i are interpreted by the sheaves $F_1, \ldots, F_n$, we shall define a subobject

$$\{ x_1, \ldots, x_n \mid \varphi \} \rightarrowtail F_1 \times \cdots \times F_n.$$

The definition and manipulation of those extensions is greatly facilitated by the symbolism of 'formal arrows'. For a sheaf F over a site $\mathbb{C}$ and an object A of $\mathbb{C}$, we write an element of $F(A)$ as an arrow $A \longrightarrow F$. Formal arrows may be composed to the left with morphisms in $\mathbb{C}$, and with morphisms in the sheaf category to the right. The fundamental properties of functors and natural transformations ensure that the associative law holds for any such composite, to the benefit of notational transparency.

Actually, the terms and atomic formulae we shall use in the sequel are so simple, that a concise definition would be confusing rather than clarifying. Just one remark about terms of type $f(t)$, f being a variable of sort (interpreted by) F^G, and t a term of sort G: in those cases, we use $f(t)$ as a shorthand for $val(f,t)$, where val is (interpreted as) the evaluation $F^G \times G \longrightarrow F$. One more remark concerns the symbol $\in$; it plays a purely metalinguistic role in our formulae. We write $\forall x \in F\ \exists y \in G \ldots$ simply as a way of indicating that we want to interpret the sort of x by the sheaf F, the sort of y by G, etc. In some occasions we shall use the notation $x \in G$, even if the variable x has been previously referred to a sheaf F, with $G \rightarrowtail F$. The correct (but clumsy) procedure would be to introduce a relation symbol R of the same sort as x, to specify that we want to interpret R by $G \rightarrowtail F$, and to write $R(x)$ instead of $x \in G$.

We thus are left with the definition of $\{ x_1, \ldots, x_n \mid \varphi \}$ for composite formulae φ. Given $\mathbf{a} := A \xrightarrow{(a_1, \ldots, a_n)} F_1 \times \cdots \times F_n$, we shall write $A \models \varphi[\mathbf{a}]$ (to be read as '$\mathbf{a}$ satisfies φ at stage A') instead of $\mathbf{a} \in \{ x_1, \ldots, x_n \mid \varphi \}(A)$. So suppose ψ, ψ_1 and ψ_2 are formulae which have their free variables among $x_1, \ldots, x_n$. We let

$A \models (\psi_1 \wedge \psi_2)[\mathbf{a}]$ iff $A \models \psi_1[\mathbf{a}]$ and $A \models \psi_2[\mathbf{a}]$;

$A \models (\psi_1 \Rightarrow \psi_2)[\mathbf{a}]$ iff for all $B \xrightarrow{\alpha} A$ in $\mathbb{C}$, if $B \models \psi_1[\mathbf{a} \circ \alpha]$, then $B \models \psi_2[\mathbf{a} \circ \alpha]$;

$A \models (\neg \psi)[\mathbf{a}]$ iff for all $B \xrightarrow{\alpha} A$ in $\mathbb{C}$, if $B \models \psi[\mathbf{a} \circ \alpha]$, then $\emptyset \in \mathrm{Cov}B$;

$A \models (\exists y \in F\ \psi)[\mathbf{a}]$ iff there is $(A_i \xrightarrow{\alpha_i} A)_{i \in I} \in \mathrm{Cov}A$, and for each $i \in I$, $A_i \models \psi[y_i, \mathbf{a} \circ \alpha_i]$ holds for some $A_i \xrightarrow{y_i} F$;

$A \models (\forall y \in F\ \psi)[\mathbf{a}]$ iff $B \models \psi[y, \mathbf{a} \circ \alpha]$ holds for all $B \xrightarrow{\alpha} A$ in $\mathbb{C}$ and $B \xrightarrow{y} F$.

For our present purposes, the main interest of the above clauses lies in the fact that the correspondence $A \mapsto \{ x_1, \ldots, x_n \mid \varphi \}(A)$ actually defines a sheaf. Of course, we may apply the logical operations directly to subobjects of sheaves instead of passing through formulae. This possibility turns sheaf semantics into a method for constructing new sheaves, or for analyzing the relations among sheaves by means of intuitionistic logic.

There are several rules that facilitate the manipulation of the above clauses. We list some of them for later use. The notation $\mathcal{E} \models \varphi$ reads 'φ holds in $\mathcal{E}$', and it means that the formula φ is satisfied for any evaluation of its free variables at any stage (under the given interpretation of $\mathcal{L}$ in $\mathcal{E}$).

0.1. A formula holds in a Grothendieck topos $\mathcal{E}$ if and only if so does its universal closure.

0.2. If two formulae φ and ψ have the same free variables, then $\mathcal{E} \models \varphi \Rightarrow \psi$ as soon as for any object A of $\mathbb{C}$ and any evaluation $\mathbf{a}$ of those variables at stage A, one has that $A \models \varphi[\mathbf{a}]$ implies $A \models \psi[\mathbf{a}]$.

0.3. If the site $\mathbb{C}$ satisfies a *Nullstellensatz*, i.e. if $\hom(\mathbf{1}, A) = \emptyset$ implies $\emptyset \in \mathrm{Cov}(A)$ for all $A \in \mathrm{Ob}\,\mathbb{C}$, then for any subobject $F \rightarrowtail G$ in the sheaf category $\mathcal{E}$ and $A \in \mathrm{Ob}\,\mathbb{C}$

$$\neg\neg F(A) = \{ A \xrightarrow{g} G \mid g \circ a \in F(\mathbf{1}) \text{ for all } \mathbf{1} \xrightarrow{a} A \}.$$

Note that when $\emptyset \in \mathrm{Cov}(\mathbf{1})$, then $\emptyset \in \mathrm{Cov}(A)$ for any $A \in \mathrm{Ob}\,\mathbb{C}$, and consequently $F(A)$ is a singleton for any sheaf F over $\mathbb{C}$. Thus $\mathcal{E}$ is equivalent to the 'absurd' topos of sheaves over the empty space, and in that case our claim is trivially satisfied. In case $\emptyset \notin \mathrm{Cov}(\mathbf{1})$, the claim follows immediately from the fact that one then has

$$\neg F(A) = \{ A \xrightarrow{g} G \mid g \circ a \notin F(\mathbf{1}) \text{ for all } \mathbf{1} \xrightarrow{a} A \}.$$

Indeed, fix $A \xrightarrow{g} G$. If $g \in \neg F(A)$, and for some $\mathbf{1} \xrightarrow{a} A$ we have $g \circ a \in F(\mathbf{1})$, then $\emptyset \in \mathrm{Cov}(\mathbf{1})$. The latter being false, we find $g \circ a \notin F(\mathbf{1})$ for all $\mathbf{1} \xrightarrow{a} A$. For the converse, suppose $g \circ a \notin F(\mathbf{1})$ for all $\mathbf{1} \xrightarrow{a} A$. We show that if $B \xrightarrow{\alpha} A$ in $\mathbb{C}$ satisfies $g \circ \alpha \in F(B)$, then $\hom(\mathbf{1}, B) = \emptyset$. If there were any $\mathbf{1} \xrightarrow{b} B$, it would satisfy $g \circ \alpha \circ b \notin F(\mathbf{1})$ by hypothesis; but $g \circ \alpha \in F(B)$ implies $g \circ \alpha \circ b \in F(\mathbf{1})$, a contradiction. •

I. Convergence in TOP

We denote by TOP the 'gros topos' of sheaves over the site consisting of the category **Top** of topological spaces and continuous maps, equipped with the open cover topology: for a space A, we define a family $(U_i \xrightarrow{\alpha_i} A)_{i \in I}$ to be covering precisely when $(U_i)_{i \in I}$ is an open cover of A, and the α_i are the inclusions. We deliberately ignore the set-theoretical difficulties that might arise from our dealing with very big categories, and we invite the sceptical reader to cut down to smaller categories, for example Grothendieck universes.

The open cover topology on **Top** is sub-canonical, i.e. for any topological space A, the presheaf h_A lives in TOP. We shall often identify A and h_A; however for the spaces $\mathbb{N}$ and $\mathbb{R}$, we indicate the sheaves they represent in TOP by N and R respectively. As customary, we assume variables such as n and m to range over elements of N, and ε and δ to have sort R. These two sheaves shall play a central role in the sequel, so let us consider some of their fundamental properties.

N is the natural number object in TOP. This is mainly because $\mathbb{N}$ is a discrete space. Indeed, for Grothendieck toposes, it is well known that the natural number object is the constant sheaf $\Delta\mathbb{N}$ associated with (the set) $\mathbb{N}$. On the other hand, Δ is left adjoint to the global section functor Γ from TOP to Sets, and the latter becomes the forgetful functor when restricted to **Top**. But 'construct the discrete space' is left adjoint to 'forget'.

R inherits several properties from $\mathbb{R}$. First of all it is a ring, and it comes equipped with an order $<$ which is represented by the usual order on $\mathbb{R}$. We write $R_>$ for the subobject of R defined by the formula $x > 0$. Since the map $\mathbb{R} \xrightarrow{|\,|} \mathbb{R}$ (absolute value) lies in **Top**, it has an analogue in TOP for which we use the same notation. Similarly, the inclusion $\mathbb{N} \hookrightarrow \mathbb{R}$ carries over to TOP, as well as the map $n \mapsto \frac{1}{n+1}$ from $\mathbb{N}$ to $\mathbb{R}$. The most relevant feature of R is the following: it is the object of Dedekind reals in TOP (for a proof, see [7], 1.6).

1. Cauchy Sequences and Limits

1.1 Definition. For sequences $(s_n)\in R^N$ we define the following (internal) properties:

(i) (s_n) *is a Cauchy sequence* if it satisfies the formula

$$(\mathrm{C}): \quad \forall \varepsilon > 0 \quad \exists n \in N \quad \forall m, k > n \quad |s_m - s_k| < \varepsilon\,;$$

(ii) (s_n) *has a limit* if it satisfies

$$(\mathrm{L}): \quad \exists \ell \in R \quad \forall \varepsilon > 0 \quad \exists n \in N \quad \forall m > n \quad |s_m - \ell| < \varepsilon\,.$$

In TOP, and more generally in any topos in which R satisfies the triangular inequality, the classical argument for establishing (L) $\Rightarrow$ (C) is valid in the internal logic. It is less trivial that one has also TOP $\models$ (C) $\Rightarrow$ (L). This is because in TOP, R is the object of Dedekind reals, and the latter is internally a complete metric space in any topos (cf. [14], Cor.4.5).

If we want an external characterization of internal convergence notions, we must first of all learn how to interpret 'sequences of real numbers'. We already noted that N is the constant sheaf $\Delta\mathbb{N}$ in any Grothendieck topos $\mathcal{E}$. When playing with Yoneda, exponentiation and the adjoint pair $\Delta \vdash \Gamma$, we thus find for any object A of the underlying site and for any sheaf F:

$$\cfrac{\cfrac{\cfrac{\cfrac{s \in F^N(A)}{A \longrightarrow F^N}}{N \times A \longrightarrow F}}{N \longrightarrow F^A}}{\mathbb{N} \longrightarrow \Gamma(F^A) \text{ in Set},}$$

where the arrows in the three intermediate lines live in the category of presheaves. Note that the presheaf F^A is always a sheaf, even when A (or rather h_A) is not; this motivates the last passage as well as the following: $\Gamma(F^A) = \hom(\mathbf{1}, F^A) = \hom(A, F) = F(A)$. Therefore one has always $F^N(A) = [F(A)]^{\mathbb{N}}$. Coming back to TOP and $F = R$, recall that $R(A) = \mathrm{C}^0(A, \mathbb{R})$. Thus

1.2 Proposition. For any topological space A, an element $s \in R^N(A)$ in TOP interprets as an ordinary sequence $(s_n)_{n \in \mathbb{N}}$ of continuous real-valued functions defined on A.

1.3 Proposition. In TOP, an element $s \in R^N(A)$ is an internal Cauchy sequence if and only if the following holds externally:

(∗) for any $\varepsilon > 0$ in $\mathbb{R}$ there is an open cover $(U_i)_{i \in I}$ of A, and on each U_i one has: there is an $n_i \in \mathbb{N}$ such that for all integers $m, k > n_i$, $\quad |s_m - s_k| < \varepsilon$.

Moreover, if s is an internal Cauchy sequence, then the s_n converge pointwise to a continuous function.

Proof. To say that s is internally Cauchy means in particular that for any (external) real number $\varepsilon > 0$, there is an open cover $(V_j)_{j \in J}$ of A and a family $\{n_j \in N(V_j)\}$ such that for each $j \in J$, letting $\varepsilon_j : V_j \longrightarrow \mathbb{R}$ denote the function with constant value ε, one has

$$V_j \models \quad \forall m, k > n_j \quad |s_m - s_k| < \varepsilon_j .$$

Now, each n_j is locally an ordinary natural number, i.e. the cover (V_j) of A may be refined so to yield an open cover $(U_i)_{i \in I}$ of A such that for each $i \in I$ and $j \in J$, the restriction of n_j to U_i is some $n_i \in \mathbb{N}$. Thus condition (∗) is necessary.

For the converse, suppose (∗) holds, and let α: $B \longrightarrow A$ and ε: $B \longrightarrow R_>$ be given in TOP. Since ε is continuous, each $b \in B$ has an open neighbourhood $U(b)$ where $\varepsilon > \varepsilon(b)/2$. We must show that $B \models \exists n \in N \quad \forall m, k > n \quad |s_m - s_k| < \varepsilon$, and since this is a local condition, we may assume that B agrees with one of the $U(b)$. From (∗) we know there is an open cover (U_i) of A and a family (n_i) of integers such that on each U_i, $|s_m - s_k| < \varepsilon(b)/2$ as soon as $m, k > n_i$. By pulling back the U_i along α, one obtains a cover (V_i) of $U(b)$ such that for all $i \in I$, $m, k > n_i$ and $x \in V_i$

$$|s_m(\alpha(x)) - s_k(\alpha(x))| < \frac{\varepsilon(b)}{2},$$

and this is easily seen to imply $V_i \models \forall m, k > n_i \quad |s_m - s_k| < \varepsilon$.

As to the last assertion, a direct proof is of course available, but in this context we prefer using the internal equivalence of (C) and (L). Thus there is an open cover (U_i) of A and a family $\{\ell_i \in R(U_i)\}$ such that for all i

$$U_i \models \quad \forall \varepsilon > 0 \quad \exists n \in N \quad \forall m > n \quad |s_m - \ell_i| < \varepsilon.$$

It is an easy guess that this forces (s_n) to converge pointwise to ℓ_i on each U_i. Therefore the various ℓ_i glue together and form a single continuous function defined on all of A. •

The 'easy guess' we just mentioned has an elegant formulation in categorical semantics, where one learns that $\varphi \Rightarrow \neg\neg\varphi$ holds for any formula φ .

1.4 Proposition. For a sequence (s_n) of real-valued continuous functions defined on a space A, one has

$$A \models \neg\neg(\mathrm{C}) \text{ in TOP} \quad \text{if and only if the } s_n \text{ converge pointwise.}$$

In other words, the formula $\neg\neg$(C) characterizes the (external) notion of pointwise convergence.

Proof. Immediate in view of 0.3. •

2. Functional Convergence

2.1 Definition. An internal sequence $(s_n) \in R^N$ is said to *converge functionally* if it satisfies

$$(\mathrm{F}): \quad \exists f \in R^{[0,1]} \quad \forall n \in N \quad s(n) = f(\frac{1}{n+1}).$$

The classical argument for (F)$\Rightarrow$(L) is based on the idea of taking $\ell = f(0)$; this works internally, provided the functions in $R^{[0,1]}$ are 'continuous'. Let us show that this is the case for TOP.

2.2 Proposition. In TOP, the following version of Brouwer's principle holds internally: any function between two metric spaces is continuous, i.e. for any two metric spaces X and Y in **Top** one has:

$$\mathrm{TOP} \models \quad \forall f \in Y^X \quad \forall x \in X \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x' \in X$$
$$\big[d(x,x') < \delta \quad \Rightarrow \quad d\big(f(x), f(x')\big) < \varepsilon\big].$$

Proof. According to 0.1 we need only show that given an object A of **Top** and arrows f: $A \longrightarrow Y^X$, x: $A \longrightarrow X$ and ε: $A \longrightarrow R_>$ in TOP, one has

$$A \models \quad \exists \delta > 0 \quad \forall x' \in X \quad \big[d(x,x') < \delta \quad \Rightarrow \quad d\big(f(x), f(x')\big) < \varepsilon\big].$$

So let $a \in A$. The map $\bar{f}$: $A \times X \longrightarrow Y$ corresponding to f by exponential adjunction is continuous in $\big(a, x(a)\big)$, hence there is an open neighbourhood U_1 of a and a real number $\delta_1 > 0$ such that for all $u \in U_1$ and $x' \in X$ satisfying $d\big(x', x(a)\big) < \delta_1$, one has $d\big(\bar{f}(a, x(a)), \bar{f}(u, x')\big) < \varepsilon(a)/4$. Similarly, the continuity of x in a yields $U_2 \in \mathcal{U}(a)$ such that $d\big(x(a), u\big) < \delta_1/2$ for $u \in U_2$; finally there is $U_3 \in \mathcal{U}(a)$ such that $\varepsilon(u) > \varepsilon(a)/2$ when $u \in U_3$. We let $U_a := U_1 \cap U_2 \cap U_3$ and define δ: $U_a \longrightarrow \mathbb{R}$ to be the function with constant value $\delta_1/2$. It is now straightforward to show that

$$U_a \models \quad \forall x' \in X \quad \big[d(x,x') < \delta \quad \Rightarrow \quad d\big(f(x), f(x')\big) < \varepsilon\big],$$

and since the U_a form an open cover of A, this concludes the proof. •

2.3 Remark. Actually, in order to establish (F) $\Rightarrow$ (L) by means of the internal logic, one needs also

$$\mathrm{TOP} \models \quad \forall \delta > 0 \quad \exists n \in N \quad \forall m > n \quad \delta > \tfrac{1}{m}.$$

But for any positive real-valued function δ, one has that $\delta > \frac{1}{n}$ for some $n \in \mathbb{N}$ holds locally; in other words

$$\mathrm{TOP} \models \quad \forall \delta > 0 \quad \exists n \in N \quad \delta > \tfrac{1}{n}.$$

On the other hand, the internal validity of $m > n \Rightarrow \frac{1}{n} > \frac{1}{m}$ in TOP is immediate.

2.4 Proposition. In TOP, conditions (C) and (F) are logically equivalent, i.e.

$$\text{TOP} \models (s_n) \text{ is a Cauchy sequence} \iff (s_n) \text{ converges functionally.}$$

Proof. We know already TOP $\models$ (F) $\Rightarrow$ (C). For the converse, the classical argument is based on the idea of connecting the single points $(\frac{1}{n+1}, s_n)$ by a broken line. Although this does not work internally, it motivates the following external construction. For each $t \in (0,1]$ there is a unique integer m such that t lies in the interval $(\frac{1}{m+2}, \frac{1}{m+1}]$, hence $t = \frac{1}{m+2} + \rho \cdot (\frac{1}{m+1} - \frac{1}{m+2})$ for a unique $\rho \in (0,1]$. If $(s_n) \in R(A)^{\mathbb{N}}$ converges internally to ℓ, we may therefore define a function $F\colon [0,1]\times A \longrightarrow \mathbb{R}$ letting

$$F(t,a) := \begin{cases} \ell(a) & \text{if } t = 0 \\ s_{m+1}(a) + \rho \cdot \big(s_m(a) - s_{m+1}(a)\big) & \text{if } t > 0. \end{cases}$$

Clearly $F(\frac{1}{n+1}, a) = s_n(a)$ for all $a \in A$ and $n \in \mathbb{N}$; this is easily seen to imply TOP $\models \forall n \in N \quad F(\frac{1}{n+1}) = s(n)$ provided it makes sense, i.e. provided F is continuous, and thus gives rise to an element of $R^{[0,1]}(A)$ by exponential adjunction. As to the continuity of F, the critical (t,a) are those with $t = 0$, and they satisfy $|F(t,b) - F(0,a)| = |\ell(b) - \ell(a)|$ if $t = 0$, while for $t \in (\frac{1}{m+2}, \frac{1}{m+1}]$,

$$\begin{aligned} |F(t,b) - F(0,a)| &= |s_{m+1}(b) + \rho \cdot \big(s_m(b) - s_{m+1}(b)\big) - \ell(a)| \\ &\le |s_{m+1}(b) - \ell(a)| + |\rho \cdot \big(s_m(b) - s_{m+1}(b)\big)| \\ &\le |s_{m+1}(b) - \ell(a)| + |s_m(b) - s_{m+1}(b)| \\ &\le |s_{m+1}(b) - s_M(b)| + |s_M(b) - s_M(a)| + \\ &\qquad + |s_M(a) - \ell(a)| + |s_m(b) - s_{m+1}(b)| \end{aligned}$$

for any given M. Let $\varepsilon > 0$. According to condition $(*)$ of 1.3, there are $U_1 \in \mathcal{U}(a)$ and $n_1 \in \mathbb{N}$ such that $|s_m - s_k| < \varepsilon/4$ on U_1 for all $m, k > n_1$. Since $s_n(a)$ converges to $\ell(a)$, there is also some n_2 such that $|s_M(a) - \ell(a)| < \varepsilon/4$ for all $M > n_2$. Thus, if $M > n_1, n_2$ and $t \in [0, \frac{1}{M})$, then $|F(t,b) - F(0,a)| < \varepsilon$ provided b lies in $U_1 \cap U_2$ with $U_2 \in \mathcal{U}(a)$ satisfying $|s_M(x) - s_M(a)| < \varepsilon/4$ and $|\ell(x) - \ell(a)| < \varepsilon$ for all $x \in U_2$ — such a U_2 exists since s_M and ℓ are continuous. Therefore F is continuous, too. •

3. Internal Convergence and Uniform Convergence on Compact Sets

3.1 Proposition. If a sequence (s_n) of continuous real-valued functions defined on a topological space A is internally a Cauchy sequence, then it converges uniformly on compact sets.

Proof. Immediate in view of condition $(*)$ of 1.3. •

We want to know to what extent the converse is true, i.e. whether sequences that converge uniformly on compact sets (henceforth said to converge UC) are internally Cauchy.

3.2 Counterexamples. We describe two sequences of real-valued functions that converge UC without being internal Cauchy sequences. Not surprisingly, the space $\mathcal{A}$ on which these sequences are defined, is poor in compact sets (only finite subsets are compact); but still, being completely regular, it is a 'good' space.

In the literature $\mathcal{A}$ is known as *Appert Space* (cf. [13], example 98). The underlying set is that of positive integers; any subset not containing 1 is declared open, while the open neighbourhoods of 1 are defined as

$$\mathcal{U}(1) := \{ S \subset \mathcal{A} \mid 1 \in S \text{ and } \lim_{n\to\infty} \frac{\varphi(n,S)}{n} = 1 \},$$

$\varphi(n,S)$ denoting the number of integers $\leq n$ in S. Note that $f: \mathcal{A} \longrightarrow \mathbb{R}$ is continuous iff for all $\varepsilon > 0$, the set $\{ a \in \mathcal{A} \mid |f(a) - f(1)| < \varepsilon \}$ lies in $\mathcal{U}(1)$.

(i) For each $n \in \mathbb{N}$, let $U_n := \{ u \in \mathbb{N} \mid u = 1 \text{ or } u > n \}$; since $U_n \in \mathcal{U}(1)$ for all n, a sequence of continuous functions from $\mathcal{A}$ to $\mathbb{R}$ is defined by the following clauses:

$$s_n(a) := \begin{cases} 0 & \text{if } a \in U_n \ , \\ 1 & \text{otherwise.} \end{cases}$$

It is readily seen that for each $a \in \mathcal{A}$,

$$\lim_{n\to\infty} \ell(a) = \begin{cases} 0 & \text{if } a = 1 \ , \\ 1 & \text{otherwise.} \end{cases}$$

Since only finite sets are compact in $\mathcal{A}$, (s_n) converges UC to ℓ. But (s_n) cannot be internally Cauchy, for ℓ is visibly discontinuous at 1, whereas we saw above that internal convergence yields a continuous limit.

(ii) Let $s_0 = s_1 \equiv 0$, and for $n > 1$ let $s_n(a)$ be constantly 0 except when $a = n$, where we set s_n equal to 1. Again, each s_n is continuous, and so is the limit this time, being constantly zero. But still, (s_n) does not converge internally, because condition $(*)$ of 1.3 is not satisfied. Indeed, if $\varepsilon < 1$, it is impossible to find any $U \in \mathcal{U}(1)$ and $n \in \mathbb{N}$ such that for all $m, k > n$ $|s_m - s_k| < \varepsilon$ on U: whatever n and U we choose, U will contain some $m > n$, and thus $|s_m(m) - s_{m+1}(m)| = |1-0| = 1$.

It seems to us that internal convergence in TOP — or the external condition $(*)$ — might be an efficient substitute for uniform convergence on compact sets. The main advantage this notion offers is of course that it ensures the continuity of the limit in any space. It shares with convergence on compact sets the nice feature of being functorial, and it is moreover local, since it defines a sheaf. [Added in proof: In the meantime we discovered a paper written some forty years ago by Arens and Dugundji (cf. [1]) which contains a deep analysis of topologies on function spaces. The authors use (a generalization of) Kuratowski's notion of 'continuous convergence' (cf. [10]) in order to classify those topologies, and they prove uniform convergence on compact sets to be in some sense the best possible convergence notion. We found moreover that our condition $(*)$ is equivalent to Kuratowski's notion — the argument is straightforward.]

For many topological spaces the two notions agree anyway:

3.3 Proposition. If the space A is first countable or locally compact, then any sequence $s \in R^N(A)$ that converges UC is internally Cauchy.

Proof. In case A is locally compact, the claim follows immediately, because condition $(*)$ then visibly holds for UC convergent sequences. So suppose A is first countable, and consider a sequence (s_n) of continuous real–valued functions on A that converge UC, but $(*)$ fails. Thus, there is $\varepsilon > 0$ and a point $a_0 \in A$ such that for all $U \in \mathcal{U}(a_0)$ and $n \in \mathbb{N}$, some $a \in U$ satisfies

$$|s_m(a) - s_k(a)| \geq \varepsilon \text{ for two integers } m, k > n.$$

If $(U_n)_{n \in \mathbb{N}}$ is a countable basis of $\mathcal{U}(a_0)$, we therefore can find for each n two integers $m_n, k_n > n$ such that

$$|s_{m_n}(a_n) - s_{k_n}(a_n)| \geq \varepsilon \quad \text{for some } a_n \in U_n.$$

We may suppose without loss that $n < n'$ implies $U_n \supset U_{n'}$. Then the a_n form together with a_0 a compact set C, hence there must be some n_0 such that for all $m, k > n_0$

$$|s_m - s_k| < \varepsilon \quad \text{on } C.$$

This is a contradiction since a_{n_0} lies in C, and $m_{n_0}, k_{n_0} > n_0$. •

4. Convergence with Subsequences

4.1 Definition. We say a sequence $(s_n) \in R^N$ *converges with its subsequences* if the following holds internally: there is an $\ell \in R$ such that any subsequence of (s_n) has some subsequence converging to ℓ. Formally those sequence are characterized by the condition

$$(\mathrm{S}): \quad \exists \ell \in R \quad \forall \varphi \in N^N_> \; \exists \psi \in N^N_> \quad \forall \varepsilon > 0 \; \exists n \in N \quad \forall m > n \quad |s_{\varphi(\psi(m))} - \ell| < \varepsilon,$$

where $N^N_>$ is defined as the object of monotone sequences of integers, i.e.

$$N^N_> := \{ \psi \in N^N \mid n > m \quad \Rightarrow \quad \psi(n) > \psi(m) \}.$$

4.2 Remark. It may be worthwhile mentioning that condition (S) is closely related to one of the three axioms Kuratowski formulated in view of his analysis of convergence (cf. [10]); it says that if ℓ is not a limit of a sequence (s_n), then some subsequence of (s_n) has none of its subsequences converging to ℓ.

4.3 Proposition. TOP $\models$ Any Cauchy sequence converges with its subsequences.

Proof. We work internally. Suppose (s_n) converges to ℓ, and consider a subsequence s' of s, i.e for some $\varphi \in N^N_>$ we have $s'_n = s_{\varphi(n)}$ for all $n \in N$. We show that s' itself converges to ℓ. Let $\varepsilon > 0$. We know there is some $n \in N$ such that $|s_m - \ell| < \varepsilon$ for all $m > n$. Therefore $|s'_m - \ell| = |s_{\varphi(m)} - \ell| < \varepsilon$ when $m > n$, for $\varphi(m) \geq m$ if φ is monotone (see below). •

4.4 Lemma. TOP $\models$ If $\varphi \in N^N$ is monotone, then $\forall\, m \in N \quad \varphi(m) \geq m$.

Proof. From 1.2 we know that $N^N(A) = N(A)^{\mathbb{N}}$ for any topological space A. Since N is represented by the space $\mathbb{N}$, we thus have

$$N^N(A) = [\mathrm{C}^0(A, \mathbb{N})]^{\mathbb{N}}.$$

So an internal sequence of integers $\varphi\colon A \longrightarrow N^N$ interprets as a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of continuous functions from A to $\mathbb{N}$. It is easy to prove that any such φ is internally monotone if and only if $\varphi_{n+1}(a) > \varphi_n(a)$ for all $n \in \mathbb{N}$ and $a \in A$, and an immediate induction shows that $\varphi_n \geq n$ then holds for all $n \in \mathbb{N}$. Consequently, for any $\alpha\colon B \longrightarrow A$ and $m\colon B \longrightarrow N$ in **Top**, one has $\varphi_{m(b)}\big(\alpha(b)\big) \geq m(b)$ for all $b \in B$, i.e. $A \models \forall m \in N \quad \varphi(m) \geq m$. •

We devote the rest of this section to the question whether the converse holds, i.e. whether convergence with subsequences implies, say Cauchy convergence. This is true classically, but the argument is by contradiction: assuming (s_n) to converge with subsequences without being Cauchy, one may produce a subsequence of (s_n) that does not admit any convergent subsequence. We shall mimick this argument below in order to establish (S) $\Rightarrow$ (C) for a restricted class of topological spaces.

4.5 Counterexamples. The use of internal sequences of integers brings in questions of connectedness, for the integers defined over a space are the more involved the more connected components it has.

For example, choose a family (C_n) of disjoint intervals of real numbers with $\frac{1}{n+1} \in C_n$ for all $n \in \mathbb{N}$, and define a sequence of functions on the space $\bigcup C_n \cup \{0\}$ by the following clauses:

$$s_n = \begin{cases} 1 & \text{on } C_n; \\ 0 & \text{elsewhere.} \end{cases}$$

More generally, consider a space X with a point x_0 such that for some family $(C_n)_{n\in\mathbb{N}}$ of disjoint clopen subsets of X, each neighbourhood of x_0 meets infinitely many of the C_n. Again, define a sequence (s_n) of functions by the above clauses. Note that example (ii) of 3.2 fits into this pattern; we shall give further examples in 4.7.

Each s_n is continuous since C_n is clopen, and the s_n clearly converge pointwise to the zero–function. But condition $(*)$ of 1.3 is not satisfied: it is impossible to find $U \in \mathcal{U}(x_0)$ and $n \in \mathbb{N}$ such that for all $m, k > n$, $\quad |s_m - s_k| < 1$ on U . We shall see however that all those sequences (s_n) satisfy condition (S) internally.

For given $Y \xrightarrow{\alpha} X$ in **Top** and $\varphi \in N^N_>(Y)$, we shall exhibit $\psi \in N^N_>(Y)$ such that the corresponding subsequence of $(s_n \circ \alpha)$ is constantly zero. We must first clarify what kind of subsequence is induced by φ. If we write (t_n) for this new sequence, we find $t_n(y) = s_{\varphi(n,y)}\big(\alpha(y)\big)$ for each $y \in Y$, and thus what we claim to exist is a sequence $(\psi_n) \in N^N_>(Y)$ such that $s_{\varphi\left(\psi(n,y),y\right)}\big(\alpha(y)\big) = 0$ for all $y \in Y$, i.e. when $\alpha(y) \in C_m$, then ψ must ensure $\varphi\big(\psi(n,y),y\big) \neq m$. Here is the definition:

$$\psi_n(y) = \psi(n,y) = \begin{cases} n & \text{if } \alpha(y) \notin C_{\varphi(k,y)} \text{ for all } k \leq n; \\ n+1 & \text{otherwise.} \end{cases}$$

Indeed, let $\alpha(y) \in C_m$. Clearly, if $\varphi(k,y) \neq m$ for all k, then $\varphi(\psi(n,y),y)) \neq m$ anyway. So suppose $\varphi(k,y) = m$ happens for some k. This k then is unique with $\alpha(y) \in C_{\varphi(k,y)}$ because the C_n were supposed to be disjoint, and because φ is monotone, hence the correspondence $k \mapsto \varphi(k,y)$ is injective. Thus $\psi(n,y) = n$ when $n < k$, and $\psi(n,y) = n+1$ for $n \geq k$. So $\psi(n,y)$ never assumes the critical value k, and therefore ψ transforms our original sequence into the most trivial one.

We are left with showing that ψ is actually an element of $N_>^N(Y)$. Note that each ψ_n assumes only two values, n and $n+1$. The continuity of ψ_n is therefore equivalent with $\psi_n^{-1}(n+1)$ being clopen. Now,

$$\begin{aligned} \psi_n^{-1}(n+1) &= \{ y \in Y \mid \exists\, k \leq n \quad \alpha(y) \in C_{\varphi(k,y)} \} \\ &= \bigcup_{k \leq n} \alpha^{-1}\big(C_{\varphi(k,y)}\big)\,. \end{aligned}$$

Since each $C_{\varphi(k,y)}$ is clopen, so is its inverse image under α; but a finite union of clopen sets is clopen. Finally, the sequence (ψ_n) is easily seen to be monotone because for fixed $y \in Y$, one has either $\psi_n(y) = n$ for all n, or there is some n_0 such that

$$\psi_n(y) = \begin{cases} n & \text{if } n < n_0 \\ n+1 & \text{if } n \geq n_0. \end{cases} \quad \bullet$$

As we saw above, an internal subsequence is quite different from a usual one in that the (external) index may vary according to the point where it is evaluated. This is what allows jumping critical points of the original sequence. However, we shall soon see that in a wide class of topological spaces, similar acrobatics are not feasible. Some preliminaries first.

4.6 Lemma. If a sequence (s_n) of continuous real-valued functions satisfies condition (S) in the internal logic of TOP, then it converges pointwise to a continuous function.

Proof. If the s_n are defined on the space A, the hypothesis says that there is an open cover $(U_i)_{i \in I}$ of A and a family $(\ell_i\colon U_i \longrightarrow \mathbb{R})$ of continuous functions such that for all $i \in I$ and $B \xrightarrow{\alpha} U_i$ in **Top**, any $\varphi \in N_>^N(B)$ satisfies

$$B \models \exists\, \psi \in N_>^N \quad \big(s_{\varphi(\psi(n))}\big) \text{ converges to } \ell_i.$$

When this is applied with $B = 1$, one finds for each point $u \in U_i$ that $\big(s_n(u)\big)$ converges to $\ell_i(u)$ in the sense of external convergence with subsequences. But classically (S) and (C) are equivalent. $\bullet$

4.7 Remark. As a consequence of the previous lemma, the internal validity of (S) is generally not implied by uniform convergence on compact sets (cf. the first example in 3.2 above).

The converse does not hold either: for a first countable space X, a family (C_n) of clopen sets as in 4.5 exists if and only if X is not locally connected. The 'if' is obvious. For the 'only if', take a countable base (U_n) of $\mathcal{U}(x_0)$, x_0 being a point where X fails to be locally connected. The U_n may be chosen such that no U_n contains a connected neighbourhood of x_0 and $U_{n+1} \subset U_n$ for all n. Starting with any clopen $C_0 \neq \emptyset$ contained in U_0 and satisfying $x_0 \notin C_0$, define inductively a family (C_n) as in 4.5 letting $C_n \subset U_n \setminus C_{n-1}$ be any clopen $\neq \emptyset$ with $x_0 \notin C_n$ (so that $U_{n+1} \setminus C_n$ is again in $\mathcal{U}(x_0)$). Using 4.5, we find for any such space a sequence satisfying (S) but not (C), whence it cannot converge UC according to 3.3.

4.8 Proposition. If the topological space A is first countable and locally connected, then a sequence $(s_n) \in R^N$ defined at stage A converges internally with its subsequences if and only if it is an internal Cauchy sequence.

Proof. In view of 4.3 we need only show (S) $\Rightarrow$ (C); as we said above, the argument is similar to the classical one. Suppose $(s_n) \in R^N(A)$ satisfies internally (S) without being internally Cauchy. According to the previous lemma the s_n converge pointwise to a continuous function ℓ: $A \longrightarrow \mathbb{R}$; but since condition $(*)$ of 1.3 cannot hold, there must be $\varepsilon > 0$ and $a_0 \in A$ such that for any $U \in \mathcal{U}(a_0)$ and $n \in \mathbb{N}$

$$|s_m(u) - \ell(u)| \geq \varepsilon \text{ for some } u \in U \text{ and } m > n.$$

Take a countable basis (U_n) of $\mathcal{U}(a_0)$, and define inductively a monotone sequence φ of integers choosing $\varphi(0)$ arbitrarily and letting $\varphi(n+1)$ be any integer $k > \varphi(n)$ such that $|s_k(u) - \ell(u)| \geq \varepsilon$ for some $u \in U_{n+1}$. Then

$$\text{for all } n > 0, \text{ some } u \in U_n \text{ satisfies } |s_{\varphi(n)}(u) - \ell(u)| \geq \varepsilon. \qquad (**)$$

The external sequence φ gives rise to an element of $N^N_>(A)$ in an obvious way, and from condition (S) we deduce that some $U \in \mathcal{U}(a_0)$ and $\psi \in N^N_>(U)$ must satisfy

$$U \models \left(s_{\varphi(\psi(n))}\right) \text{ converges to } \ell. \qquad (***)$$

Since A is locally connected, there is no loss if we assume U to be connected. Then any continuous function from U to $\mathbb{N}$ is constant, hence ψ is an ordinary monotone sequence of integers, and $\left(s_{\varphi(\psi(n))}\right)$ is an ordinary subsequence of (s_n). According to $(***)$ it converges internally to ℓ on U, whence criterion $(*)$ yields $V \in \mathcal{U}(a_0)$ and $n \in \mathbb{N}$, such that any $m > n$ satisfies

$$|s_{\varphi(\psi(m))} - \ell| < \varepsilon \text{ on } V.$$

This is incompatible with $(**)$ — at least if $U_{n+1} \subset U_n$ for all n, as we may assume — for then V contains all $U_{\psi(m)}$ for m big enough. •

4.9 Remark. An inspection of the proofs of 4.8 and 4.6 allows a strengthening of 4.8 which we shall need later on: for A first countable and locally connected, a sequence $(s_n) \in R^N(A)$ is internally Cauchy as soon as for some $\ell \in R(A)$, any external subsequence $(s_{\varphi(n)})$ of (s_n) satisfies $A \models \exists\, \psi \in N^N_> \ \left(s_{\varphi(\psi(n))}\right)$ converges to ℓ.

5. Convergence with N^*

For completeness we discuss yet another way of defining convergent sequences. Let $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$ denote the one–point compactification of the space $\mathbb{N}$. $\mathbb{N}^*$ is embedded in $\mathbb{R}$ via the (unique continuous) extension i of the map $n \mapsto \frac{1}{n+1}$ from $\mathbb{N}$ to $\mathbb{R}$. Now, suppose a sequence $s : \mathbb{N} \longrightarrow \mathbb{R}$ converges functionally, i.e. there is a function $[0,1] \xrightarrow{f} \mathbb{R}$ such that for all $n \in \mathbb{N}$, $f(\frac{1}{n+1}) = s(n)$. Then for $x \in \mathbb{N}^*$, the real number $r = f\big(i(x)\big)$ satisfies $x \in \mathbb{N} \Rightarrow r = s(x)$. Letting N^* be the sheaf represented by $\mathbb{N}^*$, we may carry over the whole argument to TOP, and we find:

5.1 Proposition. TOP $\models$ If a sequence $s \in R^N$ converges functionally, then s *converges with* N^*, i.e. the following then holds:

$$(\mathrm{N}^*): \quad \forall x \in N^* \quad \exists r \in R \quad [x \in N \Rightarrow r = s(x)]\,.$$

Condition (N^*) provides one more characterization of internal Cauchy sequences in TOP:

5.2 Proposition. TOP $\models$ Any sequence converging with N^* is a Cauchy sequence.

Proof. If $A \models (\mathrm{N}^*)$ for some $(s_n) \in R^N(A)$, then

$$A \times \mathbb{N}^* \models \exists r \in R \quad [x \in N \Rightarrow r = s(x)]\,,$$

where x is interpreted as the projection from $A \times \mathbb{N}^*$ on $\mathbb{N}^*$, and s by the projection $A \times \mathbb{N}^* \longrightarrow A$ followed by $A \xrightarrow{(s_n)} \mathbb{R}$. In particular, for each $a \in A$ there is $U_a \in \mathcal{U}(a)$ and an open neighbourhood V_a of ∞ in $\mathbb{N}^*$ such that some $U_a \times V_a \xrightarrow{r_a} R$ satisfies

$$U_a \times V_a \models x \in N \Rightarrow r_a = s(x).$$

Let $\varepsilon > 0$. Since r_a is continuous, there is $U'_a \in \mathcal{U}(a)$ and an open neighbourhood V'_a of ∞ in $\mathbb{N}^*$ such that $|r_a(u,m) - r_a(a,\infty)| < \varepsilon/2$ for all $m \in V_a \cap V'_a$ and $u \in U_a \cap U'_a$. But the $U_a \times U'_a$ form an open cover of A, and each $V_a \cap V'_a$ excludes only a finite number of $n \in \mathbb{N}$. Condition $(*)$ of 1.3 is thus seen to be satisfied for (s_n), and we may conclude by applying 0.1 and 0.2. •

II. Convergence in Smooth Toposes

1. Preliminaries

The notion of smooth topos appeared for the first time in SGA 4 [2], and it has been generalized in various directions (cf. in particular [12], exp.5, fasc.2). In this paper we shall confine our attention to those toposes that arise as the category of sheaves over a site fully embedded in the dual of the category of finitely generated germ–determined C^∞–rings with the open cover topology.

This section is devoted to a quick review of the basic notions involved. We would like to emphasize the fact that although most of our definitions refer to particular presentations of the relevant C^∞-rings, they actually admit invariant formulations. The interested reader may consult [9], as well as the forthcoming book on this subject by Moerdijk and Reyes, which promises to become an excellent reference.

1.1 Loci. We work throughout in the category $\mathbb{L}$ of *loci* or *formal C^∞-varieties* (cf. [12]), defined as the dual of the category of finitely generated C^∞-rings. Thus the objects of $\mathbb{L}$ are C^∞-rings that admit a presentation of type

$$L \cong C^\infty(\mathbb{R}^n)/I,$$

where I is an arbitrary ideal. A morphism $C^\infty(\mathbb{R}^m)/J \longrightarrow C^\infty(\mathbb{R}^n)/I$ in $\mathbb{L}$ is an equivalence class of smooth functions $\mathbb{R}^m \xrightarrow{F} \mathbb{R}^n$ with the property that

$$\mathbb{R}^n \xrightarrow{g} \mathbb{R} \in I \quad \text{implies} \quad g \circ F \in J,$$

any two such F and F' being equivalent if all their components are equivalent modulo J.

As a consequence of Whitney's embedding theorem, the category **Mf** of smooth manifolds is fully embedded in $\mathbb{L}$ via the functor that sends a manifold M into the ring $C^\infty(M)$ of smooth maps from M to $\mathbb{R}$. Since $C^\infty(M)$ is always a finitely presented C^∞-ring, we may describe $\mathbb{L}$ as having objects all $C^\infty(M)/I$ with $M \in \mathbf{Mf}$ and I an ideal of $C^\infty(M)$. Warning: a morphism $C^\infty(M)/I \longrightarrow C^\infty(M')/J$ in $\mathbb{L}$ is not necessarily represented by a smooth $M \longrightarrow M'$.

1.2 Points. A *point* of a locus L is a morphism $\mathbb{R} \cong C^\infty(\mathbb{R}^0) \longrightarrow L$ in $\mathbb{L}$. This is consistent with the usual definition of points as global sections: $\mathbb{R}$ is clearly terminal in $\mathbb{L}$. It is consistent with our geometric intuition, too: for a manifold M, ordinary points correspond precisely to points of $C^\infty(M)$ in $\mathbb{L}$, and this identification enables us to view a point of $L = C^\infty(M)/I$ as a *zero* of I, i.e. as an element of

$$Z(I) := \{\, p \in M \mid f(p) = 0 \text{ for all } f \in I \,\}.$$

We write ptL for the set of points of a locus L. This set is of course far from being invariant under presentations; however, its geometry is well defined upto 'smooth isomorphisms' in a sense we shall explain now. Note that morphisms between loci become concrete maps when restricted to points: $L \xrightarrow{\alpha} L'$ associates with a point $1 \xrightarrow{p} L$ of L the point $\bar{\alpha}(p) := \alpha \circ p$ of L'. Moreover, once we have identified ptL and ptL' with the corresponding subsets of the manifolds M and M', the map $\bar{\alpha}: ptL \longrightarrow ptL'$ turns out to be *smooth* in the sense that each point of ptL has an open neighbourhood in M, where $\bar{\alpha}$ may be extended to a C^∞-map. As a consequence, the topology of ptL is well determined; in particular, any space that arises as ptL is homeomorphic to a closed subset of some $\mathbb{R}^n$, for $Z(I)$ is always closed in M.

We say $L = C^\infty(M)/I$ is *point-determined* if a function $f \in C^\infty(M)$ belongs to I as soon as $f(p) = 0$ for all $p \in ptL$.

1.3 Germs and germ–determined loci. For a point p of a manifold M, we write $C^\infty_p(M)$ for the C^∞–ring of germs at p, i.e. for the quotient of $C^\infty(M)$ with respect to the ideal of functions that vanish on an open neigbourhood of p. For $f \in C^\infty(M)$, f_p denotes as usual the germ of f at p, i.e. the image of f in $C^\infty_p(M)$; similarly, write I_p for the image of an ideal $I \subseteq C^\infty(M)$ in the ring of germs at p. Thus

$$I_p = \{\, f_p \mid f_p = g_p \text{ for some } g \in I, \text{ i.e. } f = g \text{ in a neighbourhood of } p \,\}.$$

We say (cf.[4b]), $L = C^\infty(M)/I$ is *germ-determined* if a function $f \in C^\infty(M)$ belongs to I as soon as $f_p \in I_p$ for all points p of L. We denote by $\mathbf{G}$ the full subcategory of $\mathbb{L}$ whose objects are germ–determined loci. Note that $\mathbf{G}$ is a coreflective subcategory of $\mathbb{L}$: a right adjoint to the inclusion $\mathbf{G} \hookrightarrow \mathbb{L}$ is provided when passing from $C^\infty(M)/I$ to $C^\infty(M)/\tilde{I}$ with

$$\tilde{I} := \{\, f \mid f_p \in I_p \text{ for all } p \in Z(I) \,\}.$$

1.4 Weil algebras, infinitesimal loci, and near–points. One of the attractions of the category of loci is the existence of various infinitesimal objects surrounding points. For the sake of simplicity, we explore this micro–universe only around the point $\mathbf{0}$ of $\mathbb{R}^n$, say.

In algebraic (as opposed to geometric) terms, the origin $\mathbf{0}$ comes about as the map

$$C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R} \quad \text{'evaluate at } \mathbf{0}\text{'},$$

and it factors through the ring $C^\infty_0(\mathbb{R}^n)$ of germs as well as through the rings of jets of order r

$$J^n_r := C^\infty_0(\mathbb{R}^n)/m^{r+1},$$

$m = \{ f \mid f(\mathbf{0}) = 0 \}$ being the maximal ideal of $C^\infty_0(\mathbb{R}^n)$ (cf. [5]). We consider more generally quotients of rings of jets; these were introduced by A. Weil under the name of 'algèbres locales' (cf. [16]), and they are called *Weil algebras* since E. Dubuc used them as an essential tool for the construction of 'well adapted models' of Synthetic Differential Geometry (cf. [3]).

When looking at Weil algebras in the category $\mathbb{L}$, they appear most naturally as *infinitesimal loci*. Quotients in the category of C^∞–rings are subobjects in the dual category of loci; on the other hand, an immediate application of Hadamard's lemma shows that the ideal m is generated by the projections $x_1, \ldots, x_n$, hence m^{r+1} has generators the monomials of degree $r + 1$. Therefore the rings of jets J^n_r correspond to the loci

$$D^n_r = \{\, (x_1, \ldots, x_n) \mid \text{ the product of any } r+1 \text{ of the } x_i \text{ is zero } \,\} \rightarrowtail R^n$$

(we use set–theoretical notation for an equalizer). Thus a locus $X \rightarrowtail D^n_r$ stemming from a Weil algebra has its 'elements' so small that any $(r + 1)$–cube whose sides are coordinates of such an element has volume 0.

For a locus L, a morphism in $\mathbb{L}$ from a Weil algebra to L is called a *near-point* of L. Note that points are near-points since $\mathbf{1} = D_0$. We call L *near-point-determined* if $b \in L$ is zero as soon as $\pi(b) = 0$ for all near-points π of L. A locus $L = C^\infty(M)/I$ is near-point determined iff a function f belongs to I as soon as f is flat on ptL, i.e. all its derivatives vanish at points of L (using local coordinates around p).

1.5 Open inclusions. For a locus $L \cong C^\infty(M)/I$ and an open subset U of M, let $L|U$ denote the quotient of $C^\infty(U)$ modulo the ideal generated by the restrictions of functions in I. The following is a pull-back square in $\mathbb{L}$:

$$\begin{array}{ccc} L|U & \longrightarrow & C^\infty(U) \\ \downarrow & & \downarrow \\ L & \longrightarrow & C^\infty(M). \end{array}$$

A subcategory $\mathbb{C}$ of $\mathbb{L}$ containing both $L \longrightarrow C^\infty(M)$ and $C^\infty(U) \longrightarrow C^\infty(M)$ need not necessary contain $L|U$. Still, it may contain an object that plays in $\mathbb{C}$ the role played by $L|U$ in $\mathbb{L}$, in the sense of giving rise — in $\mathbb{C}$ — to a pull-back square as above. Since both $L \longrightarrow C^\infty(M)$ and $C^\infty(U) \longrightarrow C^\infty(M)$ are monics in $\mathbb{C}$, such an object then is the intersection (in the categorical sense) of L and $C^\infty(U)$ in $\mathbb{C}$; we therefore denote it by $L \cap U$ and call the morphism $L \cap U \rightarrowtail L$ an *open inclusion in* $\mathbb{C}$. For example, when $\mathbb{C} = \mathbf{G}$, we may take $L \cap U$ to be the germ-determined reflection of $L|U$; further examples shall be given in 1.7.

The following alternative characterization of open inclusions shows that this notion is actually invariant under presentations. With the same notation as above, let $\chi_U \in C^\infty(M)$ be a 'smooth characteristic function' for U ($\chi_U(x) \neq 0$ iff $x \in U$), and let $b \in L$ be represented by χ_U. When looking at $L \cap U \longrightarrow L$ in the dual of $\mathbb{C}$, it appears as $L \longrightarrow L\{b^{-1}\}$, i.e. as the universal solution to the problem of inverting b in $\mathbb{C}$. This is because in $\mathbb{L}^{op}$, the universal solution of inverting χ_U is provided by $C^\infty(M) \longrightarrow C^\infty(U)$, and it is readily checked that in any category of rings (with morphisms preserving multiplication and units), a commutative square

$$\begin{array}{ccc} L' & \longleftarrow & A\{a^{-1}\} \\ \uparrow & & \uparrow \\ L & \xleftarrow{f} & A \end{array}$$

is a pushout if and only if $L \longrightarrow L'$ solves universally the problem of inverting $f(a) \in L$ (hint: $A \longrightarrow A\{a^{-1}\}$ is always epic).

We therefore find that a morphism of $\mathbb{C}$ is an open inclusion if and only if the corresponding morphism in $\mathbb{C}^{op}$ is of type $L \longrightarrow L\{b^{-1}\}$ for some $b \in L$. This observation also shows that our notion generalizes the definition of open inclusions given in [9], III.7. We need however an extra condition in order to ensure that the composite of two open inclusions be again an open inclusion (I am grateful to A. Kock for having pointed out this fact to me). One such condition is the following: the morphism $L \cap U \rightarrowtail U$ corresponds in $\mathbb{C}^{op}$ to a *surjective* homomorphism of C^∞-rings, i.e. $L \cap U$ is a quotient of $C^\infty(U)$, or, in invariant style: with $b \in L$ as above, the canonical map that connects the universal solution to inverting b in

$\mathbb{L}^{op}$ with the analogous solution in $\mathbb{C}^{op}$ is surjective. This condition is satisfied in all the examples below, and it does ensure the stability of open inclusions under composition; the proof is essentially as in [9], III.7. For a subcategory $\mathbb{C}$ of $\mathbb{L}$, we therefore say it has *stable open inclusions* if (i) it contains $C^\infty(U)$ for any open subset U of any $\mathbb{R}^n$; (ii) whenever $L \cong C^\infty(\mathbb{R}^n)/I$ is in $\mathbb{C}$ and U is an open subset of $\mathbb{R}^n$, the intersection $L \cap U$ exists in $\mathbb{C}$, and it satisfies the above condition. Note that one then has $pt(L \cap U) = pt L \cap U$.

1.6 The open cover topology. For a subcategory $\mathbb{C}$ of $\mathbb{L}$, say a family $(L_i \longrightarrow L)_i$ of morphisms in $\mathbb{C}$ is an *open cover* of the locus L, if

(i) each $L_i \longrightarrow L$ is an open inclusion in $\mathbb{C}$;
(ii) $pt L \subseteq \bigcup_i pt L_i$, more precisely: each point of L factors through some $L_i \longrightarrow L$.

It follows from our considerations in the previous paragraph that with open covers as covering families, we obtain a Grothendieck–topology on any any full subcategory of $\mathbb{L}$ which has stable open inclusions. We call it the *open cover topology.*

1.7 Smooth toposes. In the present paper, by *smooth topos* we mean the category of sheaves over a site $(\mathbb{C}, \tau)$, where $\mathbb{C}$ is a full subcategory of $\mathbf{G}$ that has stable open inclusions, while τ is the open cover topology. The restriction to sites fully embedded into the category of germ–determined loci has the advantage of ensuring that the open cover topology is sub–canonical, and more generally, the presheaf $h_0(L) := \hom_{\mathbb{L}}(-, L)$ is a sheaf for any locus L. For the case $\mathbb{C} = (-, L)$ is a sheaf for any locus L. For the case $\mathbb{C} = \mathbf{G}$ this is proved in [4b], and for a proof of our more general case the argument given in [9], prop. 4 of III.7 is readily adapted.

Examples.

(i) The *topos lisse* of SGA 4 is defined as the category of sheaves over the site $(\mathbf{Mf}, j)$, where j is the open cover topolgy in the usual sense. Since the correspondence $M \mapsto C^\infty(M)$ gives rise to a full embedding of $\mathbf{Mf}$ in $\mathbf{G}$ which preserves and reflects open inclusions (cf.[9], III.7), the conditions above are satisfied. Note that a sheaf over $(\mathbf{Mf}, j)$ is completely determined once we know its effect on all open subsets of the various $\mathbb{R}^n$. Therefore an equivalent description of the 'topos lisse' is based on the site having objects all $C^\infty(U)$ with U any open subset of some $\mathbb{R}^n$.
(ii) The *Cahiers topos* of [3] has underlying site all loci of type $C^\infty(M) \times X$, with $M \in \mathbf{Mf}$ and X a Weil algebra — the product being performed in the category of loci. These loci appear naturally as 'infinitesimal extensions' of the classical manifolds (quoted from op.cit.). Arguing as we did before, one may again restrict the underlying site to have objects simply the loci of type $C^\infty(U) \times X$, with U an open subset of some $\mathbb{R}^n$.
(iii) If we choose the underlying site to have objects all finitely presented loci, we find the topos which Dubuc called that of C^∞–schemes in [4a]; this paper contains all we need to see that this choice actually yiels a smooth topos in our sense (cf. in particular proposition 14).
(iv) The *Euclidean topos* of [7] is defined as the category of sheaves over the site $(\mathbb{E}, j)$, where $\mathbb{E}$ is the category of all locally closed subspaces of the various

$\mathbb{R}^n$ together with C^∞–maps (in the sense we explained in 1.2 above) between them, while j is again the usual open cover topology. To see that this topos may be described as a smooth topos according to our definition, use first of all a partitions of unity argument in order to show that for any $E \xrightarrow{\alpha} F$ in $\mathbb{E}$, there is an open $U \supseteq E$ and an ordinary C^∞–map $\bar{\alpha}: U \longrightarrow \mathbb{R}^m$ such that $\bar{\alpha}|E \equiv \alpha$. If $E \subseteq \mathbb{R}^n$ is closed, then α extends even to a C^∞–map defined on all of $\mathbb{R}^n$, and this is easily seen to imply that the full subcategory of $\mathbb{E}$ having objects all closed $E \subseteq \mathbb{R}^n$ is fully embedded in $\mathbf{G}$ via the correspondence $E \mapsto C^\infty(E) := C^\infty(\mathbb{R}^n)/I(E)$, where $I(E)$ is the ideal of functions vanishing on E. Next note that if $U \subseteq \mathbb{R}^n$ is open, then $C^\infty(U)$ is isomorphic in $\mathbb{E}$ to $C^\infty(\hat{U})$ with $\hat{U} \subseteq \mathbb{R}^{n+1}$ closed: take a smooth characteristic function χ_U for U, and let $\hat{U} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \chi_U(x) \cdot t = 1\}$. Therefore $\mathbb{E}$ is equivalent to the full subcategory of $\mathbf{G}$ having objects all $C^\infty(E)$ with $E \subseteq \mathbb{R}^n$ closed. Yet another description of the Euclidean topos is provided taking as objects of the underlying site all point–determined loci.

(v) Take as objects of the underlying site all near–point determined loci.
(vi) Let the underlying site be $\mathbf{G}$ itself.
(vii) It is clear that from any smooth topos $\mathcal{E} = Sheaves(\mathbb{C}, j)$ we may derive another one by restricting the underlying to those loci in $\mathbb{C}$ that have their point–spaces characterized by some topological property which is inherited by open subsets, and holds for each $\mathbb{R}^n$ — for example local connectedness.

1.8 The natural number object. For a fixed smooth topos $\mathcal{E}$, we denote again by N its natural number object. We know $N = \Delta\mathbb{N}$, i.e. N is the associated sheaf of the constant presheaf $\mathbb{N}$. Now, the correspondence $L \mapsto C^0(ptL, \mathbb{N})$ defines a sheaf $\mathcal{N}$ on any full subcategory of $\mathbf{G}$ which has stable open inclusions, and this sheaf clearly has the constant presheaf associated with $\mathbb{N}$ as a sub–presheaf. Therefore $\Delta\mathbb{N}$ must agree with the closure of $\mathbb{N}$ in $\mathcal{N}$, and thus $N \leq \mathcal{N}$. But we have also $\mathcal{N} \leq N$. Indeed, for any locus L, the continuous functions from ptL to $\mathbb{N}$ are the locally constant ones, and this amounts precisely to saying that $\mathcal{N}$ is contained in the closure of $\mathbb{N} \rightarrowtail \mathcal{N}$. Thus $N = \mathcal{N}$, i.e. for any locus L in the underlying site

$$N(L) = C^0(ptL, \mathbb{N}) = \{ ptL \xrightarrow{f} \mathbb{N} \mid f \text{ is locally constant } \}.$$

1.9 The basic ring object. For a given smooth topos $\mathcal{E}$, we denote by R the sheaf represented by $C^\infty(\mathbb{R})$, and hence by $\mathbb{R}$ itself if we take the embedding of $\mathbf{Mf}$ into $\mathcal{E}$ seriously. Obviously, R is a ring in $\mathcal{E}$. Since $C^\infty(\mathbb{R})$ is the free C^∞–ring on one generator, we have $R(L) = L$ for any locus L. The order on R is again represented by $\mathbb{R}_>$, and $L \xrightarrow{f} R$ satisfies $L \models f > 0$ if and only if $f(p) > 0$ for all points p of L (for a proof cf. [12], exposé 5 of fasc.2). As to the norm on $\mathbb{R}$, it is of course not represented by any smooth map — in the present context this does not matter, since we only use formulas of type $|x| < \varepsilon$, and these may be viewed as a shorthand for $x < \varepsilon \wedge -x < \varepsilon$.

In opposition to what we had in part I, R is no longer the object of Dedekind reals in $\mathcal{E}$. As a striking illustration of this fact, there is proposition 2.2 below showing that R is far from being Cauchy complete. Using the results of [7], in particular 5.1, one can prove for various smooth toposes that the Dedekind reals are given by the sheaf $R_{Ded}(L) = C^0(ptL, \mathbb{R})$.

We define the object $[0,1]$ in a smooth topos $\mathcal{E}$ to be the sheaf represented (eventually from the outside) by $C^\infty([0,1])$, i.e. by the quotient of $C^\infty(\mathbb{R})$ modulo the ideal of functions vanishing on the ordinary unit interval.

2. Internal Cauchy Sequences and Limits

We now interpret in a smooth topos the notions of Cauchy sequence and of limit we introduced in I.1.

2.1 Proposition. In a smooth topos $\mathcal{E}$, a sequence $(s_n) \in R^N$ defined at stage L is an internal Cauchy sequence if and only if the sequence $(s_n|ptL)$ converges on ptL in the sense of uniform convergence on compact sets.

Proof. Since ptL is isomorphic to a closed subset of a manifold, it is locally compact. Propositions 1.3 and 3.1 of part I therefore reduce the problem to showing that $L \models$ (C) holds if and only if $(s_n|ptL)$ satisfies condition $(*)$ of 1.3. Next recall that both N and $<$ involve only points, hence so does the crucial subformula

$$\varphi \equiv \exists\, n \in N \quad \forall\, m, k > n \quad |s_m - s_k| < \varepsilon$$

of condition (C). On the other hand, $(s_n) \in R^N(L)$ satisfies internally condition (C) in $\mathcal{E}$ if and only if for all $L' \xrightarrow{\alpha} L$ and $L \xrightarrow{\varepsilon} R_>$ in the underlying site, one has $L' \models \varphi$ with (s_n) interpreted by $(t_n) := (s_n \circ \alpha)$. But since φ involves only points, the latter amounts to $ptL' \models \varphi$ in TOP (this makes sense because (t_n) and ε restrict to continuous functions on ptL'). Finally, with an inspection of the proof of I.1.3, the following turn out to be equivalent:

(i) $(s_n|ptL)$ satisfies condition $(*)$;

(ii) $ptL \models \varphi$ in TOP for any constant $\varepsilon > 0$;

(iii) for any $ptL' \xrightarrow{<\varepsilon,\alpha>} ptR_> \times ptL$ in **Top** one has $ptL' \models \varphi$ in TOP (with (s_n) interpreted by $(t_n|ptL')$).

The above arguments show that (iii) implies $L \models$ (C) in $\mathcal{E}$, and that the latter in turn implies (ii). Our claim is thus established. •

2.2 Proposition. In any smooth topos, a sequence $(s_n) \in R^N$ defined at stage $L = C^\infty(M)/I$ has an internal limit, i.e.

$$L \models \exists\, \ell \quad \forall\, \varepsilon > 0 \quad \exists\, n \in N \quad \forall\, m > n \quad |s(m) - \ell| < \varepsilon,$$

if and only if the external sequence $(s_n|ptL)$ converges UC on ptL to a function $ptL \xrightarrow{\ell} \mathbb{R}$ which is smooth in the sense we explained in 1.2 above.

Proof. Easy on the ground of previous work (especially the previous proof and the last part of the proof of I.1.3). •

This proposition provides a plain confirmation of what we said above: the ring R we chose in our smooth toposes is not the object of Dedekind reals. Many examples of internal Cauchy sequences of 'smooth reals' that do not admit an internal limit are readily exhibited.

2.3 Uniqueness of limits. There is another classical phenomenon that does not occur with internal limits, namely uniqueness of limits. Again, this is due to the fact that the formula $\forall \varepsilon > 0 \quad \exists\, n \in N \quad \forall m > n \quad |s(m) - \ell| < \varepsilon$ is completely decided on points. Therefore, if $\ell \in L = \mathrm{C}^\infty(M)/I$ is an internal limit of $(s_n) \in R^N(L)$, then so is any ℓ' that agrees with ℓ on ptL. Using 0.1, we may formulate this as

Proposition. For a smooth topos $\mathcal{E}$ the following are equivalent:
(i) $\mathcal{E} \models$ if ℓ and ℓ' in R are both limits of some $(s_n) \in R^N$, then $\ell = \ell'$;
(ii) any locus in the underlying site is point-determined.

3. Functional Convergence and Convergence with N^*

Were the notions of Cauchy sequence and internal limit quite weak, in so far as they capture only the behaviour of sequences on points, the notion of functional convergence is remarkably strong in a smooth topos. Indeed, consider the sequence of real numbers

$$s_n := \frac{1}{\sqrt{n+1}}.$$

Clearly, it gives rise to an element of $R^N(L)$ for any locus L, and one would presume that it be convergent in any reasonable sense. Instead it does not converge functionally: Even in the simplest case, where $L = \mathbf{1}$, it is impossible to find $f \in R^{[0,1]}(\mathbf{1}) = \mathrm{C}^\infty([0,1])$ satisfying $f(\frac{1}{n+1}) = s(n)$, because this would imply

$$f'(0) = \lim_{n\to\infty} \frac{f(\frac{1}{n+1}) - f(0)}{\frac{1}{n+1} - 0} = \lim_{n\to\infty} \frac{\sqrt{\frac{1}{n+1}}}{\frac{1}{n+1}} = \infty.$$

In order to remedy this inconvenient, one might weaken condition (F) taking instead

$$(\mathrm{F}^+) : \exists\, f \in R^{[0,1]} \quad \exists\, \lambda > 0 \quad \forall n \in N \quad s(n) = f\left(\frac{1}{(n+1)^\lambda}\right).$$

But again, you will soon find some external Cauchy sequence of real numbers for which condition (F^+) fails; for example

$$s_n := n^{-\frac{1}{\rho(n)}},$$

where $\rho(n) :=$ the smallest integer r with $n \leq (r+1)^{r+1}$. An argument similar to the previous one shows that condition (F^+) is still too strong as to capture the convergence of this sequence.

In categorical semantics one often succeeds in weakening a condition by taking a formula which is classically equivalent, but weaker in intuitionistic logic. This technique, however, would not help in the case of condition (F), since we found obstructions already at level 1, and there classical and intuitionistic logic agree. But what about convergence with N^*, i.e.

$$(\mathrm{N}^*) : \forall x \in N^* \quad \exists r \in R \quad [x \in N \Rightarrow r = s(x)].$$

In I.5 we used internal logic to infer (N^*) from (F), and all we needed was the fact that N is a subobject of N^*, and the existence of an arrow from N^* to R which extends the map $n \mapsto \frac{1}{n+1}$ from $\mathbb{N}$ to $\mathbb{R}$. Now, the most obvious analogue of the object N^* we chose in TOP is the sheaf represented (eventually from the outside) by the locus $C^\infty(\mathbb{N}^*) = C^\infty(\mathbb{R})$ modulo the ideal of functions vanishing on $\mathbb{N}^* := \{0\} \cup \{\frac{1}{n+1}\}_{n\in\mathbb{N}}$. It lives in any smooth topos, and it does satisfy the hypotheses we used in I.5: this is readily proved simply using the adjunction $\Delta \vdash \Gamma$ and elementary arithmetic. We denote it again by N^*.

3.1 Proposition. Suppose the underlying site $\mathbb{C}$ of a smooth topos $\mathcal{E}$ has all products of type $L\times C^\infty(\mathbb{N}^*)$ with $L \in \mathrm{Ob}\,\mathbb{C}$. (This is the case for the smooth toposes of examples (iv)–(vi) in 1.7, where $\mathbb{C}$ is a coreflexive subcategory of $\mathbb{L}$ containing $C^\infty(\mathbb{N}^*)$.) Then conditions (N^*) and (F) are equivalent in $\mathcal{E}$.

Proof. We proceed as in the proof of I.5.2: given a locus $L \cong C^\infty(\mathbb{R}^m)/I$ and $L \xrightarrow{(s_n)} R^N$ satisfying (N^*), we may find for each $p \in ptL$ neighbourhoods $U_p \in \mathcal{U}(p)$ and $V_p \in \mathcal{U}(0)$ and an arrow $r_p : L\times N^* \cap U_p\times V_p \longrightarrow R$ such that

$$L\times N^* \cap U_p\times V_p \models x \in N \Rightarrow r_p = s(x),$$

where x is interpreted by the obvious projection to N^*. Thus there must be $f \in C^\infty(U_p\times V_p)$ such that $f(\frac{1}{n+1}, -) \equiv s_n$ modulo $L \cap U_p$ holds for all n with $\frac{1}{n+1} \in V_p$. But there are at most finitely many n with $\frac{1}{n+1} \notin V_p$; therefore, using 'smooth Tietze', we may find $\bar{U}_p \in \mathcal{U}(p)$ and $\bar{f} \in C^\infty(\bar{U}_p\times[0,1])$ which extends f in a neighbourhood of $\{p\}\times(\mathbb{N}^* \cap V_p)$, and such that the element of $R^{[0,1]}(\bar{U}_p)$ corresponding to $\bar{f}$ satisfies indeed

$$\bar{U}_p \models \forall n \in N \quad f(\frac{1}{n+1}) = s(n). \qquad \bullet$$

3.2 Proposition. Suppose the underlying site $\mathbb{C}$ of a smooth topos $\mathcal{E}$ has ptL connected or locally connected for all $L \in \mathrm{Ob}\,\mathbb{C}$. (This is the case for the smooth toposes of examples (i) and (ii) in 1.7). Then

$$\mathcal{E} \models \text{Any sequence } s \in R^N \text{ converges with } N^*.$$

Proof. Consider a sequence $L \xrightarrow{(s_n)} R^N$ defined on a locus $L \cong C^\infty(\mathbb{R}^m)/I$ in $\mathbb{C}$, and let $L \xrightarrow{x} N^*$ be given. For a fixed connected component C_i of ptL, suppose

$x(p) = \frac{1}{n+1}$ for some $p \in C_i$; this n then is necessarily unique, and $x(p) = \frac{1}{n+1}$ holds for all $p \in C_i$. On the other hand, since the C_i are open in ptL, there are open sets U_i in $\mathbb{R}^m$ such that $\{L \cap U_i\}_i$ is a cover of L and $C_i = ptL \cap U_i$ for each i. Choose any family $\{L \cap U_i \xrightarrow{r_i} R\}$ with $r_i = s_n$ if $x(p) = \frac{1}{n+1}$ on C_i (letting r_i whatever you like otherwise). Then $L \cap U_i \models x \in N \Rightarrow r_i = s(x)$. Indeed, if some $L' \xrightarrow{\alpha} L \cap U_i$ satisfies $x \circ \alpha \in N$, i.e. $x \circ \alpha$ is locally represented by functions of type $y \mapsto \frac{1}{n+1}$, then our choice of the U_i forces $x \circ \alpha \equiv \frac{1}{n+1}$ to hold globally on ptL' for a unique n, hence the (r_i) do what we claim, namely $r_i \circ \alpha = s(x \circ \alpha)$. •

4. Convergence with Subsequences and Convergence in the Whitney Topology

There is a notion of convergence that is of special interest in differential geometry: a sequence $(f_n)_{n \in \mathbb{N}}$ of differentiable functions defined on a manifold M is said to be ***W–convergent*** to $f \in C^\infty(M)$ if (using local coordinates on M) for any multi-index k, the sequence $(D^k f_n)$ converges to $D^k f$ in the sense of uniform convergence on compacta. The 'W' here is reminiscent of Whitney: he introduced the corresponding topology on $C^\infty(M)$ under which it becomes a Fréchet space.

4.1 Definition. (i) For a manifold M, denote by $\mathcal{W}_0(M)$ the set of W–convergent sequences in $C^\infty(M)$.

(ii) Given a locus L, let $\mathcal{W}(L)$ be the set of sequences $(s_n) \in L^{\mathbb{N}}$ such that for some presentation of L as a quotient of $C^\infty(\mathbb{R}^m)$, one may represent (s_n) by a sequence in $\mathcal{W}_0(\mathbb{R}^m)$.

We shall see that W–convergence gives rise to a sheaf in any smooth topos. Since W–convergent sequences are stable under smooth maps from $\mathbb{R}^n$ to $\mathbb{R}^m$, the correspondence $L \mapsto \mathcal{W}(L)$ is certainly functorial. Thus a sequence in $\mathcal{W}(L)$ is represented by a W–convergent sequence in any of the presentations of L as a quotient of some $C^\infty(\mathbb{R}^m)$. We can do better:

4.2 Lemma. Suppose a locus L has a presentation as a quotient of $C^\infty(U)$ for some open $U \subseteq \mathbb{R}^m$. Then a sequence of elements of L is in $\mathcal{W}(L)$ if and only if it is represented by a W–convergent sequence of elements of $C^\infty(U)$. In particular $\mathcal{W}(U) = \mathcal{W}_0(U)$.

Proof. Choosing a smooth characteristic function $\chi_U \in C^\infty(\mathbb{R}^m)$ for U, we obtain a presentation of $C^\infty(U)$ as the quotient of $C^\infty(\mathbb{R}^{m+1})$ modulo the ideal of functions vanishing on $\hat{U} = \{(x,t) \in \mathbb{R}^m \times \mathbb{R} \mid \chi_U(x) \cdot t = 1\}$. In the given hypotheses, this provides also a presentation of L as a quotient of $C^\infty(\mathbb{R}^{m+1})$. Thus, for $(s_n) \in L^{\mathbb{N}}$ to be in $\mathcal{W}(L)$ means to originate from a sequence $(f_n) \in \mathcal{W}_0(\mathbb{R}^{m+1})$. But then the functions $\tilde{f}_n : x \in U \mapsto f_n(x, \frac{1}{\chi_U(x)})$ which represent the s_n in $C^\infty(U)$, clearly form a sequence in $\mathcal{W}_0(U)$. For the converse, suppose (s_n) is represented by $(g_n) \in \mathcal{W}_0(U)$. Since $\hat{U}$ is closed and contained in the open $U \times \mathbb{R}$, we can find h in $C^\infty(\mathbb{R}^{m+1})$ which is 1 on $\hat{U}$ and has support in $U \times \mathbb{R}$. Letting $f_n(x,t) := g_n(x) \cdot h(x,t)$ if $(x,t) \in U \times \mathbb{R}$

and extending by 0 outside $U \times \mathbb{R}$, we end up with a sequence which lies visibly in $\mathcal{W}_0(\mathbb{R}^{m+1})$ and represents (g_n), hence also (s_n). •

4.3 Proposition. For any smooth topos the correspondence $L \mapsto \mathcal{W}(L)$ defines a sheaf on the underlying site.

Proof. Since this correspondence is functorial, it gives rise to a subfunctor $\mathcal{W}$ of R^N. We thus are left with showing that given an open cover $(L_i \longrightarrow L)$ of a locus L, a sequence $(s_n) \in R^N(L)$ belongs to $\mathcal{W}(L)$ as soon as each $(s_n|L_i)$ is in $\mathcal{W}(L_i)$. Choosing a presentation of L as a quotient of $C^\infty(\mathbb{R}^m)$, we obtain presentations of the L_i as quotients of $C^\infty(U_i)$, with (U_i) an open cover of $ptL \subseteq \mathbb{R}^m$. In view of the previous lemma our hypothesis yields for each sequence $(s_n|L_i)$ a representative $(g_n^i) \in \mathcal{W}_0(U_i)$. Let $\{(h_i), h\}$ be a partition of unity on $\mathbb{R}^m$ subordinate to the cover $\{(U_i), \mathbb{R}^m \setminus ptL\}$, and define $g_n := \sum_i h_i \cdot g_n^i$. Since (h_i) is locally finite, (g_n) is W–convergent on $\mathbb{R}^m$. Finally, let us check that (g_n) represents (s_n) in L: we know already $h_i \cdot g_n^i \equiv h_i \cdot s_n$; thus

$$s_n \equiv s_n \cdot \textstyle\sum_i h_i \equiv \sum_i h_i \cdot g_n^i = g_n$$

on the level of germs at points of L. The claim now follows, for L is germ–determined. •

For the rest of the paper we analyze the possibility of characterizing the sheaf $\mathcal{W}$ internally. Surprisingly enough, it is the notion of convergence with subsequences that offers this possibility, at least in some cases. The key lemma is the following

4.4 Lemma. Suppose (f_n) is a W–convergent sequence of smooth functions defined on $\mathbb{R}^m$. There is a subsequence (g_n) of (f_n) and a smooth $\mathbb{R}^m \times [0,1] \xrightarrow{F} \mathbb{R}$, such that $g_n(x) = F(x, \frac{1}{n+1})$ for each n.

Proof. We prove the claim in case (f_n) converges to the zero–function — this is clearly sufficient. Choose $h \in C^\infty(\mathbb{R})$ satisfying $h(0) = 1$ and $h \equiv 0$ outside the interval $(-1, +1)$. Next fix a sequence (δ_n) of positive reals such that the intervals $I_n = (\frac{1}{n+1} - \delta_n, \frac{1}{n+1} + \delta_n)$ have disjoint closures, and for $t \in \mathbb{R}$ let $h_n(t) := h(\frac{1}{\delta_n} \cdot (t - \frac{1}{n+1}))$. Finally, given any subsequence (g_n) of (f_n), define a function $F: \mathbb{R}^m \times \mathbb{R} \longrightarrow \mathbb{R}$ letting

$$F(x,t) := \begin{cases} h_n(t) \cdot g_n(x) & \text{if } t \in I_n \text{ for some } n; \\ 0 & \text{otherwise.} \end{cases}$$

The definition is correct because we chose the I_n disjoint and h_n vanishing outside I_n. Furthermore, $F(x, \frac{1}{n+1}) = g_n(x)$ since $h_n(\frac{1}{n+1}) = h(0) = 1$. It is also clear that F is smooth in any point outside the t-axis. The problem consists in choosing (g_n) carefully, so to ensure that F becomes smooth in those points, too.

Note that $(0, \infty)$ has an open cover (J_n) such that $F(x,t) = h_n(t) \cdot g_n(x)$ for all $t \in J_n$. Therefore, if DF stands for any derivative of F involving r derivations with respect to t, and $k_1, \dots, k_m$ derivations with respect to $x_1, \dots, x_m$ (in any order), we have for $t \in J_n$

$$DF(x,t) = h_n^{(r)}(t) \cdot D^k g_n(x), \quad \text{where } D^k := \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_m^{k_m}}$$

(as usual, for a multi-index $k = (k_1, \dots, k_m)$ we let $|k| := k_1 + \cdots + k_m$). We claim (g_n) may be chosen in such a way that for all D

$$DF(x,0) \text{ exists and is } = 0.$$

We proceed by induction on the order of D. If it is 0, there is nothing to prove: $F(x,0) \equiv 0$ by definition. If D has order > 0, it is of type $\partial \bar{D}/\partial y$, and the claim follows immediately by induction in case y is one of the x_μ. Else $y = t$, and

$$DF(x,0) = \lim_{t \to 0} \frac{\bar{D}F(x,t) - \bar{D}F(x,0)}{t}.$$

Here $\bar{D}F(x,0) = 0$ by induction. As for $\bar{D}F(x,t)$, we know it is 0 for $t < 0$, whereas when $t > 0$, it agrees with $h_n^{(r-1)}(t) \cdot D^k g_n(x)$ for some n. Now, h_n vanishes in a neigbourhood of 0, hence

$$\frac{h_n^{(r-1)}(t)}{t} = \frac{h_n^{(r-1)}(t) - h_n^{(r-1)}(0)}{t} = h_n^{(r)}(t') = \frac{1}{\delta_n^r} \cdot h^{(r)}(t' - \frac{1}{n+1})$$

for some $t' \in (0,t)$. Since $h^{(r)}$ is obviously bounded, we therefore find as a sufficient condition for $DF(x,0)$ to exist:

$$\lim_{n \to \infty} D^k g_n(x) \cdot \frac{1}{\delta_n^r} = 0.$$

The latter may be satisfied in the following way. Cover $\mathbb{R}^m$ by a countable family (C_n) of compact sets. Since (f_n) is W–convergent to the zero–function, we may find a monotone sequence $(\varphi(n))$ of integers such that for each n

$$|D^k f_{\varphi(n)}| \le \frac{\delta_n^n}{n+1} \text{ on the compact } \bigcup_{\nu \le n} C_\nu$$

for all $k = (k_1, \dots k_m)$ with $|k| \le n$. Let $(g_n) := (f_{\varphi(n)})$. Once x, k and r are fixed, consider any $\varepsilon > 0$: if $x \in C_s$, then for all $n \ge \max(|k|, r, s, 1/\varepsilon)$

$$\left| D^k g_n(x) \cdot \frac{1}{\delta_n^r} \right| \le \left| D^k g_n(x) \cdot \frac{1}{\delta_n^n} \right| \le \frac{1}{n+1} < \varepsilon. \quad \bullet$$

This lemma strongly suggests that the elements of $\mathcal{W}$ satisfy internally the following condition of *functional convergence with subsequences*

$$(\mathrm{SF}) : \exists \ell \;\; \forall \varphi \in N_>^N \exists \psi \in N_>^N \;\; \exists f \in R^{[0,1]} \;\; [\ell = f(0) \wedge \forall n \in N \, s_{\psi(\varphi(n))} = f(\frac{1}{n+1})].$$

However, as we saw already in part I, internal subsequences need not always be well–behaved. Consider the following

4.5 Counterexample. Letting $s_n = 0$ when n is even, and $= \sqrt{\frac{1}{n}}$ when n is odd, we clearly obtain an element of $\mathcal{W}(L)$ for any locus L. But for some loci, (s_n) has internal subsequences at stage L that live no longer in $\mathcal{W}(L)$, e.g. when $L = C^\infty(\mathbb{N}^*)$ (cf. section 3). Indeed, letting

$$\varphi_n(x) := \begin{cases} 2n & \text{if } x < 1/n \\ 2n+1 & \text{if } x \geq 1/n, \end{cases}$$

we obtain a monotone sequence of integers at stage L, hence it gives rise to an internal subsequence of (s_n), say $(\bar{s}_n)$. For $x \in ptL$, we have:

$$\bar{s}_n(x) = s_{\varphi_n(x)}(x) = \begin{cases} 0 & \text{if } x < 1/n \\ \sqrt{\frac{1}{2n+1}} & \text{if } x \geq 1/n. \end{cases}$$

Choose any $(f_n) \in C^\infty(\mathbb{R})^{\mathbb{N}}$ representing $(\bar{s}_n)$. Since (f_n) must agree with $(\bar{s}_n)$ on ptL, we have for each n

$$\frac{f_n(\frac{1}{n}) - f_n(0)}{\frac{1}{n} - 0} = \frac{\sqrt{\frac{1}{2n+1}}}{\frac{1}{n}}.$$

Thus there must be $t_n \in (0, \frac{1}{n})$ satisfying $f_n'(t_n) = n \cdot \sqrt{\frac{1}{2n+1}} > \sqrt{\frac{n}{3}}$. Now, if the f_n' were uniformly converging to some function ℓ', this would imply $\ell'(0) = \lim_{n\to\infty} f_n'(t_n) = \infty$. So (t_n) cannot be represented by any W–convergent sequence. Similar phenomena occur only when ptL fails to be locally connected:

4.6 Proposition. Let $\mathcal{E} = \mathit{Sheaves}(\mathbb{C}, \tau)$ be a smooth topos. If all loci in $\mathbb{C}$ have ptL connected or locally connected, then any internal subsequence of a W–convergent sequence is again W–convergent; more precisely, the following then holds:

$$\mathcal{E} \models (s_n) \in \mathcal{W} \quad \Rightarrow \quad \forall \varphi \in N_>^N \quad (s_{\varphi(n)}) \in \mathcal{W}.$$

Proof. It is clearly sufficient to show that if a locus $L = C^\infty(\mathbb{R}^m)/I$ has ptL connected or locally connected, then for any $(s_n) \in \mathcal{W}(L)$ and $\varphi \in N_>^N(L)$, the corresponding $(s_{\varphi(n)})$ is again in $\mathcal{W}(L)$. Since φ is a sequence of continuous functions from ptL to $\mathbb{N}$, the φ_n are constant integers on each connected component C_i of ptL. Now, in any case the C_i are open in ptL, hence there is an open cover (U_i) of ptL in $\mathbb{R}^m$ such that the C_i are of type $ptL \cap U_i$, and each $(s_{\varphi(n)}|L \cap U_i)$ is an ordinary subsequence of $(s_n|L \cap U_i)$; thus it is in $\mathcal{W}(L \cap U_i)$ when $(s_n) \in \mathcal{W}(L)$. We conclude by 4.3. •

4.7 Proposition. Let $\mathcal{E} = \mathit{Sheaves}(\mathbb{C}, \tau)$ be a smooth topos, and suppose all loci L in $\mathbb{C}$ have ptL connected or locally connected. Then

$$\mathcal{E} \models (s_n) \in \mathcal{W} \quad \Rightarrow \quad \exists \ell \quad \forall \varphi \in N_>^N \quad \exists \psi \in N_>^N \quad \exists f \in R^{[0,1]}$$
$$\left[\ell = f(0) \quad \wedge \quad \forall n \in N \quad s_{\psi(\varphi(n))} = f(\tfrac{1}{n+1})\right].$$

In other words, in the internal logic of $\mathcal{E}$, W–convergence implies functional convergence with subsequences.

Proof. Since the formulae $(s_n) \in \mathcal{W}$ and (SF) have the same free variable (namely (s_n), a variable of sort R^N), by 0.2 it is enough to show that in the given hypothesis, for any locus L in $\mathbb{C}$ and $(s_n) \in \mathcal{W}(L)$, one has $L \models$ (SF). So suppose $(s_n) \in \mathcal{W}(L)$ converges to $b \in L$. From 4.6 we know that for any $L' \xrightarrow{\alpha} L$ and $L' \xrightarrow{\varphi} N^N_>$, the sequence $(s_{\varphi(n)} \circ \alpha)$ is in $\mathcal{W}(L')$ and has limit $b \circ \alpha$. So all we have to show is:

($\diamond$) For any locus L' in $\mathbb{C}$ and $(t_n) \in \mathcal{W}(L')$ with limit $\ell \in L'$

$$L' \models \exists\, \psi \in N^N_> \quad \exists\, f \in R^{[0,1]} \quad \left[\ell = f(0) \quad \wedge \quad \forall\, n \in N \quad t(\psi(n)) = f(\tfrac{1}{n+1})\right].$$

It will even be enough to prove ($\diamond$) in case $L' = C^\infty(\mathbb{R}^m)$, since $\mathcal{W}$ has been defined using presentations, and anything true at stage $C^\infty(\mathbb{R}^m)$ remains true at later stages. Now, we have the obvious equation $R^R(C^\infty(\mathbb{R}^m)) = C^\infty(\mathbb{R}^m \times \mathbb{R})$. Together with lemma 4.4 it implies ($\diamond$) with R^R instead of $R^{[0,1]}$. But the object $[0,1]$ of $\mathcal{E}$ is represented (eventually from the outside) by a quotient of $C^\infty(\mathbb{R})$. Therefore we have an arrow $[0,1] \longrightarrow R$ in $\mathcal{E}$, and thus an arrow $R^R \longrightarrow R^{[0,1]}$. It is now immediate that ($\diamond$) continues to hold when R^R is replaced by $R^{[0,1]}$. •

We shall see that in some cases condition (SF) characterizes W–convergence internally, i.e. we may reverse the arrow $\Rightarrow$ in the previous proposition. Again, we must restrict to loci that have locally connected point–spaces: otherwise it may happen that (SF) does not even imply condition (C), i.e. uniform convergence on compact subsets of the point–space. Consider $L = C^\infty(\bigcup C_n \cup \{0\})$, where (C_n) is any family of disjoint intervals of real numbers with $1/n \in C_n$. This locus lives in the smooth toposes of examples (iv)–(vi) in 1.7, and the clauses $s_n \equiv 1$ on C_n and $\equiv 0$ elsewhere clearly give rise to an element of $R^N(L)$. But it does not converge UC on ptL, although it satisfies (SF) with $\ell \equiv 0$: this is readily seen using what we found in I.4.5. In view of remark 4.7 of part I a similar counterexample exists for any locus whose points fail to form a locally connected space. Note however that we don't really loose the relevant examples of smooth toposes under this restriction according to 1.7 (vii).

4.8 Proposition. Suppose the underlying site $\mathbb{C}$ of a smooth topos $\mathcal{E}$ is such that

(i) for any $L \cong C^\infty(\mathbb{R}^m)/I$ in $\mathbb{C}$, the product of L with R exists in $\mathbb{C}$, and as a C^∞–ring, it is a quotient of $L \otimes_\infty C^\infty(\mathbb{R}) = C^\infty(\mathbb{R}^{m+1})/J$, where J is generated by the functions $(x_1, \ldots, x_{m+1}) \mapsto f(x_1, \ldots, x_m)$ with $f \in I$;

(ii) ptL is locally connected for any locus L in $\mathbb{C}$.

(Condition (i) is satisfied in all the examples of smooth toposes we gave in 1.7, and condition (ii), although necessary, is not really essential, as we noted above.)
Then for $L \in \mathrm{Ob}\,\mathbb{C}$, any sequence satisfying condition (SF) at stage L in $\mathcal{E}$ converges UC to a smooth function on ptL, i.e. $\quad \mathcal{E} \models (\mathrm{SF}) \Rightarrow (\mathrm{L})$.

Proof. Any sequence $L \xrightarrow{(s_n)} R$ satisfying (SF) in $\mathcal{E}$ clearly has the following property:

there is $\ell \in R(L)$ such that for any external subsequence $(s_{\varphi(n)})$

$$L \models \exists\, \psi \in N^N_> \quad \exists\, f \in R^{[0,1]} \quad \left[f(0) = \ell \wedge \forall n \in N \quad s_{\psi(\varphi(n))} = f(\frac{1}{n+1}) \right].$$

We claim this holds also in TOP, with ptL instead of L. Since in TOP, functional convergence implies internal convergence, remark 4.9 of part I then enables us to conclude that $(s_n|ptL)$ is internally Cauchy in TOP, hence it converges UC to $\ell|ptL$. All we need in order to establish this claim is that any $L \xrightarrow{f} R^{[0,1]}$ in $\mathcal{E}$ induces an arrow $ptL \xrightarrow{\bar{f}} R^{[0,1]}$ in TOP.

By exponential adjunction, $L \xrightarrow{f} R^{[0,1]}$ in $\mathcal{E}$ corresponds to a natural transformation $\hat{f}$ from $L\times[0,1]$ to R. On the other hand, the map $x \mapsto \sin^2 x$ induces an arrow $R \longrightarrow [0,1]$ in $\mathcal{E}$, hence its product with id_L provides an arrow $L\times R \xrightarrow{g} L\times[0,1]$. But the composite $\hat{f}\circ g\colon L\times R \longrightarrow R$ connects two representables, hence it must be induced by some smooth $\mathbb{R}^m\times\mathbb{R} \xrightarrow{h} \mathbb{R}$. Thus $\Gamma(\hat{f}\circ g) = \Gamma(\hat{f})\circ\Gamma(g)$ agrees with the restriction of h on $pt(L\times[0,1]) = ptL\times[0,1]$. The continuity of $\bar{f} := \Gamma(\hat{f})$ now follows because $\Gamma(g) = id\times\sin^2$ is an identification map. •

4.9 Proposition. In the 'topos lisse' (cf. example (i) in 1.7), functional convergence with subsequences characterizes W–convergence internally, i.e. the sheaf $\mathcal{W}$ agrees with the interpretation of condition (SF):

$$\mathcal{E} \models \forall (s_n) \in R^N \left[(s_n) \in \mathcal{W} \iff (s_n) \text{ converges functionally with subsequences} \right].$$

Proof. We work with the description of the topos lisse as sheaves over the site having objects all $C^\infty(U)$, for U open in some $\mathbb{R}^m$. Note first of all that in this topos, $R^{[0,1]}(C^\infty(U)) = C^\infty(U\times[0,1])$ (this may be proved adapting the arguments of [9], theorems 9.5 and 9.6 of part III: remark that the proof of Kock's lemma 9.9 terminates in this case after a few lines, since $\mathbb{C}$ has all objects point–determined.) Now suppose a sequence (s_n) of smooth functions defined on some U is such that $C^\infty(U) \models$ (SF) in the topos lisse. Then it is easy to see that for each multi–index k, the sequence $(D^k s_n)$, too, satisfies (SF) at stage $C^\infty(U)$. Thus we may conclude by using the previous proposition. •

4.10 Proposition. In the 'Cahiers topos' (cf.1.7) functional convergence with subsequences characterizes W–convergence internally.

Proof. We use the description of this topos as sheaves over the site having objects all $C^\infty(U)\times X$, for U open in some $\mathbb{R}^m$ and X a Weil algebra. The argument is exactly the same as in the previous proof, but we must convince ourselves that all the sheaves involved associate with $C^\infty(U)\times X$ simply the dth power of what they associate with U in the topos lisse, d being the dimension of X as a vector space over $\mathbb{R}$.

Consider first of all the sheaf R. By exponential adjunction we may identify $R(C^\infty(U)\times X)$ with $R^X(C^\infty(U))$. But $R^X \cong R^d$ is precisely the content of the axiom of Kock–Lawvere, which holds in this topos. With an explicit description of this isomorphism it will become clear that we have the same situation for the sheaf $\mathcal{W}$.

We know $X \cong C_0^\infty(\mathbb{R}^q)/J$ for an ideal J containing m^{r+1} (we use notation from 1.4, and for simplicity, we do only the case where X is centered at $\mathbf{0}$). Thus $C^\infty(U)\times X$ may be presented as the quotient of $C^\infty(U\times\mathbb{R}^q)$ modulo the ideal generated by J. Iterating Hadamard's lemma, one finds for any $f \in C^\infty(U\times\mathbb{R}^q)$

$$f(x,y) = \sum_{|k|\leq r} \frac{1}{k!} D^k f(x,0)\cdot y^k + \sum_{|k|=r+1} g_k(x,y)\cdot y^k$$

with the usual conventions for multi-indices $k = (k_1,\dots,k_q)$. The second sum vanishes modulo m^{r+1}, and the first one may be rearranged choosing polynomials $p_1,\dots,p_d$ in $\mathbb{R}[Y_1,\dots,Y_q]$ that represent a basis of X over $\mathbb{R}$. Then each y^k writes as $r_{1k}p_1 + \cdots + r_{dk}p_d$, whence

$$f \equiv \sum_{i=1}^{d} f_i(x)\cdot p_i(y) \quad \text{with } f_i(x) = \sum_{|k|\leq r} \frac{r_{ik}}{k!} D^k f(x,0).$$

It is the correspondence $f \mapsto (f_1,\dots,f_d)$ that operates the isomorphism between $R(-\times X)$ and R^d; the last formula therefore shows immediately how this isomorphism carries over to $\mathcal{W}$.

Finally, we have also $R^{[0,1]}(C^\infty(U)\times X) = \left[C^\infty(U\times[0,1])\right]^d$. This is established in the proof of thm. 11.5 in chapter III of [9]. •

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