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SHEAVES OF SEMIPRIME IDEALS

by S.B. NIEFIELD and K.I. ROSENTHAL

RÉSUMÉ. Cet article étudie le quantale Idl($\boldsymbol{O}$) des idéaux d'un faisceau commutatif d'anneaux $\boldsymbol{O}$ sur un local L. En particulier, on caractérise les éléments premiers et semi-premiers de ce quantale et on les utilise pour donner une description explicite externe du spectre de Zariski de $\boldsymbol{O}$. Un faisceau d'anneaux peut être défini sur ce spectre, ce qui permet d'itérer la construction. Ces résultats sont comparés au travail de Hakim et Tierney.

INTRODUCTION.

Recall that a *quantale* is a complete lattice Q together with an associative binary operation & such that $a\&-$ and $-\&a$ preserve sups, for all $a \in Q$. Quantales were first introduced by Mulvey [10] to provide a setting for non-commutative logic. Examples of quantales include the lattice Idl(R) of two-sided ideals of a ring R as well as any locale (which is just a quantale in which $\& = \wedge$, the usual lattice meet). Of course, a special case of the latter example is the lattice $\Omega(X)$ of open subsets of a topological space X. By a *morphism of quantales*, we shall mean a sup, & and τ preserving map, where τ denotes the top element of a quantale. Note that in the case of locales such a map gives a morphism in the category of frames, the dual of the category of locales.

In [11], we presented an explicit description of the universal surjective morphism of a quantale Q onto a locale L(Q). In the special case where Q is *two-sided* (i.e., $a\&\tau \le a$ and $\tau\&a \le a$ for all $a \in Q$), L(Q) is the set of semiprime elements of Q, where c is *semiprime* iff $c\&c \le a$ implies $c \le a$. Also, the surjective morphism $r\colon Q \twoheadrightarrow L(Q)$ is given by

$$r(a) = \inf \{ b \in L(Q) \mid a \le b \}.$$

This construction generalizes the well-known map

$$\mathrm{Idl}(R) \twoheadrightarrow \mathrm{RIdl}(R)$$

which takes an ideal of a commutative ring R with unit to its radical, where RIdl(R) denotes the locale of radical ideals of R.

The goal of this paper is to study the locale obtained by applying the results of [11] to the quantale Q of ideals of a sheaf of rings $\boldsymbol{O}$ on a locale L. In this case, we get a locale L(Q) over L which is the externalization (using the isomorphism [9] between the category of internal locales in the topos Sh(L) of sheaves on L and the category of locales over L) of the internal locale of "radical ideals of $\boldsymbol{O}$", in the sense of Tierney [13]. Before continuing, we present a brief account of the relevance of this internal locale.

In [13], Tierney considers several constructions of the spectrum of a ringed topos ($\boldsymbol{E}$,R), i.e., a right adjoint to the forgetful functor from local ringed toposes to ringed toposes. The existence of such an adjoint for Grothendieck toposes was first established by Hakim [5]. Tierney's constructions include one based on the internal locale of radical ideals of R, and another using "forcing topologies". Although the latter provides the most elegant presentation of the adjunction, the former is sometimes useful for calculations. For an additional reference on the spectrum of a ringed topos, the reader should consult [7].

We begin (§1) by recalling the properties we will need to study sheaves of ideals. In Sections 2 and 3, we consider the locale L(Q) obtained as above, when Q is the quantale of ideals of a sheaf of rings. We conclude the paper (§4) with a description of the ringed spaces obtained by iterating the construction of the Zariski spectrum of a commutative ring with unit.

1. PRELIMINARIES.

In this section, we recall some basic properties of ideals of a sheaf of rings. The reader should note that an ordinary ring can be viewed as a sheaf on a one point space and that the constructions we describe reduce to the usual ones in this case.

We cannot offer any references for much of what follows, as most of it can be attributed to folklore. However, most of our terminology and operations on ideals come from Grothendieck [4].

Throughout this paper $\boldsymbol{O}$ will denote a sheaf of commutative rings with unit on a locale L. If $a \in L$, then the unit element of $\boldsymbol{O}(a)$ will be denoted by 1_a. We will often write 1 for the unit element of $\boldsymbol{O}(\tau)$, where τ denotes the top element of L. Of course, $1_a = 1|_a$.

A *presheaf of ideals* of $\boldsymbol{O}$ is a presheaf I on L such that I(a) is an ideal of $\boldsymbol{O}(a)$, for all $a \in L$. If, in addition, I is a sheaf, then I is called a *sheaf of ideals* or simply an *ideal* of $\boldsymbol{O}$. The set of ideals of $\boldsymbol{O}$ will be denoted by $\mathrm{Idl}(\boldsymbol{O})$. Of course, an ideal I is just an internal ideal of the ring $\boldsymbol{O}$ in Sh(L).

Given $\{I_\alpha\} \subseteq \mathrm{Idl}(\boldsymbol{O})$, let $\cap I_\alpha$ be defined on $a \in L$ by $(\cap I_\alpha)(a) = \cap I_\alpha(a)$. Then it is not difficult to show that $\cap I_\alpha$ is a sheaf of ideals. Thus, $\mathrm{Idl}(\boldsymbol{O})$ is a complete lattice with $\inf = \cap$. Note that $\boldsymbol{O}$ is the top element of $\mathrm{Idl}(\boldsymbol{O})$.

Recall that the *sheafification* $\tilde{I}$ of a presheaf of ideals I is given by $r \in \tilde{I}(a)$ iff there is a cover $\{a_i\}$ of a such that $r|_{a_i} \in I(a_i)$, for all i. Moreover, if I is a presheaf of ideals, then

$$\tilde{I} = \cap\{J \in \mathrm{Idl}(\boldsymbol{O}) \mid I \subseteq J\}.$$

Now, if $\{I_\alpha\} \subseteq \mathrm{Idl}(\boldsymbol{O})$, then $a \mapsto \Sigma I_\alpha(a)$ is a presheaf, but need not be a sheaf. Thus, unlike in the ordinary ring case, the sup of a family of ideals is not given by their sum. However, we can consider the sheafification of ΣI_α, which will be denoted by $\tilde{\Sigma} I_\alpha$. Clearly, $\tilde{\Sigma} I_\alpha$ is the smallest ideal of $\boldsymbol{O}$ containing I_α, for all α, and so we obtain $\sup(I_\alpha)$.

For the quantale structure, suppose I and J are ideals of $\boldsymbol{O}$. Then their *sheaf product* is defined by

$$I\&J = \widetilde{IJ}\ , \text{ where } (IJ)(a) = I(a)J(a),$$

the usual ideal product in $\boldsymbol{O}(a)$. Note that the presheaf IJ can also be defined as above, if I and J are just presheaves of ideals.

LEMMA 1.1. *If* I *and* J *are presheaves of ideals, then* $\tilde{I}\&\tilde{J} = \widetilde{IJ}$.

PROOF. Since $I \subseteq \tilde{I}$, for all I, it follows that $IJ \subseteq \tilde{I}\tilde{J} \subseteq \tilde{I}\&\tilde{J}$. Thus $\widetilde{IJ} \subseteq \tilde{I}\&\tilde{J}$. For the reverse containment, it suffices to show that $\tilde{I}\tilde{J} \subseteq \widetilde{IJ}$. Suppose $r \in \tilde{I}(a)$ and $s \in \tilde{J}(a)$, for some $a \in L$. Then $r|_{a_i} \in I(a_i)$, for some cover $\{a_i\}$ of a, and $s|_{b_j} \in J(b_j)$, for some cover $\{b_j\}$ of a. Thus, $\{a_i \wedge b_j\}$ covers a and

$$rs|_{a_i \wedge b_j} \in (IJ)(a_i \wedge b_j).$$

Hence $rs \in \widetilde{IJ}(a)$, as desired. ∎

PROPOSITION 1.2. $\mathrm{Idl}(\boldsymbol{O})$ *is a two-sided quantale.*

PROOF. Clearly, & is associative. To see that I&– preserves sups, let $\{J_\alpha\} \subseteq \mathrm{Idl}(\boldsymbol{O})$. Since $I\&J_\alpha \subseteq I\&(\tilde{\Sigma}J_\alpha)$, for all α, it follows that $\tilde{\Sigma}(I\&J_\alpha) \subseteq I\&(\tilde{\Sigma}J_\alpha)$. But, $I\&(\tilde{\Sigma}J_\alpha)$ is the sheafification of $I(\Sigma J_\alpha)$, by Lemma 1.1, and $I(\Sigma J_\alpha) = \Sigma(IJ_\alpha)$. Thus

$$I\&(\tilde{\Sigma}J_\alpha) = \tilde{\Sigma}(IJ_\alpha) \subseteq \tilde{\Sigma}(I\&J_\alpha) \text{ and so } I\&(\tilde{\Sigma}J_\alpha) = \tilde{\Sigma}(I\&J_\alpha).$$

Similarly, $-\&\mathrm{I}$ preserves sups. Therefore, $\mathrm{Idl}(\boldsymbol{O})$ is a quantale. Since $\boldsymbol{O}(a)$ is a ring with unit, for all a, it easily follows that $\mathrm{Idl}(\boldsymbol{O})$ is two-sided. ∎

We conclude this section with a description of the "principal ideals" of $\boldsymbol{O}$, which turn out to be "generators" of $\mathrm{Idl}(\boldsymbol{O})$. If $r \in \boldsymbol{O}(a)$, then the *principal presheaf ideal* $\boldsymbol{O}r$ is defined by

$$\boldsymbol{O}r(b) = \begin{cases} \boldsymbol{O}(b)r|_b & \text{if } b \leq a \\ 0 & \text{if } b \not\leq a \end{cases}$$

Note that if $\mathrm{I} \in \mathrm{Idl}(\boldsymbol{O})$, then $\tilde{\boldsymbol{O}}r \subseteq \mathrm{I}$ iff $r \in \mathrm{I}(a)$, or equivalently,

$$\tilde{\boldsymbol{O}}r = \cap\{\mathrm{I} \in \mathrm{Idl}(\boldsymbol{O}) \mid r \in \mathrm{I}(a)\}.$$

Thus,
$$\mathrm{I} = \tilde{\Sigma}\{\tilde{\boldsymbol{O}}r \mid r \in \mathrm{I}(a)\}.$$

Recall also that a subset $\boldsymbol{B}$ of a complete lattice Q is said to *generate* Q if every $a \in \mathrm{Q}$ can be written $a = \sup a_i$, for some $\{a_i\} \subseteq \boldsymbol{B}$. Using the above properties of $\tilde{\boldsymbol{O}}r$, it is not difficult to show that if $\boldsymbol{B}$ generates L, then $\mathrm{Idl}(\boldsymbol{O})$ is generated by

$$\{\tilde{\boldsymbol{O}}r \mid r \in \boldsymbol{O}(b), \text{ for some } b \in \boldsymbol{B}\}.$$

It will be useful to have a description of products of principal ideals.

PROPOSITION 1.3. *If $r \in \boldsymbol{O}(a)$ and $s \in \boldsymbol{O}(b)$, then*

$$\tilde{\boldsymbol{O}}r \,\&\, \tilde{\boldsymbol{O}}s = \tilde{\boldsymbol{O}}r|_c s|_c\,, \quad \textit{where } c = a \wedge b.$$

PROOF. Clearly, $\boldsymbol{O}r\boldsymbol{O}s = \boldsymbol{O}r|_c s|_c$. Applying Lemma 1.1, the desired result follows. ∎

2. SEMIPRIME IDEALS OF A SHEAF OF RINGS.

In the previous section, we saw that $\mathrm{Idl}(\boldsymbol{O})$ is a two-sided quantale. Following [11], we obtain a locale by considering only the "semiprime" elements.

An ideal I of $\boldsymbol{O}$ is said to be *semiprime* if $\mathrm{J}\&\mathrm{J} \subseteq \mathrm{I}$ implies $\mathrm{J} \subseteq \mathrm{I}$. As in the ordinary ring case, it is not difficult to show that I is semiprime iff $\mathrm{J}\&\mathrm{K} \subseteq \mathrm{I}$ implies $\mathrm{J} \cap \mathrm{K} \subseteq \mathrm{I}$. The set of semiprime ideals of $\boldsymbol{O}$ will be denoted by $\mathrm{SIdl}(\boldsymbol{O})$.

Thus, as in [11], $\mathrm{SIdl}(\boldsymbol{O})$ is a locale and we obtain a universal surjective morphism $\mathrm{rad}\colon \mathrm{Idl}(\boldsymbol{O}) \twoheadrightarrow \mathrm{SIdl}(\boldsymbol{O})$ onto a locale. This map is given by

$$\mathrm{rad}(\mathrm{I}) = \cap\{\mathrm{J} \in \mathrm{SIdl}(\boldsymbol{O}) \mid \mathrm{I} \subseteq \mathrm{J}\}.$$

Note that SIdl($\boldsymbol{O}$) is closed under intersections and so the meet of two semiprime ideals is just their intersection. Thus, since rad preserves the multiplicative structure of the quantales, it follows that

$$\mathrm{rad}(\mathrm{I}\&\mathrm{J}) = \mathrm{rad}(\mathrm{I}) \cap \mathrm{rad}(\mathrm{J}),$$

for all ideals I and J.

Before considering the relationship between SIdl($\boldsymbol{O}$) and the internal locale of radical ideals of $\boldsymbol{O}$, we introduce some notation. If $\mathrm{I} \in \mathrm{Idl}(\boldsymbol{O})$ and $r \in \boldsymbol{O}(a)$, for some $a \in \mathrm{L}$, then

$$[\![r \in \mathrm{I}]\!] = \sup \{ b \leq a \mid r|_b \in \mathrm{I}(b) \}.$$

Since I is a sheaf, one easily shows that if $b \leq a$, then $b \leq [\![r \in \mathrm{I}]\!]$ iff $r|_b \in \mathrm{I}(b)$.

PROPOSITION 2.1. SIdl($\boldsymbol{O}$) *is a locale over* L, *via the map* p: SIdl($\boldsymbol{O}$) $\rightarrow$ L *whose inverse and direct images are given respectively by:*

$$p^*(a) = \mathrm{rad}(\tilde{\boldsymbol{O}}1_a) \text{ and } p_*(\mathrm{I}) = [\![1 \in \mathrm{I}]\!].$$

PROOF. Clearly, p^* preserves τ. For binary meets, applying Proposition 1.3 and a remark above, we get

$$\begin{aligned} p^*(a \wedge b) &= \mathrm{rad}(\tilde{\boldsymbol{O}}1_{a\wedge b}) = \mathrm{rad}(\tilde{\boldsymbol{O}}1_a \& \tilde{\boldsymbol{O}}1_b) \\ &= \mathrm{rad}(\tilde{\boldsymbol{O}}1_a) \cap \mathrm{rad}(\tilde{\boldsymbol{O}}1_b) = p^*(a) \wedge p^*(b). \end{aligned}$$

A straightforward calculation shows that p^* is left adjoint to p_*. ∎

Now we give a characterization of semiprime ideals of $\boldsymbol{O}$. Recall that an ideal I of $\boldsymbol{O}$ is *internally radical* if $r^2 \in \mathrm{I} \Rightarrow r \in \mathrm{I}$ holds in Sh(L), i.e., if $r^2 \in \mathrm{I}(a)$, for some a, then there is a cover $\{a_i\}$ of a such that $r|_{a_i} \in \mathrm{I}(a_i)$ for all i. Since I is a sheaf, it easily follows that $r \in \mathrm{I}(a)$. Thus, I is internally radical iff I(a) is a radical ideal of $\boldsymbol{O}(a)$, for all $a \in \mathrm{L}$.

PROPOSITION 2.2. *The following are equivalent for an ideal* I *of* $\boldsymbol{O}$:

(1) I *is a semiprime ideal of* $\boldsymbol{O}$.

(2) I(a) *is a radical ideal of* $\boldsymbol{O}(a)$ *for all* $a \in$ L.

(3) I *is internally radical.*

PROOF. By the above remark, it suffices to show that I is semiprime iff I(a) is a radical ideal of $\boldsymbol{O}(a)$, for all $a \in \mathrm{L}$.

Suppose I is semiprime and $r^2 \in \mathrm{I}(a)$, for some $a \in \mathrm{L}$. Then

$\tilde{O}r^2 \subseteq \mathrm{I}$ and so $\tilde{O}r \& \tilde{O}r \subseteq \mathrm{I}$, by Proposition 1.3. Thus, $\tilde{O}r \subseteq \mathrm{I}$, since I is semiprime, and hence $r \in \mathrm{I}(a)$. Therefore, $\mathrm{I}(a)$ is a radical ideal of $\boldsymbol{O}(a)$.

Conversely, suppose that $\mathrm{I}(a)$ is radical, for all a, and that $\mathrm{J}\&\mathrm{J} \subseteq \mathrm{I}$. Then, for each a, we have

$$\mathrm{J}(a)\mathrm{J}(a) \subseteq (\mathrm{J}\&\mathrm{J})(a) \subseteq \mathrm{I}(a), \text{ and so } \mathrm{J}(a) \subseteq \mathrm{I}(a),$$

since I is radical. Therefore, I is semiprime. ∎

COROLLARY 2.3. *The locale morphism* p: $\mathrm{SIdl}(\boldsymbol{O}) \to \mathrm{L}$ *is the externalization of the internal locale in* Sh(L) *of radical ideals of* $\boldsymbol{O}$.

PROOF. By Proposition 2.2, $\mathrm{SIdl}(\boldsymbol{O})$ is the locale of global sections of the internal locale $\mathrm{RIdl}(\boldsymbol{O})$ of radical ideals of $\boldsymbol{O}$. It is not difficult to show that p arises from the unique morphism $\mathrm{RIdl}(\boldsymbol{O}) \to \Omega$ of internal locales in Sh(L). ∎

We can use Proposition 2.2 to obtain a description of the morphism $\mathrm{rad}\colon \mathrm{Idl}(\boldsymbol{O}) \twoheadrightarrow \mathrm{SIdl}(\boldsymbol{O})$. As in the ordinary ring case, we can relate this map to the usual "radical" of an ideal.

PROPOSITION 2.4. *The ideal* rad(I) *is the sheafification of* $\sqrt{\mathrm{I}}$, *where*

$$\sqrt{\mathrm{I}}(a) = \sqrt{\mathrm{I}(a)} = \{r \in \boldsymbol{O}(a) \mid r^n \in \mathrm{I}(a), \text{ for some } n\},$$

for all $a \in \mathrm{L}$.

PROOF. Let K denote the sheafification of $\sqrt{\mathrm{I}}$. First, we show that K is semiprime. By Proposition 2.2, it suffices to show that $\mathrm{K}(a)$ is a radical ideal of $\boldsymbol{O}(a)$, for all $a \in \mathrm{L}$. If $r^n \in \mathrm{K}(a)$, then there is a cover $\{a_i\}$ of a such that

$$r^n|_{a_i} \in \sqrt{\mathrm{I}(a_i)} \quad \text{for all } i.$$

Since $r^n|_{a_i} = (r|_{a_i})^n$, it follows that $r|_{a_i} \in \sqrt{\mathrm{I}(a_i)}$, for all i, and so $r \in \mathrm{K}(a)$. Next, we show that if $\mathrm{I} \subseteq \mathrm{J}$ and J is semiprime, then $\mathrm{K} \subseteq \mathrm{J}$. Since $\mathrm{J}(a)$ is a radical and $\mathrm{I}(a) \subseteq \mathrm{J}(a)$, for all a, we have

$$\sqrt{\mathrm{I}(a)} \subset \mathrm{J}(a), \text{ for all } a.$$

But, then $\sqrt{\mathrm{I}} \subseteq \mathrm{J}$ and J is a sheaf, and so $\mathrm{K} \subseteq \mathrm{J}$, as desired. ∎

In Section 1, we noted that if $\boldsymbol{B}$ generates L, then $\mathrm{Idl}(\boldsymbol{O})$ is generated by

$$\{\tilde{O}r \mid r \in \boldsymbol{O}(b), \text{ for some } b \in \boldsymbol{B}\}.$$

Since rad is a sup-preserving surjection, it follows that

$$\{\mathrm{rad}(\tilde{O}r \mid r \in \boldsymbol{O}(b), \text{ for some } b \in \boldsymbol{B}\}$$

is a generating set for $\mathrm{SIdl}(\boldsymbol{O})$.

We conclude this section with an example which we will use in the next section. In [1], Banaschewski and Bhutani give an example of a sheaf of Boolean algebras (on a locale L) the lattice of ideals of which is isomorphic to L. Since every Boolean algebra can be viewed as a Boolean ring, this construction is relevant to the present setting.

Recall that if $a \in \mathrm{L}$, then the *down-segment* of a is the set $\downarrow(a) = \{b \in \mathrm{L} \mid b \leq a\}$. It is not difficult to see that $\downarrow(a)$ is a locale.

EXAMPLE 2.5. Let I_{L} denote the sheaf of rings on L defined as follows. If $a \in \mathrm{L}$, then

$$I_{\mathrm{L}}(a) = \{b \in \mathrm{L} \mid b \text{ is complemented in } \downarrow(a)\}.$$

Then, as in [1], I_{L} is a sheaf of Boolean algebras whose ideal lattice is isomorphic to L. Since every ideal of a Boolean ring is semiprime and the algebra ideals agree with the ring ideals, it follows that $\mathrm{Idl}(I_{\mathrm{L}}) = \mathrm{SIdl}(I_{\mathrm{L}}) \simeq \mathrm{L}$, as locales. Briefly, the isomorphism is obtained showing that every ideal I of I_{L} is of the form

$$\mathrm{I}(a) = \begin{cases} I_{\mathrm{L}}(a) & \text{if } a \leq [\![1 \in \mathrm{I}]\!] \\ 0 & \text{if } a \not\leq [\![1 \in \mathrm{I}]\!] \end{cases}.$$

Example 2.5 says that every locale can be expressed as the spectrum of a ringed locale, i.e., a locale L together with a sheaf of rings on L. This relates to Hochster's characterization [6] of spectral spaces. However, in [6], he also showed that the functor Spec from the category of commutative rings with unit to the category of topological spaces cannot be inverted functorially. This is not the case in the present situation. The assignment $\mathrm{L} \mapsto (\mathrm{L}, I_{\mathrm{L}})$ is easily seen to define a functor from the category **Loc** of locales to the category **RLoc** of ringed locales, whose morphisms $(\mathrm{L},\boldsymbol{O}) \to (\mathrm{L}',\boldsymbol{O}')$ are pairs (f,φ), where $f\colon \mathrm{L}\to\mathrm{L}'$ is a morphism of locales and $\varphi\colon \boldsymbol{O}' \to f_*(\boldsymbol{O})$ is a homomorphism. Moreover, using the isomorphism $\mathrm{L} \simeq \mathrm{SIdl}(I_{\mathrm{L}})$, we see that this functor provides a right (pseudo)inverse to $\mathrm{SIdl}\colon$ **RLoc** $\to$ **Loc**. In some sense, Hochster could not invert Spec functorially because the category of rings is too small.

3. PRIME IDEALS OF A SHEAF OF RINGS.

In this section, we consider some properties of the locale

SIdl($\boldsymbol{O}$). In particular, we consider conditions under which this locale is spatial.

Recall that an element p of a locale L is *prime* if $p \neq \tau$ and $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. This is the usual definition of a prime element of a lattice. If L is the locale $\Omega(X)$ of open subsets of a topological space X, then it is not difficult to show that an open set U is prime iff X\U is an irreducible closed subset of X. Moreover, a locale L is *spatial* (i.e., isomorphic to $\Omega(X)$ for some space X) iff for every $a \in L$,

$$a = \inf\{p \in L \mid a \leq p \text{ and } p \text{ is prime}\}.$$

Translating to the present section, we see that SIdl($\boldsymbol{O}$) is spatial iff every semiprime ideal is an intersection of primes, where a semiprime ideal P is prime iff $I \cap J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. Thus, we would like to consider prime elements of the locale SIdl($\boldsymbol{O}$). However, there are several other notions of prime in this context. Since we are concerned with the locale SIdl($\boldsymbol{O}$) here and not the corresponding internal locale, we consider only external notions of prime. For a treatment of internal primes in a localic topos, we refer the reader to [3].

Recall that an ideal P of $\boldsymbol{O}$ is called *prime* if $P \neq \boldsymbol{O}$, and $I \& J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. Since $I \& J \subseteq P \Leftrightarrow I \cap J \subseteq P$, for any semiprime ideal P, it follows that the two external definitions of prime agree.

THEOREM 3.1. *Suppose* $\boldsymbol{B}$ *generates* L *and* P *is an ideal of* $\boldsymbol{O}$. *Then* P *is prime iff the following conditions hold:*

(1) $P(a)$ *is a prime ideal of* $\boldsymbol{O}(a)$ *or* $P(a) = \boldsymbol{O}(a)$ *for all* $a \in \boldsymbol{B}$

(2) If $a, b \in \boldsymbol{B}$, $P(a) \neq \boldsymbol{O}(a)$, $a \leq b$, $r \in \boldsymbol{O}(b)$, *and* $r|_a \in P(a)$, *then* $r \in P(b)$.

(3) $[\![1 \in P]\!]$ *is a prime element of* L.

PROOF. Suppose P is a prime ideal of $\boldsymbol{O}$. If $a \in \boldsymbol{B}$ and $P(a) \neq \boldsymbol{O}(a)$, then to see that $P(a)$ is prime, let $r, s \in \boldsymbol{O}(a)$ and $rs \in P(a)$. Then $\tilde{\boldsymbol{O}}rs \subseteq P$ and $\tilde{\boldsymbol{O}}rs = \tilde{\boldsymbol{O}}r \& \tilde{\boldsymbol{O}}s$ by Proposition 1.3. Thus, $\tilde{\boldsymbol{O}}r \& \tilde{\boldsymbol{O}}s \subseteq P$, and so $\tilde{\boldsymbol{O}}r \subseteq P$ or $\tilde{\boldsymbol{O}}s \subseteq P$. Hence, $r \in P(a)$ or $s \in P(a)$, as desired. For (2), suppose $a, b \in \boldsymbol{B}$, $a \leq b$, and $P(a) \neq \boldsymbol{O}(a)$. Assume $r \in \boldsymbol{O}(b)$ and $r|_a \in P(a)$. Since $\tilde{\boldsymbol{O}}r \& \tilde{\boldsymbol{O}}1_a = \tilde{\boldsymbol{O}}r|_a$, by Proposition 1.3, it follows that $\tilde{\boldsymbol{O}}r \& \tilde{\boldsymbol{O}}1_a \subseteq P(a)$. Thus, $\tilde{\boldsymbol{O}}r \subseteq P$ or $\tilde{\boldsymbol{O}}1_a \subseteq P$, since P is prime. But, $1_a \notin P(a)$, since $P(a) \neq \boldsymbol{O}(a)$, and so $\tilde{\boldsymbol{O}}1_a \not\subseteq P$. Thus, $\tilde{\boldsymbol{O}}r \subseteq P$ and we have $r \in P(b)$, as desired. For [3], suppose

$$c = a \wedge b \leq [\![1 \in P]\!].$$

Then we can cover c with $\{c_i\} \subseteq \boldsymbol{B}$ such that $1_{c_i} \in P(c_i)$. Since P is a sheaf, it follows that $1_c \in P(c)$. Thus,

$$\tilde{\boldsymbol{O}}1_a \& \tilde{\boldsymbol{O}}1_b = \tilde{\boldsymbol{O}}1_c \subseteq \mathrm{P}(c)$$

and it follows that $\tilde{\boldsymbol{O}}1_a \subseteq \mathrm{P}$ or $\tilde{\boldsymbol{O}}1_b \subseteq \mathrm{P}$, i.e., $1_a \in \mathrm{P}(a)$ or $1_b \in \mathrm{P}(b)$. Clearly, $[\![1 \in \mathrm{P}]\!] \neq \tau$, since $\mathrm{P} \neq \boldsymbol{O}$. Therefore, $[\![1 \in \mathrm{P}]\!]$ is a prime element of L.

Conversely, suppose P satisfies (1), (2), (3). Since $\tau \neq [\![1 \in \mathrm{P}]\!]$, it follows that $\mathrm{P} \neq \boldsymbol{O}$. Suppose $\mathrm{I} \& \mathrm{J} \subseteq \mathrm{P}$ and $\mathrm{I} \not\subseteq \mathrm{P}$. Then there exists $r \in \mathrm{I}(a) \backslash \mathrm{P}(a)$ for some $a \in \boldsymbol{B}$. Note that $a \not\leq [\![1 \in \mathrm{P}]\!]$, since $\mathrm{P}(a) \neq \boldsymbol{O}(a)$. To show that $\mathrm{J} \subseteq \mathrm{P}$, let $b \in \boldsymbol{B}$ and suppose $s \in \mathrm{J}(b)$. If $\mathrm{P}(b) = \boldsymbol{O}(b)$, then $s \in \mathrm{P}(b)$. So, assume $\mathrm{P}(b) \neq \boldsymbol{O}(b)$, and let $c = a \wedge b$. Since

$$a \not\leq [\![1 \in \mathrm{P}]\!] \text{ and } b \not\leq [\![1 \in \mathrm{P}]\!],$$

applying (3), we know that $c \not\leq [\![1 \in \mathrm{P}]\!]$, and so $\mathrm{P}(d) \neq \boldsymbol{O}(d)$, for some $d \in \boldsymbol{B}$ such that $d \leq c$. Since $r \notin \mathrm{P}(a)$ and $d \leq a$, using (2), we get that $r|_d \notin \mathrm{P}(d)$. But, $r|_d s|_d \in \mathrm{I}(d)\mathrm{J}(d) \subseteq \mathrm{P}(d)$ and $\mathrm{P}(d)$ is prime. Thus, $s|_d \in \mathrm{P}(d)$. Since $d \leq b$, we can apply (2) again to conclude that $s \in \mathrm{P}(b)$. Thus, $\mathrm{J}(b) \subseteq \mathrm{P}(b)$. Therefore, P is a prime ideal of $\boldsymbol{O}$. ∎

Note that condition (2) could have been equivalently stated as: if $a \leq b$ and $\mathrm{P}(a) \neq \boldsymbol{O}(a)$, then $\mathrm{P}(b)$ is the inverse image of $\mathrm{P}(a)$ under the restriction map $\boldsymbol{O}(b) \to \boldsymbol{O}(a)$. Condition (3) comes from the work of Borceux, Pedicchio and Rossi [2] on sheaves of Boolean algebras.

Next we turn to the relationship between prime and semiprime ideals of $\boldsymbol{O}$. Of course, every prime ideal is semiprime. As in the ordinary ring case, we have the following result.

LEMMA 3.2. *If* I *is a semiprime ideal of a sheaf of rings on a spatial locale* L, $a \in \mathrm{L}$, *and* $r \in \boldsymbol{O}(a) \backslash \mathrm{I}(a)$, *then* $r \notin \mathrm{P}(a)$, *for some prime ideal* P *containing* I.

PROOF. Suppose I is semiprime and $r \in \boldsymbol{O}(a) \backslash \mathrm{I}(a)$. Then $a \not\leq [\![r \in \mathrm{I}]\!]$, and so $a \not\leq p$, for some prime $p \in \mathrm{L}$ such that $[\![r \in \mathrm{I}]\!] \leq p$. We will use Zorn's Lemma to obtain the desired prime ideal P.

Consider the set $\boldsymbol{S}$ of ideals J of $\boldsymbol{O}$ such that $\mathrm{I} \subseteq \mathrm{J}$ and $[\![r^n \in \mathrm{J}]\!] \leq p$ for all n. Since I is semiprime, we know that $\mathrm{I}(b)$ is a radical ideal of $\boldsymbol{O}(b)$, for all $b \in \boldsymbol{L}$. Thus,

$$r|_b \in \mathrm{I}(b) \text{ iff } r^n|_b = (r|_b)^n \in \mathrm{I}(b).$$

Hence, $[\![r \in \mathrm{I}]\!] = [\![r^n \in \mathrm{I}]\!]$, for all n, and so $\mathrm{I} \in \boldsymbol{S}$. If $\mathrm{J}_1 \subseteq \mathrm{J}_2 \subseteq \cdots$ is a chain in $\boldsymbol{S}$, consider $\mathrm{J} = \tilde{\Sigma}\mathrm{J}_i$. Note that J is the sheafification of $\cup \mathrm{J}_i$. Clearly, $\mathrm{I} \subseteq \mathrm{J}$. If $\mathrm{J} \notin \boldsymbol{S}$, then $[\![r^n \in \mathrm{J}]\!] \not\leq p$, for some n. Since J is the sheafification of $\cup \mathrm{J}_i$, there is a cover $\{b_j\}$ of

$[\![r^n \in \mathrm{J}]\!]$ such that

$$r^n|_{b_j} \in \cup \mathrm{J}_i(bj) \text{ for all } j.$$

But, $[\![r^n \in \mathrm{J}]\!] \not\leq p$ implies that $b_j \not\leq p$, for some j. Since $r^n|_{b_j} \in \cup \mathrm{J}_i(b_j)$, we know that $r^n|_{b_j} \in \mathrm{J}_i(b_j)$ for some i. Thus, $b_j \leq [\![r^n \in \mathrm{J}_i]\!]$, and since $\mathrm{J}_i \in \boldsymbol{S}$, $[\![r^n \in \mathrm{J}_i]\!] \leq p$, contradicting the fact that $b_j \not\leq p$. Hence $\mathrm{J} \in \boldsymbol{S}$, as desired. Therefore, $\boldsymbol{S}$ has a maximal element, call it P. To see that P is prime, suppose that $\mathrm{J}\&\mathrm{K} \subseteq \mathrm{P}$, $\mathrm{J} \subseteq \mathrm{P}$ and $\mathrm{K} \not\subseteq \mathrm{P}$. Let J' and K' denote the sheafification of J+P and K+P, respectively. Then $\mathrm{J}' \not\in \boldsymbol{S}$ and $\mathrm{K}' \not\in \boldsymbol{S}$, since P is maximal. Thus, there exist m and n such that

$$[\![r^m \in \mathrm{J}']\!] \not\leq p \text{ and } [\![r^n \in \mathrm{K}']\!] \not\leq p.$$

Since p is prime, we see that

$$d = [\![r^m \in \mathrm{J}']\!] \wedge [\![r^n \in \mathrm{K}']\!] \not\leq p.$$

However,

$$r^{m+n}|_d = r^m|_d\, r^n|_d \in (\mathrm{J}'\&\mathrm{K}')(d) \text{ and } \mathrm{J}'\&\mathrm{K}' \subseteq \mathrm{P}$$

since $\mathrm{J}\&\mathrm{K} \subseteq \mathrm{P}$. Thus, $r^{m+n}|_d \in \mathrm{P}(d)$, and since $\mathrm{P} \in \boldsymbol{S}$, it follows that $d \leq [\![r^{m+n} \in \mathrm{P}]\!] \leq p$, contradicting the fact that $d \not\leq p$. Therefore, P is a prime ideal of $\boldsymbol{O}$, $\mathrm{I} \subseteq \mathrm{P}$, and $[\![r^n \in \mathrm{P}]\!] \leq p$ for all n.

It remains to show that $r \not\in \mathrm{P}(a)$. If $r \in \mathrm{P}(a)$, then $a \leq [\![r \in \mathrm{P}]\!] \leq p$, contradicting the fact that $a \not\leq p$. This completes the proof. ∎

PROPOSITION 3.3. *If $\boldsymbol{O}$ is a sheaf of rings on a spatial locale then an ideal* I *of $\boldsymbol{O}$ is semiprime iff* I *is an intersection of prime ideals of $\boldsymbol{O}$.*

PROOF. Clearly, any intersection of prime ideals is semiprime. The converse follows easily from the above lemma. ∎

Combining this proposition with the description of the universal surjective quantale map rad: $\mathrm{Idl}(\boldsymbol{O}) \twoheadrightarrow \mathrm{SIdl}(\boldsymbol{O})$ given at the beginning of Section 2, we obtain the following corollary.

COROLLARY 3.4. *If $\boldsymbol{O}$ is a sheaf of rings on a spatial locale* L *and* I *is an ideal of $\boldsymbol{O}$, then*

$$\mathrm{rad}(\mathrm{I}) = \cap\{\mathrm{P} \in \mathrm{Idl}(\boldsymbol{O}) \mid \mathrm{I} \subseteq \mathrm{P},\ \mathrm{P} \text{ prime}\}.$$

Next, we use Proposition 3.3 to consider spatial locales.

THEOREM 3.5. *Let* L *be a locale. Then* $\mathrm{SIdl}(\boldsymbol{O})$ *is spatial, for all sheaves of rings $\boldsymbol{O}$ on* L, *iff* L *is spatial.*

PROOF. If L is spatial and $\mathcal{O}$ is a sheaf of rings on L, then SIdl($\mathcal{O}$) is spatial by Proposition 3.3. Conversely, if SIdl($\mathcal{O}$) is spatial, for all sheaves $\mathcal{O}$, then taking $\mathcal{O}$ to be the sheaf $\mathcal{I}_L$, defined in Example 2.5, we see that L is spatial, since $L \simeq \mathrm{SIdl}(\mathcal{I}_L)$. ∎

We conclude this section with a characterization of maximal ideals of $\mathcal{O}$ analogous to Theorem 3.1. Although we will not be considering maximal ideals in the remainder of this article, we include the following theorem for completeness.

THEOREM 3.6. *Let* M *be an ideal of a sheaf of rings* $\mathcal{O}$ *on a locale* L. *Then* M *is maximal iff the following conditions hold:*

(1) If $r \in \mathcal{O}(a)\backslash M(a)$ *for some* $a \in L$, *then there is a cover* $\{a_i\}$ *of* a *such that* $M(a_i)+\mathcal{O}(a_i)r|_{a_i} = \mathcal{O}(a_i)$ *for all* i.

(2) $[\![1 \in M]\!]$ *is a maximal element of* L.

PROOF. Suppose M is a maximal ideal of $\mathcal{O}$ and $r \in \mathcal{O}(a)\backslash M(a)$, for some $a \in L$. Let I denote the sheafification of $M+\mathcal{O}r$. Since $M \subseteq I$ and M is maximal, it follows that $I = \mathcal{O}$. Thus, $1_a \in I(a)$ and so there exists a cover $\{a_i\}$ of a with $1_{a_i} \in M(a_i)+\mathcal{O}(a_i)r|_{a_i}$, for all i, proving (1).

For (2), suppose $[\![1 \in M]\!] \leq b$ and $b \neq \tau$, for some $b \in L$. Let I denote the sheafification of $M+\mathcal{O}1_b$. We claim that $I \neq \mathcal{O}$. Suppose $I = \mathcal{O}$, then $1 \in I(\tau)$, and so there is a cover $\{t_j\}$ of τ such that $1_{t_j} \in M(t_j)+\mathcal{O}1_b(t_j)$. Then $t_j \leq b$ for all j, for if $t_j \not\leq b$ for some j, we would have

$$\mathcal{O}1_b(t_j) = 0, \; 1_{t_j} \in M(t_j)$$

(since $[\![1 \in M]\!] \leq b$), and

$$1_{t_j} \in M(t_j)+\mathcal{O}1_b(t_j).$$

But, then $\tau = \sup(t_j) \leq b$, contrary to the assumption that $b \neq \tau$. Since $M \subseteq I$ and M is maximal, it follows that $M = I$, and so $1_b \in M(b)$. Thus, $b = [\![1 \in M]\!]$, i.e., $[\![1 \in M]\!]$ is maximal.

Conversely, suppose that M satisfies (1) and (2) and $M \subsetneq I$, where I is an ideal of $\mathcal{O}$. Then $[\![1 \in M]\!] \leq [\![1 \in I]\!]$. We claim that

$$[\![1 \in M]\!] \neq [\![1 \in I]\!].$$

Suppose $[\![1 \in M]\!] = [\![1 \in I]\!]$. Since $M \subseteq I$, there is an a in L with $M(a) \neq I(a)$. Choose $r \in I(a)\backslash M(a)$. By (1), there is a cover $\{a_i\}$ of a with $M(a_i)+\mathcal{O}(a_i)r|_{a_i} = \mathcal{O}(a_i)$ for all i. But. $M(a_i) \subseteq I(a_i)$ and $r|_{a_i} \in I(a_i)$, since $r \in I(a)$, and hence

$$\mathcal{O}(a_i) = M(a_i)+\mathcal{O}(a_i)r|_{a_i} \subseteq I(a_i), \text{ for all } i.$$

Thus, $a_i \leq [\![1 \in I]\!]$, for all i, and so

$$a = \sup(a_i) \leq [\![1 \in \mathrm{M}]\!] = [\![1 \in \mathrm{I}]\!],$$

contradicting the fact that $\mathrm{M}(a) \neq \boldsymbol{O}(a)$. Hence, $[\![1 \in \mathrm{M}]\!] \neq [\![1 \in \mathrm{I}]\!]$. Since $[\![1 \in \mathrm{M}]\!]$ is maximal, by (2), it follows $[\![1 \in \mathrm{I}]\!] = \tau$, and so we have $\mathrm{I} = \boldsymbol{O}$. Therefore, M is maximal. ∎

4. THE ITERATED SPECTRUM OF A RING.

In this section, we consider the Zariski spectrum of a ring. Since the locale (or topos) in question is spatial and our construction preserves spatial locales, we shall consider this spectrum as a ringed space, rather than a ringed locale or topos. In particular, we give an explicit description of the spectrum of this ringed space, showing that it satisfies the appropriate universal property for ringed spaces.

Let R be a commutative ring with unit and consider the structure sheaf $\boldsymbol{Z}$ of the Zariski spectrum Spec(R). Recall that the points of Spec(R) are prime ideals of R and basic opens are sets of the form

$$\mathrm{D}(f) = \{\mathbf{p} \in \mathrm{Spec(R)} \mid f \not\in \mathbf{p}\}. \text{ where } f \in \mathrm{R}.$$

Moreover, $\boldsymbol{Z}(\mathrm{D}(f)) = \mathrm{R}[f^{-1}]$. We shall identify the locale of open subsets of Spec(R) with the locale RIdl(R) of radical ideals of R.

Consider the locale SIdl($\boldsymbol{Z}$) of semiprime ideals of $\boldsymbol{Z}$. We know that SIdl($\boldsymbol{Z}$) is generated by the ideals of the form $\mathrm{rad}(\tilde{\boldsymbol{Z}}r)$, where $r \in \boldsymbol{Z}(\mathrm{D}(f)) = \mathrm{R}[f^{-1}]$. Of course, we need only consider those r which are elements of R. We also know that SIdl($\boldsymbol{Z}$) is spatial and its points are prime ideals of $\boldsymbol{Z}$.

We claim that prime ideals of $\boldsymbol{Z}$ correspond to pairs $\mathbf{p} \subseteq \mathbf{q}$ of prime ideals of R. First, if P is a prime ideal of $\boldsymbol{Z}$, let

$$\mathbf{p} = \mathrm{P}((1)), \text{ and } \mathbf{q} = [\![1 \in \mathrm{P}]\!] = \sup\{\mathrm{D}(f) \mid \mathrm{P}(\mathrm{D}(f)) = \mathrm{R}(f^{-1})\},$$

considered as radical ideals of R. Then, by Theorem 3.1, P(D(1)) and $[\![1 \in \mathrm{P}]\!]$ are prime elements of RIdl(R), i.e., $\mathbf{p}$ and $\mathbf{q}$ are prime ideals of R. Note that $f \in \mathbf{q}$ iff $\mathrm{P}(\mathrm{D}(f)) = \mathrm{R}[f^{-1}]$. Clearly, $\mathbf{p} \subseteq \mathbf{q}$ for if $f \in \mathbf{p}$, then $f|_{\mathrm{D}(f)} \in \mathrm{P}(\mathrm{D}(f))$, and so $\mathrm{P}(\mathrm{D}(f)) = \mathrm{R}[f^{-1}]$. Next, we show that

$$(*) \qquad \mathrm{P}(\mathrm{D}(f) = \begin{cases} \mathrm{R}[f^{-1}] & \text{if } f \in \mathbf{q} \\ \mathbf{p}[f^{-1}] & \text{if } f \not\in \mathbf{q} \end{cases}$$

If $f \in \mathbf{q}$, we know that $\mathrm{P}(\mathrm{D}(f)) = \mathrm{R}[f^{-1}]$. Suppose that $f \not\in \mathbf{q}$. Then $\mathbf{p}[f^{-1}] \subseteq \mathrm{P}(\mathrm{D}(f))$, since $a \in \mathbf{p}$ implies $a|_{\mathrm{D}(f)} \in \mathrm{P}(\mathrm{D}(f))$. Now if $a/f^n \in \mathrm{P}(\mathrm{D}(f))$, then

$$a|_{D(f)} \in P(D(f)) \text{ or } 1/f^n \in P(D(f)),$$

since P(D(f)) is prime in R[f^{-1}]. But $P(D(f)) \neq R[f^{-1}]$ since $f \not\in \mathfrak{q}$, and so $1/f^n \not\in P(D(f))$. Thus, $a|_{D(f)} \in P(D(f))$. Applying Theorem 3.1 (2), we see that $a \in P(D(f)) = \mathfrak{p}$. Therefore, $P(D(f)) \subseteq \mathfrak{p}[f^{-1}]$, as desired.

It remains to show that every pair $\mathfrak{p} \subseteq \mathfrak{q}$ of prime ideals of R gives rise to a prime ideal of $\boldsymbol{Z}$. Given such a pair, we define P(D(f)) as in (*). Since P clearly satisfies the conditions of Theorem 3.1, it suffices to show that P is a sheaf. Since $\boldsymbol{Z}$ is a sheaf, we need only show that if

$$D(f) = \sup\{D(f_i)\},\ f \not\in \mathfrak{q}, \text{ and } (a/f^n)|_{D(f_i)} \in \mathfrak{p}[f_i^{-1}] \text{ for all } i,$$

then $a/f^n \in \mathfrak{p}[f^{-1}]$. Since $f \not\in \mathfrak{q}$ and $D(f) = \sup\{D(f_i)\}$, it follows that $f_i \not\in \mathfrak{q}$, for some i. Clearly, $f_i \not\in \mathfrak{p}$, since $\mathfrak{p} \subseteq \mathfrak{q}$. Writing $f_i^m = r_i f$, for some $m > 0$ and $r_i \in R$, we see that

$$a r_i^n / f_i^{mn} = (a/f^n)|_{D(f_i)} \in \mathfrak{p}[f_i],$$

and so $a r_i^n \in \mathfrak{p}$, since $f_i \not\in \mathfrak{p}$. But, $r_i \not\in \mathfrak{p}$, since $r_i f = f_i^m$ and $f_i^m \not\in \mathfrak{p}$. Thus, $a \in \mathfrak{p}$, and it follows that $a/f^n \in \mathfrak{p}[f^{-1}]$, as desired.

Thus, we obtain a space Spec($\boldsymbol{Z}$) whose points are pairs $\mathfrak{p} \subseteq \mathfrak{q}$ of prime ideals of R. Since SIdl($\boldsymbol{Z}$) is generated by the radicals of certain principal ideals, as remarked above, we see that Spec($\boldsymbol{Z}$) has a base consisting of open sets of the form

$$D(r,f) = \{\mathfrak{p} \subseteq \mathfrak{q} \mid r \not\in \mathfrak{p} \text{ and } f \not\in \mathfrak{q}\},$$

where $r, f \in R$. Consider the sheaf of rings $\boldsymbol{Z}'$ defined on Spec($\boldsymbol{Z}$) as follows. Let $\boldsymbol{Z}'(D(r,f)) = R[r^{-1}, f^{-1}]$. If $D(r,f) = D(s,g)$, then it is not difficult to show that $D(rf) = D(sg)$ in Spec(R), and so $R[(rf)^{-1}]$ is canonically isomorphic to $R[(sg)^{-1}]$. Thus, $R[r^{-1}, f^{-1}]$ is canonically isomorphic to $R[s^{-1}, g^{-1}]$, and as in the ordinary ring case we get a sheaf of rings on Spec($\boldsymbol{Z}$). Moreover, it is not difficult to show that the stalk of $\boldsymbol{Z}'$ over a point $\mathfrak{p} \subseteq \mathfrak{q}$ is isomorphic to the localization $R_{\mathfrak{p}}$ of R at $\mathfrak{p}$. Therefore, (Spec($\boldsymbol{Z}$), $\boldsymbol{Z}'$) is a local ringed space.

Recall that a *morphism* $(X, \boldsymbol{O}) \to (X', \boldsymbol{O}')$ *of ringed spaces* is a pair (h, φ), where $h: X \to X'$ is a continuous map and $\varphi: \boldsymbol{O}' \to h_*(\boldsymbol{O})$ is a homomorphism. Such a pair is called a *morphism of local ringed spaces* if the stalks of the ringed spaces are local rings and the inverse images of the homomorphisms on the stalks, induced by φ, preserve the maximal ideals. Let **RSp** and **LRSp** denote the categories of ringed spaces and local ringed spaces, respectively.

One way to define the spectrum of a ringed space is as the right adjoint to the inclusion of **RSp** in **LRSp** [5]. Thus, the spectrum of $(\mathrm{Spec}(R), \boldsymbol{Z})$ is a local ringed space $(X, \boldsymbol{O})$ together with a morphism $(X,\boldsymbol{O}) \to (\mathrm{Spec}(R), \boldsymbol{Z})$ of ringed spaces such that any other morphism from a local ringed space into $(\mathrm{Spec}(R), \boldsymbol{Z})$ factors uniquely through $(X, \boldsymbol{O})$ via a morphism of local ringed spaces.

PROPOSITION 4.1. $(\mathrm{Spec}(\boldsymbol{Z}), \boldsymbol{Z}')$ *is the spectrum of* $(\mathrm{Spec}(R), \boldsymbol{Z})$.

PROOF. Define $p: \mathrm{Spec}(\boldsymbol{Z}) \to \mathrm{Spec}(R)$ by $p(\mathfrak{p} \subseteq \mathfrak{q}) = \mathfrak{q}$. Then $p^{-1}(D(f)) = D(f,f)$, and so p is continuous. Let $\psi: \boldsymbol{Z} \to p_*(\boldsymbol{Z}')$ be the homomorphism which induces the canonical homomorphism $R_{\mathfrak{q}} \to R_{\mathfrak{p}}$ on the stalks corresponding to the points $\mathfrak{p} \subseteq \mathfrak{q}$ and $p(\mathfrak{p} \subseteq \mathfrak{q}) = \mathfrak{q}$.

For the universal property, suppose $(X, \boldsymbol{O})$ is a local ringed space and

$$(h,\varphi): (X, \boldsymbol{O}) \to (\mathrm{Spec}(R), \boldsymbol{Z})$$

is a morphism. First, we define $\bar{h}: X \to \mathrm{Spec}(\boldsymbol{Z})$. Given $x \in X$, let $\mathfrak{q} = h(x)$ and $\mathfrak{p} = \varphi_x^{-1}(\mathfrak{m}_x) \cap R$, where $\varphi_x: R_{\mathfrak{q}} \to \boldsymbol{O}_x$ is the map induced on the stalks and $\mathfrak{m}_x$ is the unique maximal ideal of $\boldsymbol{O}_x$. Consider $\bar{h}(x)$ to be the point $\mathfrak{p} \subseteq \mathfrak{q}$ of $\mathrm{Spec}(\boldsymbol{Z})$. Clearly, $p \circ \bar{h} = h$. To see that $\bar{h}$ is continuous, let $D(r,f)$ be a basic open of $\mathrm{Spec}(\boldsymbol{Z})$. Then

$$\bar{h}^{-1}(D(r,f)) = h^{-1}(D(f)) \cap \{x \in X \mid \varphi_x(r) \not\in \mathfrak{m}\},$$

and it is not difficult to show that the latter set is open in X. Next, we define $\bar{\varphi}: \boldsymbol{Z}' \to \bar{h}_*(\boldsymbol{O})$ to be the homomorphism such that the corresponding map on stalks is defined as follows. If $x \in X$, then the map $\varphi_x: R_{\mathfrak{q}} \to \boldsymbol{O}_x$ factors through $R_{\mathfrak{p}}$ since $\mathfrak{p} = \varphi_x^{-1}(\mathfrak{m}_x) \cap R$, where $h(x)$ is the point $\mathfrak{p} \subseteq \mathfrak{q}$. Thus, we get a map $\bar{\varphi}_x$: $R_{\mathfrak{p}} \to \boldsymbol{O}_x$ which is clearly a local homomorphism and $(p,\psi) \circ (\bar{h}, \varphi) = (h, \varphi)$, as desired. To complete the proof one checks that $(\bar{h}, \bar{\varphi})$ is the unique such map. ∎

Iterating the above construction, we get $(\mathrm{Spec}_n(R), \boldsymbol{Z}_n)$, the n^{th}-*spectrum* of R. The points of $\mathrm{Spec}_n(R)$ are chains $\mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n$ of prime ideals of R, basic opens are of the form

$$D(f_1, \dots, f_n) = \{\mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n \mid f_i \not\in \mathfrak{p}_i, \text{ for all } i\},$$

where $f_1, \dots, f_n \in R$, and

$$\boldsymbol{Z}_n(D(f_1, \dots, f_n)) = R[(f_1)^{-1}, \dots, (f_n)^{-1}].$$

As above we see that $(\mathrm{Spec}_n(R), \boldsymbol{Z}_n)$ is the spectrum of $(\mathrm{Spec}_{n-1}(R), \boldsymbol{Z}_{n-1})$.

We conclude with a remark about the Pierce spectrum [12] (Specp(R), $\boldsymbol{E}$) of R. It is possible to go through a construction similar to that above and show that the spectrum of (Specp(R), $\boldsymbol{E}$) is the Zariski spectrum of R. However, Chris Mulvey has pointed out that this result is not surprising, for the Zariski spectrum factors through the Pierce spectrum of a ring R, and so the spectra of R and (Specp(R), $\boldsymbol{E}$) can be seen to agree via their universal properties.

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