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GAMUTS AND COFIBRATIONS [1]

by R. ROSEBRUGH and R.J. WOOD

RÉSUMÉ. Soit $()_*$: $\mathcal{K} \to \mathcal{M}$ une donnée de proflèches. Si elle vérifie les deux axiomes suivants d'exactitude:

(i) les collages finis existent et se comportent bien,

(ii) il existe une coalgèbre d'Eilenberg-Moore pour toute comonade idempotente (axiome de faisceaux),

alors les cofibrations de B à A dans $\mathcal{K}$ sont équivalentes aux gamuts de B à A. La composition de cofibrations correspond à une composition de gamuts par collage. Parmi les exemples de $\mathcal{K}$, on indique les bicatégories de topoi, de catégories abéliennes et de catégories à limites finies (avec une notion de morphismes géométriques appropriée dans chaque cas).

1. INTRODUCTION.

Let $\mathcal{K}$ and $\mathcal{M}$ be bicategories. It was first proved by Street in [6] that cofibrations in $\mathcal{K} = \mathcal{V}$-cat are equivalent to certain diagrams, which he called gamuts, in $\mathcal{M} = \mathcal{V}$-mod. Arrows of gamuts (and transformations between these) involve the proarrow equipment [7] $(\ )_*: \mathcal{V}\text{-cat} \to \mathcal{V}\text{-mod}$. The authors in [2] proved the equivalence of cofibrations in

$$\mathcal{K} = \text{(toposes and geometric morphisms)}$$

with gamuts in

$$\mathcal{M} = \text{(toposes and left exact functors)}.$$

Street's definition of gamut makes sense in this context. One uses "forget the inverse image" in the role of $()_*$ above. In [4] the authors obtained the analogous result for

$$\mathcal{K} = \text{(abelian categories and "geometric morphisms")},$$
$$\mathcal{M} = \text{(abelian categories and left exact functors)}.$$

The same arguments even allow the replacement of "abelian categories" by "categories with finite limits".

The characterization of cofibrations in each of the $\mathcal{K} \to \mathcal{M}$ examples above implies that *codiscrete* cofibrations in $\mathcal{K}$ are equivalent to arrows in $\mathcal{M}$. In [5] the authors obtained this

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result, directly, for any proarrow equipment $(\ )_*: \mathcal{K} \to \mathcal{M}$ satisfying an exactness condition for collages which we refer to as Axiom C and recall below.

Here we unify these earlier results by establishing the equivalence of general cofibrations and gamuts in the context of proarrow equipment satisfying the afore mentioned Axiom C and a further Axiom S. The latter requires some comment.

By definition, $(\ )_*: \mathcal{K} \to \mathcal{M}$ being proarrow equipment means that the objects of $\mathcal{M}$ are those of $\mathcal{K}$, $(\ )_*$ is the identity on objects, $(\ )_*$ is locally fully faithful and for every arrow f in $\mathcal{K}$ we have an adjunction $f_* \dashv f^*$ in $\mathcal{M}$. It is convenient to suppress $(\ )_*$ and write $f^{\sim}: 1 \to ff^*$, respectively $f_{\sim}: f^*f \to 1$ for units, respectively counits, in $\mathcal{M}$.

AXIOM C. $\mathcal{M}$ has all finite collages. All collage injections i: $A \to C$ are in $\mathcal{K}$. An arrow $C \to X$ is in $\mathcal{K}$ iff all $A \to C \to X$ are in $\mathcal{K}$. Applying $(\ )^*$ to a collage diagram yields an opcollage diagram.

Many consequences of this axiom can be found in [8, 5], but the following is indeed independent of it:

AXIOM S. Every idempotent comonad (A, Φ) in $\mathcal{M}$ has an Eilenberg-Moore coalgebra $[i, \iota]$ with $i: A_\Phi \to A$ in $\mathcal{K}$. An arrow $X \to A_\Phi$ is in $\mathcal{M}$ iff $X \to A_\Phi \to A$ is in $\mathcal{K}$.

In the example $\mathcal{K}$ = toposes, such a Φ is an idempotent left exact triple. (Recall the variance conventions for 2-cells in topos theory.) Axiom S is satisfied by the existence of sheaf subtoposes. In fact it is a formal consequence of $i: A_\Phi \to A$ being Eilenberg-Moore that $i^{\sim}: 1 \to ii^*$ is an isomorphism. Thus, requiring that i be in $\mathcal{K}$ ensures that it is an inclusion with respect to $(\ )_*: \mathcal{K} \to \mathcal{M}$. It also follows formally that $X \to A_\Phi$ is a map iff $X \to A_\Phi \to A$ is a map. So the second sentence of the axiom is redundant if $\mathcal{K}$ = MAP($\mathcal{M}$).

All the examples of proarrow equipment which have been mentioned in this introduction satisfy Axiom C and, in addition to $\mathcal{K}$ = toposes, Axiom S holds for $\mathcal{K}$ = abelian categories and $\mathcal{K}$ = categories with finite limits. Axiom S does *not* hold without qualification for $\mathcal{K}$ = $\mathcal{V}$-cat. Counterexamples for cat and 2-cat (= ordered sets) were given in [1]. However, the specific instances of Axiom S that we need in the proof of our main theorem are indeed satisfied in the $\mathcal{K}$ = $\mathcal{V}$-cat example. Similar remarks apply to the related examples: $\mathcal{B}$-cat ($\mathcal{B}$ a bicategory), *S-in-*

dexed CAT and cat(S) (S a category with finite limits).

In Street's original proof [6] of the theorem for $\mathcal{V}$-cat, a certain full ($\mathcal{V}$-)subcategory was constructed as that determined by the set-theoretic complement of a class of objects. Our elimination of the set theory through the use of idempotent comonads seems to suggest a wider applicability. Indeed, we use it in this paper to get a further result about the composition of gamuts which generalizes that in [3].

2. GAMUTS TO COFIBRATIONS AND BACK

Assume that $(\ )_*: \mathcal{K} \to \mathcal{M}$ is a proarrow equipment satisfying Axioms C and S. For objects A and B in $\mathcal{K}$, a *gamut* from B to A is a diagram in $\mathcal{M}$ of the form

$$\begin{tikzcd} A & & B \arrow[ll, "\Theta"'] \arrow[dl, "\Phi"] \\ & X \arrow[ul, "\Psi"] \arrow[u, phantom, "\uparrow\sigma"] & \end{tikzcd}$$

A *morphism of gamuts* from $(\Theta,X,\Phi,\Psi,\sigma)$ to $(\Theta',X',\Phi',\Psi',\sigma')$ is (τ,k,γ,δ) where $\tau\colon\ \Theta\to\Theta'$, $k\colon X\to X'$ in $\mathcal{K}$, $\gamma\colon \Psi\to\ k\Psi'$ and δ: $\Phi k\to\Phi'$ satisfy

$$\begin{tikzcd} & X' \arrow[dl, "\Psi'"'] \arrow[d, phantom, "\uparrow\sigma'"] & \\ A \arrow[r, phantom, "\uparrow\gamma"] & \ & B \arrow[ll, bend right=60, "\Theta'"'] \arrow[ul, "\Phi'"'] \arrow[dl, "\Phi"] \\ & X \arrow[ul, "\Psi"] \arrow[uu, "k"] \arrow[ur, phantom, "\uparrow\delta"] & \end{tikzcd} \quad = \quad \begin{tikzcd} A & & B \arrow[ll, "\Theta"'] \arrow[ll, bend right=50, "\Theta'"'] \arrow[dl, "\Phi"] \\ & X \arrow[ul, "\Psi"] \arrow[u, phantom, "\uparrow\sigma"] & \end{tikzcd}$$

A *transformation* from (τ,k,γ,δ) to $(\tau',k',\gamma',\delta')$ is a $\varkappa\colon\ k\to\ k'$ satisfying $\gamma\cdot\varkappa\Psi' = \gamma'$ and $\Phi\varkappa\cdot\delta' = \delta$. The resulting bicategory of gamuts is denoted GAM(B,A).

Our first objective is to construct a cofibration from B to A from a gamut. The required cospan from B to A consists of the two injections, p and q, to the collage, E, of the gamut

$$\begin{tikzcd} A \arrow[ddr, "p"'] & & B \arrow[ll, "\Theta"'] \arrow[dl, "\Phi"] \arrow[ddl, "q"] \\ & X \arrow[ul, "\Psi"] \arrow[u, phantom, "\uparrow\sigma"] \arrow[d] \arrow[dl, phantom, "\psi"] \arrow[dr, phantom, "\varphi\ t"] & \\ & E \arrow[loop below, "\vartheta"] & \end{tikzcd}$$

As injections, p and q are representable (i.e., in $\mathcal{K}$) (and so is t). It will be useful to know that

$$\begin{bmatrix} p \\ t \\ q \end{bmatrix} : \mathrm{A}\oplus\mathrm{X}\oplus\mathrm{B} \longrightarrow \mathrm{E}$$

is a Kleisli object for the monad on $\mathrm{A}\oplus\mathrm{X}\oplus\mathrm{B}$ with underlying arrow defined by the first matrix below.

$$\begin{bmatrix} \mathrm{A} & 0 & 0 \\ \Psi & \mathrm{X} & 0 \\ \Theta & \Phi & \mathrm{B} \end{bmatrix} \qquad \begin{bmatrix} \mathrm{A} & 0 & 0 \\ \Psi+\Psi & 0 & 0 \\ \Theta+\Phi\Psi+\Theta & \Phi+\Phi & \mathrm{B} \end{bmatrix}$$

In the second matrix (above) the +'s denote local sums. The second matrix is of course the square of the first and multiplication for the monad in question is via codiagonals with the help of σ. The unit is built using $!: 0 \to \Psi$ etc. More on this matrix calculus can be found in [7,5]. Here we note that

$$\begin{bmatrix} p \\ t \\ q \end{bmatrix} [p^* t^* q^*] = \begin{bmatrix} pp^* & pt^* & pq^* \\ tp^* & tt^* & tq^* \\ qp^* & qt^* & qq^* \end{bmatrix} \approx \begin{bmatrix} \mathrm{A} & 0 & 0 \\ \Psi & \mathrm{X} & 0 \\ \Theta & \Phi & \mathrm{B} \end{bmatrix}$$

allows us to conclude that p, t and q are inclusions, $tp^* \approx \Psi$, $pt^* \approx 0$, etc.

PROPOSITION 1. *The cospan* $p: \mathrm{A} \to \mathrm{E} \leftarrow \mathrm{B} : q$ *is a cofibration from* B *to* A.

PROOF. From [5] we need only show that the cospan (p,q) is both a left cofibration and a right cofibration. For the first, we require a left adjoint (in $\mathcal{K}$-cospans from B to A) to the $\mathcal{K}$-cospan morphism

$$\begin{tikzcd} & \underline{p}^* \arrow[dd, "c_{\mathrm{L}} := \begin{bmatrix} p \\ p_{\sim} \\ \mathrm{E}1 \end{bmatrix}"] & \\ \mathrm{A} \arrow[ur, "i"] \arrow[dr, "p"'] & & \mathrm{B} \arrow[ul, "qj"'] \arrow[dl, "q"] \\ & \mathrm{E} & \end{tikzcd}$$

in which

$$\begin{tikzcd} \mathrm{A} \arrow[dr, "i"'] & & \mathrm{E} \arrow[ll, "\underline{p}^*"'] \arrow[dl, "j"] \\ & \underline{p}^* \arrow[u, phantom, "\xrightarrow{\varkappa}"] & \end{tikzcd}$$

is itself a collage and the specification of c_{L} is given accordingly. Thus we require a representable $\varsigma_L : E \to \underline{p}^*$ such that $p\varsigma_L \approx i$, $q\varsigma_L \approx qj$ and which is left adjoint to c_L: $\underline{p}^* \to E$, under A,B. Consider

$$\begin{tikzcd}
A \arrow[dr, "i"'] & & B \arrow[ll, "\Theta"'] \arrow[dl, "qj"] \arrow[dl, phantom, bend left=60, "\alpha"] \\
& \underline{p}^* &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[dr, "i"'] & X \arrow[l, "\Psi"'] \arrow[d, "tj"] \arrow[dl, phantom, "\beta"] \arrow[dr, phantom, "\varphi j"] \arrow[u, phantom, "\uparrow\sigma"] & B \arrow[l, "\Phi"] \arrow[dl, "qj"] \\
& \underline{p}^* &
\end{tikzcd}$$

where

$$\begin{tikzcd}
\Theta i \arrow[rr, "\alpha"] \arrow[dr] & & qj \\
& qp^* i \arrow[ur, "q\varkappa"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\Psi i \arrow[rr, "\beta"] \arrow[dr] & & tj \\
& tp^* i \arrow[ur, "t\varkappa"'] &
\end{tikzcd}$$

and φ is as in the description of E. The first diagram commutes, qua transformation diagram, yielding an arrow $E \to \underline{p}^*$, since E is a collage. Call it ς_L. It is representable since its arrow components are representable and we have $p\varsigma_L \approx i$, $q\varsigma_L \approx qj$ by construction. Now one has an isomorphism $E1 (= 1_E) \to \varsigma_L c_L$, for in calculating the composite $\varsigma_L c_L$ we invoke $ic_L \approx p$, $jc_L \approx E1$ and $\varkappa c_L = p\sim$ (modulo the previous isomorphisms), this being how c_L was defined. For the composite $c_L \varsigma_L$, consider the arrow component with domain A of the arrow component with domain E ($\underline{p}^*$ is a collage of collages). The diagrams above show that it is i: $A \to \underline{p}^*$ while the corresponding component of $\underline{p}^* 1$ is pj: $A \to \underline{p}^*$. We have a transformation $i \to pj$, namely the transpose of $\varkappa$: $p^* i \to j$. The other arrow components of $c_L \varsigma_L$ are isomorphs of the corresponding components of $\underline{p}1$ and thus $i \to pj$ provides us with a transformation $c_L \varsigma_L \to \underline{p}^* 1$. We claim that this together with the afore mentioned isomorphism $E1 \to \varsigma_L c_L$ provide the counit and unit for the required adjunction (under A,B). More informative than the check of the triangle identities, which we leave to the reader, is the picture in our Remark below.

For the right cofibration property let

$$\begin{tikzcd}
E \arrow[dr, "i"'] & & B \arrow[ll, "q"'] \arrow[dl, "j"] \arrow[ll, phantom, shift right=4, "\xrightarrow{\varkappa}"] \\
& \underline{q} &
\end{tikzcd}$$

be a collage diagram (reallocating i, x and j); then the requirement becomes a right adjoint in COSPN $\mathcal{K}(\mathrm{B},\mathrm{A})$ for

$$\begin{tikzcd}
& \underline{q} \arrow[dd, "c_{\mathbf{R}}:=\begin{bmatrix} \mathrm{E}1 \\ q1 \\ q \end{bmatrix}"'] & \\
\mathrm{A} \arrow[ur, "p"] \arrow[dr, "p"'] & & \mathrm{B} \arrow[ul, "j"'] \arrow[dl, "q"] \\
& \mathrm{E} &
\end{tikzcd}$$

From [8] we have $c_{\mathbf{R}}^* \approx [\mathrm{E}1, q_\sim, q^*]$ but to argue as in the previous paragraph on the left structure we need a description of $c_{\mathbf{R}}^*$ as an arrow whose domain is a collage. However,

$p c_{\mathbf{R}}^*$	$t c_{\mathbf{R}}^*$	$q c_{\mathbf{R}}^*$
$\approx p[\mathrm{E}1, q_\sim . q^*]$	$\approx t[\mathrm{E}1 . q_\sim , q^*]$	$\approx q[\mathrm{E}1, q_\sim , q^*]$
$\approx [p, pq_\sim, pq^*]$	$\approx [t, tq_\sim, tq^*]$	$\approx [q, qq_\sim, qq^*]$
$\approx [p, !, 0]$	$\approx [t, !, 0]$	$\approx [q, q, \mathrm{B}1]$
$\approx p[\mathrm{E}1, !, 0]$	$\approx t[\mathrm{E}1, !, 0]$	$\approx j$
$\approx pi$	$\approx ti$	

(where we have used freely our matrix result preceding this proposition) gives us a description of the components of $c_{\mathbf{R}}^*$ that completes our requirements. We will write $\text{ç}_{\mathbf{R}}$ (in $\mathcal{K}$) for $c_{\mathbf{R}}^*$.

REMARK 2. Fibrations in a bicategory are defined via birepresentability and fibrations in CAT. By duality this remark applies also to cofibrations in a bicategory, in particular to cofibrations in CAT. The main reference for this is [6] in which, amongst many other things, it was shown that in ($\mathcal{U}$-)CAT a cospan $p : \mathrm{A} \to \mathrm{E} \leftarrow \mathrm{B} : q$ is a cofibration iff it is equivalent to one of the following form:

(i) p and q are disjoint inclusions of full replete subcategories.

(ii) If X is defined to be the full subcategory of E determined by those objects which are in neither A nor B, then

$$\mathrm{E}(b,x) = \mathrm{E}(x.a) = \mathrm{E}(b,a) = \varnothing$$

for all $a \in \mathrm{A}$, $x \in \mathrm{X}$ and $b \in \mathrm{B}$.

It is convenient to visualize this as

$$\mathrm{E} = (\mathrm{A} \Rightarrow \mathrm{X} \Rightarrow \mathrm{B}).$$

The same notational convention then gives

$$\underline{p}^* = (\mathrm{A} \Rightarrow \mathrm{A} \Rightarrow \mathrm{X} \Rightarrow \mathrm{B}),$$

where the first A is a disjoint copy of A "glued" to E at A.

These allow simple schematics for c_{L} and ς_{L},

$$\varsigma_{\mathrm{L}}:\ \begin{tikzcd}[column sep=small] (A \arrow[r, Rightarrow] & A \arrow[r, Rightarrow] & X \arrow[r, Rightarrow] & B), \\ & (A \arrow[r, Rightarrow] \arrow[ul, mapsto] & X \arrow[r, Rightarrow] \arrow[u, mapsto] & B) \arrow[u, mapsto] \end{tikzcd} \qquad c_{\mathrm{L}}:\ \begin{tikzcd}[column sep=small] (A \arrow[r, Rightarrow] \arrow[dr, mapsto] & A \arrow[r, Rightarrow] \arrow[d, mapsto] & X \arrow[r, Rightarrow] \arrow[d, mapsto] & B), \arrow[d, mapsto] \\ & (A \arrow[r, Rightarrow] & X \arrow[r, Rightarrow] & B). \end{tikzcd}$$

which make $\varsigma_{\mathrm{L}} \dashv c_{\mathrm{L}}$ self-evident. Our Axiom C has the effect of making the abstract collages behave similarly. It is similar to the "sums are disjoint and universal" condition in topos theory.

PROPOSITION 3. *The glueing construction above extends to a homomorphism of bicategories*

$$\mathrm{G}\colon \mathrm{GAM}(\mathrm{B},\mathrm{A}) \longrightarrow \mathrm{COFIB}(\mathrm{B},\mathrm{A}).$$

To obtain the gamut which corresponds to a cofibration $p\colon \mathrm{A} \to \mathrm{E} \leftarrow \mathrm{B} \colon q$ we begin by recalling several results from [5].

First, since (p,q) is a left cofibration we get an idempotent comonad $(\varsigma_{\mathrm{L}} j^*, \varphi)$ on E in $\mathcal{M}$ where $\varphi\colon \varsigma_{\mathrm{L}} j^* \to \mathrm{E}1$ is induced by the canonical $\tau\colon \varsigma_{\mathrm{L}} \to j$ (Proposition 5 [6]). Further, since we have a right cofibration we get an idempotent comonad $(i\varsigma_{\mathrm{R}}^*, \psi)$ on E in $\mathcal{M}$, where

$$\psi\colon i\varsigma_{\mathrm{R}}^* \to i i^* \varsigma_{\mathrm{R}} \varsigma_{\mathrm{R}}^* \approx \mathrm{E}1$$

uses the unit for $i^* \dashv \varsigma_{\mathrm{R}}$ (Proposition 6 [6]). The composite $i\varsigma_{\mathrm{R}}^* \varsigma_{\mathrm{L}} j^*$ is also an idempotent comonad on E. To show this we need a lemma. Until Proposition 5 we write $\Phi = \varsigma_{\mathrm{L}} j^*$ and $\Psi = i\varsigma_{\mathrm{R}}^*$. ∎

LEMMA 4. *The following is a pullback in* $\mathcal{M}(\mathrm{E},\mathrm{E})$:

$$\begin{tikzcd} \Phi\Psi \arrow[r, "\Phi\psi"] \arrow[d, "\varphi\Psi"'] & \Phi \arrow[d, "\varphi"] \\ \Psi \arrow[r, "\psi"] & \mathrm{E}1 \end{tikzcd}$$

PROOF. Using $q\varsigma_{\mathrm{L}} \approx qj$ and $\mathrm{E}1 \approx \varsigma_{\mathrm{L}}\varsigma_{\mathrm{L}}^*$ we get

$$\Phi q^* = \varsigma_{\mathrm{L}} j^* q^* \approx \varsigma_{\mathrm{L}} \varsigma_{\mathrm{L}}^* q^* \approx q^*,$$

which shows that

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \Phi q^* \arrow[d] \\ 0 \arrow[r] & q^* \end{tikzcd}$$

is an absolute pullback in $\mathcal{M}(E,B)$. When q is applied the result is a pullback in $\mathcal{M}(E,E)$; in fact, with slight abuse of notation, the following is a pullback in $\mathcal{M}(E,q)/\mathcal{M}(E,E)$:

$$\begin{tikzcd}
0q \arrow[rr] \arrow[dr] \arrow[dd] & & \Phi q^* q \arrow[dr] \arrow[dd] & \\
& 0q \arrow[rr] \arrow[dd] & & q^* q \arrow[dd] \\
\Phi \arrow[rr] \arrow[dr] & & \Phi \arrow[dr] & \\
& E \arrow[rr] & & E
\end{tikzcd}$$

The category $\mathcal{M}(E,q)/\mathcal{M}(E,E)$ is equivalent to $\mathcal{M}(E,\underline{q})$, since $\underline{q}$ is a comma object in $\mathcal{M}$, and using our "matrix calculus" from [8] and [5] we write the above pullback, in $\mathcal{M}(E,\underline{q})$, as

$$\begin{tikzcd}
{[0,!,\Phi]} \arrow[r] \arrow[d] & {[\Phi q^*,\Phi q_{\sim},\Phi]} \arrow[d] \\
{[0,!,E]} \arrow[r] & {[q^*,q_{\sim},E]}
\end{tikzcd}$$

Applying the right adjoint ς_R^* to this yields a pullback in the category $\mathcal{M}(E,E)$. We claim that it is the diagram of the statement. Indeed, since

$$\varsigma_R^* = \begin{bmatrix} \Psi \\ q\psi \\ q \end{bmatrix} : \underline{q} \longrightarrow E$$

its composite with $[q^*,q_{\sim},E]$ is the local pushout of $q_{\sim}\Psi$ along $q^*q\psi$ but this is E (Lemma 12, [5]). Also $[\Phi q^*,\Phi q_{\sim},\Phi]\varsigma_R^*$ is the local pushout of $\Phi q_{\sim}\Psi$ along $\Phi q^* q\psi$ and this is Φ since it is Φ applied to the previous (composition stable) pushout. As pushouts along $0 \to 0$, the isomorphisms

$$\Psi \approx [0,!,E]\varsigma_R^* \text{ and } \Phi\Psi \approx [0,!,\Phi]\varsigma_R^*$$

are even more easily obtained. ∎

Since

$$\begin{tikzcd}
\Psi\Phi \arrow[r, "\psi\Phi"] \arrow[d, "\Psi\varphi"'] & \Phi \arrow[d, "\varphi"] \\
\Psi \arrow[r, "\psi"] & E1
\end{tikzcd}$$

also commutes we get $\delta : \Psi\Phi \to \Phi\Psi$ and, by a dual of Lemma 37 [8], δ is a distributive law with the property that a coalgebra for the composite (idempotent) comonad $\Psi_\delta\Phi$ is an arrow $X \to E$

which is simultaneously a Ψ-coalgebra and a Φ-coalgebra (the distributive requirement being automatically satisfied). We write $\Gamma = \Psi\Phi$ and $\gamma = \psi\varphi$, thus (Γ,γ) is an idempotent comonad (in $\mathcal{M}$) on E.

We now apply Axiom S to define the coalgebra $s: Y := E_\Gamma \to E$ (in $\mathcal{K}$) and note that s is necessarily an inclusion. With s available we can define the gamut corresponding to the cofibration (p,q). It is

$$\begin{tikzcd} A & & B \arrow[ll, "qp^*"'] \arrow[dl, "qs^*"] \\ & Y \arrow[ul, "sp^*"] \arrow[u, phantom, "\uparrow qs_\sim p^*"] & \end{tikzcd}$$

PROPOSITION 5. *The construction above extends to a homomorphism of bicategories*

$$D: \mathrm{COFIB}(B,A) \longrightarrow \mathrm{GAM}(B,A).$$

The purpose of the remainder of this section is to demonstrate that G and D constitute a biequivalence.

PROPOSITION 6. $GD: \mathrm{GAM}(B,A) \longrightarrow \mathrm{GAM}(B,A)$ *is equivalent to the identity.*

PROOF. As we have done above, we denote G of

$$\begin{tikzcd} A & & B \arrow[ll, "\Theta"'] \arrow[dl, "\Phi"] \\ & X \arrow[ul, "\Psi"] \arrow[u, phantom, "\uparrow\sigma"] & \end{tikzcd}$$

by

$$\begin{tikzcd} A \arrow[r, "p"] & E & B \arrow[l, "q"'] \\ & X \arrow[u, "t"] & \end{tikzcd}$$

Since

$$qp^* \approx \Theta, \; tp^* \approx \Psi \text{ and } qt^* \approx \Phi,$$

as mentioned prior to Proposition 1, and since, given these isomorphisms, σ corresponds to $qt_\sim p^*$, it is enough to show that $\Gamma(= i\varsigma_R^*\varsigma_L j^*) \approx t^*t$. For then we may conclude that $Y \sim X$, via an equivalence which identifies $s: Y \to E$ with $t: X \to E$.

We express $i\varsigma_R^*$ as an arrow into an opcollage and $\varsigma_L j^*$ as an arrow out of a collage. From our previous descriptions of ς_R

and ς_L we have the following

$$\begin{tikzcd}
 & B \arrow[d, "qt^*"] \arrow[dr, "q"] \arrow[dd, bend right=60, "qp^*"'] & \\
E \arrow[ur, "0"] \arrow[r, "t^*"] \arrow[dr, "p^*"'] & X \arrow[r, "t"] \arrow[d, "tp^*"] & E \arrow[loop right] \\
 & A \arrow[ur, "0"'] &
\end{tikzcd}$$

$$(E \xrightarrow{i\varsigma_R^*} E \xrightarrow{\varsigma_L j^*} E).$$

From the general theory in [8] it follows that the composite $(i\varsigma_R^*)(\varsigma_L j^*)$ is the colimit of the above diagram in $\mathcal{M}$(E,E). Drawn as a diagram of arrows and transformations it becomes

$$\begin{tikzcd}
t^*tp^*0_{AE} \arrow[d] \arrow[drr] & & 0_{EB}qp^*0_{AE} \arrow[dll] \arrow[drr] & & 0_{EB}qt^*t \arrow[d] \arrow[dll] \\
p^*0_{AE} & & t^*t & & 0_{EB}q
\end{tikzcd}$$

The colimit is clearly t^*t. ■

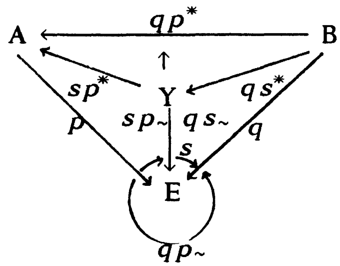
It remains to be shown that DG is equivalent to the identity. Beginning with a cofibration from B to A, $p: A \to E \leftarrow B: q$, we construct a gamut using the inclusion s:

$$\begin{tikzcd}
A & & B \arrow[ll, "qp^*"'] \arrow[dl, "qs^*"] \\
 & Y \arrow[ul, "sp^*"] \arrow[u, phantom, "\uparrow qs_{\sim}p^*"] &
\end{tikzcd}$$

and construct its collage, denoted

$$\begin{tikzcd}
A \arrow[dr, "\overline{p}"'] & & B \arrow[ll] \arrow[dl] \arrow[ddl, "\overline{q}"] \\
 & Y \arrow[ul] \arrow[d, "\overline{s}"] \arrow[u, phantom, "\uparrow"] & \\
 & \overline{E} \arrow[loop below] &
\end{tikzcd}$$

Thus, we must show that $\overline{p}: A \to \overline{E} \leftarrow B: \overline{q}$ is equivalent (qua cofibration) to the originally given cospan. To begin, we define an arrow $k: \overline{E} \to E$ in $\mathcal{K}$ (since p, q and s are) by the following



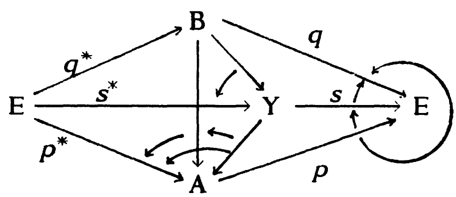
We now show that k is an equivalence.

PROPOSITION 7. $k^\sim : \overline{\mathrm{E}} \Rightarrow kk^*$.

PROOF. Both the identity on $\overline{\mathrm{E}}$ and kk^* are arrows from $\overline{\mathrm{E}}$ to $\overline{\mathrm{E}}$. As such, they may be compared by comparing the arrows from A, Y and B to A, Y and B (and the transformations) which define them. Noting that $k^*: \mathrm{E} \to \overline{\mathrm{E}}$ is defined using p^*, q^* and s^*, and that all of p, q and s are inclusions (so e.g. $\mathrm{A}1 \Rightarrow pp^*$), we easily see that all the entries are isomorphic and the desired result follows. ■

PROPOSITION 8. $k_\sim : k^*k \to \mathrm{E}$.

PROOF. We begin by noting that, as an arrow through the collage object $\overline{\mathrm{E}}$, kk^* is defined by the following diagram:



As such, k^*k is the colimit in $\mathcal{M}(\mathrm{E,E})$ of the following diagram:

$$\begin{tikzcd}
s^*sp^*p \arrow[d] \arrow[drr] & & q^*qp^*p \arrow[dll] \arrow[drr] & & q^*qs^*s \arrow[dll] \arrow[d] \\
p^*p & & s^*s & & q^*q
\end{tikzcd}$$

in which all arrows involve the counits for the adjunctions for p, q and s. We claim that this colimit is the identity on E. First, from ([5], Corollary 13) the pushout of

$$s^*s \leftarrow q^*qs^*s \to q^*q$$

is $\varsigma_L j^*$. It follows that the pushout of

$$s^* s p^* p \leftarrow q^* q s^* s p^* p \to q^* q p^* p$$

is $\varsigma_L j^* p^* p$. Simple commutativities yield arrows

$$p^* p \leftarrow \varsigma_L j^* p^* p \to \varsigma_L j^*,$$

whose pushout is the colimit of the diagram in question. According to ([5], Lemma 12) the pushout is E1. ■

Combining the propositions above we have:

THEOREM 9. G *and* D *determine an equivalence of bicategories*

$$\mathrm{COFIB}(B,A) \longrightarrow \mathrm{GAM}(B,A). \quad ■$$

Just as in the case $\mathcal{K}$ = toposes case of [2], we can in the present axiomatic framework extract considerably more information from the analysis given. For example, if $p\colon A \to E \leftarrow B\colon q$ is a cofibration, then $s\colon (Y = E_\Gamma) \to E$ appears from the description of Γ-coalgebras preceding Proposition 5 to be "the intersection of $i\varsigma_R^*$-coalgebras and $\varsigma_L j^*$-coalgebras". One can show that

$$\underline{sp}^* \approx E_{i\varsigma_R^*} \to E \text{ and } \underline{qs}^* \approx E_{\varsigma_L j^*} \to E.$$

In terms of the hieroglyphics of Remark 2 we can picture this rather simply

$$\begin{tikzcd} E = (A \arrow[r, Rightarrow] \arrow[r, bend left, "\underline{sp}^*"] & Y \arrow[r, Rightarrow] \arrow[r, bend right, "\underline{qs}^*"'] & B) \end{tikzcd}$$

3. COMPOSITION OF GAMUTS.

We will phrase our remarks in terms of the diagram below:

$$\begin{tikzcd}
A \arrow[ddr, "p"'] & & B \arrow[ll, "\Theta"'] \arrow[dl, "\Phi"] \arrow[ddl, "q"] \arrow[ddr, "u"'] & & C \arrow[ll, "\Sigma"'] \arrow[dl, "\Gamma"] \arrow[ddl, "v"] \\
& X \arrow[ul, "\Psi"] \arrow[u, phantom, "\uparrow\sigma"] & & Y \arrow[ul, "\Delta"] \arrow[u, phantom, "\uparrow\tau"] & \\
& E & & F &
\end{tikzcd}$$

Let (p,q) and (u,v) denote cofibrations and let in the diagram above each cospan denote the corresponding gamut. The composite cofibration $A \to F \otimes E \leftarrow C$ is defined as an inverter in

COSPN $\mathcal{K}(C,A)$

$$\begin{tikzcd} F\otimes E \arrow[r] & F\circ E \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\downarrow"] & F\nwarrow E \end{tikzcd}$$

where $F\circ E$ is obtained by pushing out q and u, $F \nwarrow E$ is the co-comma object "from q to u", and

$$\begin{tikzcd} F\circ E \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\downarrow"] & F\nwarrow E \end{tikzcd}$$

is most easily explained by the schematics below.

We will proceed informally. The material of the preceding section (and the authors' earlier papers) make the transition to a rigorous treatment straightforward. We note that both $F\circ E$ [5] and $F\nwarrow E$ [8] have been discussed earlier in the context of Axiom C and we have

$$E = (A \Rightarrow X \Rightarrow B), \qquad F = (B \Rightarrow Y \Rightarrow C),$$

$$\begin{tikzcd}[column sep=small] F\circ E \arrow[d, shift right] \arrow[d, shift left] \arrow[d, phantom, "\to"] & = & (A \arrow[d] & \Rightarrow & X \arrow[d] & \Rightarrow & B \arrow[d] \arrow[dr, phantom, "\mapsto"] \arrow[drr] & \Rightarrow & Y \arrow[drr] & \Rightarrow & C) \arrow[drr] & & \\ F\nwarrow E & = & (A & \Rightarrow & X & \Rightarrow & B & \Rightarrow & B & \Rightarrow & Y & \Rightarrow & C) \end{tikzcd}$$

where we have displayed only the non-trivial component of the transformation. Intuitively, it is clear that the inverter is the "full subobject of $F\circ E$ determined by deletion of the objects of B". After all,

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\downarrow"] & 2B \end{tikzcd}$$

is an inverter diagram in CAT. In fact, it is a worthwhile exercise to show that Axiom C alone implies that

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\downarrow"] & 2B \end{tikzcd}$$

is an inverter in $\mathcal{M}$.

Now the identity on $F\circ E$ can be obtained by "glueing" the identities on A, X, B, Y and C. If the identity on B is replaced by $0: B \to B$, then the result $\Gamma: F\circ E \to F\circ E$ is idempotent and $\Gamma \to (F\circ E)1$, induced by $0 \to B1$, makes it a comonad. We claim that $(F\circ E)_\Gamma$ as in Axiom S gives $F\otimes E$, the result being of course

$$A \Rightarrow X \Rightarrow Y \Rightarrow C.$$

Using gamuts the construction is easier. We get $X \Rightarrow Y$ above by glueing X and Y along $\Delta\Phi$. As usual this is denoted $\underline{\Delta\Phi}$ and we have the following diagrams

$$\begin{tikzcd}
 & C \arrow[ddl, "\Sigma\Phi"'] \arrow[d] \arrow[ddr, "\Gamma"] & \\
 & \underline{\Delta\Phi} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\leftarrow"] & \\
X \arrow[loop left, "\tau\Phi"] & \, & Y \arrow[ll, "\Delta\Phi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[dr] \arrow[ddr, "\Psi"'] & \arrow[d, phantom, "\rightarrow"] & Y \arrow[ll, "\Delta\Phi"'] \arrow[dl] \arrow[ddl, "\Delta\Theta"] \\
 & \underline{\Delta\Phi} \arrow[d] & \\
 & A \arrow[loop below, "\Delta\sigma"] &
\end{tikzcd}$$

defining arrows $A \leftarrow \underline{\Delta\Phi} \leftarrow C$, whose composite is the displayed local pushout from which we get a transformation to $\Sigma\Theta$ which we call $\tau\otimes\sigma$:

$$\begin{tikzcd}
 & \Gamma\Delta\Phi\Psi \arrow[dl, "\tau\Phi\Psi"'] \arrow[dr, "\Gamma\Delta\sigma"] & \\
\Sigma\Phi\Psi \arrow[r] \arrow[ddr, "\Sigma\sigma"'] & + \arrow[d, phantom] & \Gamma\Delta\Theta \arrow[l] \arrow[ddl, "\tau\Theta"] \\
 & \bullet \arrow[d, "\tau\otimes\sigma"] & \\
 & \Sigma\Theta &
\end{tikzcd}$$

Assembling the picture, we have

$$\begin{tikzcd}
A & & C \arrow[ll, "\Sigma\Theta"'] \arrow[dl] \\
 & \underline{\Delta\Phi} \arrow[ul] \arrow[u, phantom, "\uparrow\tau\otimes\sigma"] &
\end{tikzcd}$$

as the composed gamut. Since $F\otimes E$ etc. and $\tau\otimes\sigma$ etc. correspond under the equivalence in Theorem 9, the result extends.

THEOREM 10. G *and* D *determine an equivalence of bicategories with bicategory enriched homs,* COFIB $\rightarrow$ GAM. ∎

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