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CAHIERS DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE CATÉGORIQUES

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THE CLASSIFYING TOPOS OF A CONTINUOUS GROUPOID. II

by Ieke MOERDIJK [1]

RÉSUMÉ. Dans cet article, on construit une complétion γG pour chaque groupoïde G, et on montre que tout foncteur continu exact $BG \to BH$ entre les topos classifiants des groupoïdes continus G et H est obtenu par produit tensoriel avec un espace muni d'une action de γG à gauche et de γH à droite ("bi-espace"). On en déduit une description complète de la catégorie des topos en termes de groupoïdes continus et de tels bi-espaces.

If G is a continuous groupoid, i.e., a groupoid in the category of spaces, it is natural to consider the category of étale G-spaces. These form a topos BG, called the classifying topos of G. It arises naturally in many contexts, e.g. in foliation theory where G is a groupoid of germs of local diffeomorphisms of a foliated manifold (see e.g. [11]), and BG is the étendue associated to a foliation (already described by Grothendieck and Verdier in [12], IV.9).

The generality of the construction is beautifully demonstrated by A. Joyal and M. Tierney, who show in [6] that every Grothendieck topos is equivalent to a category of the form BG, for a suitable continuous groupoid G.

In Part I (cf. [7]), I discussed many properties of the functor $G \to BG$. This functor is not full, but it was proved there that the category of toposes can be obtained from a category of groupoids by a calculus of fractions, in the sense of Gabriel and Zisman (see [2]).

The aim of this second part is to describe the morphisms of toposes $BH \to BG$ in terms of the continuous groupoids G and H, in a way which is somewhat in the spirit of Morita theory for modules over commutative rings.

The argument proceeds in two steps. First, I construct for each continuous groupoid G a *completion* γG. γG is a continuous category (no longer a groupoid), but the étale γG-spaces are the same as the étale G-spaces, i.e., there is an equivalence of categories

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(1) $$BG \approx B(\gamma G).$$

The second step is to show that for each geometric morphism $f\colon BH \to BG$ there exists a space $R(f)$ equipped with an action by γH on the right and one by γG on the left, such that the inverse image functor f^* comes from tensoring with $R(f)$: there is a natural isomorphism

(2) $$E \otimes_{\gamma G} R(f) \approx f^*(E)$$

for each étale G-space E.

Just like the category of commutative rings can be made into a bicategory with bimodules as morphisms and the tensor product as composition (cf. [1]), such spaces equipped with an action of γG on the left and one of γH on the right – call them bispaces – form the morphisms of a bicategory with continuous groupoids as objects and tensor-product as composition. It is a formal consequence of (2) that the 2-category of toposes is equivalent to a bicategory of groupoids and such bispaces, as I will spell out in Section 6.

Although this paper is a sequel to Part I ([7]), familiarity with all of Part I is by no means necessary. However, I do assume that the reader is familiar with the preliminaries listed in Section 1 of Part I (appropriate references are given there), as well as with Sections 5 and 6 of Part I. Some of the basic facts from Part I are quickly reviewed in Section 1 below.

I should also say that the results of this paper have already been worked out for the case of continuous groups (not groupoids) in my paper [8]. The technical details are much easier for groups, and for the reader with an interest in these details, it might be more pleasant to read [8] first.

The author is much indebted to A. Kock and the referee. Both spotted numerous inaccuracies, and moreover made various suggestions which have improved the paper substantially.

CONTENTS.

1. THE TOPOS ASSOCIATED TO A CONTINUOUS GROUPOID.

In this section, we briefly review a construction of the classifying topos of a continuous groupoid, discussed in Part I.

1.1. DEFINITION OF BG (see I, 5.1-5.4). Let G be a continuous groupoid, i.e., a groupoid object in the category of spaces (in the generalized sense, see e.g. [6]). As in Part I, we write $d_0: G_1 \to G_0$ and $d_1: G_1 \to G_0$ for the domain and codomain. $m: G_1 \times_{G_0} G_1 \to G_1$ for the composition ($m(f,g) = f \cdot g$), and $s: G_0 \to G_1$ for the identity: so these are all continuous maps of spaces satisfying the usual identities. As in Part I, we will always assume that d_0 and d_1 are *open* maps (it then follows that m must be open, too).

A G-*space* is a space $p: E \to G_0$ over G_0 equipped with an action of G on the right, $\cdot: E \times_{G_0} G_1 \to E$, satisfying the usual identities. So a G-space is a triple $(E, p, \cdot)$, but we usually just write E to refer to the triple. A map of G-spaces $f: E \to E'$ is a map over G_0 which preserves the action.

A G-space E is called *open*, resp. *étale*, if the map $f: E \to G_0$ is open, resp. étale (i.e., a local homeomorphism). Note that this implies that the action $E \times_{G_0} G_1 \to E$ is an open map. BG is the full subcategory of (G-spaces) consisting of étale G-spaces. BG is a topos, called the *classifying topos of* G. The canonical geometric morphism whose inverse image is given by forgetting the action is denoted by $\pi_G: Sh(G_0) \to BG$.

The construction is functorial in G: if $\varphi: H \to G$ is a homomorphism of continuous groups, φ induces a geometric morphism $B\varphi: BH \to BG$ (I.5.4).

The construction relativizes to an arbitrary base topos: if G is a continuous groupoid in a topos *E*, the étale G-spaces in *E* form a topos over *E*, which we denote by B(*E*,G). An important property is the stability of the construction under change-of-base (not in the least because stability allows us to use point-set arguments, as was pointed out throughout Part I: cf. in particular I.5.3). We restate it explicitly here.

1.2. Stability THEOREM (I.6.7). *Let p: F→E be a geometric morphism, and let* G *be a continuous groupoid in E (we assume that* $G_1 \rightrightarrows G_0$ *is open). Then there is a canonical equivalence of toposes*

$$F \times_E B(E,G) \simeq B(F, p^*(G)).$$

1.3. Generators for BG (I.6.1). Let G be a continuous groupoid, and let $U \subset G_0$ be an open subspace. An *open* U-*congruence* is an open subspace $N \subset G_1$ such that $d_0(N), d_1(N) \subset U$ and N contains all identities $(s(U) \subset N)$ and is closed under inverse and composition. We factor out such an N to obtain an étale G-space, namely

(1) $$G_1 \cap d_1^{-1}(U)/N,$$

a space over G_0 by d_0, with action given by composition.

Intuitively one can think of $G_1 \cap d_1^{-1}(U)/N$ in point-set terms: the elements are equivalence classes $[g]$ of morphisms g: $x \to x'$ in G with $x' \in U$, where two such $g_1: x \to x_1$ and $g_2: x \to x_2$ are equivalent if $g_2 \cdot g_1^{-1} \in N$. The action of G on the right is described by

$$[g] \cdot h = [gh] = [m(g,h)].$$

Since the quotient in (1) is stable, one may use change-of-base techniques to actually exploit this point-set description of the étale G-space $G_1 \cap d_1^{-1}(U)/N$.

The étale G-spaces which are of the form (1) generate BG; the corresponding full subcategory is denoted by $\mathbf{S}_G$, or $\mathbf{S}(G)$.

1.4. Maps between generators. As explained in I.6, a section

$$a: V \to G_1 \cap d_1^{-1}(U)/N$$

(i.e., $d_0 \cdot a = \mathrm{id}_V$) where $V \subset G$ is an open subspace, induces a morphism

(2) $$\tilde{a}: G_1 \cap d_1^{-1}(V)/M \to G_1 \cap d_1^{-1}(U)/N$$

if M is a sufficiently small open V-congruence. In point-set notation, $\tilde{a}$ is described by

(3) $$\tilde{a}([g]) = [a(d_1 g) \cdot g].$$

Every morphism in $\mathbf{S}(G)$ is of the form (2). A generating G-space of the form (1) always has one distinguished section given by the identity, which we denote by

$$s: W \to G_1 \cap d_1^{-1}(U)/N$$

for any open $W \subset U$.

1.5. Inverses in S(G). Let

$$\tilde{a}: G_1 \cap d_1^{-1}(V)/M \to G_1 \cap d_1^{-1}(U)/N$$

be a map in $\mathbf{S}(G)$ coming from a section $a: V \to G_1 \cap d_1^{-1}(U)/N$ as above. Then there is a collection $\{U_i\}$ of open subspaces of

U, with open U_i-congruences $N_i \subset N$, such that there are sections $b_i\colon U_i \to G_1 \cap d_1^{-1}(V)/M$ with the property that

$$\{\tilde{b}_i : G_1 \cap d_1^{-1}(U_i)/N_i \to G_1 \cap d_1^{-1}(V)/M\}_i$$

is an epimorphic family, and for each i, $\tilde{a} \cdot \tilde{b}_i$ is the natural subquotient map $G_1 \cap d_1^{-1}(U_i)/N_i \to G_1 \cap d_1^{-1}(U)/N$.

To see this, we use a point-set argument (and implicit base extension). Let ξ be a point of $G_1 \cap d_1^{-1}(V)/M$ (in any base extension) and choose (by going to some further open surjective base extension) a point $g\colon z \to y$ of G_1 with $y \in V$ and $\xi = [g]$, and a point $h\colon y \to x$ in G_1 with $x \in U$ and $a(y) = [h]$. Let

$$b_x\colon U_x \to G_1 \cap d_1^{-1}(V)/M$$

be a section through the point $[h^{-1}\colon x \to y]$. Since the space $G_1 \cap d_1^{-1}(V)/M$ is étale over G_0, we may assume (by choosing U_x small enough) that $\tilde{a} \cdot b_x = s$ (the identity section, cf. 1.4). Choose M_x small enough for b_x to induce a map

$$G_1 \cap d_1^{-1}(U_x)/M_x \to G_1 \cap d_1^{-1}(V)/M.$$

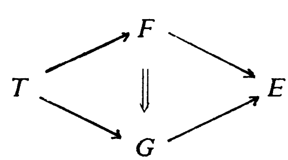
Then clearly $\tilde{a} \cdot \tilde{b}_x = \tilde{s}$; moreover

$$\tilde{b}_x[g \cdot h] = [g \cdot h \cdot b_x(x)] = [g] = \xi.$$

1.6. Continuous categories. Notice that the definition of the category BG of étale G-spaces given in 1.1 also makes sense if G is just a continuous category (a category object in the category of spaces), rather than a groupoid. Below, we will use the same notation BG for the category of étale G-spaces in the case of a continuous category G. BG is still a topos, but many of the results of Part I do not extend to this case where G is a continuous category.

2. LAX FIBERED PRODUCTS OF TOPOSES.

Let $G \to E \leftarrow F$ be geometric morphisms of S-toposes (where S is the base). The *lax fibered product*, or *lax pullback* is the universal solution (up to equivalence of hom-categories) to having a pair of geometric morphisms $T \to F$, $T \to G$, together with a natural transformation (all over S)



(between the *inverse* image functors). We write $F \Rightarrow_E G$ for this universal topos, and

(1)
$$\begin{array}{ccc} F\Rightarrow_E G & \longrightarrow & F \\ \downarrow & \overset{\xi}{\swarrow\!\!\!\swarrow} & \downarrow \\ G & \longrightarrow & E \end{array}$$

for the corresponding universal square. It can be constructed as a pullback

(2)
$$\begin{array}{ccc} F\Rightarrow_E G & \longrightarrow & F\times_S G \\ \downarrow & & \downarrow \\ E\Rightarrow_E E & \longrightarrow & E\times_S E \end{array}$$

2.1. LEMMA. *The construction of lax fibered products is stable, i.e., if $T\to S$ is an extension of the base, then*

$$(F\Rightarrow_E G)\times_S T \approx (F\times_S T) \Rightarrow_{E\times_S T} (G\times_S T).$$

PROOF. Obvious.

The following theorem was found independently by Pitts ([10], Theorem 4.5) and the author (preprint version (1986) of the present paper); our methods of proof were completely different:

2.2. THEOREM. *Let $F\to E - G$ be geometric morphisms with lax fibered product $F\Rightarrow_E G$, as in (1). If $F\to E$ is open, then so is the projection $(F\Rightarrow_E G)\to G$.*

For the proof of 2.2, we shall use the Sierpinski space S; it has two distinguished points, an open 1 and a closed one 0. For toposes T and E over S, a geometric morphism h: $T\times_S \mathrm{Sh(S)}\to E$ is equivalent to a pair of morphisms h_0, h_1: $T\to E$, together with a natural transformation $h_0^* \Rightarrow h_1^*$. Clearly the topos $F\Rightarrow_E G$ of (1) can also be constructed as the pullback

(3)
$$\begin{array}{ccc} F\Rightarrow_E G & \longrightarrow & F\times_S G \\ \downarrow & & \downarrow \\ E^{\mathrm{Sh(S)}} & \longrightarrow & E\times_S E \end{array}$$

where $E^{\mathrm{Sh(S)}}$ is the exponential of S-toposes [5].

PROOF of 2.2. Let us first observe that it suffices to prove that for an S-topos E, the evaluation $ev_1 : E^{Sh(S)} \to E$ is open. Indeed, one can construct the pullback (3) in two stages,

$$F \Rightarrow_E G = (E^{Sh(S)} \times_E F) \times_E G$$

as in

$$\begin{array}{ccc} E^{Sh(S)} \times_E F & \longrightarrow & F \\ {\scriptstyle \pi_1}\downarrow & & \downarrow \\ E^{Sh(S)} & \xrightarrow{ev_0} & E \end{array} \qquad \begin{array}{ccc} (E^{Sh(S)} \times_E F) \times_E G & \longrightarrow & G \\ \downarrow & & \downarrow \\ E^{Sh(S)} \times_E F & \xrightarrow{ev_1 \pi_1} & E \end{array}$$

and use that open maps are stable under pullback.

To prove that $ev_1 : E^{Sh(S)} \to E$ is open, we follow an idea similar to the proof of [9], 2.1. If $E = Sh(X)$, for a space X in S, then $Sh(X)^{Sh(S)} \approx Sh(X^S)$, where the exponential X^S of spaces has a presentation of the form (U_0, U_1) where $U_1 \subset U_0 \subset X$ are open subspaces of X. Then $(U_0, U_1) \subset X^S$ is the subspace defined by saying that a map $h : T \to X^S$ factors through (U_0, U_1) iff

$$h_0 \leq h_1 \text{ as maps } T \to X, \text{ and } h_0(T) \subset U_0,\ h_1(T) \subset U_1.$$

Clearly $ev_1(U_0, U_1) = U_1$, so ev_1 is open.

If E is any topos, a construction of Joyal (see [6]) gives a space Y and an open surjection p: $Sh(Y) \to E$. By considering the diagram

$$(1) \qquad \begin{array}{ccc} Sh(Y)^{Sh(S)} & \xrightarrow{p^{Sh(S)}} & E^{Sh(S)} \\ {\scriptstyle ev_1}\downarrow & & \downarrow{\scriptstyle ev_1} \\ Sh(Y) & \xrightarrow{p} & E \end{array}$$

we find that it is enough to show that $Sh(Y)^{Sh(S)} \to E^{Sh(S)}$ is a (stable) surjection. Recall that $Sh(Y) \to E$ is constructed as a pullback

$$(2) \qquad \begin{array}{ccc} Sh(Y) & \longrightarrow & Sh(X_U) \\ \downarrow & & \downarrow \\ E & \longrightarrow & S[U] \end{array}$$

where $E \to S[U]$ is a localic geometric morphism into the object classifier $S[U]$ and $Sh(X_U)$ classifies partial enumerations of the generic object U. So by exponentiating the pullback (2) by Sh(S), it is enough to show that

$$(3) \qquad Sh(X_U)^{Sh(S)} \to S[U]^{Sh(S)}$$

is a stable surjection. We will show that for any geometric morphism $f: T \to \mathcal{S}[U]^{Sh(S)}$ (T any topos over S) there is an open surjection $g: T' \to T$ and a commutative diagram

$$(4) \qquad \begin{array}{ccc} T' & \xrightarrow{f'} & Sh(X_U)^{Sh(S)} \\ {\scriptstyle g}\downarrow & & \downarrow \\ T & \xrightarrow{f} & \mathcal{S}[U]^{Sh(S)} \end{array}$$

from which it follows that $Sh(X_U)^{Sh(S)} \to \mathcal{S}[U]^{Sh(S)}$ is a stable surjection. By working in T it is enough to take the case $T = S$ (="sets"). The given map f corresponds to a pair A_0, A_1 of sets together with a function $\alpha: A_0 \to A_1$. We need to find partial enumerations $\mathbb{N} \supset U_i \xrightarrow{\varepsilon_i} A_i$ ($i = 0,1$) such that $\alpha \cdot \varepsilon_0 = \varepsilon_1$ on $U_0 \cap U_1$, in some base extension T', for this precisely defines a map $h: T \to Sh(X_U)^{Sh(S)}$ making (4) commute. There is an open surjection $T' \to T$ such that there are partial enumerations $\mathbb{N} \supset V_i \xrightarrow{\beta_i} A_i$ in T. Let $U_0 = V_0$, $U_1 = V_0 + V_1$; then

$$U_0, U_1 \subset \mathbb{N}+\mathbb{N} \approx \mathbb{N}. \text{ and } U_0 \cap U_1 = V_0 \subset \mathbb{N}+\mathbb{N}.$$

so if we define $\varepsilon_0 = \beta_0$, $\varepsilon_1 | V_0 = \alpha\beta_0$, $\varepsilon_1 | V_1 = \beta_1$, the proof is complete.

We furthermore have:

2.3. LEMMA. *Let $F \to E \leftarrow G$ be geometric morphisms with lax fibered product $F \Rightarrow_E G$. If F and G are spatial toposes (i.e.. $F \approx Sh(X)$, $G \approx Sh(Y)$ for spaces X.Y). then so is $F \Rightarrow_E G$.*

PROOF. If F and G are spatial, so is $F \times_S G$. Moreover, $E \Rightarrow_E E \to E \times_S E$ is spatial (since the left hand side classifies morphisms of E-models while the right hand side classifies pairs of E-models: the latter kind of structure bounds the former). So also the top arrow in the pullback (2) is spatial. Since the composite of spatial morphisms is spatial, $F \Rightarrow_E G$ is a spatial topos.

3. COMPLETION OF CONTINUOUS GROUPOIDS.

In this section we will construct a continuous category γG for any continuous groupoid G. γG is a kind of completion of G, which still defines the same topos; i.e., there is an equivalence $B(\gamma G) \approx BG$, induced by a continuous homomorphism $G \to \gamma G$ of continuous categories.

3.1. DEFINITION of γG. Let G be a continuous groupoid, with classifying topos BG. We define a continuous category γG as follows: the space of objects is the same as that of G, i.e., $(\gamma G)_0 = G_0$, and the space of morphisms $(\gamma G)_1$ is defined by the lax fibered product

(1)
$$\begin{array}{ccc} Sh(\gamma G_1) & \xrightarrow{d_1} & Sh(G_0) \\ {\scriptstyle d_0}\downarrow & \xi \swArrow & \downarrow{\scriptstyle \pi_G} \\ Sh(G_0) & \xrightarrow{\pi_G} & BG \end{array}$$

Notice that this lax fibered product is indeed spatial by 2.4, so γG_1 is uniquely defined as a space. The two geometric morphisms d_0 and d_1 in (1) define the domain and codomain. The identity $s\colon \gamma G_0 \to \gamma G_1$ is defined by the universal property of (1) and the identity transformation from π_G^* to itself. Composition in γG, which is a map

(2)
$$m\colon (\gamma G)_1 \times_{\gamma G_0} (\gamma G)_1 \to (\gamma G)_1$$

(where the pullback in (2) is along d_0 on the left, d_1 on the right) is defined by the universal property of $Sh(\gamma G_1)$ as follows: write π_1 and π_2 for the two projections

$$\gamma G_1 \times_{\gamma G_0} \gamma G_1 \rightrightarrows \gamma G_1$$

so that there is an isomorphism

(3)
$$d_0 \pi_1 \approx d_1 \pi_2$$

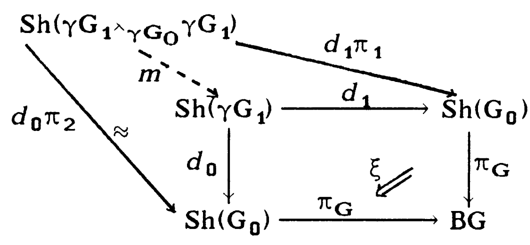
and consider the composition of 2-cells

(4)
$$\pi_G d_1 \pi_1 \xrightarrow{\xi \cdot \pi_1} \pi_G d_0 \pi_1 \approx \pi_G d_1 \pi_2 \xrightarrow{\xi \cdot \pi_2} \pi_G d_0 \pi_2 .$$

Since (1) is a lax fibered product, there is a unique continuous map of spaces

$m\colon \gamma G_1 \times_{G_0} \gamma G_1 \to \gamma G_1$ such that $d_1 m = d_1 \pi_1$, $d_0 m = d_0 \pi_2$,

and the composition (4) coincides with $\xi \cdot m$:

(5)


It is somewhat tedious but straightforward to verify that γG thus defined is indeed a continuous category, by using the universal property of γG_1. Alternatively, by stability (3.2 below) composition in γG can be described in point-set language (using change-of-base) and it is then obvious that the laws of a category hold. cf. 3.5 below.

Notice that the definition of γG is functorial in G.

3.2. Stability LEMMA. *Let p: $F \to E$ be a geometric morphism. and let G be a continuous groupoid in E. Then there is an isomorphism of continuous categories in F. $p^{\#}(\gamma G) \approx \gamma p^{\#}(G)$.*

PROOF. Obvious from 2.1.

3.3. The continuous homomorphism ϑ: $G \to \gamma G$. Let G be a continuous groupoid in the base topos, with associated continuous category γG. The action of G on étale spaces defines a natural transformation μ: $d_1^* \pi_G^* \to d_0^* \pi_G^*$:

(1)
$$\begin{array}{ccc} Sh(G_1) & \xrightarrow{d_1} & Sh(G_0) \\ {\scriptstyle d_0}\downarrow & \overset{\mu}{\swarrow\!\!\!\swarrow} & \downarrow{\scriptstyle \pi_G} \\ Sh(G_0) & \xrightarrow{\pi_G} & BG \end{array}$$

so by the universal property of 3.1 (1). there is a unique continuous map

ϑ_1: $G_1 \to (\gamma G)_1$ such that $d_0\vartheta_1 = d_0$. $d_1\vartheta_1 = d_1$ and $\mu = \xi \cdot \vartheta_1$.

Letting $\vartheta_0 : G_0 \to (\gamma G)_0$ be the identity, we obtain a continuous homomorphism

(2) $$\vartheta\colon G \to \gamma G.$$

Clearly. the definition is natural in G and stable under change-of-base. We will come back to this map ϑ in 3.9 below.

3.4. REMARK. Let $\hat{G}_1 \subset \gamma G_1$ be the subspace of invertible morphisms in the category γG. with inclusion i: $\hat{G}_1 \hookrightarrow \gamma G_1$. Then $\xi' = \xi \cdot i$ is an isomorphism and

(1)
$$\begin{array}{ccc} Sh(\hat{G}_1) & \xrightarrow{d_1} & Sh(G_0) \\ {\scriptstyle d_0}\downarrow & \overset{\xi'}{\swarrow\!\!\!\swarrow} & \downarrow{\scriptstyle \pi_G} \\ Sh(G_0) & \xrightarrow{\pi_G} & BG \end{array}$$

is a pullback of toposes. The continuous groupoid $\hat{G}$ (with $\hat{G}_0 = G_0$) is precisely the étale completion of G considered in I.7.2.

3.5. Points of γG. Let G and γG be constructed in the base topos S. A point of γG_1 is a triple (x,x',α) where x and x' are points of G_0 and α is a natural transformation $\alpha\colon ev_{x'} \to ev_x$; here $ev_x\colon BG \to S$ is the functor taking an étale G-space E to its fiber E_x over x. Codomain and domain are given by:

(1) $$d_0(x,x',\alpha) = x, \qquad d_1(x,x',\alpha) = x'.$$

Clearly if $g\colon x \to x'$ is a map in G (a point of G_1) then the action of g defines a natural transformation

(2) $$g^*\colon ev_{x'} \to ev_x$$

given in point-set notation by

(3) $$g^*_E(e) = e\cdot g.$$

So (x,x',g^*) defines a point of γG_1, and this describes precisely the map ϑ on points:

(4) $$\vartheta(g) = (d_0 g, d_1 g, g^*).$$

Notice that by stability and change-of-base, ϑ can actually be defined by formula (4), provided one interprets g as a point of G_1 (or really $p^\# G$, but we suppress base-extensions from notation) in an arbitrary base extension $p\colon E \to S$.

Composition in γG_1 can be described similarly: to define $m\colon \gamma G_1 \times_{\gamma G_0} \gamma G_1 \to \gamma G_1$, it is enough (Yoneda Lemma) to define for each test space T a function

$$m_T\colon \mathrm{Cts}(T,\gamma G_1) \times_{\mathrm{Cts}(T,G_0)} \mathrm{Cts}(T,\gamma G_1) \to \mathrm{Cts}(T,\gamma G_1),$$

natural in T. But a pair of continuous maps $T \to \gamma G_1$ in the domain of m_T is nothing but a pair of points of γG_1, two triples (x,x',α) and (x',x'',β), not in S but in the base extension Sh(T), and $m_T((x,x',\alpha),(x',x'',\beta))$ is just $(x,x'',\alpha\cdot\beta)$. So from the point of view of test spaces, it is clear that composition is associative, etc.

Let us consider a point (x,x',α) of γG_1 more closely: First of all, $\alpha\colon ev_{x'} \to ev_x$ is completely determined by its components at generators $G_1 \cap d_1^{-1}(U)/N$ of BG. Moreover, if $[g]$ is any point of $G_1 \cap d_1^{-1}(U)/N$ in any base-extension, represented by a morphism $g\colon x' \to y$ with $y \in U$, then for a small neighborhood V of x' we can find a section $a\colon V \to G_1 \cap d_1^{-1}(U)/N$ of the étale G-space $G_1 \cap d_1^{-1}(U)/N$ such that $a(x') = [g]$, and this defines a morphism in BG,

$$\tilde{a}\colon G_1 \cap d_1^{-1}(V)/M \to G_1 \cap d_1^{-1}(U)/N \text{ with } \tilde{a}(x') = [g],$$

for M small enough (see 1.4). Now $\alpha_{G_1 \cap d_1^{-1}(V)/M}([s_{x'}])$ is represented by some arrow $h\colon x \to z$ in G where z is a point of V: naturality of α as in

$$\begin{array}{ccc} (G_1 \cap d_1^{-1}(V)/M)_{x'} & \xrightarrow{\tilde{a}_{x'}} & (G_1 \cap d_1^{-1}(U)/N)_{x'} \\ \alpha \downarrow & & \downarrow \alpha = \alpha_{(G_1 \cap d_1^{-1}(U)/N)} \\ (G_1 \cap d_1^{-1}(V)/M)_{x} & \xrightarrow{\tilde{a}_{x}} & (G_1 \cap d_1^{-1}(U)/N)_{x} \end{array}$$

gives

$$\alpha_{G_1 \cap d_1^{-1}(V)/M}([g]) = \tilde{a}_x(\alpha_{(G_1 \cap d_1^{-1}(V)/M)}([s x']))$$
$$= \tilde{a}_x([h]) = [a(z) \cdot h].$$

In other words, α is completely determined by its values

$$\alpha_{G_1 \cap d_1^{-1}(V)/M}([s x']) \in (G_1 \cap d_1^{-1}(V)/M)_x,$$

where V ranges over all neighborhoods of x' and M over all open V-congruences. So α corresponds to a unique point in the space

(5) $$\varprojlim_{V,M}(G_1 \cap d_1^{-1}(V)/M)_x$$

i.e., given points x, x' of G_0, the space $\gamma G(x, x') \subset \gamma G_1$ of morphisms $x \to x'$ is precisely the inverse limit (5),

(6) $$\gamma G(x, x') \approx \varprojlim_{V,M}(G_1 \cap d_1^{-1}(V)/M)_x.$$

Therefore, we will also write points of γG_1 as triples

(7) $(x, x', \bar{g})$ where $\bar{g}$ is a sequence (8) $\bar{g} = \{[g_{V,M}]\}_{V,M}$

of equivalence classes $[g_{V,M}: x \to x_{V,M}]$ with $x_{V,M} \in V$ (V ranging over neighborhoods of x', M over open V-congruences).

If G is a continuous group ($G_0 = 1$) then (6) reduces to

$$\gamma G \approx \varprojlim_M G/M$$

where M ranges over the open subgroups of G. So γG is precisely the monoid associated to G that I considered in [8] (there γG was called M(G)).

3.6. Points of γG (bis). By the preceding discussion, we may use the following scheme to define points of $\gamma G(x, x')$ where x and x' are given points of G_0. Suppose $V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$ is a neighborhood basis at x', and that $M_i \subset G_1$ ($i = 1,2,\ldots$) is a system of open V_i-congruences (cf. 1.3) such that

$$M_{i+1} \subset M_i | V_{i+1} = M_i \cap (d_0, d_1)^{-1}(V_{i+1} \times V_{i+1})$$

which form a *cofinal system at* x' in the sense that for any neighborhood W of x and any open W-congruence $N \subset G_1$, there is an i such that $V_i \subset W$ and $M_i \subset N$, thus giving rise to a map in **S**(G),

$$\tilde{s}: G_1 \cap d_1^{-1}(V_i)/M_i \to G_1 \cap d_1^{-1}(W)/N$$

(cf. 1.3). Moreover, suppose that $U_0 \geq U_1 \geq \cdots$ is a neighborhood basis at x. Let

(1) $$a_i: U_i \to G_1 \cap d_1^{-1}(V_i)/M_i \qquad (i = 0,1,2,\ldots)$$

be a coherent family of sections of the étale G-spaces $G_1 \cap d_1^{-1}(V_i)/M_i$, coherent in the sense that for each i,

(2) $$\begin{array}{ccc} U_{i+1} & \xrightarrow{a_{i+1}} & G_1 \cap d_1^{-1}(V_{i+1})/M_{i+1} \\ \downarrow & & \downarrow \tilde{s} \\ U_i & \xrightarrow{a_i} & G_1 \cap d_1^{-1}(V_i)/M_i \end{array}$$

commutes. Then $[a_i]_i$ defines a point of $\gamma G(x,x')$: in the form of 3.5 (8), if V is any neighborhood of x and M is a V-congruence, choose i so large that $V_i \leq V$, $M_i \leq M$, so that there is a map

$$\tilde{s}: G_1 \cap d_1^{-1}(V_i)/M_i \to G_1 \cap d_1^{-1}(V)/M$$

and let

$$[g_{V,M}] = \tilde{s}(a_i(x)) \in G_1 \cap d_1^{-1}(V)/M.$$

3.7. REMARK. If one writes points of γG_1 as triples $(x,x',\bar{g})$ where $\bar{g}$ is a sequence as in 3.5 (8), then composition in γG can be described as follows: if $(x',x'',\bar{h})$ is another such point, then

$$(x',x'',\bar{h}) \cdot (x,x',\bar{g}) = (x,x'',\bar{k})$$

where the sequence $\bar{k}$ has components $[k_{UM}]$ (U a neighborhood of x'', M an open U-congruence) defined by choosing a section $a: V \to G_1 \cap d_1^{-1}(U)/M$ through $[h_{UM}]$ on a neighborhood V of x', letting N be an open V-congruence small enough for a to define a morphism of G-spaces

$$\tilde{a}: G_1 \cap d_1^{-1}(V)/N \to G_1 \cap d_1^{-1}(U)/M$$

and then setting $[k_{U,M}] = \tilde{a}([g_{V,N}])$.

3.8. A presentation of γG. From the description of points of γG_1 (in any base extension, by 3.2) it is not difficult to obtain a presentation of the space γG_1, by a preorder equipped with a stable covering system (I.1.1, [6], §III.4). The elements of the preorder **B** are triples

(1) $[U,N$

where $U \subset G_0$ is open, N is an open U-congruence, and A is an open subspace of $G_1 \cap d_1^{-1}(U)/N$.

(As an open subspace of γG_1, $[U,N,A]$ is defined by stating that a point $(x, x', \bar{g})$ (in any base extension) as in 3.5 (8) lies in $[U,N,A]$ iff $x' \in U$ and $[g_{U,N}] \in A$ (so $x \in d_0(A)$).)

The preorder on elements of the form (1) is generated by three conditions:

(i) If $U' \leq U$ then $[U',N|U', A|U'] \leq [U,N,A]$;

(ii) If $A' \leq A$ then $[U,N,A'] \leq [U,N,A]$ (for given U,N);

(iii) If for given U, N' and N are open U-congruences with $N' \subset N$, and

$$\tilde{s} : G_1 \cap d_1^{-1}(U)/N' \to G_1 \cap d_1^{-1}(U)/N$$

denotes the projection (an étale surjection), then for A contained in $G_1 \cap d_1^{-1}(U)/N$,

(a) $[U,N,A] \leq [U,N', \tilde{s}^{-1}(A)]$

and, for $B \subset G_1 \cap d_1^{-1}(U)/N'$,

(b) $[U,N',B] \leq [U,N', \tilde{s}(B)]$.

Notice that (a) and (b) imply

(2) $$[U,N,A] = [U,N', \tilde{s}^{-1}(A)]$$

since $\tilde{s}\tilde{s}^{-1}(A) = A$.

The *covering system* on this preorder **B** is generated by covers of two kinds:

(α) If $\{U_i\}_i$ covers U in G_0, then $\{[U_i, N|K_i, A|K_i]_i\}$ covers $[U,N,A]$;

(β) If $\{A_i\}_i$ covers A in the space $G_1 \cap d_1^{-1}(U)/N$, then $\{[U,N|K,A_i]_i\}$ covers $[U,N,A]$.

Notice that this is a *stable* generating system.

Alternatively, one can define **B** as a semilattice, where meets are given by:

(3) $$[U,N,A] \wedge [V,M,B] = [U \cap V, N|V \cap M|U, \tilde{s}^{-1}(A) \cap \tilde{s}^{-1}(B)]$$

where $\tilde{s}^{-1}(A)$ is the inverse under

$$\tilde{s} : G_1 \cap d_1^{-1}(U \cap V)/(N \cap M) \to G_1 \cap d_1^{-1}(U)/N,$$

and similarly for $\tilde{s}^{-1}(B)$.

To see that **B** equipped with this covering system is indeed a presentation of γG_1, it is enough to show that if $P \subseteq \mathbf{B}$ is a subset which is inhabited, closed under meets, and is such that if a cover of some $[U,N,A]$ is contained in P then so is $[U,N,A]$ itself, then P gives rise to a unique point of γG_1. Let

$$x' = \{U \mid \exists M,A : [U,M,A] \in P\}.$$

Then x' defines a point of G_0 by the covers of type (α) and order-condition (i) (or (3)). Similarly, if [U,M,A] is any element of P, then

$$\{A_i \subset G_1 \cap d_1^{-1}(U)/M \mid [U,M,A_i] \in P\}$$

defines a point $[g_{U,M}]$ of $G_1 \cap d_1^{-1}(U)/M$, and therefore a point $x = d_0(g_{U,M})$ of G_0. Now clearly from (iii), the sequence $\bar{g} = \{[g_{U,M}]\}$ where U ranges over neighborhoods of x' and M over open U-congruences, defines a point $(x,x',\bar{g})$ of γG_1, by 3.5 (8).

In the sequel, we will refer to opens of γG_1 of the form [U,M,A] as *basic* opens, and often (implicitly) use point-set notation

$$[U,M,A] = \{(x,x',\bar{g}) \mid x' \in U,\ [g_{U,M}] \in A\}$$

(where the right-hand side is considered as a set of points in a variable base extension).

3.9. PROPOSITION. *The continuous homomorphism* $\vartheta : G \to \gamma G$ *induces an equivalence of toposes* $BG \xrightarrow{\sim} B\gamma G$.

PROOF. (As usual, we freely use point-set language as everything is stable, and leave base extensions implicit.) Let E be an étale G-space, and let $g: x \to y$ be a point of G (in a base extension). The action by g gives a map $g^*: E_y \to E_x$ which only depends on $\vartheta(g)$. To see this, take $e \in E_y$ and a section $a: U \to E$ through e, where U is a neighborhood of y. By continuity, there is a neighborhood $W \subseteq G_1 \cap (d_0, d_1)^{-1}(U \times U)$ of $s(U)$ such that for any $h \in W$,

(1) $$a(d_1 h) \cdot h \in a(U).$$

Let W' be the closure of W under inverse and composition. W' is an open U-congruence and for any $h \in W'$, $a(d_1 h) \cdot h \in a(U)$. So if $g': x \to y$ is another point of G_1 such that $\vartheta(g) = \vartheta(g')$, then $[g] = [g']$ in $G_1 \cap d_1^{-1}(U)/W'$, and therefore there is (in some open surjective base extension) an $h \in W'$ with $hg = g'$. Then

$$e \cdot g' = (e \cdot h) \cdot g = e \cdot g.$$

On the other hand, the action $\cdot : E \times_{G_0} G_1 \to E$ of G on E can be extended to an action $*: E \times_{G_0} \gamma G_1 \to E$ by γG in an obvious way: if (x,x',α) is a point of γG_1 (cf. 3.5), i.e., $\alpha: ev_y \to ev_x$, then for $e \in E_y$

(2) $$e * (x,y,\alpha) = \alpha_E(y).$$

Clearly $e * \vartheta(g) = e \cdot g \in E_x$, so this indeed extends the action by G (which by the preceding can be recovered from the action of

γG). It remains to see that the extension is unique. To this end, it is easier to think of points of γG_1 as given in the form $(x,y,\bar{g})$ as in 3.5 (8). If $e \in E_y$, take a section $a: U \to E$ through e; then $e*(x,y,\bar{g})$ is the limit in E_x of the sequence

$$(3) \qquad \{a(d_1(g_{V,M}))*\vartheta(g_{V,M})\}_{V,M}$$

(V ranging over neighborhoods of y contained in U), by continuity of the action $*: E \times_{G_0} \gamma G_1 \to E$. ((3) eventually becomes constant, since E_x is discrete.) So $*$ is completely determined by what it does on γG-morphisms in the image of ϑ.

From this, the equivalence $BG \approx B\gamma G$ is clear.

3.10. REMARK. If E is an étale G-space, the action $E \times_{G_0} G_1 \to E$ is an open map, as pointed out in 1.1. It is easy to see that the extension $E \times_{G_0} \gamma G_1 \to E$ described above is again an open map.

4. BISPACES.

In this section, G and H are continuous groupoids, with associated completions γG and γH. We will discuss how certain spaces equipped with an action by γG, as well as one by γH, give rise to geometric morphisms $BH \to BG$.

4.1. γG-γH-**bispaces.** A γG-γH-*bispace* is a space R equipped with an action of γG on the left and one of γH on the right, such that these commute; so there are maps $p_G: R \to G_0$, $p_H: R \to H_0$, and actions

$$*: \gamma G_1 \times_{G_0} R \to R, \quad \cdot: R \times_{H_0} \gamma H_1 \to R$$

satisfying the usual unit- and associativity identities. When γG and γH are understood, we will just speak of bispaces.

If R and R' are two γG-γH-bispaces, a homomorphism f: $R \to R'$ of bispaces is a continuous map of spaces which is both a map of γG-spaces and one of γH-spaces; i.e., f satisfies the usual identities

$$p_G f(r) = p_G(r), \quad p_H f(r) = p_H(r),$$
$$f(\xi * r) = \xi * f(r) \text{ and } f(r \cdot \eta) = f(r) \cdot \eta.$$

This defines a category (γG-γH-bispaces).

A bispace R is called *open* if

(i) $p_H: R \to H_0$ is open;

(ii) both action maps $*$: $\gamma G_1 \times_{G_0} R \to R$ and $\cdot$: $R \times_{H_0} \gamma H_1 \to R$ are open;

(iii) the diagonal action $\mu: \gamma G_1 \times_{G_0} \gamma G_1 \times_{G_0} R \to R \times_{H_0} R$ defined by $\mu(\xi, \xi', r) = (\xi * r, \xi' * r)$ is open

(so the pullback $\gamma G_1 \times_{G_0} \gamma G_1$ in (iii) is along d_0 on both sides).

4.2. Tensor products. If E is a γG-space, with action $\cdot$: $E \times_{G_0} \gamma G_1 \to E$, and R is a bispace as above, we may construct the *tensor product* $E \otimes_{\gamma G} R$ as the coequalizer of spaces

(1) $$E \times_{G_0} \gamma G_1 \times_{G_0} R \underset{\cdot \times R}{\overset{E \times *}{\rightrightarrows}} E \times_{G_0} R \longrightarrow E \otimes_{\gamma G} R.$$

If $E \otimes *$ and $\cdot \times R$ are open maps, the coequalizer (1) is stable ([8], Lemma 1.2), so in that case we can use change-of-base techniques and point-set arguments to investigate the structure of $E \otimes_{\gamma G} R$. In particular, the right γH-space structure of R can then be used to define a right action of γH on $E \otimes_{\gamma G} R$. So if R is open, (1) defines a functor

(2) $$- \otimes_{\gamma G} R\text{: (open } \gamma G\text{-spaces)} \longrightarrow \text{(open } \gamma H\text{-spaces)}$$

where we call a γG-space E open if the action $E \otimes_{\gamma G} \gamma G_1 \to E$ is an open map.

If E is an étale G-space, we define $E \otimes_{\gamma G} R$ as the tensor product (1), where E is regarded as an open γG-space by 3.9, 3.10.

4.3. LEMMA. *If* R *is an open bispace and* E *is an étale* G*-space, then* $E \otimes_{\gamma G} R$ *is an étale* γH*-space. So* R *defines a functor* $- \otimes_{\gamma G} R: BG \to BH$.

PROOF. Since $- \otimes_{\gamma G} R$ preserves colimits, it is enough to show that $- \otimes_{\gamma G} R$ sends generators to étale γH-spaces (colimits in BG are computed just as colimits of G-spaces). Take a generator $G_1 \cap d_1^{-1}(U)/N$ (cf. 1.3). Write $[U, N, s(U)]$ for the basic open (3.8) of γG_1 given by the open set $s(U) \subset G_1 \cap d_1^{-1}(U)/N$, where s: $U \to G_1 \cap d_1^{-1}(U)/N$ is the section coming from the "identity" $s: G_0 \to G_1$. Let $R_U = R \cap p_G^{-1}(U)$, and write $R_U/[U, N, s(U)]$ for the quotient of R_U by the action of $[U, N, s(U)]$: i.e.,

(1) $$[U, N, s(U)] \otimes_{\gamma G} R_U \underset{*}{\overset{\pi_2}{\rightrightarrows}} R_U \xrightarrow{\ q\ } R_U/[U, N, s(U)]$$

is a coequalizer. Clearly

(2) $$(G_1 \cap d_1^{-1}(U)/N) \otimes_{\gamma G} R \approx R_U/[U, N, s(U)].$$

Now consider the following diagram, in which μ is open by assumption (cf. 4.1 (iii)), as is the quotient map q (by [8], Lemma 1.2), so the diagonal Δ must be open.

$$\begin{array}{ccc}
[U,N,s(U)]\times_{G_0}[U,N,s(U)]\times_{G_0}R_U & \xrightarrow{\ \mu\ } & R_U\times_{H_0}R_U \\
\downarrow & & \downarrow{\scriptstyle q\times q} \\
R_U/[U,N,s(U)] & \xrightarrow{\ \Delta\ } & R_U/[U,N,s(U)]\times_{H_0}R_U/[U,N,s(U)]
\end{array}$$

Since $p_H\colon R\to H_0$ is open, it follows that $p_H\colon R_U/[U,N,s(U)]\to H_0$ is étale ([6], § V.5). This proves the lemma.

4.4. Left flat bispaces. A γG-γH-bispace R is called *left-flat* if R is open (4.1) and $-\otimes_{\gamma G}R$ preserves finite limits of étale G-spaces, i.e., (cf. 4.3) the functor $-\otimes_{\gamma G}R\colon BG\to \mathbf{B}H$ is left-exact.

So a left-flat bispace R induces a geometric morphism $g(R)$: $BH\to BG$ given by

(1) $$g(R)^*(E) = E\otimes_{\gamma G}R.$$

If R and R' are both left-flat bispaces and $\gamma\colon R\to R'$ is a homomorphism of bispaces, then clearly we obtain a natural transformation $-\otimes_{\gamma G}R \to -\otimes_{\gamma G}R'$, i.e., a 2-cell $g(R)\to g(R')$. So if we write $Flat(\gamma G,\gamma H)$ for the full subcategory of bispaces whose objects are left-flat, and $Hom_S(BH,BG)$ for the category of geometric morphisms $BH\to BG$ over the base topos S, we obtain a functor $g\colon Flat(\gamma G,\gamma H)\to Hom_S(BH,BG)$.

4.5. PROPOSITION. *Let* G, H, K *be continuous groupoids, and let* R *be an open* γG-γH-*bispace,* S *an open* γH-γK-*bispace. Then*

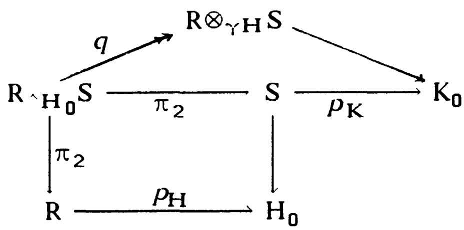
(i) $R\otimes_{\gamma G}S$ *is an open* γG-γK-*bispace.*

(ii) for any open γG-*space* E, *there is a canonical isomorphism*

$$(E\otimes_{\gamma G}R)\otimes_{\gamma H}S \approx E\otimes_{\gamma G}(R\otimes_{\gamma H}S).$$

(iii) if R *and* S *are left-flat, so is* $R\otimes_{\gamma H}S$.

PROOF. (i) Consider the diagram



where q is a quotient map. Since p_H is open, so is π_2, and hence since p_K is open, $p_K\pi_2$ is open. Since q is a surjection, it follows that $R\otimes_{\gamma G}S \to K_0$ is open.

To see that the action $*$ of γG on $R\otimes_{\gamma G}S$ is open, consider the diagram

$$\begin{array}{ccc}
\gamma G_1\times_{G_0}R\times_{H_0}S & \xrightarrow{\;*\times S\;} & R\times_{H_0}S \\
\Big\downarrow{\scriptstyle \gamma G_1\times q} & & \Big\downarrow{\scriptstyle q} \\
\gamma G_1\times_{G_0}(R\otimes_{\gamma H}S) & \xrightarrow{\;*\;} & R\otimes_{\gamma H}S
\end{array}$$

Since $*$ and q are open surjections, so are $q\cdot(*\times S)$ and $\gamma G_1\times q$. Hence

$$*: \gamma G_1\times_{G_0}(R\otimes_{\gamma H}S) \to R\otimes_{\gamma H}S$$

is open.

The proof that the action $(R\otimes_{\gamma H}S)\times_{K_0}\gamma K_1 \to R\otimes_{\gamma H}S$ is open is similar.

Finally, we show that the diagonal action

$$\mu: \gamma G_1\times_{G_0}\gamma G_1\times_{G_0}(R\otimes_{\gamma H}S) \to R\otimes_{\gamma H}S$$

is open. Consider the diagram

$$(1)\qquad \begin{array}{ccc}
(R\times R)\times_{H_0\times H_0}(\gamma H_1\times_{H_0}\gamma H_1)\times_{H_0}S & \xrightarrow{\;\psi\;} & (R\times_{H_0}R)\times_{H_0}S \\
\Big\downarrow{\scriptstyle (R\times R)\times\mu} & & \\
(R\times R)\times_{H_0\times H_0}(S\times_{K_0}S) & & \swarrow{\scriptstyle \varphi} \\
\Big\downarrow{\scriptstyle (q\times q)\cdot\tau} & & \\
(R\otimes_{\gamma H}S)\times_{K_0}(R\otimes_{\gamma H}S) & &
\end{array}$$

where in (1), μ is the diagonal of S (an open surjection by hypothesis), τ interchanges the second and third coordinates, q: $R\times_{H_0}S \to R\otimes_{\gamma H}S$ is the quotient map, and φ, ψ are described in point-set notation by

$$\psi(r,r',h,h',s) = (r\cdot h, r'\cdot h', s), \quad \varphi(r,r',s) = (r\otimes s, r'\otimes s).$$

The diagram commutes (by the identity $r\cdot h\otimes s = r\otimes hs$ for points of $R\otimes_{\gamma H}S$), ψ is surjective (it splits), while $(R\times R)\times\mu$ and $(q\times q)\cdot\tau$ are open surjections, so φ is an open surjection. Next, consider the following diagram where μ' is the diagonal action of R and μ that of $R\otimes_{\gamma H}S$.

$$
(2)\qquad
\begin{array}{ccc}
\gamma G_1\times_{G_0}\gamma G_1\times_{G_0}(R\times_{H_0}S) & \xrightarrow{1\times q} & \gamma G_1\times_{G_0}\gamma G_1\times_{G_0}(R\otimes_{\gamma H}S) \\
\downarrow{\scriptstyle \mu'\times s} & & \\
R\times_{H_0}R\times_{H_0}\times S & & \swarrow{\scriptstyle \mu} \\
\downarrow{\scriptstyle \varphi} & & \\
(R\otimes_{\gamma H}S)\times_{K_0}(R\otimes_{\gamma H}S) & &
\end{array}
$$

Since φ and μ' are open surjections, while q is a surjection, μ must be an open surjection.

(ii) The coequalizer

$$
E\times_{G_0}\gamma G_1\times_{G_0}R \rightrightarrows E\times_{G_0}R \longrightarrow E\otimes_{\gamma G}R
$$

is stable if E is an open γG-space (as noted in 4.2): therefore one obtains some other coequalizers from it, and similarly for the coequalizer defining $R\otimes_{\gamma H}S$. These fit together in a 3×3 diagram, and the argument then proceeds as in [4] p. 60 (proof of the associativity of composition of profunctors).

(iii) This follows from (i) and (ii).

5. BISPACES INDUCED BY GEOMETRIC MORPHISMS.

In this section we will prove our main result, namely that every geometric morphism comes from tensoring with a suitable bispace.

5.1. Construction of the functor R. Let G and H be continuous groupoids, with completions γG and γH respectively, as discussed in §3, and let $f\colon BH\to BG$ be a geometric morphism. The *representation* $R(f)$ of f as a bispace is the space defined by the lax fibered product

$$
(1)\qquad
\begin{array}{ccccc}
Sh(R(f)) & & \xrightarrow{\quad p_G \quad} & & Sh(G_0) \\
\downarrow{\scriptstyle p_H} & & \xi(f)\,\Leftarrow & & \downarrow{\scriptstyle \pi_G} \\
Sh(H_0) & \xrightarrow{\pi_H} & BH & \xrightarrow{f} & BG
\end{array}
$$

Usually, we will just write ξ for the universal natural transformation $\xi(f)$: $p_G^*\pi_G^* \to p_H^*\pi_H^* f^*$. Notice that the lax fibered product square (1) indeed defines a unique space R(f), by 2.4.

The aim of this section is to show that R(f) is an open γG-γH-bispace, and that there is a natural isomorphism $-\otimes_{\gamma G} R(f) \approx f^*$ of functors BG$\to$BH.

5.2. Bispace structure of R(f). We will show that R(f) has a bispace structure given by an action of γG on the left and one of γH on the right. There are two possible approaches: one is to use the universal property of R(f) (similarly to the approach in 3.1). Alternatively, one can use change-of-base techniques and work with points of R(f); we shall follow the latter approach. It is clear from the definition that a point of R(f) is given as a triple

(1) $$(x, y, \sigma)$$

where $x \in G_0$ and $y \in H_0$ are points, and σ is a natural transformation

(2) $$\sigma:\ ev_x \to ev_y \cdot f^*$$

(recall that ev_x: BG$\to$S takes an étale G-space E to its fiber E_x over x).

In 3.5, we described points of γG_1 as triples (x, x', α) where $\alpha: ev_{x'} \to ev_x$. The action ν of γG on R(f) on points is simply described by

(3) $$(x, x', \alpha) * (x, y, \sigma) = (x', y, \sigma\alpha)$$

(where we write $-*-$ for $\nu(-,-)$). Similarly, denoting the action μ of γH on R(f) by a dot, the map $\mu: R(f) \times_{H_0} \gamma H_1 \to R(f)$ is given on points by

(4) $$(x, y', \sigma) \cdot (y, y', \beta) = (x, y, \sigma \cdot (\beta \cdot f^*)).$$

It is important to note that (3) and (4) apply to points of γG, γH and R(f) defined over *any* base extension, since by stability of the construction involved, $p^{\#}(R(f)) = R(p^{\#} f)$, in analogy with 3.2. Therefore, using the familiar method of test-spaces and base-extensions, (3) and (4) can be applied to actually *define* the action maps μ and ν. From this point of view, it is clear that the bispace identities hold for μ and ν.

We remark that the construction of R(f) is functorial in f: if f, g: BH$\to$BG are two geometric morphisms, and τ: $f^* \to g^*$ is a natural transformation, the universal property of R(g) gives a unique continuous map $R(\tau): R(f) \to R(g)$ such that $p_G \cdot R(\tau) = p_G$,

$p_H \cdot R(\tau) = p_H$, and

$$\pi_G p_G \xrightarrow{\xi(f)} f\pi_H p_H \xrightarrow{\tau\cdot\pi_H p_H} g\pi_H p_H = \pi_G p_G \xrightarrow{\xi(g)} g\pi_H p_H.$$

It is not difficult to check that $R(\tau)$ is a homomorphism of bi spaces.

Let us take a closer look at a point (x,y,σ) of $R(f)$. Just as in 3.5, the natural transformation $\sigma: ev_x \to ev_y \cdot f^*$ is completely determined by its values $\sigma_{G_1\cap d_1^{-1}(W)/M}([sx])$ of identities at generators — here U ranges over open neighborhoods of x and M over open U-congruences, as in 3.5. So a point (x,y,σ) of $R(f)$ can alternatively be represented as a triple

(5) $$(x,y,\bar{r})$$

where $\bar{r} = \{r_{U,M}\}_{U,M}$ is a sequence of points

$$r_{U,M} \in f^*(G_1\cap d_1^{-1}(U)/M)_y$$

coherent in the sense that for any $U' \leq U$ and $M' \leq M|U$, with associated map

$$\tilde{s}: G_1\cap d_1^{-1}(U')/M' \to G_1\cap d_1^{-1}(U)/M$$

(cf. 1.4) we have

(6) $$f^*(\tilde{s}(r_{U',M'})) = r_{U,M}.$$

In other words, the fiber $R(f)(x,y)$ over the pair of points x of G_0, y of $\mathbf{H}_0$ can be described as an inverse limit

(7) $$R(f)(x,y) \approx \varprojlim_{U,M} f^*(G_1\cap d_1^{-1}(U)/M)_y.$$

Writing points of $R(f)$ in the form (5), and points of γG, γH in the form 3.5 (7), the actions $\mu: R(f)\times_{H_0}\gamma H_1 \to R(f)$ (denoted by $\cdot$) and $\nu: \gamma G\times_{G_0} R(f) \to R(f)$ (denoted by $*$) can be described as follows: For $(x,y,\bar{r}) \in R(f)$, $(x,x',\bar{g}) \in \gamma G$ and $(y',y,\bar{h}) \in \gamma H$,

(8) $$(x,y,\bar{r})\cdot(y',y,\bar{h}) = (x,y',\bar{r}\cdot\bar{h})$$

where for $x \in U$ and open U-congruence M, $(\bar{r}\cdot\bar{h})_{U,M}$ is defined by

(9) $$(\bar{r}\cdot\bar{h})_{U,M} = r_{U,M}\cdot(y',y,\bar{h}),$$

the action on the right-hand side of (9) coming from the fact that $f^*(G_1\cap d_1^{-1}(U)/M) \in BH$ has a γH-space structure by 3.9; for the action of γG, we have

(10) $$(x,x',\bar{g})*(x,y,\bar{r}) = (x,y',\bar{g}*\bar{r})$$

where for $x' \in U'$ and open U'-congruence M', $(\bar{g}*\bar{r})_{U',M'}$ is defined by choosing a section

(11) $$b: U \to G_1\cap d_1^{-1}(U')/M'$$

with $x \in U$, $b(x) = [g_{U',M'}]$, and choosing M small enough for b

to define a map of étale G-spaces

(12) $$\tilde{b} : G_1 \cap d_1^{-1}(U)/M \to G_1 \cap d_1^{-1}(U')/M' \; ;$$

then

(13) $$(\bar{g} * \bar{r})_{U',M'} = f^*(b)(r_{U,M}).$$

5.3. Points of R(f). Analogously to 3.6, points of R(f) may be defined in the following way. Let $x \in G_0$ and $y \in H_0$ be points, let (V_i, M_i), $i = 0,1,2,\ldots$ be a cofinal system at x (as in 3.6, so $\{V_i\}_i$ is a neighborhood basis at x, M_i is an open V_i-congruence, etc.), and let $U_0 \geq U_1 \geq \cdots$ be a neighborhood basis at y. A coherent system of sections b_i: $U_i \to f^*(G_1 \cap d_1^{-1}(U_i)/M_i)$, coherent in the sense that each square

$$\begin{array}{ccc} U_{i+1} & \xrightarrow{b_{i+1}} & f^*(G_1 \cap d_1^{-1}(U_{i+1})/M_{i+1}) \\ \downarrow & & \downarrow f^*(\tilde{s}) \\ U_i & \xrightarrow{b_i} & f^*(G_1 \cap d_1^{-1}(U_i)/M_i) \end{array}$$

commutes, defines a point r of R(f)(x,y) in the form 5.2 (5). If V is any open neighborhood of x and M any V-congruence, then for i large enough, $V_i \subset V$ and s: $V_i \to G_1 \cap d_1^{-1}(V)/M$ induces a map $\tilde{s}$: $G_1 \cap d_1^{-1}(U_i)/M_i \to G_1 \cap d_1^{-1}(V)/M$,

and we put $r_{V,M} = f^*(\tilde{s})(b_i(y))$.

5.4. Basic opens of R(f). Analogously to 3.8, one can show that the space R(f) is generated by opens of the form

(1) $$[U,M,B]$$

where $U \subset G_0$ is open, $M \subset G_1$ is an open U-congruence, and B is an open subspace of the étale H-space $f^*(G_1 \cap d_1^{-1}(U)/M)$. Writing points of R($f$) in the form 5.2 (5), $[U,M,B] \subset R(f)$ is defined as the subspace consisting of those points (in any base-extension) $(x,y,\bar{r})$ such that $x \in U$ and $r_{U,M} \in B$. Suggestively, we put

(2) $$[U,M,B] = \{(x,y,\bar{r}) \mid x \in U, \; r_{U,M} \in B\}.$$

One can define a presentation of R(f) by a preorder equipped with a stable covering system, completely analogous to 3.8, and details are left to the reader.

5.5. THEOREM. *Let be a geometric morphism. Then the associa-*

ted bispace R(f) *is open, i.e. (see 4.1):*

(i) p_H: R(f)→H_0 *is open.*

(ii) Both actions $\gamma G_1 \times_{G_0} R(f) \to R(f)$ *and* $R(f) \times_{H_0} \gamma H_1 \to R(f)$ *are open.*

(iii) The diagonal action (also denoted by μ*)*

$$\mu: \gamma G_1 \times_{G_0} \gamma G_1 \times_{G_0} R(f) \to R(f) \times_{H_0} R(f)$$

is open.

PROOF. (i) is a special case of 2.2. For the second part of (ii), i.e., openness of the action

(1) $$\cdot: R(f) \times_{H_0} \gamma H_1 \to R(f),$$

consider a basic open of the domain space, say

(2) $$[U,M,B] \times_{H_0} [V,N,A]$$

where we may assume that B is the image of a section

(3) $$b: V \to f^*(G_1 \cap d_1^{-1}(U)/M)$$

and A the image of a section

(4) $$a: W \to H_1 \cap d_1^{-1}(V)/N :$$

moreover, we may assume that N is so small that b induces a map

(5) $$\tilde{b}: H_1 \cap d_1^{-1}(V)/N \to f^*(G_1 \cap d_1^{-1}(U)/M).$$

We claim that the image of (2) under the action (1) is the basic open

(6) $$[U,M,\tilde{b}(A)].$$

To prove this, take a point $(x,z,\bar{q})$ of $[U,M,\tilde{b}(A)]$ (in *any* base-extension!). So

(7) $$q_{U,M} = \tilde{b}(a(z)),\ z \in W,\ x \in U.$$

By going to a further open surjective base extension, we may choose a point $\xi: z \to z'$ of G_1 with $z' \in V$ so that $[\xi] = a(z)$ in the space $H_1 \cap d_1^{-1}(V)/N$. Now let $\bar{r} = \bar{q} \cdot \vartheta(\xi^{-1})$. Then $(x,z,\bar{r})$ is a point of R(f), $(z,z',\vartheta(\xi))$ one of γH_1, and

$$(x,z',\bar{r}) \cdot (z,z',\vartheta(\xi)) = (x,z,\bar{q}).$$

Moreover clearly $(z,z',\vartheta(\xi)) \in [V,N,A]$. So it remains to show that $(x,z',\bar{r}) \in [U,M,B]$. By definition of the action of γH on R(f), $r_{U,M} = (\bar{q} \cdot \vartheta(\xi^{-1}))_{U,M}$ is constructed as follows: take a section $c: W' \to f^*(G_1 \cap d_1^{-1}(U)/M)$ through $q_{U,M}$, i.e., $c(z) = q_{U,M}$, where W' is some neighborhood of z, and one extends to a map

$$\tilde{c}: H_1 \cap d_1^{-1}(W')/N' \to f^*(G_1 \cap d_1^{-1}(U)/M)$$

for small enough N': then

$$r_{U,M} = (\bar{q}\cdot\vartheta(\xi^{-1}))_{U,M} = \tilde{c}([\xi^{-1}]).$$

But choosing $W = W'$ and $c = \tilde{b}\cdot a : W \to f^*(G_1\cap d_1^{-1}(U)/M)$, we find that $\tilde{c}([\xi^{-1}]) = b(z')$, i.e., $r_{U,M} \in \mathrm{im}(b) = B$.

This proves that (1) is open.

The first half of (ii) follows from (iii) by considering the diagram

$$\begin{array}{ccc} \gamma G_1\times_{G_0}\gamma G_1\times_{G_0}R(f) & \xrightarrow{\ \mu\ } & R(f)\times_{H_0}R(f)\\ \Big\downarrow{\scriptstyle\bar{\pi}_{13}} & & \Big\downarrow{\scriptstyle\pi_1}\\ \gamma G_1\times_{G_0}R(f) & \xrightarrow{\ *\ } & R(f) \end{array}$$

because π_1 is an open surjection by (i).

Rather than proving (iii), we prove the following stronger assertion.

5.6. PROPOSITION. *Let* $R(f)$ *be as in 5.5. and let* $U\subset G_0$ *be an open subspace, with associated basic open* $[U.N, s(U)]$ *(where* $s: U\to G_1\cap d_1^{-1}(U)/N$ *). Then the diagonal action defines an open surjection*

(1) $[U.N.s(U)]\times_{G_0}[U.N.s(U)]\times_{G_0}R(f) \to R(f)\times_{f^*(G\cap d_1^{-1}(U))}R(f)$.

PROOF. Consider a basic open in the domain of (1),

(2) $$[U_1.N_1.A_1]\times_{G_0}[U_2.N_2.A_2]\times_{G_0}[V.M.B]$$

where A_1, A_2, B are so small that they can be written as images of sections of appropriate étale spaces, say $A_i = \mathrm{im}(a_i)$, $B = \mathrm{im}(b)$, where

$$a_1: V \to G_1\cap d_1^{-1}(U_1)/N_1,\quad a_2: V\to G_1\cap d_1^{-1}(U_2)/N_2$$
$$b: W \to f^*(G_1\cap d_1^{-1}(V)/M).$$

By choosing V small enough in (2), we may assume that a_i is defined on V; moreover by choosing M small enough, we may assume that the a_i define maps of G-spaces

(3) $$\tilde{a}_i: G_1\cap d_1^{-1}(V)/M \to G_1\cap d_1^{-1}(U_i)/N_i.$$

We claim that the image of the basic open (2) in

$$R(f)\times_{f^*(G_1\cap d_1^{-1}(U)/N)}R(f)$$

is

(4) $$[U_1.N_1.f^*(\tilde{a}_1)(B)]\times_{H_0}[U_2.N_2.f^*(\tilde{a}_2)(B)].$$

Clearly this is enough to prove the proposition.

First, assume $S = \text{Sets}$ and $G_i, H_i, R(f)$ are all countably presented (hence have enough points). Take a pair of points

(5) $$(x_1, y, \bar{p}_1),\ (x_2, y, \bar{p}_2)$$

in the open subspace (4). So $x_j \in U_j$ and $(p_j)_{U_j,N_j} \in f^*(a_j)(B)$. We will define a new point $(x, y, \bar{r}) \in R(f)(x,y)$ and morphisms

$$x \xrightarrow{\ \bar{g}_1\ } x_1 \xrightarrow{\ \bar{g}_2\ } x_2$$

in γG such that

(6) $$(x, y, \bar{r}) \in [V, M, B],$$

(7) $$(x, x_j, \bar{g}_j) \in [U_j, N_j, A_j],$$

(8) $$(x, x_j, \bar{g}_j) * (x, y, \bar{r}) = (x_j, y, \bar{p}_j)$$

where $j = 1, 2$.

Fix a neighborhood basis

(9) $$W^0 \geq W^1 \geq \cdots$$

at $y \in H_0$, and fix a descending cofinal system at x_j

(10) $$U_j = U_j^0 \geq U_j^1 \geq \cdots, \qquad N_j = N_j^0 \geq N_j^1 \geq \cdots$$

(as in 3.6, so $\{U_j^k\}_k$ is a basis at x_j and N_j^k is an open U_j^k-congruence, etc.). Moreover, let $\mathbf{B}$ be a basis for G_0 and let for $B \in \mathbf{B}$

(11) $$\mathcal{U}^0(B) \geq \mathcal{U}^1(B) \geq \cdots$$

be an enumeration of the basic covers of B in a stable generating system. (G_0 is countably presented by assumption.) In (11), $\geq$ means "is refined by". Finally, fix for each $B \in \mathbf{B}$ a cofinal system of open B-congruences.

(12) $$M^1(B) \geq M^2(B) \geq \cdots$$

We now construct by induction sequences

$$\{m_k\}_k,\ \{V^k\}_k,\ \{M_k\}_k,\ \{a_j^k\}_k\ (j = 1, 2),\ \{b_k\}_k$$

where

(i) $\{V^k\}_k$ is a descending sequence of basic opens of G_0 (i.e., elements of $\mathbf{B}$) defining a neighborhood base at some point $x \in G_0$.

(ii) for $j = 1, 2$, $\{a_j^k : V_k \to G_1 \cap d_1^{-1}(U_j^k)/N_j^k\}_k$ is a coherent system of sections.

(iii) M^k is an open V^k-congruence such that $\{(V^k, M^k)\}_k$ is a cofinal system at x, and M^k is so small that a_j^k defines a map

$$\tilde{a}_j^k : G_1 \cap d_1^{-1}(V^k)/M^k \to G_1 \cap d_1^{-1}(U_j^k)/N_j^k.$$

(iv) $m_0 < m_1 < \cdots$ is a strictly increasing sequence of natural numbers, and

$$\{b_k : W^{m_k} \to f^*(G_1 \cap d_1^{-1}(V^k)/M^k)\}_k$$

is a coherent system of sections, and

(v) $$f^*(\tilde{a}_j^k)(b_k(y)) = (p_k)_{U_j^k, N_j^k} .$$

Then by 3.6. 5.3, $\{b_k\}_k$ defines a point $(x, y, \bar{r})$ of $R(f)(x, y)$ with

$$r_{V^k, M^k} = b_k(y) \in f^*(G_1 \cap d_1^{-1}(V^k)/M^k)_y$$

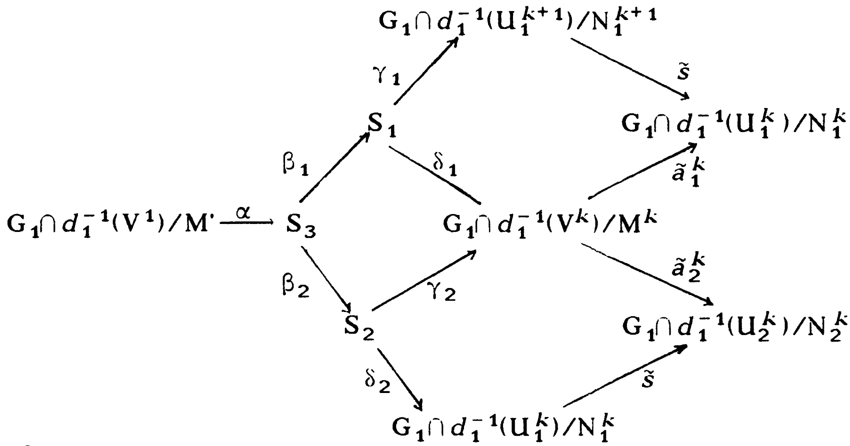
and $\{a_j^k\}_k$ defines a point $(x, x_j, \bar{g}_j)$ of γG with

$$[(g_j)_{U_j^k, N_j^k}] = a_j^k(x),$$

and (v) implies that (8) holds. (6) and (7) are obvious from the initial step of the construction. being the following: Since the pair $(x_1, y, \bar{p})$, $(x_2, y, \bar{p}_2)$ is in the open (4), we have $f^*(\tilde{a}_j)(b(y)) = (p_j)_{U_j, N_j}$ so we take (assuming that $V \in \mathbf{B}$, which one can do without loss of generality)

$$V^0 = V, \ M^0 = M, \ b^0 = b, \ a_j^0 = a_j .$$

Now we suppose the sequences are defined up to k. For the next step. we first use flatness of f^* to construct a diagram in $\mathbf{S}(G)$ (cf. 1.3-1.5) of the form



as follows: since

$$\tilde{s}((p_1)_{U_j^{k+1}, N_j^{k+1}}) = (p_j)_{U_j^k, N_j^k} = \tilde{a}_j^k(b^k(y)),$$

there are elements S_j of $\mathbf{S}(G)$ and maps

$$G_1 \cap d_1^{-1}(V^k)/M^k \xleftarrow{\delta_j} S_j \xrightarrow{\gamma_j} G_1 \cap d_1^{-1}(U_j^{k+1})/N_j^{k+1}$$

and an element $\xi_j \in f^*(S_j)$ such that

$$\tilde{s} \circ \gamma_j = \tilde{a}_j^k \circ \delta_j, \quad f^*(\gamma_j)(\xi_j) = (p_j)_{U_j^k, N_j^k}, \quad f^*(\gamma_j)(\xi_j) = b^k(y).$$

Again by flatness of f^*, there are an object S_3 of $\mathbf{S}(G)$, maps

$\beta_j: S_3 \to S_j$ and a point $\zeta \in f^*(S_j)$ such that

$$\delta_1\beta_1 = \gamma_2\beta_2, \; f^*(\beta_j)(\zeta) = \xi_j \qquad (j = 1,2).$$

By 1.5, there exist a $V' \geq V^k$, an $M' \leq M^k$, and a map $\alpha: G_1 \cap d_1^{-1}(V')/M' \to S_3$ in $\mathbf{S}(G)$ such that $\delta_1\beta_1\alpha = \delta_2\beta_2\alpha$ is induced by the "identity section" $s: V' \to G_1 \cap d_1^{-1}(V^k)/M^k$ and such that ζ is in the image of α, say $\zeta = \alpha(\zeta')$.

Now let $V^{k+1} \leq V^k$ be an element of a common refinement of the covers $U^k(V^k), \ldots, U^k(V^0)$ such that

(13) $$\zeta' \in f^*(G_1 \cap d_1^{-1}(V^{k+1})/M^k | V^{k+1})):$$

such a V^{k+1} exists since $U^k(V^k)$, $U^k(V^{k-1}), \ldots, U^k(V^0)$ all cover V^k, so writing

$$V = U^k(V^k) \wedge U^k(V^{k-1}) \wedge \cdots \wedge U^k(V^0)$$

for their common refinement, $\{f^*(G_1 \cap d_1^{-1}(V)/(M^k|V))\}_{V \in V}$ is a cover of $f^*(G_1 \cap d_1^{-1}(V^k)/M^k)$. Next, let

(14) $$M^{k+1} = M^k(V^k) \cap M^k(V^{k-1}) \cap \cdots \cap M^k(V^0)$$

and let ζ'' be an element of $f^*(G_1 \cap d_1^{-1}(V^{k+1})/M^{k+1})$ projecting to ζ', i.e., $f^*(\tilde{s}(\zeta'')) = \zeta'$.

We are now ready to define the next stage of the sequences: let m_{k+1} be so large that there is a section

$$b^{m_{k+1}}: W^{k+1} \to f^*(G_1 \cap d_1^{-1}(V^{k+1})/M^{k+1})$$

through ζ'', i.e., $b_{k+1}(y) = \zeta''$, and let $a_j^{k+1}: V^{k+1} \to G_1 \cap d_1^{-1}(V^k)/M^k$ be a section such that $\gamma_j \cdot \beta_j \cdot \alpha = \tilde{a}_j^{k+1}$. By choosing m_{k+1} large enough, these are compatible with earlier defined b^k and a_j^k. This completes the description of the induction step.

Now consider $F = \{O \in \mathbf{B} \mid \exists k: V^k \leq O\}$. If $O \in F$ and $\{O_i\}_i$ is a basic cover (in the presentation $\mathbf{B}$) of O, then there is a k with $V^k \leq O$ and a k' with $U^{k'}(V^k) \leq \{O_i \cap V^k\}_i$. Thus $V^{\max(k,k')+1} \leq$ some O_i. This shows that F defines a point x of G_0 such that $\{V^k\}$ is a neighborhood basis at x, i.e., (i) holds. Moreover, it is clear from the construction that (ii)-(v) hold.

This completes the proof of 5.6 in case G, H, R(f) are all countably presented, and S = Sets. In the general case, one can pass to an open surjective base extension $E \to S$ where all enumerations that we have used ((9)-(12)) exist, by [9], Lemma B.

One then builds a tree of finite initial segments of the sequences $\{m_k\}$, $\{V^k\}$, $\{M^k\}$, $\{a_j^k\}$, $\{b^k\}$ satisfying (i)-(v), and proves that the tree has an infinite path in a further open surjective base extension $E' \to E$, by [9], Lemma C.

This completes the proof of 5.6.

6. EQUIVALENCE BETWEEN TOPOSES AND GROUPOIDS.

In this section, we will show how the results of Sections 4 and 5 give an equivalence of bicategories between toposes and groupoids.

Let G and H be continuous groupoids. As before, we write $Hom_S(BH, BG)$ for the category whose objects are geometric morphisms $f: BH \to BG$ over S, and whose morphisms $f \to f'$ are natural transformations $f^* \to f'^*$ over S. Moreover $Flat(\gamma G, \gamma H)$ denotes the category of left-flat γG-γH-bispaces and homomorphisms of bispaces. So there are functors

$$Flat(\gamma G, \gamma H) \underset{R}{\overset{g}{\rightleftarrows}} Hom_S(BH, BG)$$

defined by

$$g(R)^* = -\otimes_{\gamma G} R, \quad Sh(R(f)) = Sh(G_0) \Rightarrow_{BG} Sh(H_0)$$

as discussed in 4.4, 5.1, 5.2.

6.1. THEOREM. *Let* G *and* H *be continuous groupoids, with induced functors*

$$Flat(\gamma G, \gamma H) \underset{R}{\overset{g}{\rightleftarrows}} Hom_S(BH, BG).$$

Then R *is fully faithful, and right-adjoint to* g.

PROOF. Recall that $g(T)^* = -\otimes_{\gamma G} T$ while

$$R(f)_{x,y} = \varprojlim_{x \in U, M} f^*(G_1 \cap d_1^{-1}(U)/M)_y$$

(cf. 5.2 (7)). We define the counit and unit of the adjunction, where by stability it is enough to work with points (in some unspecified base-extension). The *unit* $\eta = \eta_T : T \to Rg(T)$ is defined by taking as fiber $\eta_{x,y}$ over $x \in G_0$, $y \in H_0$ the map $\eta_{x,y}(t) = (x, y, ([s x]_{U,M} \otimes t))$ where $[s(x)]_{U,M}$ denotes the class of $s(x)$ in $G_1 \cap d_1^{-1}(U)/M$; so

$$([s(X)] \otimes t)) \in \varprojlim_{U,M} f^*(G_1 \cap d_1^{-1}(U)/M \otimes_{\gamma G} T) = Rg(T)_{x,y}.$$

The counit $\varkappa = \varkappa_f : -\otimes_{\gamma G} R(f) \to f^*(-)$ has components at generators $G_1 \cap d_1^{-1}(U)/M$,

$$\varkappa_{U,M}([g: x \to x'] \otimes (x, y, F)) = (\vartheta(g) \cdot \bar{F})_{U,M}.$$

It is trivial to check that the triangular identities

$$\varkappa_{g(T)} \cdot g(\eta_T) = id_{g(T)} \text{ and } R(\varkappa_f) \cdot \eta_{R(f)} = id_{R(f)}$$

hold (it is enough to check this on points, by stability and base extension). So $g \dashv R$.

We now prove that $\varkappa$ is an isomorphism. Again, by stability it is enough to show that $\varkappa$ is 1-1 on points, and that for any point r in $f^*(G_1 \cap d_1^{-1}(U)/M)$ there exists a point $\xi \in (G_1 \cap d_1^{-1}(U)/M) \otimes_{\gamma G} R(f)$ in some surjective base extension which is mapped to r by $\varkappa$.

To see that $\varkappa$ is 1-1 on points, suppose that

$$[g_1: x_1 \to x'_1] \otimes (x_1, y, \bar{r}_1) \text{ and } [g_2: x_2 \to x'_2] \otimes (x_2, y, \bar{r}_2)$$

have the same image under $\varkappa$, i.e.

$$(x'_1, y, \vartheta(g_1) * \bar{r})_{U,M} = (x'_2, y, \vartheta(g_2) * \bar{r})_{U,M}$$

in $f^*(G_1 \cap d_1^{-1}(U)/M)$. By 5.6. there are $x \in G_0$ and $\bar{a}_1: x \to x'_1$, $\bar{a}_2: x \to x'_2$, $(x, y, \bar{q}) \in R(f)$, all in some open surjective base extension, such that $(a_j)_{U,M} = [s(x'_j)]$ and $(\bar{a}_j * \bar{q} = \vartheta(g_j) * \bar{r}_j$. Thus for $j = 1, 2$.

$$[g_j: x_j \to x'_j] \otimes (x_j, y, \bar{r}_j) = [s(x'_j)] \otimes (x'_j, y, \vartheta(g_j) * \bar{r}_j)$$
$$= [s(x'_j)] \otimes (x'_j, y, \bar{a}_j * \bar{q}) = [s(x'_j)] \cdot \bar{a}_j * (x, y, \bar{q}) = s(x) \otimes (x, y, \bar{q})$$

(since $\bar{a}_j \in [U,M,s(U)]$). So $[g_1] \otimes (x_1, y, \bar{r}_1) = [g_2] \otimes (x_2, y, \bar{r}_2)$.

To see that $\varkappa$ is "onto", let $r \in f^*(G_1 \cap d_1^{-1}(U)/M)$. This is an étale space over H_0, by π say, so we get a point $y = \pi(r) \in H_0$ over which r lies. The problem is to find a point x of G_0. By going to an open surjective base extension, we may assume that G is countably presented. Let **B** be a basis for G_0, and let for every basic open $B \in \mathbf{B}$, $\mathcal{U}^0(B) \geq \mathcal{U}^1(B) \geq \cdots$ be an enumeration of the covers in some stable generating system for G_0; and let for each B, $M^0(B) \geq M^1(B) \geq \cdots$ be a cofinal system of open B-congruences, just like in the proof of 5.6. We may assume that $U \in \mathbf{B}$ and $M = M^0(U)$. Pick $U_0 \in \mathcal{U}^0(U)$ such that r is in $f^*(G_1 \cap d_1^{-1}(U)/M|U_0))$: such an U_0 exists since $\mathcal{U}_0(U)$ is a cover of U, and therefore

$$\{f^*(\tilde{s}): f^*(G_1 \cap d_1^{-1}(W)/(M|W)) \to f^*(G_1 \cap d_1^{-1}(U)/M)\}_{W \in \mathcal{U}_0(U)}$$

is an epimorphic family. Let $M_0 = M^0(U) \cap M^0(U_0)$, then $\tilde{s}$ induces a projection

$$\pi: f^*(G_1 \cap d_1^{-1}(U_0)/M_0) \to f^*(G_1 \cap d_1^{-1}(U_0)/(M|U_0))$$

so there is an $r_0 \in f^*(G_1 \cap d_1^{-1}(U_0)/M_0)$ such that $\pi(r_0) = r$.

Proceeding in this way, we can construct an open surjective base extension (cf. [9], Lemma C) in which there are sequences $\{U_k\}_k$ and $\{r_k\}_k$ with

$$U_0 \in \mathcal{U}^0(U),\ U_1 \in \mathcal{U}^1(U) \cap \mathcal{U}^1(U_0), \ldots,\ U_k \in \mathcal{U}^k(U) \cap \cdots \cap \mathcal{U}^k(U_{k-1})$$

and $r_k \in f^*(G_1 \cap d_1^{-1}(U_k)/M_k)$, where

$$M_k = M^k(U_k) \cap \cdots \cap M^k(U_0) \cap M^k(U)$$

such that the projection

$$f^*(\tilde{s}): f^*(G_1 \cap d_1^{-1}(U_k)/M_k) \to f^*(G_1 \cap d_1^{-1}(U_{k-1})/M_{k-1})$$

maps r_k to r_{k-1}. Put $F = \{V \in \vartheta(G_0) \mid \exists k : U_k \leq V\}$. F is closed under $\cap$, and if a cover $\vee V_j$ is in F, then some V_j must be in F, as is clear from the construction. So F defines a point x by

$$x \in V \Leftrightarrow \exists k \ (U_k \leq V).$$

Moreover by construction, $\{(U_k, M_k)\}_k$ is a cofinal system at x (cf. 3.6), so that (5.3)

$$R(f)_{x,y} \approx \lim_k f^*(G_1 \cap d_1^{-1}(U_k)/M_k)_y,$$

and hence $(x, y, \{r_k\}_k)$ defines a point of $R(f)$ such that $x([s(x)] \otimes (x, y, \{r_k\}_k) = r$.

This completes the proof of 6.1.

6.2 COROLLARY. *Let* G *and* H *be continuous groupoids. Then every geometric morphism* f: BH→BG *comes from tensoring by a flat* γG-γH-*bispace: namely, there is a natural isomorphism:*

$$f^*(E) \xrightarrow{\sim} E \otimes_{\gamma G} R(f).$$

6.3. Complete bispaces. Call a flat γG-γH-bispace T complete if η: T→Rg(T) is an isomorphism, i.e., if

$$T_{x,y} \approx \varprojlim_{U,M} f^*(G_1 \cap d_1^{-1}(U)/M) \otimes_{\gamma G} T)_y$$

where U ranges over neighborhoods of x and M over open U-congruences. We write *CFlat*(γG,γH) for those flat bispaces which are complete.

By 6.1, there is an equivalence of categories

(1) $$CFlat(\gamma G, \gamma H) \approx Hom_S(BH, BG)$$

and every flat γG-γH-bispace T has a completion $\hat{T} = Rg(T)$.

The complete bispaces give rise to a bicategory (cf. [1]) *(Groupoids)* whose objects are continuous groupoids (with open domain and codomain maps d_0 and d_1), whose 1-*cells* H→G are complete flat γG-γH-bispaces, and whose 2-*cells* are homomorphisms of bispaces. The composition of 1-cells is given by the completion of the tensor product: if

$$K \xrightarrow{\ T\ } H \xrightarrow{\ S\ } G$$

i.e., T is a complete flat γH-γG-bispace and S is a complete flat γG-γH-bispace, then $S \cdot T = S \hat{\otimes}_{\gamma G} T$. The functors

$$g: CFlat(\gamma G, \gamma H) \xrightarrow{\sim} Hom_S(BH, BG)$$

then define a homomorphism of bicategories

(2) $$(Groupoids) \rightarrow (Toposes)$$

given on objects by the classifying topos construction $G \mapsto BG$ of Section 1. That this is a homomorphism follows from 4.5, which obviously gives

$$g(S\cdot T)^* \approx g(S\otimes_{YH} T)^* \approx g(T)^*\cdot g(S)^* = (g(S)\cdot g(T))^*.$$

This homomorphism of bicategories (2) is essentially surjective on objects, by [6], § VIII.3, and an equivalence on Hom-categories, cf. (1) above. So we conclude

6.4. COROLLARY. *The homomorphism (Groupoids) → (Toposes) is an equivalence of bicategories.*

This result is valid over any base topos.

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