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GRAHAM J. ELLIS

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RELATIVE DERIVED FUNCTORS AND THE HOMOLOGY OF GROUPS

by Graham J. ELLIS

RÉSUMÉ. Dans cet article on étudie l'homologie d'un groupe relative à une famille de sous-groupes distingués. On obtient: une description du premier et du deuxième groupe d'homologie relative; deux généralisations de la formule de Hopf; une nouvelle suite exacte d'homologie.

0. INTRODUCTION.

Let N be a normal subgroup of a group G. It is not difficult to define relative homology groups $H_n(G{:}N)$ for $n \geq 1$ (see §2) which fit into a natural long exact sequence

$$\to H_{n+1}(G/N) \to H_n(G{:}N) \to H_n(G) \to H_n(G/N) \to \cdots \to H_1(G/N) \to 0$$

where $H_n(G)$ is the n^{th}-homology of G with integer coefficients. More generally to m normal subgroups $N_1, \dots, N_m$ of G one can associate hyper-relative homology groups $H_n(G{:}N_1, \dots, N_m)$ for $n \geq 1$ which fit into a natural long exact sequence

$$\begin{aligned}\to H_{n+1}(G/N_m{:}N_1N_m/N_m, \dots, N_{m-1}N_m/N_m) &\to H_n(G{:}N_1, \dots, N_m)\\ \to H_n(G{:}N_1, \dots, N_{m-1}) \to H_n(G/N_m{:}N_1N_m/N_m, &\dots, N_{m-1}N_m/N_m)\\ \to \cdots \to H_1(G/N_m{:}N_1N_m/N_m, \dots, N_{m-1}N_m/N_m) &\to 0.\end{aligned}$$

In [2] topological methods were used to give a computational description of the first hyper-relative homology group. In the present article we shall use algebraic methods to recover this description, and then to give a new computational description of the second hyper-relative homology group. Our methods are sufficiently general to apply in other algebraic settings such as the homology of Lie algebras and commutative algebras, and the details of this will be given in a subsequent article.

The description of the first hyper-relative homology group which we reprove is:

THEOREM 1 [2]. *For $m \geq 1$ there is a natural isomorphism*

$$H_1(G;N_1,\ldots,N_m) \cong \{\textstyle\bigcap_{i\in\langle m\rangle} N_i\}/\{\prod_{\alpha\subseteq\langle m\rangle}[\bigcap_{i\in\alpha} N_i, \bigcap_{i\notin\alpha} N_i]\}$$

where $\langle m\rangle = \{1,\ldots,m\}$, $\bigcap_{i\in\varnothing} N_i = G$ and $\prod$ denotes the group product in G.

Thus for $m = 2$ this formula reads

$$H_1(G;N_1,N_2) \cong \{N_1 \cap N_2\}/\{[N_1,N_2][G,N_1\cap N_2]\}.$$

The new description of the second hyper-relative homology is in terms of a "hyper-relative version" of the non-abelian exterior product of groups introduced in [3], which we denote by $\wedge(G;N_1,\ldots,N_m)$.

The group $\wedge(G;N_1,\ldots,N_m)$ has a presentation with generators $x \wedge y$ for $x, y \in G$ such that $x \in \bigcap_{i\in\alpha} N_i$ and $y \in \bigcap_{i\notin\alpha} N_i$ for some $\alpha \subseteq \langle m\rangle$: the relations are

$$xx' \wedge y = (xx'x^{-1} \wedge xyx^{-1})(x \wedge y),$$

$$x \wedge yy' = (x \wedge y)(yxy^{-1} \wedge yy'y^{-1}), \quad x \wedge y = (y \wedge x)^{-1},$$

$$(x \wedge y)(u \wedge v)(x \wedge y)^{-1} = [x,y]u[x,y]^{-1} \wedge [x,y]v[x,y]^{-1}, \quad z \wedge z = 1$$

for $z \in \bigcap_{i\in\langle m\rangle} N_i$ and $x, x' \in \bigcap_{i\in\alpha} N_i$, $y, y' \in \bigcap_{i\notin\alpha} N_i$,

$$u \in \textstyle\bigcap_{i\in\beta} N_i, \quad v \in \bigcap_{i\notin\beta} N_i, \quad \alpha, \beta \subseteq \langle m\rangle.$$

Here $[x,y] = xyx^{-1}y^{-1}$. There is a homomorphism

$$\wedge(G;N_1,\ldots,N_m) \rightarrow G$$

defined on generators by $x \wedge y \mapsto [x,y]$.

THEOREM 2. *There is a natural isomorphism*

$$H_2(G;N_1,\ldots,N_m) \cong \mathrm{Ker}(\wedge(G;N_1,\ldots,N_m) \rightarrow G).$$

On taking $m = 2$ and $N_1 = N_2 = G$ we find

$$H_2(G) \cong H_2(G;G,G) \cong \mathrm{Ker}(\wedge(G;G,G) \rightarrow G)$$

where the first isomorphism follows immediately from the above long exact sequences, and the second follows from Theorem 2. This description of the second integral homology of G is also given in [3] where it is shown to be a reformulation of a result of Miller [11] (also cf. [6]).

With ingenuity one can use Theorems 1 and 2 to obtain other results on the ordinary integral homology of a group. We shall cite some examples, the proofs of which are given in §6.

PROPOSITION 3. *Let* $N_1,\dots,N_m$ *be normal subgroups of* G *such that* $H_k(G/N_iN_j)=0$ *for all* $k=1,2,\dots,n+m-1$ *and* $i\neq j$. *Then there is a natural exact sequence with* $(3n-1)$ *terms*:

$$\begin{gathered} H_n(G)\to H_n(G/N_1)\oplus\cdots\oplus H_n(G/N_m)\to H_{n-1}(G:N_1,\dots,N_m)\to \\ H_{n-1}(G)\to H_{n-1}(G/N_1)\oplus\cdots\oplus H_{n-1}(G/N_m)\to H_{n-2}(G:N_1,\dots,N_m)\to \\ \cdots\to H_1(G/N_1)\oplus\cdots\oplus H_1(G/N_m)\to 0. \qquad\blacksquare \end{gathered}$$

This proposition together with Theorems 1 and 2 generalises the eight term exact sequence of [3] Theorem 4.5 in two respects: firstly our result is in terms of m normal subgroups of G and not just two; secondly for the case of two normal subgroups N_1 and N_2 the requirement in [3] that $G=N_1N_2$ can be weakened to a requirement that $H_i(G/N_1N_2)=0$ for $i=1,2,3$. Note that if $G=N_1N_2$ then our group $\wedge(G:N_1N_2)$ coincides with the group $N_1\wedge N_2$ in [3].

Proposition 3 together with Theorems 1 and 2 also generalises the five term exact sequence of [13] Theorem 2 in two respects: firstly it extends the five term sequence to eight terms; secondly the requirement in [13] that $G=N_k(\bigcap_{i\neq k}N_k)$ for all k can be very much weakened to a requirement that

$$H_i(G/N_kN_j)=0 \text{ for } i=1,2,3, \text{ and all } k\neq j.$$

As further instances of how Theorems 1 and 2 can be used to give information about the ordinary integral homology of a group we cite two more theorems, the first of which has already appeared in [2]. Again we defer the proofs to §6.

THEOREM 4 [2]. *Let* $R_1,\dots,R_n$ *be normal subgroups of a group* F *such that*

$$H_2(F)=0,\ H_r(F/\textstyle\prod_{i\in\alpha}R_i)=0 \textit{ for } r=|\alpha|+1,\ r=|\alpha|+2$$

with α *a non-empty proper subset of* $\langle n\rangle$ *(for example these* $F/\prod_{i\in\alpha}R_i$ *could be free) and* $F/\prod_{i\in\langle n\rangle}R_i\cong G$. *Then there is an isomorphism*

$$H_{n+1}\cong\{\textstyle\bigcap_{i\in\langle n\rangle}R_i\cap[F,F]\}/\{\prod_{\alpha\subset\langle n\rangle}[\bigcap_{i\in\alpha}R_i,\bigcap_{i\notin\alpha}R_i]\}. \quad\blacksquare$$

THEOREM 5. *Let* $R_1,\dots,R_n$ *be normal subgroups of a group* F *such that*

$$H_r(F/\prod_{i\in\alpha}R_i) = 0 \quad \textit{for } r = |\alpha|+2,\ r = |\alpha|+3$$

with $\alpha \neq \langle n\rangle$ *a non-empty proper subset of* $\langle n\rangle$ *and* $F/\prod_{i\in\langle n\rangle}R_i \approx G$. *Then there is an isomorphism*

$$H_{n+2}(G) \approx \mathrm{Ker}(\wedge(F:R_1,\dots,R_n) \to F). \qquad \blacksquare$$

For $n = 1$ Theorem 4 is the well-known Hopf formula for the second integral homology of a group, and Theorem 5 is the description of the third integral homology given in [3] Corollary 4.7. For $n = 2$ Theorem 5 is new, and the isomorphisms of Theorems 4 and 5 read

$$H_3(G) \approx \{R_1\cap R_2\cap [F,F]\}/\{[R_1,R_2][F,R_1\cap R_2]\},$$
$$H_4(G) \approx \mathrm{Ker}(\wedge(F;R_1,R_2)\to F).$$

As was pointed out in [2], for any group G an F and R_i can always be found to satisfy the hypothesis of Theorem 4 (or 5). One method is analogous to methods of [12,14] and is best illustrated for $n = 2$. Choose any surjection $F_i \to G$ with F_i free, $i = 1,2$. Let P be the pullback of these surjections and choose a surjection $F \to P$ with F free. Let R_i be the kernel of the composite map $F \to P \to F_i$. In general one constructs inductively an n-cube of groups F such that, for $\alpha \subset \langle n\rangle$:

(i) $F_{\langle n\rangle}$ is G, and

(ii) the morphism $F_\alpha \to \lim_{\beta\supset\alpha} F_\beta$ is surjective.

The organization of this article is as follows. In §1 and §2 we recall basic definitions and facts on simplicial resolutions and derived functors. In §3 we prove some abstract technical results on relative derived functors. Theorems 1 and 2 are proved in §4 and §5 respectively. In §6 we prove Proposition 3 and Theorems 4 and 5.

1. DERIVED FUNCTORS.

In this section we recall from [1,8] facts on cotriples and derived functors. A useful reference for the details of simplicial objects is [4].

Let $\boldsymbol{C}$ be an arbitrary category and $\boldsymbol{E} = (E,\varepsilon,\delta)$ a cotriple on $\boldsymbol{C}$. That is $E: \boldsymbol{C} \to \boldsymbol{E}$ is an endofunctor and $\varepsilon: E \to 1_{\boldsymbol{C}}$, $\delta: E \to E^2$ are natural transformations such that

$$(E\varepsilon)\delta = \mathrm{id},\ (\varepsilon E)\delta = \mathrm{id} \text{ and } (E\delta)\delta = (\delta E)\delta.$$

EXAMPLE 1. Let U: $Gp \to Set$ be the functor from groups to sets which takes a group to its underlying set, and let F: $Set \to Gp$ be the functor which takes a set to the free group generated by the set. Then $E = FU: Gp \to Gp$ is an endofunctor, and the obvious natural transformations provide us with a cotriple $\boldsymbol{E} = (E,\varepsilon,\delta)$.

To each object C of $\boldsymbol{C}$ one can associate a simplicial object $E(C)_{\#}$ in $\boldsymbol{C}$ by setting

$$E(C)_n = E^{n+1}C,\ \varepsilon_i = E^i\varepsilon E^{n-i}: E(C)_n \to E(C)_{n-1},$$
$$\delta_i = E^i\delta E^{n-i}: E(C)_n \to E(C)_{n+1}, \text{ for } 0 \le i \le n.$$

Now let $T: \boldsymbol{C} \to Gp$ be a functor from $\boldsymbol{C}$ to groups. By applying T dimensionwise to the simplicial object $E(C)_{\#}$ we obtain a simplicial group $TE(C)_{\#}$. The homotopy groups of this simplicial group are the *derived functors of* T *with respect to the cotriple* $\boldsymbol{E}$, and we write

$$L_n^T(C) = \pi_n(TE(C)_{\#}).$$

The homotopy groups of $TE(C)_{\#}$ are isomorphic to the homology groups of the associated *Moore complex* (cf. [4])

$$\cdots \to M_n \xrightarrow{d_n} M_{n-1} \longrightarrow \cdots \xrightarrow{d_1} M_0 \xrightarrow{d_0} 0$$

where $M_0 = TE(C)_0$, $M_n = \bigcap_{0 \le i < n} \mathrm{Ker}(\varepsilon_i: TE(C)_n \to TE(C)_{n-1})$ for $n \ge 1$ and d_n is the restriction of ε_n. Thus

$$L_n^T(C) = \mathrm{Ker}\, d_n / \mathrm{Im}\, d_{n+1},\ n \ge 0.$$

EXAMPLE 2. Let $\boldsymbol{E}$ be the cotriple described in Example 1, and let $T = (-)^{ab}: Gp \to Gp$ be the functor which takes a group G to its abelianization $G^{ab} = G/[G,G]$. It is shown in [11] that $L_n^T(G) = H_{n+1}(G)$ for $n \ge 0$.

2. HYPER-RELATIVE DERIVED FUNCTORS.

Throughout this and the next section we shall suppose that $\boldsymbol{E} = (E,\varepsilon,\delta)$ is a cotriple on a category $\boldsymbol{C}$ such that $E: \boldsymbol{C} \to \boldsymbol{C}$ factors through a pair of functors

$$\begin{tikzcd}
\boldsymbol{C} \arrow[rr, "E"] \arrow[dr, "U"'] & & \boldsymbol{C} \\
& \boldsymbol{D} \arrow[ur, "F"'] &
\end{tikzcd}$$

with U a right adjoint to F, with $\varepsilon: FU \to 1_C$ the counit of the adjunction, with $\nu: 1_D \to UF$ the unit of the adjunction and $\delta = F\nu U$ (cf. [5]).

We shall say that a morphism $f: C \to A$ in ***C*** is a *fibration* if there exists a morphism $\lambda: UA \to UC$ in ***D*** such that $(Uf)\lambda = 1_{UA}$. It is readily seen that a fibration $f: C \to A$ induces a dimensionwise surjective homomorphism of simplicial groups $f_\#$: $TE(C)_\# \to TE(A)_\#$. We denote the homotopy groups of the kernel of this simplicial map by

$$L_n^{RT}(f: C \to A) = \pi_n(\operatorname{Ker} f_\#)\ ,\ n \geq 0.$$

Since a surjection of simplicial groups yields a long exact sequence of homotopy groups (cf. [4]) we have immediately

PROPOSITION 6. *A fibration f:* $C \to A$ *in* ***C*** *yields a natural long exact sequence*

$$\cdots \to L^T_{n+1}(A) \to L^{RT}_n(C \to A) \to L^T_n(C) \to L^T_n(A) \to \cdots \to L^T_0(A) \to 0. \ \blacksquare$$

Now the functors L_n^{RT} can be regarded as derived functors. To see this let ***RC*** denote the category whose objects are the fibrations $C \to A$ in ***C*** and whose morphisms are commutative squares in ***C***:

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & A \arrow[d] \\
C' \arrow[r] & A'
\end{tikzcd}$$

The cotriple ***E*** = (E, ε, δ) on ***C*** extends in an obvious way to a cotriple ***RE*** = $(RE, \varepsilon, \delta)$ on ***RC*** : on objects RE is defined by

$$RE(C \to A) = EC \to EA.$$

Let RT: ***RC*** $\to$ *Gp* be the functor which maps a fibration $C \to A$ to the kernel of the induced group homomorphism $TC \to TA$, that is

$$RT(C \to A) = \operatorname{Ker}(TC \to TA).$$

Clearly the functors L_n^{RT} defined above are the derived functors of the functor RT with respect to the cotriple ***RE*** on ***RC***.

Note that the endofunctor RE factors through a pair of adjoint functors

$$\begin{tikzcd}
\boldsymbol{RC} \arrow[rr, "\mathrm{RE}"] \arrow[dr, "\mathrm{RU}"'] & & \boldsymbol{RC} \\
& \boldsymbol{RD} \arrow[ur, "\mathrm{RF}"'] &
\end{tikzcd}$$

where ***RD*** is the category whose objects are the morphisms of ***D*** which possess at least one splitting, and whose morphisms are commutative squares in ***D***: the adjoint pair is the obvious one. Thus we can define inductively, for $m \geq 0$,

$$\boldsymbol{R}^{m+1}\boldsymbol{C} = \boldsymbol{R}(\boldsymbol{R}^{m}\boldsymbol{C}),\ \boldsymbol{R}^{0}\boldsymbol{C} = \boldsymbol{C}\ ,\ \mathrm{R}^{m+1}\mathrm{T} = \mathrm{R}(\mathrm{R}^{m}\mathrm{T}),\ \mathrm{R}^{0}\mathrm{T} = \mathrm{T}.$$

Therefore we have derived functors $L_n^{\mathrm{R}^m\mathrm{T}}: \boldsymbol{R}^m\boldsymbol{C} \to \mathit{Gp}$, $m, n \geq 0$. We call these functors the *hyper-relative derived functors of* T.

EXAMPLE 3. Let G be a group with normal subgroups $N_1, \dots, N_m$. This data gives rise to an m-cubical diagram consisting of groups $G_\alpha = G/\prod_{i \in \alpha} N_i$ for each $\alpha \subseteq \langle m \rangle$, and of quotient homomorphisms $G_\alpha \to G_{\alpha \cup \langle i \rangle}$ for each $i \in \alpha$. We denote this m-cube by $\{G_\alpha\}$. Thus for example if $m = 2$ we have

$$\{G_\alpha\} = \begin{tikzcd}
G \arrow[r] \arrow[d] & G/N_1 \arrow[d] \\
G/N_2 \arrow[r] & G/N_1N_2
\end{tikzcd}$$

It is clear that in general $\{G_\alpha\}$ is an object in the category $\boldsymbol{R}^m\mathit{Gp}$.

Letting $\mathrm{T} = (-)^{\mathrm{ab}}$ as in Example 2, we define

$$H_{n+1}(G : N_1, \dots, N_m) = L_n^{\mathrm{R}^m(-)^{\mathrm{ab}}}\{G_\alpha\}.$$

It follows immediately from Proposition 6 that these hyper-relative homology groups fit together in exact sequences as described in the Introduction.

3. SOME TECHNICAL RESULTS;

Extending a definition of [9] we shall say that a diagram

$$(\dagger) \qquad \begin{tikzcd}
C'' \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & C' \arrow[r, "p"] & C
\end{tikzcd}$$

in ***C*** satisfying $q_1p = q_2p$ is *split over* ***D*** if there exist morphisms $\lambda : UC \to UC'$ and $\mu : UC' \to UC''$ in $\boldsymbol{D}$ such that

$$(Up)\lambda = 1_{UC},\ (Uq_1)\mu = 1_{UC'} \text{ and } (Uq_2)\mu = \lambda(Up).$$

We shall say that a functor $W : \boldsymbol{C} \to Gp$ *is right exact* if it maps any split diagram (*) to a diagram in *Gp* which is split over *Set* with set maps $\lambda : WC \to WC'$, $\mu : WC' \to WC''$ which preserve the group identity elements.

LEMMA 7. *For any functor* $T : \boldsymbol{C} \to Gp$ *the zeroth derived functor* $L_0^T : \boldsymbol{C} \to Gp$ *is right exact.*

PROOF. A split diagram (*) gives rise to a diagram of group homomorphisms

$$\begin{tikzcd} TFUC'' \arrow[r, shift left=2] \arrow[r] & TFUC' \arrow[l, shift left=4] \arrow[r, shift right] & TFUC \arrow[l, shift right] \end{tikzcd} .$$

Since $L_0^T(C''')$ is a quotient of $TFUC'''$ for $C''' = C, C', C''$ it follows that the diagram induced by the arrows going from left to right

$$\begin{tikzcd} L_0^T(C'') \arrow[r, shift left, "L_0^T(q_1)"] \arrow[r, shift right, "L_0^T(q_2)"'] & L_0^T(C') \arrow[r, "L_0^T(p)"] & L_0^T(C) \end{tikzcd}$$

is split over *Set*, and that the splittings can be chosen so that they preserve the group identity elements.

LEMMA 8. *If*

$$\begin{tikzcd} G'' \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & G' \arrow[r, "p"] & G \end{tikzcd}$$

is a diagram of groups such that $q_1 p = q_2 p$ *and which is split over Set with splittings which preserve identity elements, then* $q_1(\mathrm{Ker}\, q_2)$ *is a normal subgroup of* G' *and* G *is isomorphic to* $G'/q_1(\mathrm{Ker}\, q_2)$.

PROOF. The homomorphism q_1 is surjective as it has a set theoretic splitting μ. Thus the image of any normal subgroup of G'' is normal in G'. Now if g is in $\mathrm{Ker}\, p$ then

$$q_1(\mu g) = g \text{ and } q_2(\mu g) = \lambda p(g) = 1.$$

Hence $\mathrm{Ker}\, p = q_1(\mathrm{Ker}\, q_2)$. Finally p is surjective as it has a set theoretic splitting λ. ■

Let us now suppose that the category $\boldsymbol{C}$ contains pullbacks. Thus given a fibration $C \to A$ in $\boldsymbol{C}$ we can consider the diagram

$$\begin{tikzcd}
C\times_A C\times_A C \arrow[r, shift left=2, "q_1"] \arrow[r, "q_2"] \arrow[r, shift right=2, "q_3"'] & C\times_A C \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & C
\end{tikzcd}$$

where p_i and q_i "forget the i^{th} copy of C". (Thus for example p_1 denotes the projection onto the second copy of C.) On applying the functor L_0^T we obtain a diagram

$$\begin{tikzcd}
L_0^T(C\times_A C\times_A C) \arrow[r, shift left=2, "\bar q_1"] \arrow[r, "\bar q_2"] \arrow[r, shift right=2, "\bar q_3"'] & L_0^T(C\times_A C) \arrow[r, shift left, "\bar p_1"] \arrow[r, shift right, "\bar p_2"'] & L_0^T(C)
\end{tikzcd}$$

where we write $\bar p_i$ and $\bar q_i$ instead of $L_0^T(p_i)$ and $L_0^T(q_i)$. With this notation we have

LEMMA 9. *For any fibration f: C→A there is an isomorphism*

$$L_0^{RT}(f\colon C\to A) \approx \{\mathrm{Ker}\,\bar p_2\}/\{\bar q_1(\mathrm{Ker}\,\bar q_2 \cap \mathrm{Ker}\,\bar q_2)\}.$$

PROOF. We can consider

$$\begin{tikzcd}
C\times_A C\times_A C \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] \arrow[d] & C\times_A C \arrow[r, "p_1"] \arrow[d] & C \arrow[d] \\
C\times_A C \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & C \arrow[r, "f"] & A
\end{tikzcd}$$

as a diagram in ***RC*** with the vertical maps the objects. As such the diagram is split over ***RD***. To see this note that a map λ: UA→UC exists by virtue of the fact that f is a fibration: the other maps of the splitting are

$$(\lambda Uf,1)\colon UC \to U(C\times_A C) \text{ and } (\lambda Uf,1)\times 1\colon U(C\times_A C) \to U(C\times_A C\times_A C).$$

Since $\bar q_3$ and $\bar p_2$ are split homomorphisms we have

$$L_0^{RT}(\bar q_3) \approx \mathrm{Ker}\,\bar q_3 \text{ and } L_0^{RT}(\bar p_2) \approx \mathrm{Ker}\,\bar p_2.$$

Hence the proof is completed by applying Lemmas 7 and 8 to the following diagram of groups:

$$\begin{tikzcd}
\mathrm{Ker}\left[\begin{array}{c} L_0^T(C\times_A C\times_A C) \\ \Big\downarrow \bar q_3 \\ L_0^T(C\times_A C) \end{array}\right] \arrow[r, shift left, "\bar q_1"] \arrow[r, shift right, "\bar q_2"'] & \mathrm{Ker}\left[\begin{array}{c} L_0^T(C\times_A C) \\ \Big\downarrow \bar p_2 \\ L_0^T(C) \end{array}\right] \arrow[r, "\bar p_1"] & L_0^{RT}(C\to A).
\end{tikzcd}$$

4. PROOF OF THEOREM 4.

If N is a normal subgroup of a group G and if $Q = G/N$, then the diagram

$$\begin{tikzcd}
G\times_Q G\times_Q G \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G\times_Q G \arrow[r, shift left] \arrow[r, shift right] & G
\end{tikzcd}$$

is isomorphic to the diagram

$$\begin{tikzcd}
N\rtimes(N\rtimes G) \arrow[r, shift left=2, "\bar q_1"] \arrow[r, "\bar q_2"'] \arrow[r, shift right=2, "\bar q_3"'] & N\rtimes G \arrow[r, shift left, "\bar p_1"] \arrow[r, shift right, "\bar p_2"'] & G
\end{tikzcd}$$

where $\rtimes$ denotes a semi-direct product: the action of G on N is ${}^{g}n = gng^{-1}$; the action of $N\rtimes G$ on N is ${}^{(n,g)}n' = gn'g^{-1}$; the homomorphisms are

$$p_1(n,g) = ng,\ p_2(n,g) = g,\ q_1(n',n,g) = (n'n^{-1}, ng),$$
$$q_2(n',n,g) = (n',g) \text{ and } q_3(n',n,g) = (n,g).$$

Now the induced maps

$$\bar q_i\colon (N\rtimes N\rtimes G)^{ab} \to (N\rtimes G)^{ab}$$

clearly satisfy $\mathrm{Ker}\,\bar q_2 \cap \mathrm{Ker}\,\bar q_3 = 1$. Thus from Examples 2 and 3 and Lemma 9 we get an isomorphism

$$H_1(G;N) \approx \mathrm{Ker}\{\bar p_2\colon (N\rtimes G)/[N\rtimes G, N\rtimes G] \to G/[G,G]\}.$$

It is thus readily seen that $H_1(G;N) \approx N/[G,N]$ and this proves Theorem 1 for $m = 1$.

More generally suppose that $N_1,\dots,N_m$ are normal subgroups of a group G ($m \geq 2$). For $\alpha \subseteq \langle m-1\rangle$ let

$$G_\alpha = G/\{\textstyle\prod_{i\in\alpha} N_i\},\ \bar G_\alpha = G/\{N_m \textstyle\prod_{i\in\alpha} N_i\}.$$

Then the following pullback diagram in $\mathbf{R}^{m-1}Gp$

$$\begin{tikzcd}
\{G_\alpha\}\times_{\langle\bar G_\alpha\rangle}\{G_\alpha\}\times_{\langle\bar G_\alpha\rangle}\{G_\alpha\} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \{G_\alpha\}\times_{\langle\bar G_\alpha\rangle}\{G_\alpha\} \arrow[r, shift left] \arrow[r, shift right] & \{G_\alpha\}
\end{tikzcd}$$

is isomorphic to the diagram

$$\begin{tikzcd}
\{(N_\alpha\cap G_\alpha)\rtimes(N_\alpha\rtimes G)\rtimes G_\alpha\} \arrow[r, shift left=2, "\bar q_1"] \arrow[r, "\bar q_2"'] \arrow[r, shift right=2, "\bar q_3"'] & \{(N_\alpha\cap G_\alpha)\rtimes G_\alpha\} \arrow[r, shift left, "\bar p_1"] \arrow[r, shift right, "\bar p_2"'] & \{G_\alpha\}.
\end{tikzcd}$$

Let us suppose that Theorem 1 has been proved for $m-1$ normal subgroups of G. Clearly $\mathrm{Ker}\,\bar q_2 \cap \mathrm{Ker}\,\bar q_3 = 1$. So Lemma 9 gives us an isomorphism

$$H_1(G;N_1,\dots,N_m) \approx \mathrm{Ker}\Big(\bar{p}_2\colon$$
$$H_1(N_m \hat{\times} G;(N_m\cap N_1)\hat{\times} N_1,\dots.(N_m\cap N_{m-1})\times N_m) \to H_1(G;N_1,\dots,N_m)\Big)$$

and this kernel is isomorphic to the kernel of

$$(\textstyle\bigcap_{i\in<m-1>}(N_m\cap N_i)\hat{\times} N_i)/\prod_{\alpha\subset<m-1>}[\bigcap_{i\in\alpha}(N_mN_i)\hat{\times} N_i, \bigcap_{i\not\in\alpha}(N_m\cap N_i)\hat{\times} N_i]$$
$$\to \textstyle\bigcap_{i\in<m-1>}N_i/\prod_{\alpha\subset<m-1>}[\bigcap_{i\in\alpha}N_i, \bigcap_{i\not\in\alpha}N_i].$$

Theorem 1 follows by induction. ■

5. PROOF OF THEOREM 2.

Let G be a group and let $E(G)_\#$ be the simplicial group obtained from the cotriple of Example 1. Applying the functors $(-)^{ab}$ and $[\,.\,]\colon Gp\to Gp$ dimensionwise to $E(G)_\#$ yields a short exact sequence of simplicial groups

$$[E(G)_\#, E(G)_\#] \to E(G)_\# \to E(G)^{ab}.$$

Since $\pi_0(E(G)_\#) \approx G$, $\pi_n(E(G)_\#) = 0$ for $n \geq 1$, and

$$\pi_n(E(G)_\#^{ab}) \approx H_{n+1}(G) \text{ for } n \geq 0.$$

the resulting long exact homotopy sequence provides us with isomorphisms

$$H_2(G) \approx \mathrm{Ker}(L_0^{[,]}(G)\to G),\ H_{n+2}(G) \approx L_n^{[,]}(G),\ n\geq 1.$$

Recall from the Introduction the definition of the group $\wedge(G;N_1,\dots,N_m)$ where N_i are normal subgroups of G. The special case $\wedge(G;G)$ coincides with the group denoted in [3] by $G\wedge G$. It is known from [3] (see [6] for an algebraic proof) that for a free group F there is an isomorphism

$$\wedge(F;F) \approx [F,F],\ x\wedge y \mapsto [x,y].$$

Thus if we let $\wedge$: $Gp\to Gp$ be the functor which takes a group G to the group $\wedge(G;G)$, then we have an isomorphism $L_n^{\wedge}(G) \approx L_n^{[,]}(G)$ for $n\geq 0$. So in particular

$$H_2(G) \approx \mathrm{Ker}(L_0^{\wedge}(G)\to G).$$

If $\{G_\alpha\}$ is the m-cubical diagram of Example 3 arising from the subgroups $N_1,\dots,N_m$ then we find that

$$H_2(G;N_1,\dots,N_m) \approx \mathrm{Ker}(L_0^{R^m\wedge}\{G_\alpha\}\to G).$$

Thus to prove Theorem 2 we need to show that

$$L_0^{R^m\wedge}\{G_\alpha\} \approx \wedge(G;N_1,\dots,N_m).$$

This isomorphism follows from an inductive use of Lemma 9 and

Proposition 11 below, together with the fact (proved in Proposition 3 of [7]) that $\hat{L}_0(G) \approx \wedge(G;G)$. The proof of Proposition 11 uses the following lemma.

LEMMA 10. *For any normal subgroups* $N_1,\dots,N_m$ *of* G $(m \geq 2)$ *the canonical sequence of groups*

$$\wedge(G;N_1,\dots,N_m) \xrightarrow{\iota} \wedge(G;N_1,\dots,N_{m-1}) \xrightarrow{\pi}$$
$$\wedge(G/N_m;N_1N_m/N_m,\dots,N_{m-1}N_m/N_m) \longrightarrow 0.$$

is exact.

PROOF. Clearly $\pi\colon x \wedge y \mapsto xN_m \wedge yN_m$ is surjective. The image of the homomorphism $\iota\colon x \wedge y \mapsto y$ clearly lies in the kernel of π, and moreover $\mathrm{Im}\,\iota$ is normal in $\wedge(G;N_1,\dots,N_{m-1})$. Finally it is readily verified (cf. Proposition 1 in [6]) that there is a homomorphism

$$\psi\colon \wedge(G/N_m:N_1N_m/N_m,\dots,N_{m-1}N_m/N_m) \to \wedge(G:N_1,\dots,N_{m-1})/\mathrm{Im}\,\iota,$$
$$xN_m \wedge yN_m \mapsto \{x \wedge y\}\mathrm{Im}\,\iota$$

and that π induces an inverse to ψ. ■

Now the homomorphisms q_i and p_i of §4 induce homomorphisms

$$\bar{q}_i\colon \wedge(N_m \hat{\times} N_m \hat{\times} G;$$
$$\{N_1 \cap N_m\} \hat{\times} \{N_1 \cap N_m\} \hat{\times} N_1,\dots,\{N_{m-1} \cap N_m\} \hat{\times} \{N_{m-1} \cap N_m\} \hat{\times} N_{m-1})$$
$$\to \wedge(N_m \hat{\times} G;\{N_1 \cap N_m\} \hat{\times} N_1,\dots,\{N_{m-1} \cap N_m\} \hat{\times} N_{m-1}),$$
$$\bar{p}_i\colon \wedge(N_m \hat{\times} G;\{N_1 \cap N_m\} \hat{\times} N_1,\dots, N_{m-1} \cap N_m\} \hat{\times} N_{m-1})$$
$$\to \wedge(G:N_1,\dots,N_{m-1}).$$

PROPOSITION 11. *With the above notation and* $m \geq 2$, *there is an isomorphism*

$$\wedge(G;N_1,\dots,N_m) \xrightarrow{\approx} \mathrm{Ker}\ \bar{p}_2/\bar{q}_1\,(\mathrm{Ker}\,\bar{q}_2 \cap \mathrm{Ker}\,\bar{q}_3).$$

PROOF. From Lemma 10 we see that $\mathrm{Ker}\,\bar{q}_2$ is generated by the elements of the form $(n.1.1) \wedge (n'.n'',x)$. Also $\mathrm{Ker}\,\bar{q}_3$ is generated by the elements of the form $(1,n,1) \wedge (n',n'',x)$. Hence $\mathrm{Ker}\,\bar{q}_2 \cap \mathrm{Ker}\,\bar{q}_3$ is generated by the elements of the form $(n,1,1) \wedge (1,m,1)$. It follows that $\bar{q}_1(\mathrm{Ker}\,\bar{q}_2 \cap \mathrm{Ker}\,\bar{q}_3)$ is generated by the elements of the form $(n,1) \wedge (m^{-1},m)$. Now $\mathrm{Ker}\,\bar{p}_2$ is generated by elements of the form $(n,1) \wedge (m\,x)$, and there is a surjective homomorphism

$$\varphi\colon \operatorname{Ker}\bar{p}_2 \longrightarrow \wedge(G;N_1,\dots,N_m),\ (n,1)\wedge(m,g) \mapsto n\wedge mg.$$

By Lemma 10 we have $\operatorname{Ker}\varphi = \bar{q}_1(\operatorname{Ker}\bar{q}_2\cap\operatorname{Ker}\bar{q}_3)$. ■

6. REMAINING PROOFS.

Proposition 3 is certainly true for $m = 1$. Suppose then that for some $m \geq 2$ the proposition is true for any suitable collection of m-1 subgroups. Let $N_1,\dots,N_m$ be subgroups of G satisfying the hypothesis of the proposition, and consider the following commutative diagram in which $k \leq n$ and the rows and columns are exact:

$$\begin{tikzcd}
& 0 = \overset{1\le i<m}{\oplus} H_{k+1}(G/N_mN_i) \arrow[d] & \\
H_k(G;N_1,\dots,N_m) \arrow[r] & H_k(G/N_m;N_1N_m/N_m,\dots,N_{m-1}N_m/N_m) \arrow[r] \arrow[d, "\approx"] & H_{k-1}(G.N_1,\dots,N_m) \\
& H_k(G/N_m) \arrow[d] & \\
& 0 = \overset{1\le i<m}{\oplus} H_k(G/N_mN_i) &
\end{tikzcd}$$

We thus obtain an isomorphism

$$\varphi_m\colon H_k(G/N_m) \longrightarrow H_{k-1}(G;N_1,\dots.N_m).$$

Similarly we obtain homomorphisms

$$\varphi_i\colon H_k(G/N_i) \longrightarrow H_{k-1}(G;N_1,\dots,N_m)$$

which we can combine to get a homomorphism

$$\varphi\colon H_k(G/N_1)\oplus\cdots\oplus H_k(G/N_m) \longrightarrow H_{k-1}(G;N_1,\dots,N_m),$$
$$(x_1,\dots,x_m) \mapsto \varphi_1(x_1)\cdots\varphi_m(x_m).$$

The canonical homomorphisms $\psi_i\colon H_k(G)\to H_k(G/N_i)$ combine to give a homomorphism

$$(\psi_1,\dots,\psi_m)\colon H_k(G) \longrightarrow H_k(G/N_1)\oplus\cdots\oplus H_k(G/N_m)\ .$$

Finally, using our inductive hypothesis, there is a composite map

$$H_k(G;N_1,\dots,N_m) \longrightarrow H_k(G;N_1,\dots,N_{m-1}) \longrightarrow H_k(G).$$

Thus the sequence of Proposition 3 exists for m subgroups. The verification of its exactness is a routine exercise.

Theorems 4 and 5 follow from Theorems 1 and 2 together with the following lemmas.

LEMMA 12. *Let* $R_1,\dots,R_n$ *be normal subgroups of a group* F *such that the hypothesis of Theorem 4 is satisfied. Then there is an exact sequence*

$$0 \longrightarrow H_{n+1}(G) \longrightarrow H_1(F;R_1,\dots,R_n) \longrightarrow H_1(F). \quad \blacksquare$$

LEMMA 13. *Let* $R_1,\dots,R_n$ *be normal subgroups of a group* F *such that the hypothesis of Theorem 5 is satisfied. Then there is an exact sequence*

$$0 \longrightarrow H_{n+2}(G) \longrightarrow H_1(F;R_1,\dots,R_n) \longrightarrow H_2(F) = 0. \quad \blacksquare$$

Lemma 12 is precisely Proposition 5 of [2], and Lemma 13 is proved in a similar fashion.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE GALWAY,
IRELAND