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COMPACT OBJECTS, SURJECTIVITY OF EPIMORPHISMS AND COMPACTIFICATIONS

by Gabriele CASTELLINI

RÉSUMÉ. La notion d'opérateur de fermeture est utilisée pour généraliser dans une catégorie abstraite la notion de compacité pour les espaces topologiques. La plupart des résultats classiques sur la compacité peut être prouvée dans ce cadre plus général. Par exemple, la compactification de Stone-Čech peut être généralisée. Dans les sous-catégories de groupes abéliens cette notion de compacité est liée à la surjectivité des épimorphismes.

INTRODUCTION.

Herrlich, Salicrup and Strecker in [11] presented a generalization of the classical notion of compactness for topological spaces. Such a generalization provides in an abstract category $\underline{X}$ a concept of compactness with respect to a factorization structure on $\underline{X}$. Because of a bijective correspondence between factorization structures and weakly hereditary idempotent closure operators on $\underline{X}$ (cf. [7]), this naturally yields a notion of compactness with respect to such closure operators. Unfortunately, it rules out all those that fail to satisfy the weakly hereditary condition. To avoid this inconvenience, we present here a concept of compactness directly with respect to the notion of closure operator.

In §1 we reprove in this general setting most of the classical results about compactness in topological spaces.

In §2 we relate this generalized notion of compactness to the surjectivity of epimorphisms in subcategories of AB or TOP.

In §3 we present a generalization of Stone-Čech compactification.

PRELIMINARIES.

Throughout the paper we consider a category $\underline{X}$ and a fixed class $\underline{M}$ of $\underline{X}$-monomorphisms, which contains all $\underline{X}$-isomorphisms. It is assumed that:

1. M is closed under composition.

2. Pullbacks of M-morphisms exist and belong to M, and multiple pullbacks of (possibly large) families of M morphisms with common codomain exist and belong to M.

One of the consequences of the above assumptions is that there is a (uniquely determined) class E of morphisms in X such that X is an (E,M)-category (cf. [7]).

The class M will be considered as the class of objects of a comma category which is denoted again by M. Its morphisms $(f,g)\colon m \to n$ are commutative squares, i.e., $nf = gm$. Since the codomain functor $U\colon M \to X$ $(U(f,g) = g)$ is faithful, M is concrete over X.

As in [7], by a closure operator on the category X with respect to the class of subobjects M we mean a concrete functor $F\colon M \to M$, i.e., $UF = U$ together with a natural transformation $\gamma\colon Id_M \to F$ such that $U\gamma = 1_U$.

We write

$$[m]_F^X\colon [M]_F^X \longrightarrow X \quad = \quad F(m\colon M \to X)$$

for every $m \in M$. $[m]_F^X$ will be called the F-*closure of m*. Subscripts and superscripts will be omitted when not necessary. The conglomerate of all closure operators on X with respect to M with the preorder defined by

$$F \le G \text{ iff } F(m) \le G(m) \text{ for every } m \in M,$$

will be denoted by CL(X,M).

For $F \in CL(X,M)$, any $m \in M$ with $m \approx [m]_F$ is called F-*closed* and any X-morphism $f\colon X \to Y$ such that its (E,M)-factorization (e,m) satisfies $[m]_F \approx 1_Y$ is called F-dense.

F is called *idempotent* if for every $m \in M$, $[m]_F$ is F-closed.

If $m_{[m]}\colon M \to [M]_F$ is the morphism induced by the natural transformation γ, then F is called *weakly hereditary* if $m_{[m]}$ is F-dense.

The class of F-closed M-subobjects and the class of F-dense morphisms will be denoted by M^F and E^F, respectively.

If m and n are M-subobjects of the same object X, with $m \le n$ and m_n denotes the morphism such that $nm_n = m$, then F is called *hereditary* if $n[m_n] \approx n \cap [m]$ holds for every $X \in X$ and M-subobjects m, n of X with $m \le n$.

We observe that the pullback of each F-closed subobject is F-closed (cf. [7]). Therefore if we consider $\underline{X}$ as a concrete category over itself via the identity functor, the above definition of closure operator is equivalent to the notion of global closure operator which appears in [4].

If $\underline{C}$ is a subcategory of $\underline{X}$, we call a morphism $r: X \to Y$ $\underline{C}$-*regular* if it is the equalizer of two morphisms $f, g: Y \to Z$, $Z \in \underline{C}$.

If $\underline{M}$ contains all regular monomorphisms of $\underline{X}$, then for every $\underline{M}$-morphism $m: M \to X$ we define

$$[m]_C = \cap\{r \mid r \geq m \text{ and } r \text{ is } \underline{C}\text{-regular}\}.$$

The functor $F_C: \underline{M} \to \underline{M}$ defined by $F_C(m) = [m]_C$ is an idempotent closure operator on $\underline{X}$ (cf. [15, 8, 1]).

All the subcategories considered will be full and isomorphism-closed.

1. BASIC DEFINITIONS AND RESULTS.

In what follows F will always denote a closure operator with respect to the given class $\underline{M}$ of monomorphisms and $\underline{X}$ will be a category with finite products.

DEFINITION 1.1. An $\underline{X}$-morphism $f: X \to Y$ is called F-*closed preserving* if, for every F-closed $\underline{M}$-subobject $m: M \to X$, in the $(\underline{E}, \underline{M})$-factorization $m_1 e_1 = fm$, m_1 is F-closed.

DEFINITION 1.2. An $\underline{X}$-object X is called F-*compact* if for each $\underline{X}$-object Z, the projection $P_Z: X \times Z \to Z$ is F-closed preserving.

Comp(F) will denote the subcategory of $\underline{X}$ whose objects are the F-compact ones. If F is induced by a subcategory $\underline{C}$, we will write Comp($\underline{C}$).

Clearly, such a concept generalizes the classical notion of compactness in Topology, since if $\underline{X}$ = TOP and F is the closure operator induced by the topology, then Comp(F) = Compact topological spaces (cf. [12]).

LEMMA 1.3. *Every $\underline{X}$-isomorphism is* F-*closed preserving.*

LEMMA 1.4. *The composition of* F-*closed preserving morphisms is* F-*closed preserving.*

PROPOSITION 1.5. Comp(F) *is closed under the formation of finite products.*

PROPOSITION 1.6. *Let* $\underline{X}$ *have products and suppose that the pullback of* $\pi_i \times 1_Z : \Pi X_i \times Z \to X_i \times Z$ *along any* F*-closed subobject belongs to* $\underline{E}$ *for every i. Then, if the* ΠX_i *is* F*-compact, each factor is too.*

PROOF. Let us consider the commutative diagram

$$\begin{array}{ccccc} \Pi X_i \times Z & \xrightarrow{\pi_i \times 1_Z} & X_i \times Z & \xrightarrow{\pi_Z} & Z \\ \uparrow{\scriptstyle n} & & \uparrow{\scriptstyle m} & & \uparrow{\scriptstyle m_1} \\ P & \xrightarrow{p} & M & \xrightarrow{e_1} & M_1 \end{array}$$

where (e_1, m_1) is the $(\underline{E}, \underline{M})$-factorization of $\pi_Z m$, m is F-closed and (P, n, p) is a pullback. Since $p \in \underline{E}$, then $(e_1 p, m_1)$ is the $(\underline{E}, \underline{M})$-factorization of $\pi_Z(\pi_i \times 1_Z) n$. n being F-closed (as pullback of an F-closed subobject) implies that m_1 is F-closed, i.e., X_i is F-compact.

LEMMA 1.7. *If* F *is weakly hereditary and idempotent and if* $m : M \to X$ *is* F*-closed, then* $m \times 1_Z : M \times Z \to X \times Z$ *is* F*-closed for every* $Z \in \underline{X}$.

PROOF. Since $(\underline{E}, \underline{M})$ is a factorization structure on $\underline{X}$, $1_Z \in \underline{M}$ implies that $m \times 1_Z \in \underline{M}$ too (cf. [11], Proposition 1.4). Since F is weakly hereditary, $(\underline{E}^F, \underline{M}^F)$ is also a factorization structure on $\underline{X}$ (cf. [7], Proposition 3.2). Thus, being m and 1_Z both F-closed, we get that $m \times 1_Z$ is F-closed.

THEOREM 1.8. *If* F *is idempotent and weakly hereditary, then the* F*-compact objects are closed under* F*-closed* $\underline{M}$*-subobjects.*

PROOF. Let us consider the commutative diagram

$$\begin{array}{ccc} X \times Z & \xrightarrow{\pi_Z} & Z \\ \uparrow{\scriptstyle m \times 1_Z} & & \\ M \times Z & & \uparrow{\scriptstyle n_1} \\ \uparrow & & \\ N & \xrightarrow{e_1} & N_1 \end{array}$$

where X is F-compact, n is F-closed in $M \times Z$, m is F-closed in X and (e_1, n_1) is the $(\underline{E},\underline{M})$-factorization of $\pi_Z(m \times 1_Z)n$.

From Lemma 1.7, $m \times 1_Z$ is F-closed in $X \times Z$ and so is $(m \times 1_Z)n$ (cf. [7], Proposition 3.2). Since X is F-compact, we have that n_1 is F-closed. Let $P_Z : M \times Z \to Z$ be the usual projection and let (e_2, n_2) be the $(\underline{E},\underline{M})$-factorization of $P_Z n$ with $n_2 : N_2 \to Z$. Since (e_2, n_2) is another $(\underline{E},\underline{M})$-factorization of $\pi_Z(m \times 1_Z)n$, we get that $N_1 \approx N_2$ and so n_2 is F-closed, i.e., M is F-compact.

PROPOSITION 1.9. *Suppose that for $e \in \underline{E}$, the pullback of $e \times 1$ along any F-closed subobject belongs to $\underline{E}$. If f: $X \to Y$ is an $\underline{X}$-morphism and (e,m) is its $(\underline{E},\underline{M})$-factorization, then if X is F-compact so is $f(X)$ (where $f(X)$ is the middle object of the $(\underline{E},\underline{M})$-factorization).*

PROOF. Let us consider the commutative diagram

$$\begin{array}{ccccc}
N & \xrightarrow{\quad n \quad} & f(X) \times Z & \xrightarrow{\quad \pi_Z \quad} & Z \\
\uparrow{\scriptstyle (e \times 1_Z)^\circ} & & \uparrow{\scriptstyle e \times 1_Z} & & \uparrow{\scriptstyle 1_Z} \\
(e \times 1_Z)^{-1}(N) & \xrightarrow{\quad n^\circ \quad} & X \times Z & \xrightarrow{\quad P_Z \quad} & Z
\end{array}$$

where n° is the pullback of the F-closed $\underline{M}$-subobject n. Let (e_1, m_1) be the $(\underline{E},\underline{M})$-factorization of $\pi_Z n$ and (e_2, m_2) the $(\underline{E},\underline{M})$-factorization of $P_Z n^\circ$. m_2 is F-closed, since n°, as pullback of an F-closed monomorphism, is F-closed and X is F-compact. From the hypothesis $(e \times 1_Z)^\circ$ belongs to $\underline{E}$, i.e., $(e_1(e \times 1_Z)^\circ, n_1)$ is an $(\underline{E},\underline{M})$-factorization of $P_Z n^\circ$. Thus, there exists an isomorphism i such that $m_2 i = m_1$, which implies that m_1 is F-closed. Hence $f(X)$ is F-compact.

REMARK 1.10. In TOP (GR), the (Epi,Strong monomorphism)-factorization satisfies the requirement of the above proposition.

DEFINITION 1.11. Let f: $X \to Y$ be an $\underline{X}$-morphism. The morphism $\langle 1_X, f \rangle : X \to X \times Y$ is called the *graph of f.*

DEFINITION 1.12. Let $\underline{X}$ have equalizers. For every $Y \in \underline{X}$, $(\Delta_Y, \delta_Y) = \mathrm{equ}(\pi_1, \pi_2)$, $\pi_1, \pi_2 : Y \times Y \to Y$ being the usual projections, is called the *diagonal* of Y.

Notice that if $\underline{M}$ contains all regular monomorphisms, then $(\Delta_Y, \delta_Y) \in \underline{M}$ for every $Y \in \underline{X}$. In such a case, let us call F-separated all those objects $Y \in \underline{X}$ satisfying the condition that (Δ_Y, δ_Y) is F-closed in $Y \times Y$.

If F is induced by a subcategory $\underline{C}$, then the objects of $\underline{C}$ are F-separated (cf. [4]).

It is easy to prove the following

PROPOSITION 1.13. *Let $\underline{X}$ have equalizers and let $\underline{M}$ contain all regular monomorphisms. An $\underline{X}$-object* Y *is* F*-separated iff for every $\underline{X}$-morphism f:* $X \to Y$, $(X, \langle 1_X, f \rangle)$ *is* F*-closed in* $X \times Y$.

THEOREM 1.14. *Let $\underline{X}$ have equalizers and let $\underline{M}$ contain all regular monomorphisms. An* F*-compact subobject of an* F*-separated object is* F*-closed.*

PROOF. Let $m: M \to \underline{X}$ be F-compact and let X be F-separated. From Propositioin 1.13, $\langle 1_M, m \rangle : M \to M \times X$ is F-closed. Let us consider the commutative diagram

$$\begin{array}{ccc} M \times X & \xrightarrow{\pi_Z} & X \\ {\scriptstyle \langle 1_M, m \rangle} \uparrow & & \uparrow {\scriptstyle m} \\ M & \xrightarrow{1_M} & M \end{array}$$

By uniqueness of factorization structures, $(1_M, m)$ is the $(\underline{E}, \underline{M})$-factorization of $\pi_X \langle 1_M, m \rangle$ and since M is F-compact, π_X is F-closed preserving. This implies that m: $M \to X$ is F-closed.

COROLLARY 1.15. *Let $\underline{X}$ have equalizers and let $\underline{M}$ contain all regular monomorphisms. Let* F *be idempotent and weakly hereditary and suppose that for $e \in \underline{E}$. the pullback of $e \times 1$ along any* F*-closed subobject belongs to $\underline{E}$. If* X *is* F*-compact and* Y *is* F*-separated, then any $\underline{X}$-morphism f:* $X \to Y$ *is* F*-closed preserving.*

PROOF. Let us consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle m} \uparrow & & \uparrow {\scriptstyle m_1} \\ M & \xrightarrow{e_1} & M_1 \end{array}$$

where m is F-closed in X and (e_1, m_1) is the $(\underline{E}, \underline{M})$-factorization of fm. From Theorem 1.8, (M,m) is F-compact and from Proposition 1.9, so is (M_1,m_1). Finally from Theorem 1.14, m_1 is F-closed.

The following two results follow directly from Propositions 3.2 and 3.1 of [7].

PROPOSITION 1.16. *Let $\underline{X}$ be a regular well-powered category with products and let $\underline{C}$ be a subcategory of $\underline{X}$ closed under the formation of products and $\underline{M}$-subobjects. Then $[\ ]_C$ is weakly hereditary in $\underline{C}$ iff the regular monomorphisms in $\underline{C}$ are closed under composition.*

PROPOSITION 1.17. *Let $\underline{X}$ be a regular wellpowered (epi. strong monomorphism)-category with product and let $\underline{M}$ be the class of all strong monomorphisms. Let $\underline{C}$ be a subcategory of $\underline{X}$ closed under products and strong subobjects. If $[\ ]_C$ is weakly hereditary in $\underline{C}$. then the extremal monomorphisms in $\underline{C}$ agree with the regular monomorphisms in $\underline{C}$*

Further results concerning the coincidence of regular monomorphisms with extremal monomorphisms in $\underline{C}$ can be found in [17] and [18].

2. COMPACTNESS AND SURJECTIVITY OF EPIMORPHISMS.

In what follows, we present some applications to the categories AB (abelian groups) and TOP (topological spaces) in which the class $\underline{M}$ will be the class of all monomorphisms in the case $\underline{X}$ = AB and the class of all embeddings for $\underline{X}$ = TOP.

LEMMA 2.1. *Let $\underline{C}$ be epireflective in* AB *and let* M *be a subgroup of* $X \in$ AB. M *is $\underline{C}$-closed in* X *iff* $X/M \in \underline{C}$.

PROOF. ($\Rightarrow$) If M is $\underline{C}$-closed, then $M = \text{equ}(f,g)$ with $f, g\colon X \to Y$ and $Y \in \underline{C}$ (cf. [1] Proposition 1.6). Clearly $X/M \in \underline{C}$, since $M = \ker(f-g)$.

($\Leftarrow$) If $X/M \in \underline{C}$, then $M = \text{equ}(q,0)$ with $q, 0\colon X \to X/M$, i. e., M is $\underline{C}$-closed.

PROPOSITION 2.2. *Let $\underline{C}$ be epireflective in* AB, *then an object* $X \in$ AB *is $\underline{C}$-compact iff for every* $Z \in$ AB *and subgroup* M *of* $X \times Z$ *such that* $X \times Z/M \in \underline{C}$, *we have* $Z/\pi_2(M) \in \underline{C}$.

PROOF. It follows directly from Lemma 2.1.

PROPOSITION 2.3. *If $\underline{C}$ is an epireflective subcategory of* AB, *closed under the formation of quotients, then the projections are $\underline{C}$-closed preserving and the regular monomorphisms in $\underline{C}$ are closed under composition.*

PROOF. Let $X, Z \in AB$. Let us consider $\pi_Z: X \times Z \rightarrow Z$ and let M be $\underline{C}$-closed in $X \times Z$. Then, $M = equ(f,g)$ with $f, g: X \times Z \rightarrow Y$ and $Y \in \underline{C}$. Also $X \times Z/M$ is a subgroup of Y and so it belongs to $\underline{C}$. Clearly, there exists an epimorphism $e: X \times Z/M \rightarrow Z/\pi_Z(M)$ and since $\underline{C}$ is closed under the formation of quotients, $Z/\pi_Z(M) \in \underline{C}$, i.e., $\pi_Z(M)$ is $\underline{C}$-closed.

Let $m: M \rightarrow N$ and $n: N \rightarrow X$ be regular monomorphisms in $\underline{C}$. Since $\underline{C}$ is closed under subgroups, the monomorphisms in $\underline{C}$ are injective. Clearly,

$$(M, nm) = equ(q,0), q, 0: X \longrightarrow X/nm(M), \; X/nm(M) \in \underline{C}.$$

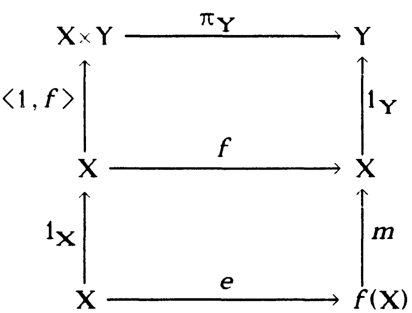
Thus, the regular monomorphisms in $\underline{C}$ are closed under composition.

COROLLARY 2.4. *If $\underline{C}$ is an epireflective subcategory of* AB *closed under the formation of quotients, then $[\;]_C$ is weakly hereditary in $\underline{C}$.*

PROOF. Apply Propositions 2.3 and 1.16.

PROPOSITION 2.5. *If $\underline{C}$ is a subcategory of* AB *or* TOP *and* Comp($\underline{C}$) *contains $\underline{C}$, then the epimorphisms in $\underline{C}$ are surjective.*

PROOF. Let $f: X \rightarrow Y$ be an epimorphism in $\underline{C}$, i.e., $[(f(X), m)]_C = Y$ (cf. [1], Theorem 1.11) and let (e, m) be the (epi,$\underline{M}$)-factorization of $f1_X$. Let us now consider the commutative diagram



Since $Y \in \underline{C}$, from Proposition 1.13, $(X, \langle 1, f \rangle)$ is $\underline{C}$-closed and since X is $\underline{C}$-compact, we get that $(f(X), m)$ is $\underline{C}$-closed, i.e.,

$$(f(X), m) = [(f(X), m)]_C = Y.$$

Thus, f is surjective.

We finally obtain

THEOREM 2.6. *Let* $\underline{C}$ *be epireflective in* AB. *The following statements are equivalent:*

a) $\underline{C}$ *is closed under the formation of quotients.*

b) The projections are $\underline{C}$*-closed preserving and the regular monomorphisms in* $\underline{C}$ *are closed under composition.*

c) Comp($\underline{C}$) = AB *and the regular monomorphisms in* $\underline{C}$ *are closed under composition.*

d) $\underline{C}$ *is contained in* Comp($\underline{C}$) *and the regular monomorphisms in* $\underline{C}$ *are closed under composition.*

e) The epimorphisms in $\underline{C}$ *are surjective and* $[\,]_C$ *is weakly hereditary in* $\underline{C}$.

f) Each subobject of a $\underline{C}$*-object is* $\underline{C}$*-closed.*

PROOF. a) ⇒ b). Proposition 2.3.

b) ⇒ c). Straightforward.

c) ⇒ d). Straightforward.

d) ⇒ e). Propositions 2.5 and 1.16.

e) ⇒ f). Let (M, m) be an $\underline{M}$-subobject of $X \in \underline{C}$ and let us consider the factorization $m = [m]_C m_{[m]}$. By the weakly hereditary hypothesis. $m_{[m]}$ is $\underline{C}$-dense. From [1], Theorem 1.11, it is an epimorphism in $\underline{C}$. Since the epimorphisms are surjective in $\underline{C}$, we get that $[m]_C \approx m$.

f) ⇒ a). Suppose there exists an AB-epimorphism $q: X \to Q$ such that $X \in \underline{C}$ but $Q \notin \underline{C}$ (clearly $Q \neq 0$). Let us consider the $\underline{M}$-subobject $i: \ker(q) \to X$. $X/\ker(q) \approx Q \notin \underline{C}$ contradicts the fact that i is $\underline{C}$-closed.

COROLLARY 2.7. *If* $\underline{C}$ *is epireflective in* AB *and closed under the formation of quotients, then* $\underline{C}$ *is the subcategory of the compact-separated objects of* F_C.

PROOF. Since $\underline{C}$ is epireflective in AB, we have that the objects of $\underline{C}$ are exactly the separated objects of F_C (cf. [4], Corollary 1.13 and [2], Theorem 2.1). From Theorem 2.6 we get that $\underline{C}$ is contained in Comp($\underline{C}$).

REMARK 2.8. We observe that the previous theorem shows that, although for a subcategory $\underline{C}$ of AB being closed under quotients always implies that the epimorphisms in $\underline{C}$ are surjective, the converse is not true, unless we put further assumptions on the subcategory $\underline{C}$.

If $\underline{C}$ is the subcategory of divisible abelian groups (DIV), algebraically compact abelian groups (AC), cotorsion abelian groups (COT) or the category AB itself, then the closure operator induced by $\underline{C}$ is weakly hereditary, simply because in all these cases every subobject is $\underline{C}$-closed This also implies that Comp($\underline{C}$) = AB.

From Proposition 2.5. we can conclude that the epimorphisms in AC are surjective. Although AC is not closed under quotients, this does not contradict Theorem 2.6, since AC is not epireflective in AB.

If $\underline{C}$ is the subcategory of torsion-free abelian groups (TF), then

$$\mathrm{Comp(TF)} \cap \mathrm{TF} = \text{torsion free divisible}$$

(cf. [11], Examples 2.3 and 4.9).

Further results about weakly hereditary closure operators in AB can be found in [3].

PROPOSITION 2.9. *If* F *is a closure operator on* Ab *or* TOP, *then the epimorphisms in* Comp(F) *are surjective.*

PROOF. If $X \in$ Comp(F) and $e: X \to Q$ is an epimorphism in AB or TOP. then from Proposition 1.9. $Q \in$ Comp(F). This clearly implies that the epimorphisms in Comp(F) are surjective.

For $\underline{X}$ = TOP. Theorem 2.6 does not hold. For instance, if $\underline{C} = TOP_1$ then every extremal subobject of a TOP_1-object is TOP_1-closed (cf. [5], Theorem 1.10) but $\underline{C}$ is not closed under quotients.

However. from Proposition 2.5 we get that if Comp($\underline{C}$) = TOP then the epimorphisms in $\underline{C}$ are surjective. This immediately implies that if $\underline{C}$ is one the subcategories Top_0. TOP_2, Top3 1/2 then Comp($\underline{C}$) ≠ TOP, because in these subcategories the epimorphisms are not surjective.

PROPOSITION 2.10. *If* $\underline{B}$ *is bireflective in* TOP. *then we have:* Comp($\underline{B}$) = TOP.

PROOF. From [5], Theorem 1.10 and [6], Lemma 2.1, we have that for every bireflective subcategory $\underline{B}$ of TOP, the $\underline{B}$-closure is the identity. This clearly implies that Comp($\underline{B}$) = TOP.

3. F-COMPACTIFICATION.

Let us assume that in the ($\underline{E}$,$\underline{M}$)-factorization structure of $\underline{X}$, $\underline{E}$ is a class of epimorphisms and $\underline{M}$ contains all regular monomorphisms. Let us also assume that $\underline{X}$ is an $\underline{E}$-co-well-powered category with products.

D(F) will denote the subcategory of F-separated objects of $\underline{X}$ (cf. [4]).

DEFINITION 3.1. Let F be a closure operator on $\underline{X}$ and let $X \in D(F)$. An *F-compactification* of X is a pair

$$(\beta, \beta_F X) \in \mathrm{Comp}(F) \cap D(F)$$

with $\beta: X \to \beta_F X$ D(F)-dense and such that for every morphism $f: X \to Y$ with $Y \in \mathrm{Comp}(F) \cap D(F)$, there exists a morphism

$$g: \beta_F X \to Y \text{ with } g\beta = f.$$

DEFINITION 3.2. A closure operator F is called *compactly productive* if Comp(F) is closed under arbitrary products.

LEMMA 3.3. *Each* F*-dense morphism is also* D(F)*-dense.*

PROOF. Let $f: X \to Y$ be an F-dense morphism and let (e, m) be its $(\underline{E}, \underline{M})$-factorization. From [4], Theorem 1.8, we get that

$$F \le KD(F), \text{ i.e., } [m]_F \le [m]_{D(F)}.$$

Since $[m]_F \approx [1_Y]$, we get that $[m]_{D(F)} \approx [1_Y]$, i.e., f is D(F)-dense.

PROPOSITION 3.4. *Let* F *be an idempotent and weakly hereditary closure operator such that* D(F) *is co-well-powered. If* F *is compactly productive, then the subcategory* $\mathrm{Comp}(F) \cap D(F)$ *is* $\underline{E}^F \cap \mathrm{Mor}\,D(F)$*-reflective in* D(F).

PROOF. Since Comp(F) and D(F) are both closed under products (cf. [2], Proposition 1.6), we get that $\mathrm{Comp}(F) \cap D(F)$ is closed under products in D(F) and under F-closed subobjects (cf. Theorem 1.8). Since F is weakly hereditary, from [7], Proposition 3.2, $(\underline{E}^F, \underline{M}^F)$ is a factorization structure on $\underline{X}$ and so

$$(\underline{E}^F \cap \mathrm{Mor}\,D(F), \underline{M}^F \cap \mathrm{Mor}\,D(F))$$

is a factorization structure on D(F). From Lemma 3.3, every morphism in $\underline{E}^F \cap \mathrm{Mor}\,D(F)$ is an epimorphism in D(F) (cf. [1], Theorem 1.11). Thus D(F) is $\underline{E}^F \cap \mathrm{Mor}\,D(F)$-co-well-powered and from [10], Theorem 37.1, we get that $\mathrm{Comp}(F) \cap D(F)$ is $\underline{E}^F \cap \mathrm{Mor}\,D(F)$-reflective in D(F).

So, we can conclude with the following

THEOREM 3.5. *Let* F *be an idempotent and weakly hereditary closure operator such that* D(F) *is co-well-powered. If* F *is compactly productive, then every* $X \in D(F)$ *has an* F*-compactification.*

PROOF. From Proposition 3.4, the subcategory Comp(F) $\cap$ D(F) is $\underline{E}^F \cap$ MorD(F)-reflective in D(F). For every X $\in$ D(F), let $\beta: X \to \beta_F X$ be the reflection morphism. β is F-dense and from Lemma 3.3, it is D(F)-dense.

EXAMPLE 3.6. If F is the closure operator induced in TOP by the (dense, closed embedding)-factorization structure, then

Comp(F) = compact and Comp(F) $\cap$ D(F) = compact Hausdorff.

If we restrict our attention to Tychonoff spaces, the induced F-compactification is the Stone-Čech compactification. Notice that Top_2 is co-well-powered (cf. [6], Corollary 3.4).

EXAMPLE 3.7. If TF is the subcategory of torsion-free abelian groups, then TFD, the subcategory of torsion-free divisible abelian groups is the subcategory of TF-compact TF-separated objects (cf. [4], [9] and [11], Examples 2.3 and 4.9). We want to construct the $[\ ]_{TF}$-compactification. Let G $\in$ TF and let D_G be its injective hull. $i: G \to D_G$ is -TF-dense, as it can be easily seen, and so an epimorphism in TF. So, given Z $\in$ TFD and $f: G \to Z$, there exists a unique $f': D_G \to Z$ such that $f'i = f$. Since D_G is torsion-free (cf. [14], pp. 21-22), the pair (i, D_G) is the wanted $[\ ]_{TF}$-compactification.

In particular, if G = Z, the additive group of integers, then D_G/T = Q, the additive group of rationals.

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