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KIMMO I. ROSENTHAL

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A NOTE ON GIRARD QUANTALES

by Kimmo I. ROSENTHAL *

RÉSUMÉ. Les quantales de Girard s'introduisent dans la sémantique pour la logique linéaire au sens introduit par J. Y. Girard. Les liens entre quantales et logique linéaire (non commutative) ont été précisés par D. Yetter. Cette note établit quelques résultats sur les quantales de Girard, en particulier en caractérisant le type le plus général de quantale de Girard et en clarifiant leur relation avec les algébres de Boole complètes.

INTRODUCTION.

In [4], J. Y. Girard introduced a new system of logic called "linear logic", which it is hoped will provide a suitable logical underpinning for the study of parallelism in computer science. This logic involves a linear negation operator $(\ )^{\perp}$, wich gives the logic a Boolean (classical) flavor. In developing what he calls the phase semantics for linear logic, Girard considers certain partially ordered monoids, which turn out to be quantales. Quantales were introduced by Mulvey [7] as a possible approach to a constructive foundation for quantum meachanics. (It should be pointed out that similar algebraic structures appear much earlier in the literature.) Quantales have been extensively studied by Niefield and Rosenthal in [6]. Recently, Yetter [8] has clarified the use of quantales in studying linear logic and he has introduced the term Girard quantale.

In this note we shall make a few observations about Girard quantales and tie up some loose ends, as well as clarify some examples. In particular, we shall point out that Girard's phase quantales are in fact the most general type of Girard quantale, in that every Girard quantale is isomorphic to a phase quantale. We also characterize when the phase quantale construction yields a complete Boolean algebra and indicate how every (unital) quantale can be embedded in a Girard quantale. Finally, we point out how the minimal completion of a Boolean algebra and the Dedekind completion of a partially ordered group are related to Girard quantales.

GIRARD QUANTALES.

We begin with some definitions.

DEFINITION 1. A *quantale* is a complete lattice Q together with an associative binary operation & satisfying

$$a\&(\sup_\alpha b_\alpha) = \sup_\alpha(a\& b_\alpha) \text{ and } (\sup_\alpha b_\alpha)\&a = \sup_\alpha(b_\alpha\&a)$$

for all $a \in Q$, $\{b_\alpha\} \subset Q$.

Note that since $a\&-$ and $-\&a$ are sup-preserving, they have right adjoints, denoted $\xrightarrow{r}\!\circ\, -$ and $\xrightarrow{\lambda}\!\circ\, -$ respectively.

Examples of quantales include frames (and hence complete Boolean algebras) and various ideal lattices of rings and C*-algebras. For more details and references, the reader is referred to [6].

DEFINITION 2. If Q, Q' are quantales, a function $f\colon Q \to Q'$ is a *homomorphism* iff it preserves sups and &.

We do not require that a homomorphism preserve the top element T (as was done in [6]). A surjective homomorphism will automatically preserve top elements. The category of quantales and homomorphisms will be denoted *Quant*.

A quantale Q is called *unital* iff there is an element 1 such that $1\&a = a = a\&1$ for all $a \in Q$. We shall denote the category of unital quantales and homomorphisms by *UnQuant*.

The following example is central to the discussion of Girard quantales. Let M be a monoid, written multiplicatively, and let P(M) denote the power set of M. P(M) is a quantale with

$$A\&B = \{a\,b \mid a \in A,\ b \in B\} \text{ for } A, B \subset M.$$

Unions play the role of sups and

$$A \xrightarrow{r}\!\circ\, B = \{m \in M \mid a\,m \in B \text{ for all } a \in A\}$$

and

$$A \xrightarrow{\lambda}\!\circ\, B = \{m \in M \mid m\,a \in B \text{ for all } a \in A\}.$$

If $f\colon M \to Q$ is a monoid homomorphism with Q a quantale, then $\bar{f}\colon PM \to Q$ defined by $\bar{f}(A) = \sup_{a \in A} f(a)$ is a quantale homomorphism. If *Mon* denotes the category of monoids and monoid homomorphisms, this construction describes the left adjoint to the forgetful functor *UnQuant* $\to$ *Mon*.

We need a few more definitions to get started.

DEFINITION 3. Let Q be a quantale. $d \in Q$ is called a *dualizing element* iff

$$(a \xrightarrow{\lambda}\!\circ\, d) \xrightarrow{r}\!\circ\, d = (a \xrightarrow{r}\!\circ\, d) \xrightarrow{\lambda}\!\circ\, d = a \text{ for all } a \in Q.$$

$c \in Q$ is called *cyclic* iff $a \xrightarrow{\lambda}\!\circ\, c = a \xrightarrow{r}\!\circ\, c$ for all $a \in Q$.

Note that c is cyclic is equivalent to $a_1 \& \cdots \& a_n \leq c$ iff $a_{\sigma(1)} \& \cdots \& a_{\sigma(n)} \leq c$ for all cyclic permutations σ of $\{1, 2, \cdots, n\}$.

We adopt the following terminology from Yetter [8].

DEFINITION 4. A quantale Q is called a *Girard quantale* iff it has a cyclic dualizing element d.

We shall denote $a \xrightarrow{\lambda}\!\circ\, d = a \xrightarrow{r}\!\circ\, d$ by $a \multimap d$ or more frequently $a^{\perp}$. Note that $a = a^{\perp\perp}$ and $a \leq b^{\perp}$ iff $b \leq a^{\perp}$; $(\,)^{\perp}$ is what Girard calls "linear negation" in his linear logic [4]. Thus, a Girard quantale has a "Boolean" aspect to it and a fundamental example of a Girard quantale is a complete Boolean algebra.

PROPOSITION 1. *Let Q be a Girard quantale with cyclic dualizing element d. Let $a, b \in Q$. Then:*

(1) $a \xrightarrow{\lambda}\!\circ\, b = (a \& b^{\perp})^{\perp}$,

(2) $a \xrightarrow{r}\!\circ\, b = (b^{\perp} \& a)^{\perp}$,

(3) $a \& b = (a \xrightarrow{\lambda}\!\circ\, b^{\perp})^{\perp}$,

(4) $b \& a = (a \xrightarrow{r}\!\circ\, b^{\perp})^{\perp}$.

PROOF. For (1), if $c \in Q$,

$$c \leq (a \& b^{\perp})^{\perp} \text{ iff } a \& b^{\perp} \leq c^{\perp} \text{ iff } c \& a \& b^{\perp} \leq d \text{ iff } c \& a \leq (b^{\perp})^{\perp}.$$

By uniqueness of adjoints, this proves that

$$a \xrightarrow{\lambda}\!\circ\, b = (a \& b^{\perp})^{\perp}.$$

(2) is proved similarly, and then (3) and (4) follow from (1) and (2) respectively. ∎

COROLLARY. *A Girard quantale is unital with unit $d^{\perp}$.*

PROOF. If $a \in Q$, by (3) above

$$a \& d^{\perp} = (a \xrightarrow{\lambda}\!\circ\, (d^{\perp})^{\perp})^{\perp} = (a \xrightarrow{\lambda}\!\circ\, d)^{\perp} = a^{\perp\perp} = a.$$

$d^{\perp} \& a = a$ is proved similarly using (4). ∎

MAIN EXAMPLE. In considering the semantics of his linear logic [4], Girard considers the following type of quantale. Let M be a commutative monoid and let $D \subset M$. If $A \subset M$, let

$$A^{\perp} = \{ m \in M \mid ma \in D \text{ for all } a \in A \}$$

Girard calls the elements of M "phases" and $A \in P(M)$ is called a "fact" iff $A = A^{\perp\perp}$. It follows that $\{ A \in P(M) \mid A = A^{\perp\perp} \}$ is a Girard quantale with dualizing element D. If A, B are facts, $A \&_D B = (A \& B)^{\perp\perp}$ and if $\{A_i\}$ is a collection of facts, $\sup_i A_i = (\cup_i A_i)^{\perp\perp}$.

To make the situation non-commutative, we could allow M to be non-commutative and require that D is cyclic in P(M). One can easily see that this amounts to $ab \in D$ iff $ba \in D$ for all $a, b \in M$. We shall denote the quantale thus constructed by $P(M)_D$ and shall call it a *phase quantale.*

Phase quantales are constructed according to a general procedure for producing Girard quantales as quotients in *Quant*, as observed by Yetter [8]. We shall add to this observation by pointing out that in fact these are the only possible Girard quotients of a given quantale. First we need to recall a definition.

DEFINITION 5. Let Q be a quantale. A closure operator $j: Q \to Q$ is a *quantic nucleus* iff $j(a) \& j(b) \leq j(a \& b)$ for all $a, b \in Q$.

If j is a quantic nucleus, $Q_j = \{a \in Q \mid j(a) = a\}$ is a quantale with

$$a \&_j b = j(a \& b) \text{ and } \sup_j a_i = j(\sup a_i) \text{ for } a, b \in Q_j,\ \{a_i\} \subset Q_j.$$

The Q_j are precisely the quotients in *Quant* and were extensively studied in [6]. There, it is also pointed out (Prop. 2.6) that:

(∗) if Q is a unital quantale, j is a quantic nucleus iff

$$a \xrightarrow{\lambda}\!\circ\, j(b) = j(a) \xrightarrow{\lambda}\!\circ\, j(b) \text{ and } a \xrightarrow{r}\!\circ\, j(b) = j(a) \xrightarrow{r}\!\circ\, j(b)$$

for all $a, b \in Q$.

THEOREM 1. *Let* Q *be a unital quantale and let* j *be a quantic nucleus on* Q . *Then,* Q_j *is a Girard quantale iff* j *is of the form* $(- \multimap d) \multimap d$ *for a cyclic element* d *of* Q.

PROOF. ⇐ This appears in [8] and is straightforward.

⇒ Suppose Q_j is a Girard quantale with cyclic dualizing element $d = j(d)$. By (∗), since

$$j(a) \xrightarrow{\lambda}\!\circ\, j(d) = j(a) \xrightarrow{r}\!\circ\, j(d)$$

it follows that

$$a \xrightarrow{\lambda}\!\circ\, j(d) = a \xrightarrow{r}\!\circ\, j(b) \text{ for all } a \in Q,$$

thus d is in fact cyclic in Q. Also, by (∗), if $a \in Q$,

$$j(a) = (j(a) \multimap d) \multimap d = (j(a) \multimap j(d)) \multimap d$$
$$= (a \multimap j(d)) \multimap d = (a \multimap d) \multimap d. \blacksquare$$

The special case of this result, where "unital quantale" is replaced by "frame" and "Girard quantale" is replaced by "complete Boolean algebra" is well known. (See [5], Exer. II.2.4 (ii).)

Every phase quantale $P(M)_D$ is obtained by this construction by considering the quantic nucleus $(- \multimap D) \multimap D$ on $P(M)$. In [8], one is left with the impression that due to Theorem 1, there might be more general Girard quantales than phase quantales. However, this is not the case.

THEOREM 2. *If* Q *is a Girard quantale, then it is isomorphic to a phase quantale.*

PROOF. Let d be a cyclic dualizing element of Q. Let us view Q as a monoid (forgetting its order structure), and let

$$D = d{\downarrow} = \{a \in Q \mid a \le d\}.$$

We shall show that $Q \simeq P(Q)_D$. Let $A \subset Q$ and let $a = \sup A$.

$$A^{\perp} = \{q \in Q \mid q \& c \le d \text{ for all } c \in A\}.$$

$q\&c \le d$ iff $q \le c^{\perp}$ for all $c \in A$. It follows that $q \in A^{\perp}$ iff $q \le a^{\perp}$. Thus $A^{\perp} = (a^{\perp}){\downarrow}$. So, $A^{\perp\perp} = (a^{\perp\perp}){\downarrow} = a{\downarrow}$. Hence, A is a fact iff $A = a{\downarrow}$, where $a = \sup A$.
Define $e\colon Q \to P(Q)_D$ by $e(a) = a{\downarrow}$. e is clearly bijective and preserves sups. (Note $e(T) = Q$.)

$$e(a\&b) = (a\&b){\downarrow} = (a{\downarrow}\& b{\downarrow})^{\perp\perp} = a{\downarrow}\&_D b{\downarrow}. \blacksquare$$

Let us consider some examples. As mentioned before, complete Boolean algebras are examples of Girard quantales. In this case $d = 0$ (the bottom element), however that is not sufficient to guarantee that one has a Boolean algebra. A lot depends on the choice of $D \subset M$ in the phase quantale construction.

It is easy to see that $D^{\perp} = D \multimap D$ is a submonoid of M. D will be the bottom element of $P(M)_D$ iff $D^{\perp} = M$. The following proposition can be easily established.

PROPOSITION 2. D *is the bottom element of* $P(M)_D$ *iff* D *is an ideal of* M. *In this case. every fact* A *is also an ideal of* M.

Thus, in this situation we are considering quantales of ideals of the monoid M. To determine when such a quantale is in fact a complete Boolean algebra, we need to recall a defintion from [6].

DEFINITION 6. If Q is a quantale, $a \in Q$ is *semiprime* iff $b\&b \leq a$ implies $b \leq a$ for all $b \in Q$.

If every element of Q is two-sided (that is

$$T\&a = a = a\&T),$$

then the semiprime elements form the largest quotient of Q which is a frame. This clearly holds if $M = D^{\perp}$.

THEOREM 3. *If Q is a unital quantale in which every element is two-sided and if d is a cyclic element, then Q_j is a frame where $j = (- \multimap d) \multimap d$, iff d is semiprime.*

PROOF. This is the content of Theorem 4.3 in [6] adapted to this particular context. ∎

COROLLARY 3.1. *If M is a monoid and $D \subset M$ then $P(M)_D$ is a complete Boolean algebra iff D is a semiprime ideal of M (iff D is semiprime in P(M)).*

PROOF. By Proposition 2, in this context we have that $P(M)_D = Idl(M)_D$ where Idl(M) is the subquantale of P(M) consisting of the ideals of M. Thus we can apply Proposition 2 and Theorem 3. A frame is a Girard quantale iff it is a complete Boolean algebra.

REMARKS. 1. If $M = \mathbb{Z}$ under multiplication, we obtain the result that given $n \in \mathbb{Z}$, the divisors of n form a complete Boolean algebra iff $n\mathbb{Z}$ is a radical ideal of $\mathbb{Z}$, i.e., n has no repeated prime factors.

2. If Q is a Girard quantale with dualizing element 0, then by Corollary 3.1, Q is a complete Boolean algebra iff Q has no nilpotents, i.e., $a\&a \neq 0$ for $a \neq 0$.

3. The Boolean algebra one obtains from a semiprime ideal may be trivial. If D is a prime ideal of M

$$(ab \in D \Rightarrow a \in D \text{ or } b \in D),$$

then $P(M)_D = \{D, M\}$.

Another type of interesting Girard quantale is one in which the dualizing object d is self-dual. Thus, $d^{\perp} = d$ and hence d is also the unit for &. It is not hard to see that D is self-dual in $P(M)_D$ iff D is a submonoid of M. We shall see that such quantales arise in completing partially ordered abelian groups.

We have mentioned before that one should view Girard quantales as the "Boolean" quantales. It is a well known theorem that every frame can be embedded in a complete Boolean algebra [5]. The analogous result for quantales and Girard quantales is easier to prove and follows a general construction of Chu's for $*$-autonomous categories [2]. We have made the necessary adjustments for the non-commutative case.

THEOREM 4. *Let Q be a unital quantale. Then, there exists a Girard quantale $\tilde{Q}$ and an embedding $\varepsilon: Q \to \tilde{Q}$ of quantales.*

PROOF. Let $\tilde{Q} = Q \times Q^{op}$. Thus, elements of $\tilde{Q}$ are (a,a') of elements of Q and the ordering is

$$(a,a') \le (b,b') \text{ iff } a \le b \text{ and } b' \le a'.$$

Let $A = (a,a')$, $B = (b,b')$. Define

$$A\&B = (a\&b,\ a \xrightarrow{\lambda}\!\!\circ\ b' \wedge b \xrightarrow{r}\!\!\circ\ a'),$$

$$A \xrightarrow{\lambda}\!\!\circ B = (a \xrightarrow{\lambda}\!\!\circ\ b \wedge b' \xrightarrow{r}\!\!\circ\ a',\ a\&b')$$

and

$$A \xrightarrow{r}\!\!\circ B = (a \xrightarrow{r}\!\!\circ\ b \wedge b' \xrightarrow{\lambda}\!\!\circ\ a',\ b'\&a).$$

Let $C = (c,c')$:

$$A\&B \le C \text{ iff } (a\&b,\ a \xrightarrow{\lambda}\!\!\circ\ b' \wedge b \xrightarrow{r}\!\!\circ\ a') \le (c,c')$$

iff $\quad a\&b \le c$ and $c' \le a \xrightarrow{\lambda}\!\!\circ\ b' \wedge b \xrightarrow{r}\!\!\circ\ a'$

iff $\quad a \le b \xrightarrow{\lambda}\!\!\circ\ c$, $c'\&a \le b'$ and $b\&c' \le a'$

iff $\quad a \le b \xrightarrow{\lambda}\!\!\circ\ c$, $a \le c' \xrightarrow{r}\!\!\circ\ b'$ and $b\&c' \le a'$

iff $\quad a \le b \xrightarrow{\lambda}\!\!\circ\ c \wedge c' \xrightarrow{r}\!\!\circ\ b'$ and $b\&c' \le a'$ iff $A \le B \xrightarrow{\lambda}\!\!\circ C$.

One similarly checks that $A\&B \le C$ iff $B \le A \xrightarrow{r}\!\!\circ C$. $\tilde{Q}$ is clearly complete since Q is. Let $D = (T,1)$. One can check that if we have $A = (a,a')$, then

$$A \xrightarrow{\lambda}\!\!\circ D = A \xrightarrow{r}\!\!\circ D \text{ and } (A \multimap D) \multimap D = A.$$

Thus, D is a cyclic dualizing element.
Define $\varepsilon: Q \to \tilde{Q}$ by $\varepsilon(a) = (a,T)$.

$$\varepsilon(a)\&\varepsilon(b) = (a,T)\&(b,T) = (a\&b,\ a \xrightarrow{\lambda}\!\!\circ T \wedge b \xrightarrow{r}\!\!\circ T)$$
$$= (a\&b,T) = \varepsilon(a\&b).$$

ε is one-to-one and clearly preserves sups and thus is an embedding of quantales. ∎

Note that if L is a frame, then the dualizing element of $\tilde{L}$ is self-dual since $T = 1$, and therefore $D = (1,1)$. Then, $\varepsilon(L) \approx D\downarrow$.

Also, we get that every frame can be obtained from a Girard quantale by a coclosure operator satisfying the strong Girard axiom for his "of course" modal operator [4]. The idempotent elements of $\tilde{L}$ form a frame containing $\varepsilon(L)$ and are characterized as pairs $(a.b)$ with $a \to b = b$ (where $\to$ is implication in L).

Finally, we wish to point out the following construction. If we remove the completeness assumption from Definition 1 for a quantale, we get what might be called a **-autonomous poset.* These are the partially ordered examples of *-autonomous categories, in the terminology of Barr [1]. Two primary examples of *-autonomous posets are any Boolean algebra B and any partially ordered group G. Note that in G,

$$a \xrightarrow{r}\!\!\circ\; b = a^{-1}b \text{ and } a \xrightarrow{\lambda}\!\!\circ\; b = ba^{-1}.$$

Let Q be a *-autonomous poset with cyclic dualizing element d. Consider $D = d\!\downarrow$ and form the Girard quantale $P(Q)_D$. As in Theorem 2, one can easily establish that $a\!\downarrow$ is a fact for all $a \in Q$ and thus we have an embedding $e\colon Q \to P(Q)_D$ given by $e(a) = a\!\downarrow$. If $A \subset Q$, let $L(A)$ and $U(A)$ denote the set of all lower bounds of A and the set of all upper bounds of A, respectively.

PROPOSITION 3. *Let Q be a *-autonomous poset with cyclic dualizing element d. Let $D = d\!\downarrow$. Then. $A \subset Q$ belongs to $P(Q)_D$ (that is $A = A^{\perp\perp}$) iff $A = L(U(A))$.*

PROOF. If $A \subset Q$, let $A^* = \{a^\perp \mid a \in A\}$.

$$A^\perp = \{c \in Q \mid c \le a^\perp \text{ for all } a \in A\},$$

thus $A^\perp = L(A^*) = U(A)^*$. Thus

$$A^{\perp\perp} = L((A^\perp)^*) = L(U(A)^{**}) = L(U(A)).$$

Hence, $A = A^{\perp\perp}$ iff $A = L(U(A))$. ∎

It is not hard to show that if B is a Boolean algebra, this construction yields the minimal completion of B (in the sense of [5]), also known as the *Macneille completion.* If G is a partially ordered group, then our construction gives rise to the *Dedekind completion* of G [3].

Thus, there are many interesting examples of Girard quantales arising in mathematics and some further study of them as well as their connection with linear logic may prove useful.

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DEPARTMENT OF MATHEMATICS
UNION COLLEGE
SCHENECTADY, N.Y. 12308. U.S.A.