CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 31, nº 1 (1990), p. 21-52

<http://www.numdam.org/item?id=CTGDC_1990__31_1_21_0>

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VOL. XXXI-1 (1990)

LOCAL ANALYTIC RINGS [1]

by Jorge C. ZILBER

RÉSUMÉ. Cet article développe certains aspects de la théorie des anneaux anaytiques locaux introduite par Dubuc-Taubin. Un anneau analytique local est un morphisme local $A \to \mathbb{C}$; on montre qu'il peut aussi être défini comme un foncteur $U \mapsto A(U)$ préservant les recouvrements ouverts arbitraires. Ceci est utilisé pour construire le topos classifiant de la théorie des anneaux analytiques locaux: c'est le topos des faisceaux sur le site dont les objets sont les modèles locaux d'espaces analytiques, avec la topologie de Grothendieck des recouvrements ouverts. Enfin on construit le spectre d'un anneau analytique finiement présentable.

INTRODUCTION.

In this article we develop some aspects of the theory of analytic rings introduced by E. Dubuc and G. Taubin in [3]. The principal results are the following:

(1) In [3] an analytic ring A in *Ens* was defined to be local if it had a local morphism $A \to \mathbb{C}$ into the ring of complex numbers. Here we define A to be local if, viewed as a functor $U \mapsto A(U)$, it preserves arbitrary open coverings. Then, we show the equivalence of these two definitions. This shows that the purely algebraic notion of being local (notice that in the case of ordinary rings preservation of arbitrary coverings by Zarisky opens is equivalent to the axiom "$x \vee (1-x)$ invertible") is in this case equivalent to the existence of a morphism $A \to \mathbb{C}$, which is of higher order.

(2) Based on the previous result, we explicitely construct the classifying topos of the theory of local analytic rings. More explicitely, we show that it is the topos of sheaves on the site whose objects are the local models of analytic spaces (in the usual sense, see for example Malgrange [6]), with the Grothendieck topology of the open coverings. This result shows that the notion of analytic ring and local analytic ring is the adequa-

[1] Research partially supported by "Subsidio UBACYT EX 068, Proyecto: Anillos analiticos y espacios analiticos, de la Universidad de Buenos-Aires.

te algebraic tool for the study of analytic spaces.

(3) We explicitely construct the spectrum (in Cole's sense) of a finitely presentable analytic ring. In particular, we show that the spectrum of an analytic ring of the form $O_n(\mathrm{U})/(h_1,\dots,h_k)$ is the local (special) model associated to the ideal $(h_1,\dots,h_k)$. In this case the construction is more complicated than in the cases corresponding to ordinary or C^∞-rings, because analytic rings is an algebraic theory where the quotients are not surjective.

We thank Eduardo Dubuc for valuable conversations during the research and writing of the paper.

0. BASIC DEFINITIONS.

0.1. DEFINITION. We denote with $\boldsymbol{C}$ the category of open subsets of $\mathbb{C}^n$ (all n) and holomorphic functions.

0.2. DEFINITION. Let U, V and W be open subsets of $\mathbb{C}^n, \mathbb{C}^m$ and $\mathbb{C}^k$, respectively; $f: \mathrm{U} \to \mathrm{V}$ and $g: \mathrm{W} \to \mathrm{V}$ holomorphic functions. We say that a diagram in $\boldsymbol{C}$:

$$\begin{array}{ccc} \mathrm{X} & \longrightarrow & \mathrm{W} \\ \downarrow & & \downarrow g \\ \mathrm{U} & \xrightarrow{\ f\ } & \mathrm{V} \end{array}$$

is a transversal pullback, if it is a pullback in $\boldsymbol{C}$ and f and g are transversal (in the usual sense, see for example [4]).

0.3. DEFINITION. A diagram in $\boldsymbol{C}$:

$$\mathrm{E} \longrightarrow \mathrm{U} \ \underset{0}{\overset{h}{\rightrightarrows}}\ \mathbb{C}^k$$

is an independent equalizer if it is an equalizer in $\boldsymbol{C}$ and the components of h are independent (in the usual sense).

0.4. PROPOSITION. *Let $\boldsymbol{E}$ be a category and A: $\boldsymbol{C} \to \boldsymbol{E}$ a functor. Then, the following statements are equivalent:*

1. A preserves transversal pullbacks and terminal object.

2. A preserves independent equalizers, finite products (including the terminal object) and open inclusions.

PROOF. Cf. Dubuc & Taubin [3]. ■

0.5. DEFINITION [3]. Let $\boldsymbol{E}$ be a category with finite limits; a functor A: $\boldsymbol{C} \to \boldsymbol{E}$ is an analytic ring in $\boldsymbol{E}$ if any of the two equivalent conditions in Proposition 0.4 holds. A morphism between analytic rings is a natural transformation, as functors.

0.6. OBSERVATION. By an abuse of notation, we will write A for A($\mathbb{C}$), and if $\varphi: A \to B$ is a morphism, we will write φ for $\varphi_{\mathbb{C}}$; let *Ens* be the category of sets, and A an analytic ring in *Ens*. If V and U are open subsets of $\mathbb{C}^n$ and $V \subset U$, then A(V) is a subset of A(U). This is so because A preserves open inclusions. Moreover, if $f: U \to W$ is a holomorphic function, we have that $A(f|V) = A(f)|A(V)$, and if $\varphi: A \to B$ is a morphism and $a \in A(V)$, then $\varphi_U(a) = \varphi_V(a)$.

We also have that $A(\mathbb{C}^n) = A^n$ and

$$\varphi_{\mathbb{C}^n}(a_1, \dots, a_n) = (\varphi(a_1), \dots, \varphi(a_n)) \text{ for } (a_1, \dots, a_n) \in A^n.$$

Note that if $\varphi, \psi: A \to B$ are morphisms such that $\varphi_{\mathbb{C}} = \psi_{\mathbb{C}}$, then $\varphi_V = \psi_V$ for all $V \in \boldsymbol{C}$.

Thus, an analytic ring may be thought as a $\mathbb{C}$-algebra A, with the additional structure given by operations with domain of definition $A(V) \subset A^n$, one for each holomorphic function $V \to \mathbb{C}$, for all V open in some $\mathbb{C}^n$.

0.7. EXAMPLE. If U is an open subset of $\mathbb{C}^n$, we denote with $O_n(U)$ the analytic ring of holomorphic functions in U; that is, if V is an open subset of $\mathbb{C}^m$, we define

$$(O_n(U))(V) = \{g: U \to V \mid g \text{ is holomorphic}\},$$

and if W is an open subset of $\mathbb{C}^r$ and $f: V \to W$ is a holomorphic function, we define

$$(O_n(U)(f))(g) = f \circ g \text{ for } g \in (O_n(U))(V).$$

0.8. OBSERVATION. Notice that $O_n(U)$ is the representable functor [U,−]. Thus, any morphism $\varphi: O_n(U) \to A$ into an analytic ring A is characterized by the element $\varphi_U(\mathrm{id}_U) \in A(U)$. (Yoneda's Lemma, see [5].)

0.9. EXAMPLE. Note that C has a structure of analytic ring in *Ens* given by $\mathbb{C}(U) = U$, $\mathbb{C}(f) = f$.

It is immediate to check the following:

0.10. PROPOSITION. *If* A *is an analytic ring, then* A *preserves finite intersections.* ■

0.11. OBSERVATION. Given an analytic ring A in *Ens*, the canonical morphism $\mathbb{C} \to A$ is a morphism of analytic rings. This is so because if U is an open subset of $\mathbb{C}$ and $\alpha \in U$, then the interpretation α_A of α in the $\mathbb{C}$-algebra A belongs to $A(U)$, and this result extends immediately to open subsets of $\mathbb{C}^n$. We abuse the notation and denote $\alpha_A = \alpha$. Moreover, it is clear that this defines a morphism of analytic rings.

It is immediate to check the following:

0.12. LEMMA. *Let* A *be an analytic ring.* U *and* V *open subsets of* $\mathbb{C}^n$ *and* $\mathbb{C}^m$ *respectively, and* $f\colon U \to V$ *a holomorphic function. Then, if* W *is open,* $W \subset V$,

$$A(f)^{-1}(A(W)) = A(f^{-1}(W)). \blacksquare$$

1. LOCAL ANALYTIC RINGS.

1.1. DEFINITION. Let A be an analytic ring in a category $\boldsymbol{E}$. We say that A is a *local analytic ring* if A preserves open coverings, that is, if for each open covering $(U_\alpha)_{\alpha \in I}$ of an open set U of $\mathbb{C}^n$, the family $(A(U_\alpha) \hookrightarrow A(U))_{\alpha \in I}$ is a universal effective epimorphic family in $\boldsymbol{E}$. Remark that this notion is stronger that to say that A is an analytic ring which is local as a ring.

1.2. OBSERVATION. When $\boldsymbol{E} = \mathit{Ens}$, it means that if $U = \bigcup_{\alpha \in I} U_\alpha$, then $A(U) = \bigcup_{\alpha \in I} A(U_\alpha)$.

1.3. OBSERVATION. Since the empty family covers the empty set, it follows that if A is a local analytic ring in a topos $\boldsymbol{E}$, then $A(\emptyset) = \emptyset$.

1.4. DEFINITION [3]. Let A and B be analytic rings in a category $\boldsymbol{E}$, $\varphi\colon A \to B$ a morphism of analytic rings. We say that φ is *local* if for all open inclusions $V \subset U$ in $\boldsymbol{C}$, the square

$$\begin{array}{ccc} A(V) & \hookrightarrow & A(U) \\ {\scriptstyle \varphi_V}\downarrow & & \downarrow{\scriptstyle \varphi_U} \\ B(V) & \hookrightarrow & B(U) \end{array}$$

is a pullback in $\boldsymbol{E}$. It is equivalent to ask this condition only for $U = \mathbb{C}^n$, all n. When $\boldsymbol{E} = \mathit{Ens}$, it is equivalent to the condition:

"if $a \in A(U)$ and $\varphi_U(a) \in B(V)$, then $a \in A(V)$".

1.5. OBSERVATION [3]. It is easy to see that if A is an analytic ring in *Ens* and there exists a local morphism $\pi\colon A \to \mathbb{C}$, then A

preserves open coverings, that is, A is a local analytic ring.

1.6. THEOREM. *Let A be a local analytic ring in Ens. Then, there exists a local morphism $\pi: A \to \mathbb{C}$.*

PROOF. Let a be an element of $A = A(\mathbb{C})$. For each $j \in \mathbb{N}$, we write $\mathbb{C} = \bigcup_{p\in\mathbb{C}} B(p,1/j)$, where $B(p,1/j)$ is the open ball in $\mathbb{C}$ of center p and radius $1/j$. Then since A preserves open coverings, we have $A = \bigcup_{p\in\mathbb{C}} A(B(p,1/j))$. Hence, there exists an element $\alpha_j \in \mathbb{C}$ such that $a \in A(B(\alpha_j,1/j))$. It is immediate that $a-\alpha_j \in A(B(0,1/j))$ (1) (here we think $\alpha_j \in A$). Then we have

(2) $$\alpha_j-\alpha_k = (\alpha_j-a)+(a-\alpha_k) \in A(B(0,(1/j)+(1/k)))$$

(where we use the fact that if $a_1 \in A(B(0,r_1))$ and $a_2 \in A(B(0,r_2))$, then $a_1+a_2 \in A(B(0,r_1+r_2))$, which is immediate considering

$$+: B(0,r_1)\times B(0,r_2) \longrightarrow B(0,r_1+r_2).$$

Now, we shall prove that if $\beta \in \mathbb{C}$ and $\beta \in A(B(0,r))$, then $|\beta| \le r$ (3). In fact, if $|\beta| > r$, then $\beta \in U$, where $U = \{z \in \mathbb{C} \mid |z| > r\}$. Then, by 0.10 and 1.3, we have

$$A(U) \cap A(B(0,r)) = A(U \cap B(0,r)) = A(\varnothing) = \varnothing .$$

Now, by 0.11, $\beta \in A(U)$; hence $\beta \notin A(B(0,r))$. Using this fact we obtain by (2) that $|a_j-\alpha_k| \le (1/j)+(1/k)$, and thus, $(\alpha_j)_{j\in\mathbb{N}}$ is a Cauchy's sequence of complex numbers; therefore, there exists $\alpha = \lim_{j\to\infty}\alpha_j$.

We show now that $a-\alpha \in A(B(0,\varepsilon))$ for all $\varepsilon > 0$: In fact, given $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that $|\alpha-\alpha_j| < \varepsilon/2$ for $j \ge j_0$. Let j_1 be such that $1/j_1 < \varepsilon/2$. If $j_2 = \max\{j_0, j_1\}$, then

$$a-\alpha = (a-\alpha_{j2})+(\alpha_{j2}-\alpha), \text{ where } a-\alpha_{j2} \in A(B(0,1/j_2))$$

(by (1)), and since $1/j_2 < \varepsilon/2$, then $B(0,1/j_2) \subset B(0,\varepsilon/2)$. Hence,

$$A(B(0,1/j_2)) \subset A(B(0,\varepsilon/2));$$

thus $a-\alpha_{j2} \in A(B(0,\varepsilon/2))$ (4).

On the other hand, since $j_2 \ge j_0$, then $|\alpha-\alpha_{j2}| < \varepsilon/2$, that is, $\alpha_{j2}-\alpha \in B(0,\varepsilon/2)$, and (by 0.11), $\alpha_{j2}-\alpha \in A(B(0,\varepsilon/2))$ (5). Hence, by (4) and (5) we obtain that $a-\alpha \in A(B(0,\varepsilon))$. Hence, we have established that given $a \in A$, there exists $\alpha \in \mathbb{C}$ such that

$$a-\alpha \in A(B(0,\varepsilon)) \text{ for all } \varepsilon > 0.$$

Now we shall see that α is unique: Let β be a complex number such that $a-\beta \in A(B(0,\varepsilon))$ for all $\varepsilon > 0$. Given $\varepsilon > 0$, then

$$\beta-a \in A(B(0,\varepsilon/2)) \text{ and } a-\alpha \in A(B(0,\varepsilon/2));$$

hence

$$(\beta-a)+(a-\alpha) = \beta-\alpha \in A(B(0,\varepsilon));$$

therefore, by (3), $|\beta-\alpha| \leq \varepsilon$. Then, $\beta=\alpha$.

In conclusion, we have that, given $a \in A$, there exists a unique $\alpha \in \mathbb{C}$ such that $a-\alpha \in A(B(0,\varepsilon))$ for all $\varepsilon > 0$. This defines a function $\pi: A \to \mathbb{C}$ which verifies $\pi(a)=\alpha$ iff $a-\alpha \in A(B(0,\varepsilon))$ for all $\varepsilon > 0$ (6). We are going to prove now that π defines a morphism of analytic rings. For this, let us see first that if U is an open subset of $\mathbb{C}^n$, and $a = (a_1,\dots,a_n) \in A(U)$, then $(\pi(a_1),\dots,\pi(a_n)) \in U$. Since U is open, we can write:

$$U = \bigcup_{j\in J} B_{j,1}\times\dots\times B_{j,n},$$

where each $B_{j,k}$ is an open ball in $\mathbb{C}$ and $\overline{B_{j,1}\times\dots\times B_{j,n}} \subset U$. Hence,

$$A(U) = \bigcup_{j\in J} A(B_{j,1}\times\dots\times B_{j,n}) = \bigcup_{j\in J} A(B_{j,1})\times\dots\times A(B_{j,n}).$$

Since $a = (a_1,\dots,a_n) \in A(U)$, there exists $j \in J$ such that

$$a \in A(B_{j,1})\times\dots\times A(B_{j,n}), \text{ that is, } a_k \in A(B_{j,k}) \ (1 \leq k \leq n).$$

Therefore, if $B_{j,k} = B(z_{j,k}, r_{j,k})$, we have that

$$a_k - z_{j,k} \in A(B(0,r_{j,k})).$$

Moreover, by (6), if $\alpha_k = \pi(a_k)$ $(1 \leq k \leq n)$, then $\alpha_k - a_k \in A(B(0,\varepsilon))$ for all $\varepsilon > 0$. Then, it follows that $\alpha_k - z_{j,k} \in A(B(0,r_{j,k}+\varepsilon))$. Therefore, by (3), we have

$$|\alpha_k - z_{j,k}| \leq r_{j,k}+\varepsilon \text{ for all } \varepsilon > 0.$$

Then $|\alpha_k - z_{j,k}| \leq r_{j,k}$, that is,

$$\alpha_k \in \overline{B(z_{j,k},r_{j,k})} = \overline{B_{j,k}} \ (1 \leq k \leq n).$$

Then,

$$(\alpha_1,\dots,\alpha_n) \in \overline{B_{j,1}}\times\dots\times\overline{B_{j,n}} \subset \overline{B_{j,1}\times\dots\times B_{j,n}} \subset U,$$

that is, $(\pi(a_1),\dots,\pi(a_n)) \in U$. This defines, for each open subset U of $\mathbb{C}^n$, a function $\pi_U: A(U)\to U$, given by

$$\pi_U(a_1,\dots,a_n) = (\pi(a_1),\dots,\pi(a_n)).$$

Finally, in order to prove that π is a morphism of analytic rings, we have to prove that if $f: U\to V$ is a holomorphic functor (where U and V are open subsets of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively), then the square:

$$\text{(7)} \qquad \begin{array}{ccc} A(U) & \xrightarrow{A(f)} & A(V) \\ \Big\downarrow{\scriptstyle \pi_U} & & \Big\downarrow{\scriptstyle \pi_V} \\ U & \xrightarrow{f} & V \end{array}$$

commutes. Suppose that $V \subset \mathbb{C}$, and let a be an element of A(U), $a = (a_1,\dots,a_n)$. Then $\pi_U(a) = (\alpha_1,\dots,\alpha_n)$, where $\alpha_k = \pi(a_k)$, $1 \leq k \leq n$.

If $b = A(f)(a)$, we have $\pi_V(b) = \pi(b)$ $(V \subset \mathbb{C})$. and then, we have to see that $f(\alpha_1, \dots, \alpha_n) = \pi(b)$. By (6), it is sufficient to prove that

(8) $$b - f(\alpha_1, \dots, \alpha_n) \in A(B(0, \varepsilon)) \text{ for all } \varepsilon > 0.$$

If $\varepsilon > 0$. since f is continuous. there exists $\delta > 0$ such that. if

$$|z_1 - \alpha_1| < \delta, \; \dots, \; |z_n - \alpha_n| < \delta,$$

then $|f(z) - f(\alpha)| < \varepsilon$ $(\alpha = (\alpha_1, \dots, \alpha_n))$. By (6), we have that $a_k - \alpha_k \in A(B(0,\delta))$, that is. $a_k \in A(B(\alpha_k, \delta))$. Then

$$a = (a_1, \dots, a_n) \in A(B(\alpha_1, \delta) \times \dots \times A(B(\alpha_n, \delta)) = A(B(\alpha_1, \delta) \times \dots \times B(\alpha_n, \delta)).$$

But we have f: $B(\alpha_1, \delta) \times \dots \times B(\alpha_n, \delta) \to B(f(\alpha), \varepsilon)$ and then

$$A(f)\colon A(B(\alpha_1, \delta) \times \dots \times B(\alpha_n, \delta)) \longrightarrow A(B(f(\alpha), \varepsilon)) .$$

Then, $A(f)(a) \in A(B(f(\alpha), \varepsilon))$, that is $A(f)(a) - f(\alpha) \in A(B(0, \varepsilon))$. Then, by (8) (since $b = A(f)(a)$), this proves that the square (7) commutes.

When $V \subset \mathbb{C}^m$, the proof is the same that for the case $m = 1$, working in each coordinate.

This proves that π is a morphism of analytic rings. Finally, we have to prove that π is local. Let $a = (a_1, \dots, a_n)$ be an element of A^n, and U be an open subset of $\mathbb{C}^n$ such that $\pi_{\mathbb{C}^n}(a) \in U$. If $\alpha = \pi_{\mathbb{C}^n}(a)$, then $\alpha_k = \pi(a_k)$ $(1 \le k \le n)$; then, there exists $r > 0$ such that

$$B(\alpha_1, r) \times \dots \times B(\alpha_n, r) \subset U.$$

By (6), $a_k - \alpha_k \in A(B(0,r))$, that is, $a_k \in A(B(\alpha_k, r))$. Thus,

$$a = (a_1, \dots, a_n) \in A(B(\alpha_1, r)) \times \dots \times A(B(\alpha_n, r)) = A(B(\alpha_1, r) \times \dots \times B(\alpha_n, r)).$$

Since

$$A(B(\alpha_1, r) \times \dots \times B(\alpha_n, r)) \subset A(U),$$

we have $a \in A(U)$. This proves that π is local ■

1.7. OBSERVATION [3]. It is immediate that if $\pi\colon A \to B$ is a local morphism of analytic rings in a category $\boldsymbol{E}$ and B is local, then A is local.

1.8. EXAMPLE [3]. Let E be a topological space and let *Sh*(E) be the topos of sheaves on E. We consider the sheaf $\mathbb{C}_E$ of germs of continuous complex-valued functions defined in E.

Then, $\mathbb{C}_E$ is an analytic ring in *Sh*(E), and it preserves open coverings. That is, $\mathbb{C}_E$ is local. Here, $\mathbb{C}_E\colon \boldsymbol{C} \to Sh(E)$ is the functor defined by $\mathbb{C}_E(U)(H) = \text{continuous}(H, U)$ for H open in E and $U \in \boldsymbol{C}$.

1.9. EXAMPLE [3]. Let U be an open subset of $\mathbb{C}^n$, and let $h_1, \dots, h_k$ be holomorphic functions defined in U. Let $E = Z(h_1, \dots, h_k)$, that is,

$$E = \{p \in U \mid h_i(p) = 0 \text{ for } 1 \leq i \leq k\}.$$

Let O_E be the sheaf on E whose fiber in a point $p \in E$ is $O_{n.p}/(h_{1.p}, \dots, h_{k.p})$, where $O_{n,p}$ is the ring of germs of holomorphic functions in p, and $h_{i.p}$ is the germ of h_i in p (for $1 \leq i \leq k$). If S is a section of O_E over an open subset $H \subset E$, then for each $p \in H$, there exists an open subset V of U such that $p \in V$ and a holomorphic function f defined in V such that $S(x) = \bar{f}_x$ for $x \in V \cap E$, where $\bar{f}_x$ is the equivalent class of f_x,

$$\bar{f}_x \in O_{n.x}/(h_{1x}, \dots, h_{kx}).$$

We define the value of S at p by $VS(p) = f(p)$. This defines a continuous function $VS: H \to \mathbb{C}$.

For each open subset W of $\mathbb{C}^m$, we define the sheaf $O_E(W)$ on E by:

$$O_E(W)(H) = \{(S_1, \dots, S_m) \mid S_i \text{ is a section of } O_E \text{ over H and } (VS_1(x), \dots, VS_m(x)) \in W \text{ for } x \in H\}.$$

Then, O_E is an analytic ring in *Sh*(E). Moreover, we have the morphism $\pi: O_E \to \mathbb{C}_E$, defined by

$$\pi_{W,H}(S_1, \dots, S_m) = (VS_1, \dots, VS_m).$$

Here, since $(S_1, \dots, S_m) \in O_E(W)(H)$, then $(VS_1, \dots, VS_m)$ is a continuous function $H \to W$.

It is easy to see that π is local: then, since C_E is local, we obtain, by (1.7), that O_E is local.

2. GERMS AND ZEROS OF HOLOMORPHIC FUNCTIONS.

2.1. DEFINITION-PROPOSITION [3]. For each $p \in \mathbb{C}^n$, we will denote with $O_{n.p}$ the ring of holomorphic functions in p. Then, $O_{n.p}$ is an analytic ring in *Ens*, and, for each open subset V of $\mathbb{C}^m$, $O_{n.p}(V)$ is the set of germs of holomorphic functions with values in V, or, equivalently, the set of germs of holomorphic functions f such that $f(p) \in V$.

2.2. LEMMA. 1. *Let* U *and* V *be open susbsets of* $\mathbb{C}^n$ *such that* $U \supset V$. *Then, the restriction morphism* $O_n(U) \to O_n(V)$ *is an epimorphism in the category of analytic rings in Ens.*

2. *Let* U *be an open subset of* $\mathbb{C}^n$ *and* $p \in$ U; *then, the canonical morphism* $O_n(\mathrm{U}) \to O_{n.p}$ *is an epimorphism of the category of analytic rings in Ens.*

PROOF. 1 follows from Yoneda's Lemma [5] (recall that $O_n(\mathrm{V})$ is the representable functor $[\mathrm{V},-]$).

2. It is an immediate consequence of 1 and the fact that $O_{n.p}$ is the filtered colimit of the analytic rings $O_n(\mathrm{V})$ with p in $\mathrm{V} \subset \mathrm{U}$ (see [3], Proposition 1.13). ∎

2.3. LEMMA (*Basic Universal Property of* $O_n(\mathrm{U}) \to O_{n,p}$). *Let* U *be an open subset of* $\mathbb{C}^n$ *and* $p \in$ U. *Consider the diagram*

$$\begin{array}{ccc} O_n(\mathrm{U}) & \xrightarrow{\quad r \quad} & O_{n.p} \\ {\scriptstyle\varphi}\big\downarrow & \swarrow {\scriptstyle\tilde{\varphi}} & \\ \mathrm{A} & & \end{array}$$

where r is the canonical morphism. Let a be the element which characterizes φ (that is, $a = \varphi_{\mathrm{U}}(\mathrm{id}_{\mathrm{U}}) \in \mathrm{A}(\mathrm{U})$, see 0.8).

Then, a necessary and sufficient condition for the existence of $\tilde{\varphi}: O_{n,p} \to \mathrm{A}$ *such that* $\tilde{\varphi} r = \varphi$ *is that* $a \in \mathrm{A}(\mathrm{V})$ *for all open neighborhoods* V *of* p. *Moreover, if this happens,* $\tilde{\varphi}$ *is unique.*

PROOF. Suppose that there exists $\tilde{\varphi}: O_{n,p} \to \mathrm{A}$ such that $\tilde{\varphi} r = \varphi$. Let α be the element $r_{\mathrm{U}}(\mathrm{id}_{\mathrm{U}}) \in O_{n,p}(\mathrm{U})$. It is clear that $\alpha \in O_{n,p}(\mathrm{V})$ for all open neighborhoods V of p (1). Let V be an open neighborhood of p, $\mathrm{V} \subset \mathrm{U}$; then, by (1) and (0.6), it follows that $\tilde{\varphi}_{\mathrm{U}}(\alpha) = \tilde{\varphi}_{\mathrm{V}}(\alpha)$. Moreover,

$$a = \varphi_{\mathrm{U}}(\mathrm{id}_{\mathrm{U}}) = \tilde{\varphi}_{\mathrm{U}}(r_{\mathrm{U}}(\mathrm{id}_{\mathrm{U}})) = \tilde{\varphi}_{\mathrm{U}}(\alpha).$$

Then, $a = \tilde{\varphi}_{\mathrm{V}}(\alpha)$, with $\tilde{\varphi}_{\mathrm{V}}: O_{n,p}(\mathrm{V}) \to \mathrm{A}(\mathrm{V})$. Hence, $a \in \mathrm{A}(\mathrm{V})$.

Conversely, suppose that $a \in \mathrm{A}(\mathrm{V})$ for all open neighborhoods V of p. For $\beta \in O_{n,p}$, there exists an open neighborhood V of p, $\mathrm{V} \subset \mathrm{U}$, and $f \in O_n(\mathrm{V})$ such that $\beta = f_p$ (the germ of f in p): we define $\tilde{\varphi}(\beta) = \mathrm{A}(f)(a)$ (recall that we have $a \in \mathrm{A}(\mathrm{V})$ and $\mathrm{A}(f): \mathrm{A}(\mathrm{V}) \to \mathrm{A}$). It is easy to see that this definition does not depend on the representant f of β. We define

$$\tilde{\varphi}_{\mathbb{C}^m}(\beta_1, \dots, \beta_m) = (\tilde{\varphi}(\beta_1), \dots, \tilde{\varphi}(\beta_m)).$$

We shall prove that if $\beta = (\beta_1, \dots, \beta_m) \in O_{n,p}(\mathrm{W})$, where W is an open subset of C^m, then $\tilde{\varphi}_{\mathbb{C}^m}(\beta) \in \mathrm{A}(\mathrm{W})$.

In fact, there exists a holomorphic function $f: \mathrm{V} \to \mathrm{W}$, where V is an open neighborhood of p, such that $\beta = f_p$: if $f = (f_1,\dots,f_m)$, then $\beta_i = f_{i,p}$ $(1 \le i \le m)$, and

$$\tilde{\varphi}_{\mathbb{C}^m}(\beta) = (\tilde{\varphi}(\beta_1),\dots,\tilde{\varphi}(\beta_m)) = (A(f_1)(a),\dots,A(f_m)(a)) = A(f)(a).$$

Since $f\colon V\to W$, then $A(f)\colon A(V)\to A(W)$. Hence, $A(f)(a) \in A(W)$, that is, $\tilde{\varphi}_{\mathbb{C}^m}(\beta) \in A(W)$. Thus, for each W, $\tilde{\varphi}_W\colon O_{n,p}(W) \to A(W)$.

Finally, it is easy to see that the family $(\tilde{\varphi}_W)_{W\in C} = \tilde{\varphi}$ is a natural transformation, that is, $\tilde{\varphi}: O_{n,p}\to A$ is a morphism of analytic rings. It is clear that $\tilde{\varphi}\, r = \varphi$, since if $f \in O_n(U)$,

$$\tilde{\varphi}(r(f)) = \tilde{\varphi}(f_p) = A(f)(a),$$

and, by Yoneda's Lemma [5], $\varphi(f) = A(f)(a)$ for all $f \in O_n(U)$. Moreover, since r is an epimorphism (by (2.2)), $\tilde{\varphi}$ is unique. ▪

2.4. COROLLARY (*Universal Property of $O_n(U)\to O_{n,p}$*). *$O_{n,p}$ is the analytic ring which solves the universal problem defined by: "to send f into all the neighborhoods of f(p)".*

More explicitely:

1. $r\colon O_n(U)\to O_{n,p}$ verifies that if $f\colon U\to\mathbb{C}$ is a holomorphic function, then $r(f) \in O_{n,p}(X)$ for all open neighborhoods X of $f(p)$.

2. If $\varphi: O_n(U)\to A$ is a morphism of analytic rings which verifies that "if $f\colon U\to\mathbb{C}$ is a holomorphic function, then $\varphi(f) \in A(X)$ for all open neighborhoods X of $f(p)$", then there exists a unique $\tilde{\varphi}: O_{n,p}\to A$ such that $\tilde{\varphi}\, r = \varphi$.

PROOF. 1. Let $f\colon U\to\mathbb{C}$ be a holomorphic function and X an open neighborhood of $f(p)$. Then, since f is continuous, there exists an open neighborhood V of p, $V \subset U$, such that $f(V) \subset X$. Then, $r(f) = f_p = g_p$ where $g = f|V$, $g: V\to X$. Hence, $g_p \in O_{n,p}(X)$, that is $r(f) \in O_{n,p}(X)$.

2. Let $z_j\colon U\to\mathbb{C}$ be the j-th projection $\mathbb{C}^n\to\mathbb{C}$ restricted to U and let $a = \varphi_U(\mathrm{id}_U)$ be the element which characterizes φ (see 0.8). Then, by (0.6),

$$a = \varphi_{\mathbb{C}^n}(\mathrm{id}_U) = \varphi_{\mathbb{C}^n}(z_1,\dots,z_n) = (\varphi(z_1),\dots,\varphi(z_n)).$$

Let V be an open neighborhood of p, $V \subset U$. There exist $X_1,\dots,X_n$, where X_j is an open neighborhood of p_j $(1 \le j \le n)$, such that $X_1\times\dots\times X_n \subset V$. Then, since X_j is an open neighborhood of $p_j = z_j(p)$, we have that $\varphi(z_j) \in A(X_j)$ $(1 \le j \le n)$ (by the hypothesis made on φ). Hence,

$$a = (\varphi(z_1),\dots,\varphi(z_n)) \in A(X_1)\times\dots\times A(X_n) = A(X_1\times\dots\times X_n) \subset A(V).$$

Then, by (2.3), there exists a unique $\tilde{\varphi}: O_{n,p}\to A$ such that $\tilde{\varphi}\, r = \varphi$. ▪

2.5. OBSERVATION. In the case of C^∞-rings, the ring C_p^∞ has the universal property of making invertible all $f \in C^\infty(U)$ such that $f(p) \neq 0$ (U is an open subset of $\mathbb{R}^n$ and $p \in U$) (see [2]) (1). Let

$$\mathbb{R}^* = \{x \in \mathbb{R} \mid x \neq 0\}.$$

Then, given $\varphi: C^\infty(U) \to A$ and $f \in C^\infty(U)$, $\varphi(f)$ is invertible if and only if $\varphi(f) \in A(\mathbb{R}^*) = A^*$, where $A^* \subset A$ is the set of invertible elements of A (see [2]).

Thus, Corollary 2.4 means that in the analytic case, instead of $\mathbb{C}^*$ we have to consider all the neighborhoods of $f(p)$. The result (1) follows from the fact that, in the C^∞-case, all open subsets of $\mathbb{R}^n$ are Zarisky open sets. That is, for each open subset X of $\mathbb{R}^n$, there exists $f \in C^\infty(\mathbb{R}^n)$ such that $X = f^{-1}(\mathbb{R}^*)$. This is no longer true in the analytic case. However, if we assume that analytic rings preserve arbitrary intersections (that is, if U is an open subset of $\mathbb{C}^n$ and $U = \bigcap_{\alpha \in I} U_\alpha$, where U_α is an open subset of $\mathbb{C}^n$ $(\alpha \in I)$, then $A(U) = \bigcap_{\alpha \in I} A(U_\alpha)$), then the corresponding result holds. In fact:

2.6. PROPOSITION. *Assume that analytic rings preserve arbitrary intersections. Then, $O_{n,p}$ is the analytic ring which solves the universal problem of making invertible all $f \in O_n(U)$ such that $f(p) \neq 0$.*

More explicitely:

1. $r: O_n(U) \to O_{n,p}$ verifies that, if $f(p) \neq 0$, then $r(f)$ is invertible in $O_{n,p}$.
2. If $\varphi: O_n(U) \to A$ verifies that $\varphi(f) \in A^*$ for all $f \in O_n(U)$ such that $f(p) \neq 0$, then there exists a unique $\tilde{\varphi}: O_{n,p} \to A$ such that $\tilde{\varphi}\, r = \varphi$.

PROOF. 1. It is clear from (2.4) and the previous observation.

2. We will use the fact that if V is an open ball in $\mathbb{C}^n$ and $q \notin V$, then there exists $f \in O_n(\mathbb{C}^n)$ such that $f(q) = 0$ and $f(z) \neq 0$ for all $z \in V$. Hence, if V is an open ball in $\mathbb{C}^n$, there exists a family $(f_\alpha)_{\alpha \in I}$, where $f_\alpha \in O_n(\mathbb{C}^n)$, such that

$$V = \bigcap_{\alpha \in I} f_\alpha^{-1}(\mathbb{C}^*).$$

(It is enough to consider for each $q \in \mathbb{C}^n - V$ a function $f^{(q)} \in O_n(\mathbb{C}^n)$ such that $f^{(q)}(q) = 0$ and $f^{(q)}(z) \neq 0$ for all $z \in V$; then,

$$V = \bigcap_{q \in \mathbb{C}^n - V} f^{(q)-1}(\mathbb{C}^*).)$$

Let $a = \varphi_U(\mathrm{id}_U)$ and let V be an open ball in $\mathbb{C}^n$ centered at p, $V \subset U$. Let $(f_\alpha)_{\alpha \in I}$ be a family, with $f_\alpha \in O_n(\mathbb{C}^n)$ such that $V =$

$\bigcap_{\alpha\in I} f_\alpha^{-1}(\mathbb{C}^*)$. Then, $V = \bigcap_{\alpha\in I} h_\alpha^{-1}(\mathbb{C}^*)$, where $h_\alpha = f_\alpha|U$. Hence

(1) $$A(V) = \bigcap_{\alpha\in I} A(h_\alpha^{-1}(\mathbb{C}^*)).$$

Since $p \in V$, then $p \in h_\alpha^{-1}(\mathbb{C}^*)$ for each $\alpha \in I$. That is, $h_\alpha(p) \neq 0$; hence $\varphi(h_\alpha) \in A^*$. Moreover, by Yoneda's Lemma we have $f(h_\alpha) = A(h_\alpha)(a)$. Then, we obtain that $A(h_\alpha)(a) \in A^* = A(\mathbb{C}^*)$, that is, $a \in A(h_\alpha)^{-1}(A(\mathbb{C}^*))$. But, by (0.12),

$$A(h_\alpha)^{-1}(A(\mathbb{C}^*)) = A(h_\alpha^{-1}(\mathbb{C}^*)).$$

Hence, $a \in A(h_\alpha^{-1}(\mathbb{C}^*))$ for each $\alpha \in I$. Then, by (1), $a \in A(V)$. Hence, $a \in A(V)$ for all open balls V centered at p, $V \subset U$; then $a \in V$ for all open neighborhoods V of p, and, by (2.3), it follows that there exists a unique $\tilde{\varphi}: O_{n,p} \to A$ such that $\tilde{\varphi} r = \varphi$. ∎

2.7. PROBLEM. We do not know whether analytic rings preserve arbitrary intersections in general. Neither we have a counter-example.

2.8. DEFINITION. Let $h_1, \dots, h_k$ be holomorphic functions in an open subset U of $\mathbb{C}^n$ and let A be an analytic ring in a category $\boldsymbol{E}$ with finite limits. We denote with $Z_A(h_1, \dots, h_k)$ the equalizer in $\boldsymbol{E}$ of

$$A(U) \underset{0}{\overset{A(h)}{\rightrightarrows}} A^k$$

where $h = (h_1, \dots, h_k)$, $h: U \to \mathbb{C}^k$.

2.9. OBSERVATION. If $\boldsymbol{E} = Ens$,

$$Z_A(h_1, \dots, h_k) = \{a \in A(U) \mid A(h_i)(a) = 0 \text{ for } 1 \le i \le k\}.$$

2.10. THEOREM. *With the hypothesis of Definition 2.8, if* V *is an open subset of* $\mathbb{C}^n$ *such that* $Z(h_1, \dots, h_k) \subset V \subset U$ *(where*

$$Z(h_1, \dots, h_k) = \{x \in U \mid h_i(x) = 0 \text{ for } 1 \le i \le k\});$$

then the canonical inclusion $\iota: Z_A(h_1, \dots, h_k) \hookrightarrow A(U)$ *factorizes*

(∗) $$\begin{array}{ccc} Z_A(h_1, \dots, h_k) & \overset{\iota}{\hookrightarrow} & A(U) \\ & \searrow & \uparrow \\ & & A(V) \end{array}$$

where $A(V) \hookrightarrow A(U)$ *is the natural inclusion given by the structure of analytic rings.*

PROOF. Let $g\colon U \to \mathbb{R}$ be the function defined by:

$$g(z) = |h(z)| + \mathrm{dist}(z, U-V),$$

where $\mathrm{dist}(z,U-V)$ is the distance of z to $U-V$. g is a continuous function which satisfies:

1. $g(z) > 0$ for all $z \in U$.

In fact, if $z \in V$, since V is open, then $\mathrm{dist}(z,U-V) > 0$, and if z is in $U-V$, since $V \supset Z(h_1,\dots,h_k)$, then $z \notin Z(h_1,\dots,h_k)$. Hence, there exists j, $1 \le j \le k$, such that $h_j(z) \neq 0$.

2. $g|U-V = |h||U-V$, that is, if $z \in U-V$, $g(z) = |h(z)|$.
3. $g|V > |h||V$, that is, if $z \in V$, $g(z) > |h(z)|$.

In fact, if $z \in V$, since V is open, then $\mathrm{dist}(z,U-V) > 0$.

Let U_1 be the set

$$U_1 = \{(z,w) \mid z \in U,\ w \in \mathbb{C}^k \text{ and } |w| < g(z)\}.$$

Since g is continuous, then U_1 is an open subset of $U \times \mathbb{C}^k$. Let $l\colon V \to U_1$ be the function defined by $l(z) = (z, h(z))$. (If $z \in V$, then, by 3, $|h(z)| < g(z)$; hence, $(z,h(z)) \in U_1$.) Let $t\colon U_1 \to \mathbb{C}^k$ be the function defined by $t(z,w) = h(z) - w$. It is easy to see, by 2 and 3, that

$$V \xrightarrow{\ l\ } U_1 \underset{0}{\overset{t}{\rightrightarrows}} \mathbb{C}^k$$

is an independent equalizer. Then, we deduce that

$$(4) \qquad A(V) \xrightarrow{\ A(l)\ } A(U_1) \underset{0}{\overset{A(t)}{\rightrightarrows}} A^k$$

is an equalizer in $\boldsymbol{E}$. If we define $m\colon U \to U_1$ by $m(z) = (z,0)$ (by 1, $g(z) > 0$ for all $z \in U$; hence $(z,0) \in U_1$), and

$$\varphi\colon Z_A(h_1,\dots,h_k) \to A(U_1) \text{ by } \varphi = A(m)\iota,$$

then it is easy to see that $A(t)\varphi = 0$. Thus, by 4, there exists a unique

$$p\colon Z_A(h_1,\dots,h_k) \longrightarrow A(V) \text{ such that } A(l)p = \varphi.$$

Finally, it is straightforward to check that diagram (+) above commutes. ∎

3. CLASSIFYING TOPOS OF LOCAL ANALYTIC RINGS.

3.1. DEFINITION. Let P be the category of pairs (U,h), where U is an open subset of $\mathbb{C}^n$ (all n) and $h\colon U \to \mathbb{C}^k$ is a holomorphic function. We note $h = (h_1,\dots,h_k)$ and $Z(h_1,\dots,h_k)$ the set of common zeros of $h_1,\dots,h_k$.

An arrow $(U,h)\to(V,g)$ in P is a collection $(f_\alpha)_{\alpha\in I}$ of holomorphic functions $f_\alpha: U_\alpha\to V$, where U_α is open, $U_\alpha\subset U$ and $Z(h_1,\dots,h_k)\subset\bigcup_{\alpha\in I}U_\alpha$, which satisfies

1. for each $\alpha\in I$, $x\in U_\alpha\cap Z(h_1,\dots,h_k)$ and j,

$$(g_j f_\alpha)_x \in (h_{1,x},\dots,h_{k,x})$$

(ideal generated by $(h_{1,x},\dots,h_{k,x})$ in $O_{n,x}$).

2. for each $x\in U_\alpha\cap U_\beta\cap Z(h_1,\dots,h_k)$, $f_\alpha(x)=f_\beta(x)$.

3. for each $x\in U_\alpha\cap U_\beta\cap Z(h_1,\dots,h_k)$ and for each holomorphic function t in an open neighborhood of $f_\alpha(x)=f_\beta(x)$,

$$(tf_\alpha)_x-(tf_\beta)_x\in(h_{1,x},\dots,h_{k,x}).$$

We define $(f_\alpha)_{\alpha\in I}$ and $(f_{\alpha'})_{\alpha'\in J}$ to be equivalent when:

1'. for each $x\in U_\alpha\cap U_{\alpha'}\cap Z(h_1,\dots,h_k)$, $f_\alpha(x)=f_{\alpha'}(x)$ $(\alpha\in I,$ $\alpha'\in J)$.

2'. for each $x\in U_\alpha\cap U_{\alpha'}\cap Z(h_1,\dots,h_k)$ and for each holomorphic function t in an open neighborhood of $f_\alpha(x)=f_{\alpha'}(x)$,

$$(tf_\alpha)_x-(tf_{\alpha'})_x\in(h_{1,x},\dots,h_{k,x}).$$

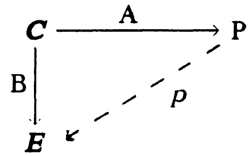
$\alpha\in I$, $\alpha'\in J$.

The arrows in P are equivalent classes of these collections.

3.2. OBSERVATION. It is sufficeint to verify 3 and 2' for $t=z_i$, $1\le i\le m$ (V is an open subset of $\mathbb{C}^m$), where $z_i: V\to\mathbb{C}$ is the i-th projection $\mathbb{C}^m\to\mathbb{C}$ restricted to V. This follows easily from the (local) Hadamard's Lemma (see [3], Proposition 0.9).

Let $A:\boldsymbol{C}\to P$ be the functor such that: $A(U)=(U,0)$; if f: $U\to V$ is a holomorphic function, $A(f):(U,0)\to(V,0)$ is the canonical arrow in P induced by f.

3.3. PROPOSITION. *Let $\boldsymbol{E}$ be a category with finite limits and let B be a local analytic ring in $\boldsymbol{E}$. Then, there exists a functor p: $P\to\boldsymbol{E}$ such that $pA=B$. Explicitely, $p(U,h)=Z_B(h_1,\dots,h_k)$.*



PROOF. We define $p(U,h)=Z_B(h_1,\dots,h_k)$. Let $f:(U,h)\to(V,g)$ be the collection $(f_\alpha)_{\alpha\in I}$, which saitsfies 1, 2 and 3. We are going to define

$$p(f): Z_B(h_1,\dots,h_k) \longrightarrow Z_B(g_1,\dots,g_r).$$

We have

$$Z(h_1,\dots,h_k) \subset \tilde{U} \subset U, \text{ where } \tilde{U} = \bigcup_{\alpha\in I} U_\alpha.$$

By Theorem 2.10, $Z_B(h_1,\dots,h_k)$ is a subobject of $B(\tilde{U})$. More explicitely, we have $j: Z_B(h_1,\dots,h_k) \hookrightarrow B(\tilde{U})$ which satisfies $tj = i$, where $t: B(\tilde{U}) \hookrightarrow B(U)$ is the canonical inclusion and i: $Z_B(h_1,\dots,h_k) \hookrightarrow B(U)$ is the inclusion of the equalizer. It is immediate to check the existence of an arrow

$$m_\alpha: Z_B(h_1|U_\alpha,\dots,h_k|U_\alpha) \hookrightarrow Z_B(h_1,\dots,h_k)$$

(for $\alpha \in I$), which makes the following diagram a pullback in $\boldsymbol{E}$:

$$\begin{array}{ccc} Z_B(h_1|U_\alpha,\dots,h_k|U_\alpha) & \xhookrightarrow{m_\alpha} & Z_B(h_1,\dots,h_k) \\ \downarrow & & \downarrow j \\ B(U_\alpha) & \hookrightarrow & B(\tilde{U}) \end{array} \tag{1}$$

Since B is local and $\tilde{U} = \bigcup_{\alpha\in I} U_\alpha$, then the family $(B(U_\alpha) \hookrightarrow B(\tilde{U}))_{\alpha\in I}$ is universal effective epimorphic in $\boldsymbol{E}$; hence, by (1), the family

$$m_\alpha: Z_B(h_1|U_\alpha,\dots,h_k|U_\alpha) \xhookrightarrow[(\alpha\in I)]{m_\alpha} Z_B(h_1,\dots,h_k)$$

is effective epimorphic in $\boldsymbol{E}$ (2). Let

$$j_\alpha: Z_B(h_1|U_\alpha,\dots,h_k|U_\alpha) \hookrightarrow B(U_\alpha)$$

be the inclusion of the equalizer (for each $\alpha \in I$), and consider $B(f_\alpha): B(U_\alpha) \to B(V)$. It is easy to see that the family

$$B(f_\alpha) j_\alpha: Z_B(h_1|U_\alpha,\dots,h_k|U_\alpha) \xhookrightarrow[(\alpha\in I)]{} B(V)$$

is compatible. Hence, by (2), we deduce that there exists a unique arrow τ: $Z_B(h_1,\dots,h_k) \to B(V)$ such that $\tau m_\alpha = B(f_\alpha) j_\alpha$ for each $\alpha \in I$. Likewise, it is easy to verify that if $(f_\alpha)_{\alpha\in I}$ and $(f_{\alpha'})_{\alpha'\in J}$ are equivalent, then we obtain the same arrow τ.

Finally it is straightforward to prove that $B(g)\tau = 0$; therefore, by definition (2.8) there exists a unique

$$\tilde{\tau}: Z_B(h_1,\dots,h_k) \longrightarrow Z_B(g_1,\dots,g_r).$$

such that $\tilde{i}\tilde{\tau} = \tau$, where $\tilde{i}$: $Z_B(g_1,\dots,g_r) \hookrightarrow B(V)$ is the inclusion of the equalizer. We define $p(f) = \tilde{\tau}$. It is immediate to check that p is a functor: moreover, for each $U \in \boldsymbol{C}$,

$$p(A(U)) = p(U,0) = Z_B(0) = B(U),$$

and it is clear that $p(A(f)) = B(f)$ for each holomorphic function $f\colon U \to V$. Hence $pA = B$. ∎

3.4. DEFINITION. Recall (see [3]) the following definitions: An A-ringed space is a pair (X, O_X), where X is a topological space and O_X is an analytic ring in $Sh(X)$ (the topos of sheaves on X), furnished with a local morphism $l_X\colon O_X \to C_X$ of analytic rings in $Sh(X)$ (where $\mathbb{C}_X$ is the sheaf of germs of continuous complex-valued functions defined in X).

It can be seen that $\mathbb{C}_X$ is a local analytic ring in $Sh(X)$. Hence, by (1.7), O_X is local.

A morphism $(X, O_X) \to (Y, O_Y)$ of one A-ringed space into another is a pair (f, φ), where $f\colon X \to Y$ is a continuous function and $\varphi\colon f^* O_Y \to O_X$ is a morphism of analytic rings in $Sh(X)$ ($f^* O_Y$ is the inverse image in $Sh(X)$ of the sheaf O_Y).

3.5. OBSERVATION. Let U be an open subset of $\mathbb{C}^n$ and let $\boldsymbol{I}$ be a coherent sheaf of ideals in O_U (the sheaf of germs of holomorphic functions defined in U). Let

$$E = \{p \in U \mid h(p) = 0 \ \forall h_p \in \boldsymbol{I}_p\}.$$

Let O_E be the restriction of $O_U/\boldsymbol{I}$ to E (E has the topology of subspace of U). It can be seen (see [3]) that O_E is an analytic ring in $Sh(E)$ and that (E, O_E) is an A-ringed space.

3.6. DEFINITION. We shall call *local model* the A-ringed space defined in (3.5). The coherence of $\boldsymbol{I}$ means that every point p of $E \subset U$ of a local model has an open neighborhood V in U such that $E \cap V$ is cut out of V by the vanishing of finitely many holomorphic functions defined in V.

A *special model* is a local model (E, O_E), $E \subset U$, cut out by the vanishing of the same (finitely many) holomorphic functions globally defined in U. That is, $E = Z(h_1, \ldots, h_k)$ and for each $p \in E$, $\boldsymbol{I}_p$ is generated by $(h_{1,p}, \ldots, h_{k,p})$.

We will say that (U, h) is a *presentation* of E, where $h = (h_1, \ldots, h_k)$, $h\colon U \to \mathbb{C}^k$.

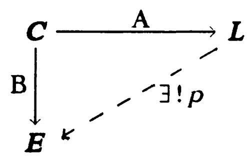
3.7. PROPOSITION. *The category* $\boldsymbol{L}$ *of local models has all finite limits.* (We remark that an equalizer of special models is not a special model. This forces the consideration of local models.)

PROOF. It follows with no difficulty; see [6] and [3], Proposition 2.12. ∎

3.8. OBSERVATION. It is straightforward to verify that an arrow in P, say $(U,h)\to(V,g)$, is equivalent to a morphism of A-ringed spaces $(E,O_E)\to(F,O_F)$, where $E=Z(h_1,\dots,h_k)$, $F=Z(g_1,\dots,g_r)$ (1). Then, if (E,O_E) is a special model, we can define $p(E,O_E)$, where $p\colon P\to \boldsymbol{E}$ is the functor constructed in (3.3).

Let $\boldsymbol{L}$ be the site of local models with the Grothendieck topology given by the open coverings. Let A be the analytic ring in $\boldsymbol{L}$ given by $A(U)=(U,O_U)$ for $U\in \boldsymbol{C}$; that is, $A=(\mathbb{C},O_{\mathbb{C}})$ (see [3], Corollary 2.9).

3.9. THEOREM. *Let* $\boldsymbol{E}$ *be a Grothendieck topos and* B *a local analytic ring in* $\boldsymbol{E}$. *Then, there exists a unique functor* $p\colon\boldsymbol{L}\to\boldsymbol{E}$ *which preserves finite limits and is a point of the site* $\boldsymbol{L}$ *such that* $pA=B$.



PROOF. By (3.3) and (3.8), if $E=Z(h_1,\dots,h_k)$ is a special model, then $p(E,O_E)=Z_B(h_1,\dots,h_k)$ is well defined and p is a functor from the category of special models to $\boldsymbol{E}$ (1). Now, let (E,O_E) be a local model given by a coherent sheaf of ideals $\boldsymbol{R}$ in O_U, with $E\subset U$. The coherence of $\boldsymbol{R}$ means that there exists a covering $(E_i)_{i\in I}$ of E, $E=\bigcup_{i\in I}E_i$, where each E_i is a special model, $E_i=Z(h_1^{(i)},\dots,h_k^{(i)})$, $h_j^{(i)}$ is a holomorphic function in an open subset U_i of U, and $E_i=E\cap U_i$. Moreover, $\boldsymbol{R}_x$ is generated by $(h_{1,x}^{(i)},\dots,h_{k,x}^{(i)})$ for each $x\in E_i$. We define:

$$p(E,O_E) = \bigcup_{i\in I}p(E_i,O_{E_i}).$$

It can be seen, using that B is local, that this definition does not depend on the election of the covering of E. Now, let $(f,\varphi)\colon(E,O_E)\to(F,O_F)$ be an arrow between local models. Writing $F=\bigcup_{i\in I}F_i$, where each F_i is a special model, it follows that $f^{-1}(F_i)$ is open in E. Then, $f^{-1}(F_i)$ is a local model; hence, $f^{-1}(F_i)=\bigcup_{j\in J_i}E_j$, where each E_j is a special model. Therefore,

$$E = \bigcup_{i\in I}f^{-1}F_i = \bigcup_{i\in I}\bigcup_{j\in J_i}E_j.$$

Hence, $E=\bigcup_{j\in J}E_j$, where each E_j is a special model, and for each $j\in J$, there exists $i_j\in I$ such that $E_j\subset f^{-1}(F_{i_j})$. Then, we have the following arrow between special models:

$$(E_j, O_{E_j}) \xrightarrow{(f|E_j, \varphi|f^*(O_{F_{ij}}))} (F_{ij}, O_{F_{ij}})$$

Therefore, by (1), we have

$$p(E_j, O_{E_j}) \xrightarrow{p(f|E_j, \varphi|f^*(O_{F_{ij}}))} p(F_{ij}, O_{F_{ij}}) \longrightarrow p(F, O_F)$$

that is, we have for each $j \in J$, an arrow $p(E_j, O_{E_j}) \to p(F, O_F)$. It is easy to see, using that B is local, that this family of arrows is compatible: hence, this defines a unique arrow τ: $p(E, O_E) \to p(F, O_F)$. Similarly, it is easy to see that τ does not depend on the election of E_i and F_j. It is clear that p is a functor and, like in (3.8), $p(E, O_E)$ does not depend on the presentation of E. Also, it is straightforward to prove that p preserves finite limits, p is a point of the site $\boldsymbol{L}$ and $pA = B$. Uniqueness of p follows from the fact that every object in $\boldsymbol{L}$ is a union of special models, and these are obtained from open subsets in $\boldsymbol{C}$ by means of equalizers (see [3], Proposition 2.12, and 1 in (4.7) ahead).

3.10. OBSERVATION (*Classifying topos of local analytic rings*). Let $\tilde{\boldsymbol{L}}$ be the topos of sheaves on the site $\boldsymbol{L}$. It is straightforward to see that the topology of $\boldsymbol{L}$ is subcanonical. Then, we have $\boldsymbol{L} \hookrightarrow \tilde{\boldsymbol{L}}$ which is full and faithful. Considering the composite

$$\boldsymbol{C} \xrightarrow{A} \boldsymbol{L} \hookrightarrow \tilde{\boldsymbol{L}}$$

which we denote again by A, it follows from general topos theory that Theorem (3.9) implies that given a local analytic ring B in a Grothendieck topos $\boldsymbol{E}$, there exists a unique functor p^*: $\tilde{\boldsymbol{L}} \to \boldsymbol{E}$ which preserves colimits and finite limits such that $p^*A = B$ (see [11]). Moreover, it is obvious that $A: \boldsymbol{C} \to \tilde{\boldsymbol{L}}$ is local.

$$\begin{array}{ccc} \boldsymbol{C} & \xrightarrow{A} & \tilde{\boldsymbol{L}} \\ {\scriptstyle B}\downarrow & \swarrow {\scriptstyle p^*} & \\ \boldsymbol{E} & & \end{array}$$

This means that $\tilde{\boldsymbol{L}}$ is the classifying topos of local analytic rings.

4. FINITELY PRESENTABLE ANALYTIC RINGS.

4.1. PROPOSITION. *Let* $\boldsymbol{A}$ *be the category of analytic rings in Ens. Then, the following statements hold:*

1. $\boldsymbol{A}$ *has finite elements and they are computed pointwise.*
2. $\boldsymbol{A}$ *has filtered colimits, they are computed pointwise and commute with finite limits* (see [3]).

Recall:

4.2. DEFINITION. We say that an analytic ring A is *finitely presentable* if the representable functor [A,—] preserves filtered colimits. We denote with $\boldsymbol{A}_{p.f}$ the category of finitely presentable analytic rings in *Ens*.

4.3. OBSERVATION. By a general category construction we know that if $A \in \boldsymbol{A}$, then $A \in \boldsymbol{A}_{p.f}$ if and only if A is retract of a quotient of $O_n(U)$; that is, there exists $n \geq 0$, an open subset U of $\mathbb{C}^n$, holomorphic functions $h_1, \ldots, h_k$ in U and a retraction

$$w, v \colon A \rightleftarrows O_n(U)/(h_1, \ldots, h_k), \quad wv = \mathrm{id}_A .$$

Here, the quotient $O_n(U)(h_1, \ldots, h_k)$ is defined by the usual universal property. In order to fix the notation, we stretch this property in the following diagram:

$$\begin{array}{ccc} O_n(U) & \xrightarrow{\;\; r \;\;} & O_n(U)/(h_1, \ldots, h_k) \\ \varphi \big\downarrow & \swarrow_{\exists ! \hat{\varphi}} & \\ B & & \end{array}$$

$$\hat{\varphi}\, r = \varphi, \quad \varphi(h_i) = 0 \text{ for } 1 \leq i \leq k.$$

φ is equivalent to an element $b \in B(U)$ (0.8) and $\varphi(f) = B(f)(b)$ for all $f \in O_n(U)$. Hence, the condition $\varphi(h_i) = 0$ is equivalent to $B(h_i)(b) = 0$, $1 \leq i \leq k$; that is, $b \in Z_B(h_1, \ldots, h_k)$ (2.8 and 2.9). Therefore, there exists a bijection:

$$[O_n(U)/(h_1, \ldots, h_k), B] \approx Z_B(h_1, \ldots, h_k)$$

(where $[O_n(U)/(h_1, \ldots, h_k), B]$ is the set of arrows

$$O_n(U)/(h_1, \ldots, h_k) \longrightarrow B).$$

4.4. OBSERVATION. Consider the dual category $\boldsymbol{A}_{p.f}^{\mathrm{op}}$.

Let $j \colon \boldsymbol{C} \to \boldsymbol{A}_{p.f}^{\mathrm{op}}$ be the functor given by $j(U) = \overline{O_n(U)}$. Let $f \colon U \to V$ be a holomorphic function: we have $f^* \colon O_m(V) \to O_n(U)$, and hence we have an arrow $\overline{O_n(U)} \to \overline{O_m(V)}$. Then, j is an analytic ring in $\boldsymbol{A}_{p.f}^{op}$ (see [3]). By the general theory we know that j has the following universal property: if $\boldsymbol{E}$ is a category with finite limits and B is an analytic ring in $\boldsymbol{E}$, then there exists a unique finite limits preserving functor F: $\boldsymbol{A}_{p.f}^{op} \to \boldsymbol{E}$ such that we have Fj = B:

$$\begin{array}{ccc} \boldsymbol{C} & \xrightarrow{\quad j \quad} & \boldsymbol{A}_{p.f}^{op} \\ {\scriptstyle B}\downarrow & \swarrow {\scriptstyle \exists! F} & \\ \boldsymbol{E} & & \end{array}$$

Furthermore, there exists an equivalence of categories between the analytic rings in $\boldsymbol{E}$ and the functors $\boldsymbol{A}_{p.f}^{op} \to \boldsymbol{E}$ that preserves finite limits.

When $\boldsymbol{E} = Ens$, F is given by F($\bar{A}$) = [A,B] (the set of morphisms A→B of analytic rings). This then generalizes to arbitrary $\boldsymbol{E}$ considering Yoneda's Lemma $\boldsymbol{E} \hookrightarrow (Ens^{\boldsymbol{E}})^{op}$.

4.5. OBSERVATION. Let (X, O_X) be a local model (see 3.6) and let U be an open subset of $\mathbb{C}^n$. It is easy to see that there exists a bijection

$$\Gamma(X, O_X(U)) \approx [(X, O_X), (U, O_U)],$$

where Γ denotes "global sections" and $[(X, O_X), (U, O_U)]$ is the set of arrows $(X, O_X) \to (U, O_U)$ in $\boldsymbol{L}$. Since

$$\Gamma(X, O_X(U)) \approx \Gamma(X, O_X)(U),$$

(see [3]) and

$$\Gamma(X, O_X)(U) \approx [O_n(U), \Gamma(X, O_X)]$$

(Yoneda), we obtain a bijection

$$[O_n(U), \Gamma(X, O_X)] \approx [(X, O_X), (U, O_U)],$$

Now, we shall prove a generalization of this result.

4.6. LEMMA. *Let* U *be an open subset of* $\mathbb{C}^n$, $h_1, \dots, h_k \in O_n(U)$ *and* $(X, O_X) \in \boldsymbol{L}$. *There exists a natural bijection*

$$[O_n(U)/(h_1, \dots, h_k), \Gamma(X, O_X)] \approx [(X, O_X), (E, O_E)]$$

where $E = Z(h_1, \dots, h_k)$ *and* O_E *is the sheaf on* E *defined in* (1.9).

4.7. OBSERVATION. The naturality means that if

$$\varphi: O_n(U)/(h_1, \dots, h_k) \longrightarrow \Gamma(X, O_X)]$$

and

$$\psi: O_m(V)/(l_1, \dots, l_q) \longrightarrow O_n(U)/(h_1, \dots, h_k)$$

are morphisms of analytic rings, and we denote $\hat{\varphi}$ and $\widehat{\varphi\psi}$ the arrows given by the bijection:

$$\hat{\varphi}: (X, O_X) \to (E, O_E) \;.\; \widehat{\varphi\psi}: (X, O_X) \to (F, O_F)$$

(where $F = Z(l_1, \dots, l_q)$), then $\widehat{\varphi\psi} = i(\bar{\psi})\hat{\varphi}$. Here, $i: A^{op}_{p.f} \to L$ is the unique finite limits preserving functor such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\;j\;} & A^{op}_{p.f} \\ \downarrow & \swarrow{\scriptstyle i} & \\ L & & \end{array}$$

commutes (4.4), where $C \to L$ is the analytic ring given by $W \mapsto (W, O_W)$ for $W \in C$; that is, it is the analytic ring $(\mathbb{C}, O_{\mathbb{C}})$ considered before Theorem (3.8). Remark that, since i preserves finite limits, then

$i(\overline{O_n(U)/(h_1, \dots, h_k)}) = (E, O_E)$ and $i(\overline{O_m(V)/(l_1, \dots, l_q)}) = (F, O_F)$;

hence, $i(\bar{\psi}): (E, O_E) \to (F, O_F)$.

PROOF OF THE LEMMA.

1. Construction of the bijection. We have the following equalizer in L (see [3]):

$$(E, O_E) \hookrightarrow (U, O_U) \underset{(0,0^*)}{\overset{(h,h^*)}{\rightrightarrows}} (\mathbb{C}^k, O_{\mathbb{C}^k})$$

where $h = (h_1, \dots, h_k)$. Hence, a morphism $(X, O_X) \to (E, O_E)$ in L is equivalent to a morphism

$$\varphi: (X, O_X) \to (U, O_U) \text{ such that } (h, h^*)\varphi = (0, 0^*)\varphi \qquad (1).$$

$\varphi = (f, \eta)$, where $f: X \to U$ is a continuous function and $\eta: f^* O_U \to O_X$. By (4.5), φ is equivalent to a section $S \in \Gamma(X, O_X)(U)$. Explicitely: $S = (S_1, \dots, S_n)$ and $S_i(x) = \eta_x(z_{i, f(x)})$ $(1 \leq i \leq n)$, where z_i is the i-th projection $\mathbb{C}^n \to \mathbb{C}$ (see [3]). By (1), we have $hf = 0$ and $\eta_x h^*_{f(x)} = \eta_x 0^*_{f(x)}$ (2). Here,

$$h^*_p: O_{\mathbb{C}^k, h(p)} \to O_{U, p} \text{ for } p \in U.$$

Equation (2) applied to $w_1, \dots, w_k$ (where w_i is the i-th projection $\mathbb{C}^k \to \mathbb{C}$), shows that

(3) $\qquad \eta_x(h_{i, f(x)}) = 0$ in $O_{X, x}$ $(x \in X, 1 \leq i \leq k)$.

Given $x \in X$ (fixed), there exists an open neighborhood W of x and a local extension $\tilde{f}$ of f in W such that $\eta_{x'}(t_{f(x')}) = (\overline{t\tilde{f}})_{x'}$

for t holomorphic in a neighborhood of $f(x')$, $x' \in X \cap W$ (see [3]). Hence.

$$S_i(x') = \eta_{x'}(z_{i,f(x')}) = \overline{(z_i\tilde{f})_{x'}} = \overline{\tilde{f}_{i,x'}};$$

therefore, $S(x') = \overline{\tilde{f}_{x'}}$ for $x' \in X \cap W$, that is S is given by $\tilde{f}$ around x. Hence, for $1 \le i \le k$,

$$\Gamma(X, O_X)(h_i) \colon \Gamma(X, O_X)(U) \longrightarrow \Gamma(X, O_X)$$

satisfies, by (3), that

$$(\Gamma(X.O_X)(h_i))(S)(x) = \overline{(h_i\tilde{f})_x} = \eta_x(h_{i,f(x)}) = 0.$$

Hence, $(\Gamma(X, O_X)(h_i))(S) = 0$ $(1 \le i \le k)$, which means that

$$S \in Z_{\Gamma(X, O_X)}(h_1, \dots, h_k).$$

Therefore, by (4.3), S is equivalent to a morphism

$$O_n(U)/(h_1, \dots, h_k) \longrightarrow \Gamma(X, O_X).$$

Conversely, a morphism $O_n(U)/(h_1, \dots, h_k) \longrightarrow \Gamma(X, O_X)$ is equivalent, by (4.3), to a section

$$S \in Z_{\Gamma(X, O_X)}(h_1, \dots, h_k) \subset \Gamma(X, O_X)(U).$$

Then, since $S \in \Gamma(X, O_X)(U)$, S is equivalent, by (4.5), to a morphism $\varphi = (f, \eta) \colon (X, O_X) \to (U, O_U)$. Explicitely, we have $f(x) = VS(x)$ for $x \in X$, where $VS(x)$ is the value of the section S in the point x, and $\eta_x(z_{i,f(x)}) = S_i(x)$ $(1 \le i \le n)$ (see [3]). Hence, if S is given by $\tilde{f}$ around $x \in X$ and t is a holomorphic function defined around $f(x)$, we have that

$$\eta_x(t_{f(x)}) = \overline{(t\tilde{f})_x}.$$

Since $S \in Z_{\Gamma(X, O_X)}(h_1, \dots, h_k)$, we have that

$$0 = (\Gamma(X, O_X)(h_j)(S))(x) = \overline{(h_j\tilde{f})_x} \quad (1 \le j \le k),$$

which means that $(h_j\tilde{f})_x \in I_x$ (5), where I is the sheaf of ideals which defines X. Hence $h_j\tilde{f}(x) = 0$ $(1 \le j \le k)$, that is $h(f(x)) = 0$. Now, if g is a holomorphic function in a neighborhood of 0 in $\mathbb{C}^k$, we can write

$$g(w) - g(0) = \sum_{j=1}^{k} w_j \gamma_j(w)$$

with γ_j holomorphic in a neighborhood of 0 in $\mathbb{C}^k$. Hence, for x' in a neighborhood of x in $\mathbb{C}^p$, some p, $X \subset \mathbb{C}^p$, we have:

$$g(h\tilde{f}(x')) - g(0) = \sum_{j=1}^{k} (h_j\tilde{f})(x') \cdot \gamma_j(h\tilde{f}(x')).$$

Therefore.

$$(gh\tilde{f})_x - (g(0))_x \in ((h_1\tilde{f})_x, \dots, (h_k\tilde{f})_x),$$

which is the ideal generated by $(h_1 \tilde{f})_x, \dots, (h_k\tilde{f})_x$. Hence, since $(h_j\tilde{f})_x \in I_x$ (by (5)), we obtain that

$$(gh\tilde{f})_x - (g(0))_x \in I_x, \text{ that is, } \overline{(gh\tilde{f})_x} = \overline{(g(0))_x}$$

(here, $g(0)$ is the constant function). Therefore, by (4),

$$\eta_x((gf)_{f(x)} = \eta_x(g(0)_{f(x)}),$$

where now $g(0)$ is the constant function defined in $\mathbb{C}^n$. Therefore, we obtain that

$$\eta_x(h_{f(x)}^*(g_0)) = \eta_x(0_{f(x)}^*(g_0)),$$

and this holds for all holomorphic function g in a neighborhood of 0 in $\mathbb{C}^k$. Hence $\eta_x h_{f(x)}^* = \eta_x 0_{f(x)}^*$; moreover, we have seen that $hf = 0$. This means that $\varphi = (f,\eta)$ satisfies the equation $(h,h^*)\varphi = (0,0_*)\varphi$; hence, by (1), φ is equivalent to a morphism $(X,O_X) \to (E,O_E)$. It is easy to see that this establishes the required bijection.

2. Naturality: we sketch the proof. First, one proves that if

$$\psi: O_m(V)/(l_1, \dots, l_q) \longrightarrow O_n(U)/(h_1, \dots, h_k)$$

is a morphism of analytic rings,

$$\varepsilon: O_n(U)/(h_1, \dots, h_k) \longrightarrow \Gamma(E,O_E)$$

is the canonical morphism (obtained from the canonical morphism $O_n(U) \to \Gamma(E,O_E)$) and $\widehat{\varepsilon\psi}: (E,O_E) \to (F,O_F)$ is given by the bijection established in 1, then $\widehat{\varepsilon\psi} = i(\bar{\psi})$ (see (4.7). Then, given

$$\varphi: O_n(U)/(h_1, \dots, h_k) \longrightarrow \Gamma(X,O_X),$$

φ is equivalent, by (4.3), to a section $S \in Z_{\Gamma(X,O_X)}(h_1, \dots, h_k)$. Hence, $S \in \Gamma(X,O_X)(U)$ is a section which takes values in E. Moreover, if

$$\vartheta = \varphi\psi: O_m(V)/(l_1, \dots, l_q) \longrightarrow \Gamma(X,O_X),$$

then, by (4.3), ϑ is equivalent to a section

$$\sigma \in Z_{\Gamma(X,O_X)}(l_1, \dots, l_q) \subset \Gamma(X,O_X)(V):$$

hence, σ is a section of (X,O_X) which takes values in F. Finally, $i(\bar{\psi}): (E,O_E) \to (F,O_F)$, that is, $i(\bar{\psi}) = (\tau,\lambda)$, where $\tau: E \to F$ is a continuous function. In order to prove the naturality, that is, $\hat{\vartheta} = i(\bar{\psi})\hat{\varphi}$, first it is shown that $V\sigma = \tau\cdot(VS)$ (here, $VS: X \to E$ and $V\sigma: X \to F$). From this fact and the equation $\widehat{\varepsilon\psi} = i(\bar{\psi})$, it can be seen that $\hat{\vartheta} = i(\bar{\psi})\hat{\varphi}$. ∎

4.8. OBSERVATION. Let

$$A \underset{v}{\overset{w}{\rightleftarrows}} O_n(U)/(h_1,\dots,h_k), \qquad wv = \mathrm{id}_A$$

be a finitely presentable analytic ring in *Ens*, and let $i\colon \boldsymbol{A}^{op}_{p.f} \to \boldsymbol{L}$ be the functor considered in (4.7). It follows that the diagram

$$i(\overline{A}) \xrightarrow{i(\overline{w})} (E, O_E) \underset{i(\overline{vw})}{\overset{\mathrm{id}}{\rightrightarrows}} (E, O_E)$$

is an equalizer in $\boldsymbol{L}$. We denote

$$i(\overline{A}) = (X_A, O_{X_A}) \in \boldsymbol{L} \text{ and } i(\overline{vw}) = (f, \eta).$$

By (1), we have $X_A = \{x \in E \mid f(x) = x\}$. Now, if B is an analytic ring in *Ens*, we define the following function

$$f_B\colon Z_B(h_1,\dots,h_k) \longrightarrow Z_B(h_1,\dots,h_k):$$

if $b \in Z_B(h_1,\dots,h_k)$, by (4.3), b is equivalent to a morphism

$$\varphi\colon O_n(U)/(h_1,\dots,h_k) \longrightarrow B\ :$$

then, we consider

$$\varphi vw\colon O_n(U)/(h_1,\dots,h_k) \longrightarrow B$$

which, by (4.3), is equivalent to an element $c \in Z_B(h_1,\dots,h_k)$. We define $f_B(b) = c$. Now, it is easy to see that there exists a bijection:

$$[A,B] \approx \{b \in Z_B(h_1,\dots,h_k) \mid f_B(b) = b\}:$$

that is,

$$[A,B] \approx \{\varphi\colon O_n(U)/(h_1,\dots,h_k) \to B \mid \varphi vw = \varphi\}.$$

4.9. OBSERVATION. As before, it can be proved that there exists a natural bijection

$$[A, \Gamma(X, O_X)] \approx [(X, O_X), (X_A, O_{X_A})] \qquad ((X, O_X) \in \boldsymbol{L}),$$

where the naturality means that if B is a finitely presentable analytic ring in *Ens*, $\psi\colon B \to A$ is a morphism of analytic rings and $\varphi\colon A \to \Gamma(X, O_X)$, then $\widehat{\varphi\psi} = i(\overline{\psi})\hat{\varphi}$. This result is obtained from (4.6) and (4.8).

5. SPECTRUM OF A FINITELY PRESENTABLE ANALYTIC RING.

Here, first we shall compute the spectrum of a quotient $O_n(U)/(h_1,\dots,h_k)$.

Recall that the spectrum of an analytic ring A in *Ens* is a

Grothendieck topos S_A together with a local analytic ring *Spec* A in S_A and a morphism $\Delta^*A \to \mathit{Spec}A$, such that for all Grothendieck topos $\boldsymbol{E}$ and all local analytic ring B in $\boldsymbol{E}$ furnished with a morphism $\Delta^*A \to B$, there exists a unique pair (p^*, π), where p^* is the inverse image of a geometric morphism,

$$p^*: S_A \to \boldsymbol{E} \text{ and } \pi: p^*\mathit{Spec}\ A \to B$$

is a local morphism, such that the following diagrams commute:

$$\begin{array}{ccc} \mathit{Ens} & \xrightarrow{\Delta^*} & \boldsymbol{E} \\ {\scriptstyle\Delta^*}\downarrow & \nearrow_{p^*} & \\ S_A & & \end{array} \qquad \begin{array}{ccc} \Delta^*A & \longrightarrow & B \\ \downarrow & \nearrow_{\pi} & \\ p^*\mathit{Spec}\ A & & \end{array}$$

5.1. OBSERVATION. Here Δ^* is the constant sheaf, and the morphism $\Delta^*A \to \mathit{Spec}A$ gives rise to a morphism $p^*(\Delta^*A) \to p^*(\mathit{Spec}A)$, that is, $\Delta^*A \to p^*\mathit{Spec}A$. In this case the diagram

$$\begin{array}{ccc} \mathit{Ens} & \xrightarrow{\Delta^*} & \boldsymbol{E} \\ {\scriptstyle\Delta^*}\downarrow & \nearrow_{p^*} & \\ S_A & & \end{array}$$

always commutes, since there exists a unique geometric morphism $\boldsymbol{E} \to \mathit{Ens}$, whose direct image is "global sections" and whose inverse image is Δ^*.

Let U be an open subset of $\mathbb{C}^n$ and $h_1, \dots, h_k$ holomorphic functions in U. We denote $E = Z(h_1, \dots, h_k) \subset U$; we have the local analytic ring O_E in $Sh(E)$ (see (1.9)).

5.2. THEOREM. *The spectrum of* $O_n(U)/(h_1, \dots, h_k)$ *is the Grothendieck topos* $Sh(E)$ *together with the local analytic ring* O_E *in* $Sh(E)$. *That is.* $\mathit{Spec}(O_n(U)/(h_1, \dots, h_k)) = O_E$.

PROOF. We denote by $\Gamma(E, O_E)$ the analytic ring of global sections of the sheaf O_E. Let $\rho: O_n(U) \to \Gamma(E, O_E)$ be the canonical morphism (if $f \in O_n(U)$, $\rho(f)$ is the global section of O_E given by f). Let $r: O_n(U) \to O_n(U)/(h_1, \dots, h_k)$ be the projection to the quotient. Since $\rho(h_i) = 0$ for $1 \le i \le k$, there exists a unique

$$\varphi: O_n(U)/(h_1, \dots, h_k) \longrightarrow \Gamma(E, O_E)$$

such that $\varphi r = \rho$. φ is equivalent to a morphism

$$\varphi: \Delta^*(O_n(\mathrm{U})/(h_1,\dots,h_k)) \longrightarrow O_{\mathrm{E}}$$

of analytic rings in $Sh(\mathrm{E})$. Let B be a local analytic ring in a Grothendieck topos $\boldsymbol{E}$ and let

$$\lambda: \Delta^*(O_n(\mathrm{U})/(h_1,\dots,h_k)) \longrightarrow \mathrm{B}$$

be a morphism of analytic rings in $\boldsymbol{E}$. By (5.1), we have to prove that there exists a unique pair (ψ^*,π), where $\psi^*: Sh(\mathrm{E}) \to \boldsymbol{E}$ is the inverse image of a geometric morphism and $\pi: \psi^* O_{\mathrm{E}} \to \mathrm{B}$ is a local morphism of analytic rings, such that the following diagram commutes:

$$\begin{array}{ccc} \Delta^*(O_n(\mathrm{U})/(h_1,\dots,h_k)) & \xrightarrow{\quad\lambda\quad} & \mathrm{B} \\ {\scriptstyle \psi^*(\varphi)}\big\downarrow & \nearrow_{\pi} & \\ \psi^* O_{\mathrm{E}} & & \end{array}$$

$\pi\psi^*(\varphi) = \lambda$. Here, λ is equivalent to a morphism of analytic rings $O_n(\mathrm{U})/(h_1,\dots,h_k) \to \Gamma(\mathrm{B})$, and, by (4.3), this morphism is equivalent to an element $b \in \mathrm{Z}_{\Gamma(\mathrm{B})}(h_1,\dots,h_k) \subset \Gamma\mathrm{B}(\mathrm{U})$. It follows that b is equivalent to an arrow $b: 1 \to \mathrm{B}(\mathrm{U})$ where 1 is the terminal object of $\boldsymbol{E}$. In fact, we actually have $b: 1 \to \mathrm{Z}_{\mathrm{B}}(h_1,\dots,h_k) \hookrightarrow \mathrm{B}(\mathrm{U})$ (1).

Now, we shall define a functor $\psi: O(\mathrm{E}) \to \boldsymbol{E}$, where $O(\mathrm{E})$ is the category of open subsets of E and open inclusions, with the Grothendieck topology given by the open coverings. Let H be an open susbset of E, $\mathrm{H} = \mathrm{V} \cap \mathrm{E}$, where V is an open subset of $\mathbb{C}^n$ and $\mathrm{V} \subset \mathrm{U}$. It can be easily seen that if $\mathrm{H} = \mathrm{V}_1 \cap \mathrm{E}$, where V_1 is an open subset of $\mathbb{C}^n$, $\mathrm{V}_1 \subset \mathrm{U}$, then

$$\mathrm{Z}_{\mathrm{B}}(h_1|\mathrm{V}_1,\dots,h_k|\mathrm{V}_1) = \mathrm{Z}_{\mathrm{B}}(h_1|\mathrm{V},\dots,h_k|\mathrm{V}).$$

Hence, we can define $\psi(\mathrm{H})$ as the following pullback in $\boldsymbol{E}$:

$$\begin{array}{ccc} \psi(\mathrm{H}) & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow {\scriptstyle b} \\ \mathrm{Z}_{\mathrm{B}}(h_1|\mathrm{V},\dots,h_k|\mathrm{V}) & \hookrightarrow & \mathrm{Z}_{\mathrm{B}}(h_1,\dots,h_k) \end{array}$$

It is easy to see that if $\mathrm{H}_1 \hookrightarrow \mathrm{H}_2$, then there exists an arrow $\psi(\mathrm{H}_1) \to \psi(\mathrm{H}_2)$ in $\boldsymbol{E}$, and that ψ is a functor which preserves finite limits. We are going to prove that ψ is a point of the site $O(\mathrm{E})$; that is, if $\mathrm{H} = \bigcup_{\alpha\in\mathrm{I}} \mathrm{H}_\alpha$, where H_α is an open subset of E, then the family $\psi(\mathrm{H}_\alpha) \to \psi(\mathrm{H})$ is an epimorphic family in $\boldsymbol{E}$. Here, $\mathrm{H}_\alpha = \mathrm{V}_\alpha \cap \mathrm{E}$ where V_α is an open subset of $\mathbb{C}^n$, $\mathrm{V}_\alpha \subset \mathrm{U}$; hence $\mathrm{H} = \mathrm{V} \cap \mathrm{E}$ where $\mathrm{V} = \bigcup_{\alpha\in\mathrm{I}} \mathrm{V}_\alpha$. By (3.9) there exists a unique functor B^*: $\boldsymbol{L} \to \boldsymbol{E}$ which preserves finite limits and is a point of the site $\boldsymbol{L}$ such that $B^*A = B$, where $\boldsymbol{L}$ is the site of local models and A is the analytic ring in $\boldsymbol{L}$ given by $A(W) = (W, \mathcal{O}_W)$.

We have the following pullback in $\boldsymbol{L}$ ($\alpha \in I$):

$$(2) \qquad \begin{array}{ccc} (H_\alpha, \mathcal{O}_{H_\alpha}) & \hookrightarrow & (H, \mathcal{O}_H) \\ \downarrow & & \downarrow \\ (V_\alpha, \mathcal{O}_{V_\alpha}) & \hookrightarrow & (V, \mathcal{O}_V) \end{array}$$

From the fact that

$$H_\alpha = V_\alpha \cap E = V_\alpha \cap Z(h_1, \dots, h_k) = Z(h_1|V_\alpha, \dots, h_k|V_\alpha)$$

and $H = Z(h_1|V, \dots, h_k|V)$, it follows that

$$B^*(H_\alpha, \mathcal{O}_{H_\alpha}) = Z_B(h_1|V_\alpha, \dots, h_k|V_\alpha)$$

and

$$B^*(H, \mathcal{O}_H) = Z_B(h_1|V, \dots, h_k|V);$$

moreover,

$$B^*(V_\alpha, \mathcal{O}_{V_\alpha}) = B^*A(V_\alpha) = B(V_\alpha) \text{ and } B^*(V, \mathcal{O}_V) = B(V).$$

Hence, applying B^* to the pullback (2), we obtain that the following square

$$\begin{array}{ccc} Z_B(h_1|V_\alpha, \dots, h_k|V_\alpha) & \hookrightarrow & Z_B(h_1|V, \dots, h_k|V) \\ \downarrow & & \downarrow \\ B(V_\alpha) & \hookrightarrow & B(V) \end{array}$$

is a pullback in $\boldsymbol{E}$ (for all $\alpha \in I$) (3). Since B is local, we have that $(B(V_\alpha) \hookrightarrow B(V))_{\alpha \in I}$ is an epimorphic family in $\boldsymbol{E}$; hence, by (3), we obtain that

$$Z_B(h_1|V_\alpha, \dots, h_k|V_\alpha) \xrightarrow[(\alpha \in I)]{} Z_B(h_1|V, \dots, h_k|V)$$

is an epimorphic family in $\boldsymbol{E}$ (4). Moreover, it is easy to see that

$$\begin{array}{ccc} \psi(H_\alpha) & \longrightarrow & \psi(H) \\ \downarrow & & \downarrow \\ Z_B(h_1|V_\alpha, \dots, h_k|V_\alpha) & \hookrightarrow & Z_B(h_1|V, \dots, h_k|V) \end{array}$$

is a pullback in $\boldsymbol{E}$ (for each $\alpha \in I$). Hence, by (4), it follows that $(\psi(H_\alpha) \to \psi(H))_{\alpha \in I}$ is an epimorphic family in $\boldsymbol{E}$. This proves that ψ is a point of the site $\mathcal{O}(E)$. Hence, $\psi: \mathcal{O}(E) \to \boldsymbol{E}$ extends to a functor

$$\psi^* : \widetilde{O(E)} = Sh(E) \to \boldsymbol{E}$$

which preserves colimits and finite limits [1].

Now, we have to define a local morphism of analytic rings $\pi: \psi^* O_E \to B$. For each open subset W of $\mathbb{C}^m$, we have that

$$O_E(W) = \operatorname*{colim}_{[-,H] \xrightarrow{S} O_E(W)} [-,H]$$

where $[-,H]$ is the contravariant representable functor associated to $H \in O(E)$; then,

(5) $$\psi^*(O_E(W)) = \operatorname*{colim}_{[-,H] \xrightarrow{S} O_E(W)} \psi(H)$$

Let $S: [-,H] \to O_E(W)$ be an arrow. By Yoneda's Lemma, S is equivalent to an element $S \in O_E(W)(H)$, that is, S is a section of $O_E(W)$ defined in H. Thus, $S \in \Gamma(H, O_H(W))$, and, by (4.5), S is equivalent to an arrow in $\boldsymbol{L}$, $S: (H, O_H) \to (W, O_W)$. Hence, we have

$$B(S): B^*(H, O_H) \longrightarrow B^*(W, O_W).$$

Here, $H = V \cap E$ where V is an open subset of $\mathbb{C}^n$, $V \subset U$; then

$$B^*(H, O_H) = Z_B(h_1|V, \dots, h_k|V); \text{ and } B^*(W, O_W) = B^*A(W) = B(W).$$

That is, $B^*(S): Z_B(h_1|V, \dots, h_k|V) \to B(W)$. Moreover, by definition of ψ, we have an arrow

$$m_H: \psi(H) \longrightarrow Z_B(h_1|V, \dots, h_k|V).$$

We define $\gamma_S = B^*(S)\, m_H: \psi(H) \to B(W)$. It is easy to see that the collection (γ_S), $S: [-,H] \to O_E(W)$, $H \in O(E)$, induces, by (5), a unique arrow $\pi_W: \psi^*(O_E(W)) \to B(W)$. Moreover, it is straightforward to prove that the collection $(\pi_W)_{W \in C}$ defines a local morphism of analytic rings $\pi: \psi^* O_E \to B$ such that $\pi\psi^*(\varphi) = \lambda$.

Now we are going to prove that (ψ^*, π) is unique. Let ψ_1^*: $Sh(E) \to \boldsymbol{E}$ be the inverse image of a geometric morphism and let $\pi': \psi_1^* O_E \to B$ be a local morphism of analytic rings. It is easy to see, using the fact that π' is local, that for each open subsets H of E ($H = V \cap E$ where V is an open subset of $\mathbb{C}^n$, $V \subset U$), there exists an arrow

$$\pi'_H: Z_{\psi_1^* O_E}(h_1|V, \dots, h_k|V) \longrightarrow Z_B(h_1|V, \dots, h_k|V)$$

such that the square

$$\begin{array}{ccc} Z_{\psi_1^* O_E}(h_1|V, \dots, h_k|V) & \hookrightarrow & Z_{\psi_1^* O_E}(h_1, \dots, h_k) \\ \downarrow \pi'_H & & \downarrow \pi'_E \\ Z_B(h_1|V, \dots, h_k|V) & \hookrightarrow & Z_B(h_1, \dots, h_k) \end{array}$$

is a pullback in $\mathcal{E}$ (6). Moreover, the identity $id_V: V \to V$ induces a section of $O_E(V)$ defined in H; and this section induces an arrow $[-,H] \to Z_{O_E}(h_1|V,\dots,h_k|V)$. Similarly, we have an arrow $[-,E] \to Z_{O_E}(h_1,\dots,h_k)$, and it is easy to see that the square

$$\begin{array}{ccc} [-,H] & \longrightarrow & [-,E] = 1 \\ \downarrow & & \downarrow \\ Z_{O_E}(h_1|V,\dots,h_k|V) & \hookrightarrow & Z_{O_E}(h_1,\dots,h_k) \end{array}$$

is a pullback in $Sh(E)$. Applying ψ_1^*, which preserves finite limits, we obtain that the square

$$(7)\qquad \begin{array}{ccc} \psi_1^*([-,H]) & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ Z_{\psi_1^* O_E}(h_1|V,\dots,h_k|V) & \hookrightarrow & Z_{\psi_1^* O_E}(h_1,\dots,h_k) \end{array}$$

is a pullback in $\mathcal{E}$ (where

$$\psi_1^*(Z_{O_E}(h_1|V,\dots,h_k|V)) = Z_{\psi_1^* O_E}(h_1|V,\dots,h_k|V)$$

and

$$\psi_1^*(Z_{O_E}(h_1,\dots,h_k)) = Z_{\psi_1^* O_E}(h_1,\dots,h_k)).$$

By (6) and (7), it follows that the square

$$\begin{array}{ccc} \psi_1^*([-,H]) & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ Z_B(h_1|V,\dots,h_k|V) & \hookrightarrow & Z_B(h_1,\dots,h_k) \end{array}$$

is a pullback in $\mathcal{E}$, and it is easy to see that, in this square, the arrow $1 \to Z_B(h_1,\dots,h_k)$ is b (see (1)). Then, from the definition of ψ, it follows that

$$\psi_1^*([-,H]) = \psi(H) = \psi^*([-,H])$$

(for $H \in O(E)$). Hence, by preservation of colimits, it follows that $\psi_1^* = \psi^*$. Finally, it is straightforward that $\pi' = \pi$. ∎

5.3. OBSERVATION. Let A be a retract of a quotient of $O_n(U)$. Recall that we have the object $(X_A, O_{X_A}) \in \mathcal{L}$ (4.8). It can be seen that the spectrum of A is the Grothendieck topos $Sh(X_A)$ with the local analytic ring O_{X_A} in $Sh(X_A)$. This result follows by Theorem (5.2) and the natural bijections (4.6) and (4.9). Hence we have:

5.4. THEOREM. *If* A *is a finitely presentable analytic ring in Ens. then the spectrum of* A *is the Grothendieck topos* $Sh(X_A)$ *with the local analytic ring* O_{X_A} *in* $Sh(X_A)$. *That is. we have* $Spec\ A = O_{X_A}$. ■

OPEN PROBLEMS.

The following are three problems of the theory of analytic rings that had resisted our effort to solve them. We think that a satisfactorily answer is of interest to the theory, and that would require some deep ideas.

PROBLEM A. Preservation of infinite intersections by analytic rings. This is relevant to a classical algebraic characterization of the rings of germs of holomorphic functions.

PROBLEM B. The computation of a general quotient of finitely presentable analytic rings. See Appendix.

PROBLEM C. Problem of the epimorphic-effective and universal character of the open coverings in the dual of the category of finitely presentable analytic rings (or the theory of analytic rings).

Problems B and C are relevant to characterize this category. Namely, is or is not a category of ringed spaces?

APPENDIX.

In general. we don't know how to compute the quotients of analytic rings: in particular. we don't have any description of the ring $O_n(U)/(h_1,\dots,h_k)$. However. we have the following result. obtained with the collaboration of A. Dickenstein and C. Sessa:

Let $h_1,\dots,h_k$ *be holomorphic functions in an open subset* U *of* $\mathbb{C}^n$ *and* $E = Z(h_1,\dots,h_k)$. *If* E *has a base of Stein neighborhoods* (see Grauert and Remmert. *Theory of Stein spaces*, Springer 1979). *then*

$$O_n(U)/(h_1,\dots,h_k) = \Gamma(E.O_E)$$

where (E,O_E) *is the special model defined in 3.6 and* $\Gamma(E,O_E)$ *is considered with its structure of analytic ring* (see [3]. Theorem 2.10). *Moreover. the kernel of the map*

$$\rho_{\mathbb{C}}:\ O_n(U)(\mathbb{C}) \longrightarrow O_n(U)/(h_1,\dots,h_k)$$

is the usual ideal as $\mathbb{C}$-algebras and $\rho_{\mathbb{C}}$ is surjective.

PROOF. It is easy to see that if U_0 is an open subset of $\mathbb{C}^n$ such that $U \supset U_0 \supset E$, then

$$O_n(U)/(h_1,\dots,h_k) = O_n(U_0)/(h_1|U_0,\dots,h_k|U_0).$$

Then, taking U_0 Stein, we may assume that U is Stein. Let $\rho: O_n(U) \to \Gamma(E, O_E)$ be the canonical morphism; we will prove that given $\varphi: O_n(U) \to B$ (B an analytic ring) such that $\varphi(h_i) = 0$, $1 \le i \le k$, then there exists a unique $\tilde{\varphi}: \Gamma(E, O_E) \to B$ such that we have $\tilde{\varphi}\rho = \varphi$. Let $S \in \Gamma(E, O_E)$. Since U is Stein, we can take a holomorphic function $f \in O_n(U)$ such that $S = \rho(f)$; we define $\tilde{\varphi}(S) = \varphi(f)$. This is well defined; in fact, if $\rho(f-g) = 0$, then

$$f_p - g_p \in (h_{1,p},\dots,h_{k,p}) \text{ for all } p \in E,$$

and, again, since U is Stein, we have that $f-g \in (h_1,\dots,h_k)$; then $\varphi(f-g) = 0$.

This essentially shows that $\Gamma(E, O_E)$ is the quotient as $\mathbb{C}$-algebras. It remains to see that it actually is the quotient as analytic rings.

We will prove now that if $S \in \Gamma(E, O_E)(W)$, where W is an open subset of $\mathbb{C}$, then $\tilde{\varphi}(S) \in B(W)$. Clearly, we have $S = \rho(f)$ with $f(E) \subset W$; hence, there exists an open subset V of $\mathbb{C}^n$, $E \subset V \subset U$ such that $f(V) \subset W$. φ is equivalent to an element b of B(U) such that $\varphi(t) = B(t)(b)$ for $t \in O_n(U)$, and since $\varphi(h_i) = 0$ $(1 \le i \le k)$, we have

$$B(h_i)(b) = 0 \ (1 \le i \le k), \text{ that is, } b \in Z_B(h_1,\dots,h_k).$$

By (2.10), we have that $Z_B(h_1,\dots,h_k) \subset B(V)$; then, $b \in B(V)$ and by considering $f|V: V \to W$, we have:

$$\varphi(f) = B(f)(b) = B(f|V)(b) \in B(W),$$

that is, $\tilde{\varphi}(S) \in B(W)$. This defines an arrow $\tilde{\varphi}: \Gamma(E, O_E)(W) \to B(W)$. Similarly, we define $\tilde{\varphi}_W$ for W an open subset of $\mathbb{C}^m$.

Finally, from the fact that E has a base of Stein neighborhoods, it will follow that $\tilde{\varphi}$ defines a morphism of analytic rings. Let g: $W \to \mathbb{C}$ be a holomorphic function: we have to prove that the square

$$\begin{array}{ccc} \Gamma(E, O_E)(W) & \xrightarrow{\ g^*\ } & \Gamma(E, O_E) \\ \Big\downarrow{\scriptstyle \tilde{\varphi}_W} & & \Big\downarrow{\scriptstyle \tilde{\varphi}} \\ B(W) & \xrightarrow{\ B(g)\ } & B \end{array}$$

commutes (we suppose again that $W \subset \mathbb{C}$). Let $S \in \Gamma(E, O_E)(W)$ and $f \in O_n(U)$ be such that $S = \rho(g)$. Take as above an open subset V of $\mathbb{C}^n$, $E \subset V \subset U$, such that $f(V) \subset W$. Then, $g^*(S) = \rho'(g \circ f|V)$, where $\rho' : O_n(V) \to \Gamma(E, O_E)$ is the canonical morphism. On the other hand, there exists $t \in O_n(U)$ such that $g^*(S) = \rho(t)$; hence

$$\tilde{\varphi}(g^*(S)) = \varphi(t) = B(t)(b);$$

moreover, $\tilde{\varphi}_W(S) = \varphi(f) = B(f)(b)$; then

$$B(g)(\tilde{\varphi}_W(S)) = B(g)(B(f)(b)) = B(g)(B(f|V)(b)) = B(g \circ f|V)(b).$$

Then, we have to prove that $B(t)(b) = B(g \circ f|V)(b)$ (1). By hypothesis, there exists an open subset V_0 of $\mathbb{C}^n$ such that $E \subset V_0 \subset V$ and V_0 is Stein; and, by (2.10), $Z_B(h_1, \dots, h_k) \subset B(V_0)$. Hence, $b \in B(V_0)$. Hence,

(2) $\quad B(t)(b) = B(t|V_0)(b)$ and $B(g \circ f|V)(b) = B(g \circ f|V_0)(b)$.

Since $t|V_0$ and $g \circ f|V_0$ define the same section $g^*(S)$ in (E, O_E), then using that V_0 is Stein we obtain that

$$g \circ f|V_0 - t|V_0 \in (h_1|V_0, \dots, h_k|V_0).$$

Then it follows (by evaluation at b) that

$$B(g \circ f|V_0)(b) = B(t|V_0)(b);$$

thus, using (2), we obtain (1).

The proof for $W \subset \mathbb{C}^n$ is the same working in each coordinate.

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