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The natural number bialgebra

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THE NATURAL NUMBER BIALGEBRA

by John L. MACDONALD and Art STONE

RÉSUMÉ. Dans cet article, on montre que les définitions familières de Peano de l'addition et de la multiplication en terme de l'opération successeur, plutôt considérée comme co-opération, s'expliquent mieux comme exemple d'une loi distributive de bialgèbre. Cet exemple de Peano est un cas particulier d'une situation souvent rencontrée en programmation informatique où les données peuvent être pensées comme faisant intervenir des co-opérations de même que des opérations. Ces co-opérations retiennent de l'espace (pour la mémoire, dans l'ordinateur) alors que les opérations libèrent de l'espace.

ABSTRACT.

Where an Eilenberg-Moore *algebra* is a pair (X,a) with a: $XT \longrightarrow X$ (satisfying axioms), a *bialgebra* is a triple (X,a,c) for which the pentagon

$$
\begin{array}{ccc}
XT & \xrightarrow{\cdot cT} & XGT \\
 & & \downarrow{\scriptstyle \cdot X\lambda} \\
\downarrow{\scriptstyle \cdot a} & & XTG \\
 & & \downarrow{\scriptstyle \cdot aG} \\
X & \xrightarrow[\cdot c]{} & XG\ .
\end{array} \tag{0.1}
$$

commutes, that is, $\cdot cT \cdot X\lambda \cdot aG = \cdot a \cdot c$. Here T is the endofunctor of a monad, G is the endofunctor of a co-monad, and λ is a *bialgebra distributive law* as defined in the third section (cf. Beck [3], Van Osdol [11]). The main point of this paper is that the familiar Peano definitions of addition and multiplication in terms of the successor (co-)operation can perhaps best be understood as defining an instance of a morphism $\cdot X\lambda$ such as

appears in (0.1). Further, this *Peano example* is an instance of something we see often in computer programming where data structures can be thought of as involving *co-operations* as well as operations. Peano's successor operation, being a unary operation, can be thought of as either an operation or a co-operation. But we can give a clearer interpretation of the definitions of addition and multiplication regarding it as a co-operation. In general, the *co-operations* of programming data structures *create* space (mark storage space, or set it aside, in the machine), while *operations release* or *free* space.

The first section introduces some background material on categories of adjunctions. $\underline{\mathbf{Adj}}(\mathbf{Cat})$ is introduced as the category whose objects are adjunctions in **Cat** and whose morphisms are pairs commuting with right adjoints. The definitions of allo natural transformation and modification are recalled and **Adj**(**X**) is described as the 2-category whose objects are strict 2-functors **Adj**→**X**, whose 1-cells are allo natural transformations and whose 2-cells are modifications for **X** a 2-category and **Adj** the "free 1-adjunction". We show how **Adj**(**X**), although described differently from $\underline{\mathbf{Adj}}(\mathbf{Cat})$, differs only slightly when **X** = **Cat**. This is because we can show that **Adj**(**Cat**) has for 1-cells those pairs of functors commuting with right adjoints up to "coherent" isomorphism rather than pairs strictly commuting as in $\underline{\mathbf{Adj}}(\mathbf{Cat})$. Various generalizations are possible at this point, namely, we could use para instead of allo to give 1-cells commuting (up to isomorphism) with left adjoints or we could start from "free" structures other than **Adj** but given by objects, 1-cells and 2-cells subject to certain equations.

The second section on distributive squares and n-cubes gives Beck's description of a distributive square as a commutative adjoint square in which a certain induced map is an isomorphism. The objects of **Adj**(**Adj**(**X**)) are then shown to be simply the distributive squares in **X** for **X** = **Cat**. The 2-category structure of **Adj**(**Adj**(**X**)) carries with it a "built in" definition of morphisms (and 2-cells) between distributive squares. The same remarks hold for

$$\mathbf{Adj}^n(\mathbf{X}) = \mathbf{Adj}(\mathbf{Adj}^{n-1}(\mathbf{X}))$$

whose objects are defined to be the *distributive n-cubes* in **X**. This section also refers to the distributive laws generated by a distributive square, one type at each vertex.

The third section on bialgebras and the natural numbers describes a bialgebra distributive law $\lambda: GT \to TG$ associated with a monad (T,η,μ) and a comonad (G,ε,δ) on $\underline{X}$ in $\mathbf{X}$. This law can be used to build a distributive square on $\underline{X}$ in which the missing vertex (the "ghost category") is that of the category of bialgebras (cf. [3,11]). A few general propositions about bialgebras and a description of augmented bisimplicial objects are given before looking at the example of the natural number bialgebra over $Set_\perp$ in which the λ given is derived from the Peano postulates. Finally we have indicated how these ideas may be applied to computer science.

We use the following notation. The vertical composite of 2-cells in a 2-category is denoted $\varphi\cdot\cdot\gamma$ and displayed as in

$$\begin{array}{c} \underline{X} \xrightarrow{\quad Y \quad} \underline{Y} \\ \Downarrow\varphi \\ \underline{X} \xrightarrow{\quad Y' \quad} \underline{Y} \\ \Downarrow\gamma \\ \underline{X} \xrightarrow{\quad Y'' \quad} \underline{Y} \end{array} \tag{0.2}$$

The composite of 1-cells X and Y or the horizontal composite of a 1-cell X and a 2-cell π are denoted by the symbols X;Y or X;π respectively and displayed as

$$\underline{W} \xrightarrow{\quad X \quad} \underline{X} \xrightarrow{\quad Y \quad} \underline{Y} \quad \text{or} \quad \underline{W} \xrightarrow{\quad X \quad} \underline{X} \;\begin{array}{c} \xrightarrow{\quad Y \quad} \\ \Downarrow\pi \\ \xrightarrow[\quad Z \quad]{} \end{array}\; \underline{Y} \tag{0.3}$$

We further indicate the context by the symbol by letting $\cdot b$ denote a morphism in a 1-category and the composition of such morphisms by $\cdot a\cdot c$ (or $a\cdot$c). $Sets_\perp$ is the category of pointed sets (with point $=\perp$).

1. CATEGORIES OF ADJUNCTIONS.

Let **Adj(Cat)** be the following 2-category. It has objects

$$S_\square = \; FS \left\uparrow \begin{array}{c} \underline{Q}S \\ \\ \underline{P}S \end{array} \right\downarrow US \tag{1.1}$$

where $S_\square$ is an adjunction in **Cat** with left adjoint FS and 1-cells $(\underline{Q}\Gamma.\underline{P}\Gamma)$

(1.2)
$$S_\square = \begin{array}{ccc} \underline{Q}S & \xrightarrow{\underline{Q}\Gamma} & \underline{Q}T \\ FS\big\uparrow\big\downarrow US & & FT\big\uparrow\big\downarrow UT \\ \underline{P}S & \xrightarrow[\underline{P}\Gamma]{} & \underline{P}T \end{array}$$

satisfying $\underline{Q}\Gamma;UT = US;\underline{P}\Gamma$ and 2-cells $(\underline{Q}s,\underline{P}s)$

(1.3)
$$S_\square = \begin{array}{ccc} \underline{Q}S & \begin{array}{c}\xrightarrow{\underline{Q}\Phi} \\ \Downarrow \underline{Q}s \\ \xrightarrow[\underline{Q}\Gamma]{}\end{array} & \underline{Q}T \\ FS\big\uparrow\big\downarrow US & & FT\big\uparrow\big\downarrow UT \\ \underline{P}S & \begin{array}{c}\xrightarrow{\underline{P}\Phi} \\ \Downarrow \underline{P}s \\ \xrightarrow[\underline{P}\Gamma]{}\end{array} & \underline{P}T \end{array}$$

satisfying $\underline{Q}s;UT = US;\underline{P}s$.

Let S and T be strict 2-functors $\mathbf{X}\to\mathbf{Y}$. We recall that an *allo* natural transformation $\Gamma: S \Rightarrow\!\!\backslash\, T$ assigns to each $Y:\underline{X}\to\underline{Y}$ in $\mathbf{X}$ a diagram

(1.4)
$$\begin{array}{ccc} \underline{X}S & \xrightarrow{YS} & \underline{Y}S \\ \underline{X}\Gamma\big\downarrow & Y\Gamma \Nearrow & \big\downarrow\underline{Y}\Gamma \\ \underline{X}T & \xrightarrow[YT]{} & \underline{Y}T \end{array}$$

such that $Y\Gamma:\underline{X}\Gamma;YT \Rightarrow YS;\underline{Y}\Gamma$ is a 2-cell and the following 3 axioms hold.

(1.5) Given $1_{\underline{X}}:\underline{X}\to\underline{X}$ it is required that the morphism $1_{\underline{X}}\Gamma$ be the identity 2-cell $\underline{X}\Gamma \Rightarrow \underline{X}\Gamma$.

Secondly for each 2-cell

$$\underline{X} \begin{array}{c}\xrightarrow{Y} \\ \Downarrow \pi \\ \xrightarrow[Y']{}\end{array} \underline{Y}$$

it is required that

(1.6)
$$\begin{array}{ccc} \underline{X}\Gamma;YT & \xRightarrow{Y\Gamma} & YS;\underline{Y}\Gamma \\ \underline{X}\Gamma;\pi T \Big\Downarrow & & \Big\Downarrow \pi S;\underline{Y}\Gamma \\ \underline{X}\Gamma;Y'T & \xRightarrow[Y'\Gamma]{} & Y'S;\underline{Y}\Gamma \end{array}$$

be a commutative diagram of 2-cells.

(1.7) Given

$$\underline{W} \xrightarrow{X} \underline{X} \xrightarrow{Y} \underline{Y}$$

in **X** it is required that $X\Gamma;YT \cdot\cdot XS;Y\Gamma = [X;Y]\Gamma$.

Suppose that a diagram

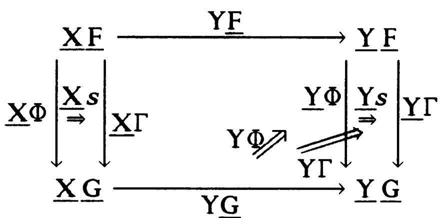
(1.8) $$\mathbf{X} \;\underset{\underline{G}}{\overset{\underline{F}}{\rightrightarrows}}\; \mathbf{A} \qquad \Downarrow\Phi \quad \Gamma\Downarrow$$

is given where $\underline{F}$ and $\underline{G}$ are strict and Φ and Γ are allo natural transformations. Then a modification $s:\Phi \Longrightarrow \Gamma$ consists of 2-cells

(1.9) $$\underline{X}\underline{F} \;\underset{\underline{X}\Gamma}{\overset{\underline{X}\Phi}{\rightrightarrows}}\; \underline{X}\underline{G} \qquad \Downarrow \underline{X}s$$

in **A**, one for each object $\underline{X}$ of **X**, such that for each 1-cell $Y: \underline{X}\to\underline{Y}$ of **X** the associated diagram

(1.10)



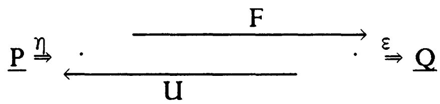
(2,0)-*commutes* in **A**. This means that the diagram

(1.11) $$\begin{array}{ccc} Y\underline{F};\underline{Y}\Phi & \xrightarrow[]{\;Y\underline{F};\underline{Y}s\;} & Y\underline{F};\underline{Y}\Gamma \\ Y\Phi \Uparrow & & \Uparrow Y\Gamma \\ \underline{X}\Phi;Y\underline{G} & \xrightarrow[\underline{X}s;Y\underline{G}]{} & \underline{X}\Gamma;Y\underline{G} \end{array}$$

of 2-cells commutes in **A**. In equational form we have

(1.12) $$Y\Phi \cdot\cdot Y\underline{F};\underline{Y}s = \underline{X}s;Y\underline{G} \cdot\cdot Y\Gamma.$$

Now let **Adj** denote the "free 1-adjunction" - the 2-category

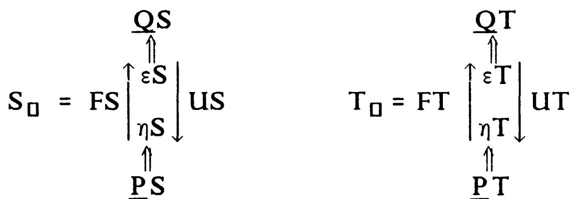
(1.13) 

with

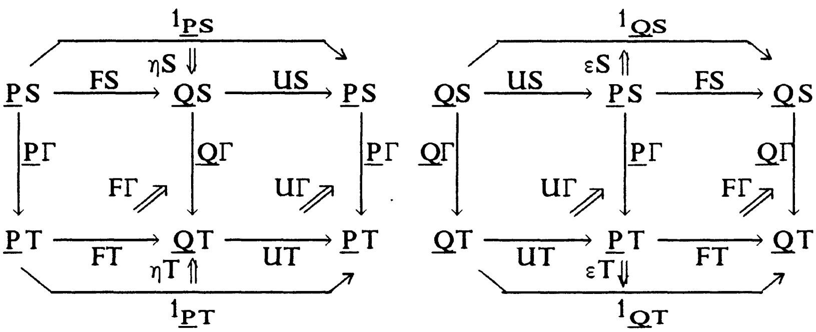
$$\eta F \cdot\cdot F\varepsilon = \cdot\cdot 1_F \text{ and } U\eta \cdot\cdot \varepsilon U = \cdot\cdot 1_U$$

described in Schanuel-Street [10] (cf. Auderset [1]) and denote by **Adj**(**X**) the 2-category of strict 2-functors **Adj** → **X** with allo natural (lax natural) transformations for 1-cells and modifications for 2-cells.

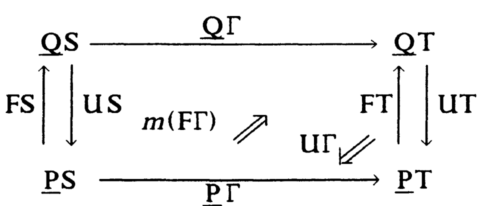
Let S and T be strict 2-functors **Adj**→**X** pictured

(1.14) 

in **X** and let Γ: S⇒T be an allo natural transformation. Then using (1.4) and (1.14) we can extract the picture

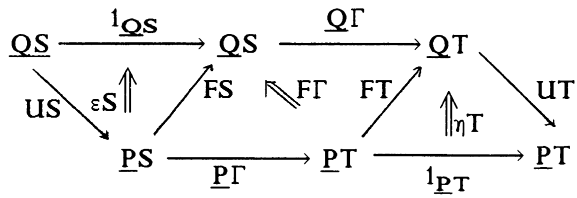
(1.15) 

From this diagram we extract a key part, namely

(1.16) 

where $m(F\Gamma)$: US;$\underline{P}\Gamma$ ⇒ $\underline{Q}\Gamma$;UT, called the *mate* of FΓ, is the composite

(1.17)

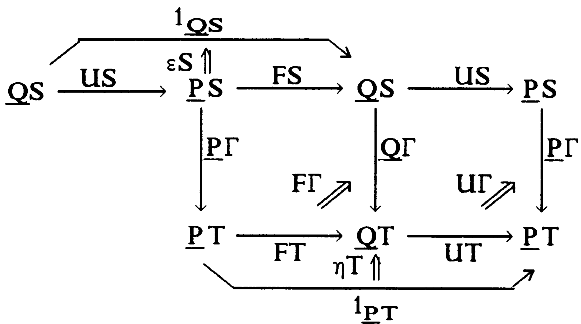


In Beck's terminology ([3], p. 139), FΓ and *m*(FΓ) are *adjoint* morphisms.

PROPOSITION 1.18. *The 2-cell* *m*(FΓ) *of* (1.16) *is the inverse of* UΓ.

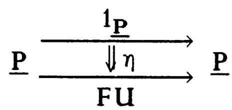
PROOF. *m*(FΓ) followed by UΓ may be pictured

(1.19)

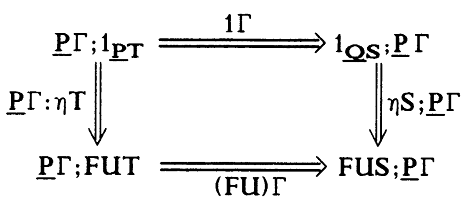


Then given

(1.20)

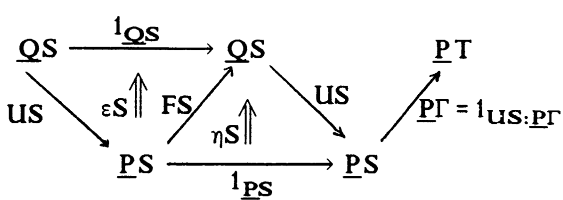


(1.21)



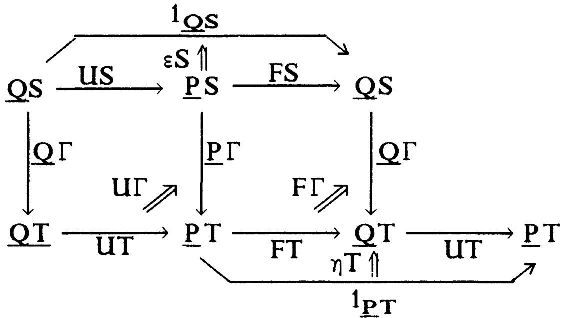
commutes by (1.6) where (FU)Γ = FΓ:UT· · FS;UΓ (by (1.7)). Thus (1.19) becomes

(1.22)



As in (1.19) $U\Gamma$ followed by $m(F\Gamma)$ may be pictured

(1.23)



which is, using successively that $U\Gamma$ and ηT can be composed in either order and (1.7):

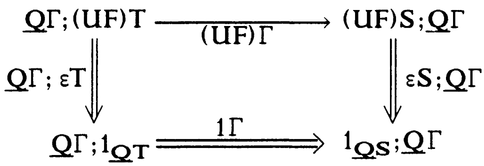
$$\begin{aligned} U\Gamma\cdot\cdot US;\underline{P}\Gamma;\eta T\cdot\cdot US;F\Gamma;UT\cdot\cdot\varepsilon S;\underline{Q}\Gamma;UT \\ = \underline{Q}\Gamma;UT;\eta T\cdot\cdot U\Gamma;FT;UT\cdot\cdot US;F\Gamma;UT\cdot\cdot\varepsilon S;\underline{Q}\Gamma;UT \\ = \underline{Q}\Gamma;UT;\eta T\cdot\cdot(UF)\Gamma;UT\cdot\cdot\varepsilon S;\underline{Q}\Gamma;UT \end{aligned}$$

which equals

(1.24) $$\underline{Q}\Gamma;UT;\eta T\cdot\cdot Q\Gamma;\varepsilon T;UT$$

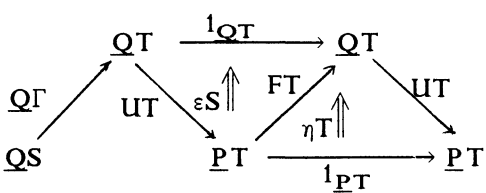
by the commutativity of

(1.25)



Then we combine (1.24) and (1.25) to obtain

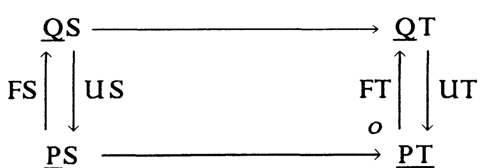
(1.26)



which equals $1_{\underline{Q}\Gamma;UT}$. ∎

In particular (1.16) becomes

(1.27)



where o is the isomorphism $\mathrm{U}\Gamma$ whose inverse is $m(\mathrm{F}\Gamma)$.

Thus the description above of $\underline{\mathbf{Adj}}(\mathbf{Cat})$ turns out to be a simplified version of $\mathbf{Adj}(\mathbf{Cat})$ in which the isomorphic 2-cell

$$\mathrm{U}\Gamma\colon\ \underline{\mathrm{Q}}\Gamma;\mathrm{UT} \Rightarrow \mathrm{US};\underline{\mathrm{P}}\Gamma$$

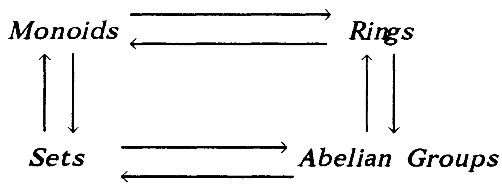
of (1.16) is the identity, as in (1.2).

The functor $\underline{\mathrm{U}}\colon \mathbf{Adj}(\mathbf{X})\to\mathbf{X}$ taking each adjunction (1.1) to $\underline{\mathrm{PS}}$ has a soft left adjoint $\underline{\mathrm{F}}$ taking $\underline{\mathrm{PS}}$ to its identity adjunction (cf. [7,8]).

2. DISTRIBUTIVE SQUARES AND n-CUBES.

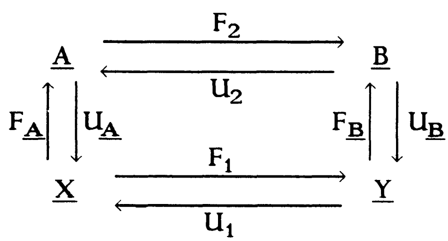
An example of a distributive square appearing in Beck ([3], p. 135) is

(2.1)



An *adjoint square* is a diagram

(2.2)

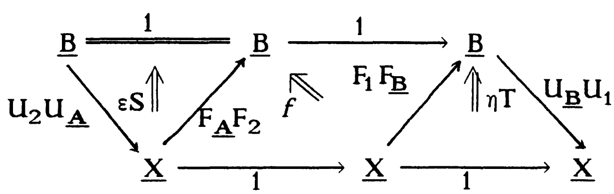


of 4 adjunctions as pictured. It is *commutative* if there are natural isomorphisms

(2.3) $$u\colon \mathrm{U}_2\mathrm{U}_{\underline{A}}\to\mathrm{U}_{\underline{B}}\mathrm{U}_1 \quad\text{and}\quad f\colon \mathrm{F}_1\mathrm{F}_{\underline{B}}\to \underline{\mathrm{F}}_{\underline{A}}\mathrm{F}_2$$

which are mates, that is, as in (1.17), $u = m(f)$ is the composite

(2.4)



A *distributive square* is a commutative adjoint square such that the induced map $e\colon U_{\underline{A}}F_1 \to F_2U_{\underline{B}}$ is an isomorphism where e is defined by

$$e = U_{\underline{A}}F_1\eta_{\underline{B}};U_{\underline{A}}fU_{\underline{B}};\varepsilon_{\underline{A}}F_2U_{\underline{B}} = \eta_2U_{\underline{A}}F_1;F_2uF_1;F_2U_{\underline{B}}\varepsilon_1 \tag{2.5}$$

(cf. [3], p. 130, 139).

PROPOSITION 2.6. *The objects of* **Adj(Adj(X))** *are the distributive squares in* **X**, *namely*

$$\begin{array}{ccccc} \underline{A} & \underset{U_2}{\overset{F_2}{\rightleftarrows}} & & & \underline{B} \\ F_{\underline{A}}\uparrow\ \downarrow U_{\underline{A}} & o & & o & F_{\underline{B}}\uparrow\ \downarrow U_{\underline{B}} \\ & & & o & \\ \underline{X} & \underset{U_1}{\overset{F_1}{\rightleftarrows}} & & & \underline{Y} \end{array} \tag{2.7}$$

where the o's at the lower left and upper right denote the isomorphisms u and f of (2.3) *and the lower right that of* (2.5).

PROOF. An object of **Adj(Adj(X))** is an adjunction

$$\underline{A}_\Box \underset{U_\Box}{\overset{F_\Box}{\rightleftarrows}} \underline{B}_\Box \tag{2.8}$$

where

$$\underline{A}_\Box = \begin{array}{c} \underline{A} \\ F_{\underline{A}}\uparrow\ \downarrow U_{\underline{A}} \\ \underline{X} \end{array} \quad \text{and} \quad \underline{B}_\Box = \begin{array}{c} \underline{B} \\ F_{\underline{B}}\uparrow\ \downarrow U_{\underline{B}} \\ \underline{Y} \end{array} \tag{2.9}$$

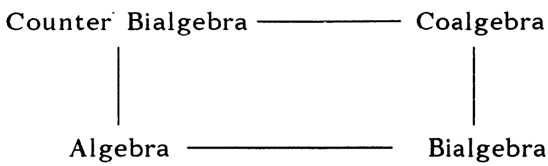
are adjunctions in **X** and $F_\Box = (F_2, F_1)$ and $U_\Box = (U_2, U_1)$ are **Adj(X)** morphisms. Thus we have a diagram of the form (2.7) in which the vertical pairs are adjunctions and $U_\Box = (U_2, U_1)$ and $F_\Box = (F_2, F_1)$ are **Adj(X)** morphisms. This implies that there are isomorphisms

$$u\colon U_2U_{\underline{A}} \to U_{\underline{B}}U_1 \text{ and } g\colon F_2U_{\underline{B}} \to U_{\underline{A}}F_1$$

determined as in (1.16). The usual adjunction equations hold for $F_\Box = (F_2, F_1)$ left adjoint to $U_\Box = (U_2, U_1)$ and these yield equations showing F_2 left adjoint to U_2 and F_1 to U_1. Thus we have an adjoint square. It is commutative since u, being an isomorphism of right adjoints, has a mate $f\colon F_1F_{\underline{B}} \to F_{\underline{A}}F_2$ which is also an isomorphism. It is not hard to show using allo natural transformation rules that the inverse of the isomorphism g is the same as the morphism e defined from u and f in (2.5). ■

In general a distributive square generates a different kind of distributive law at each vertex as pictured in the diagram

(2.8)



An object of $\mathbf{Adj}^n(\mathbf{X}) = \mathbf{Adj}(\mathbf{Adj}^{n-1}(\mathbf{X}))$ will be called a *distributive n-cube.*

3. BIALGEBRAS AND THE NATURAL NUMBERS.

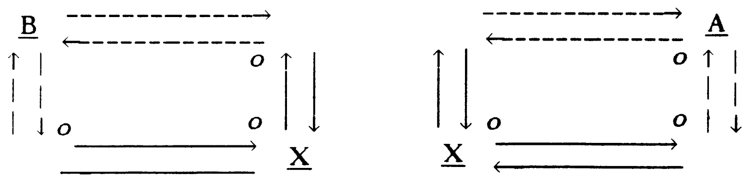
Where (T,η,μ) is a monad and (G,ε,δ) is a comonad on $\underline{X}$ in $\mathbf{X}$, a *bialgebra distributivity of* (T,η,μ) *over* (G,ε,δ) is a natural transformation $\lambda : GT \Rightarrow TG$ for which

(3.1) $$\begin{cases} G\eta\cdot\cdot\lambda = \eta G,\ \lambda\cdot\cdot T\varepsilon = \varepsilon T, \\ \lambda T\cdot\cdot T\lambda\cdot\cdot\mu G = G\mu\cdot\cdot\lambda \text{ and } \delta T\cdot\cdot G\lambda\cdot\cdot\lambda G = \lambda\cdot\cdot T\delta \end{cases}$$

These axioms (two triangles and two pentagons) are analogues of Beck's (cf. [3] as well as Van Osdol [11]).

The "ghost category" problem is one of the following type. Given

(3.2)

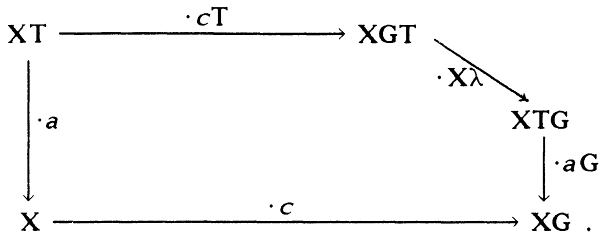


fill in the dotted arrows and describe $\underline{B}$ on the left hand side or $\underline{A}$ on the right so that the resulting squares are distributive. There are analogous problems at the other two vertices as well as higher order analogues when there are three or more structures at a given vertex. For a description of $\underline{A}$ see Beck [3].

Where λ is a bialgebra distributivity of (T,η,μ) over (G,ε,δ) on a category $\underline{X}$, a *bialgebra* for λ is a triple:

(3.3) (X,a,c) in which (X,a) is a (T,η,μ)-algebra and (X,c) is a (G,ε,δ)-coalgebra and $a\cdot c = cT\cdot X\lambda\cdot aG$.

(3.4)



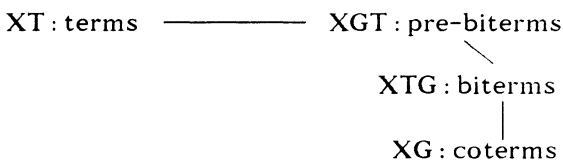
Bi-homomorphisms are morphisms which are algebra and coalgebra homomorphisms.

PROPOSITION 3.5 (*The Bialgebras of the Pentagon*). *If* (X,*a*,*c*) *is a bialgebra, then so are*

$$(3.6)\quad \begin{cases} (XT,\cdot X\mu,\ \cdot cT\cdot X\lambda),\ (XGT,\cdot XG\mu,\cdot X\delta T\cdot XG\lambda), \\ (XTG,\cdot XT\lambda\cdot X\mu G,\cdot XT\delta)\ \text{and}\ (XG,\cdot X\lambda\cdot aG,\cdot X\delta). \end{cases} \qquad \blacksquare$$

The underlying objects of these bialgebras are vertices in the pentagon (3.4) for (X,*a*,*c*). The elements of these bialgebras may be called

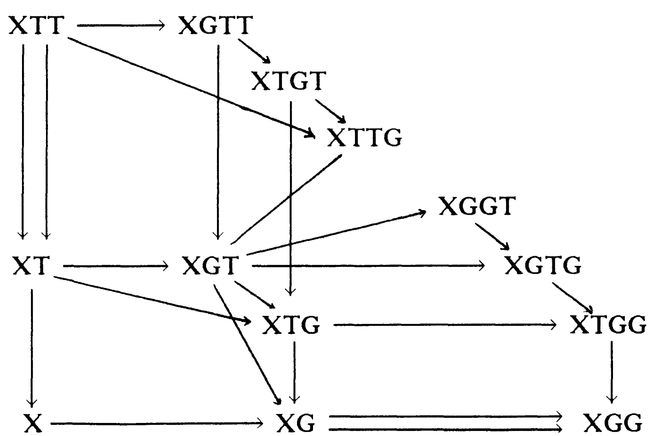
(3.7)



and over λ .

Repeated application of the Bialgebras of the Pentagon Proposition 3.5 gives us an infinite diagram, an *augmented bi-simplicial object.*

(3.8)



In the following paragraphs we show how the Peano axioms for the natural numbers, with addition and multiplication, give us an example over $Set_{\perp}$.

Let 0 and $\cdot s$ denote Peano's zero and successor operation. The Peano axioms for addition and multiplication

(3.9) $a+0 = a,\ a+b\cdot s = [a+b]\cdot s,\ a*0 = 0,\ a*b\cdot s = [a*b]+a$

are such that, over $\varnothing$ in *Set* or over $\{\perp\}$ in $Set_{\perp}$, the binary operations produce no new terms. Peano's set of elements is determined by 0 and $\cdot s$ alone.

We will think of $\cdot s$ as a co-operation. (T,η,μ) will be the monad for the category $CS_{\perp}$ of (pointed) commutative semi-rings with zero. Explicitly this means that $CS_{\perp}$ has binary operations + and $*$ and constants 0 and $\perp$ such that + is associative and commutative with identity 0 and $*$ is associative and distributive with respect to + and

$$0\cdot 0 = 0,\ a+\perp=\perp \text{ and } a*\perp = \perp.$$

(G,ε,δ) will be similar to the product comonad of Lambek as described in [5], page 285 and in [6], page 62.

To define λ we need recursive definitions of T and G. Let X be a pointed set. Then XT_f is by definition the set containing X as a subset, with further elements

(3.10) 0 (zero element), $a+b$ and $a*b$ for all $a,b\in XT_f$,

Let XT be the set of equivalence classes of XT_f determined by the axioms of $CS_{\perp}$. The definition of η and μ (and the extension of T to a functor) is straightforward. We define XG_f recursively by

(3.11) X is contained in XG_f and $a\cdot s$ is in XG_f for all $a\in XG_f$.

Then XG is the set of equivalence classes respecting the operation s determined by $\perp\sim\perp\cdot s$. We let $[a]$ denote the equivalence class of a.

Let $\cdot X\varepsilon: XG\to X$ and $\cdot X\delta: XG\to XGG$ be the identity on X, and

(3.12) $[x\cdot s^n]\cdot X\varepsilon = x$ and $[x\cdot s^n]\cdot X\delta = [[x\cdot s^n]\cdot \underline{s}^n]$

for $x\in X$, where $\cdot\underline{s}$ denotes the successor (co-)operation of XGG.

Let $\cdot X\lambda: XGT\to XTG$ be the map induced from $X\lambda_f: XGT_f\to XTG$ where first we define $X\lambda_f$ on XG by

(3.13) $\begin{cases} [x]\cdot X\lambda_f = [[x]] \text{ for } x \text{ in the subset X of } XG_f \\ \qquad\qquad (x \text{ is in } XT_f \text{ as well}) \text{ and} \\ [a\cdot s]\cdot X\lambda_f = ([a]\cdot X\lambda_f)\cdot s, \\ \qquad\qquad \text{if } a \text{ is in } XG_f \text{ and } [a] \text{ is in } \mathrm{Dom}(\cdot X\lambda_f), \end{cases}$

then we extend the definition of $X\lambda_f$ to XGT_f by:

(3.14) $0 \cdot X\lambda_f = [[0]]$, and by letting $a+b\cdot s$ be in $\mathrm{Dom}(\cdot X\lambda_f)$ if $a+b$ is, and

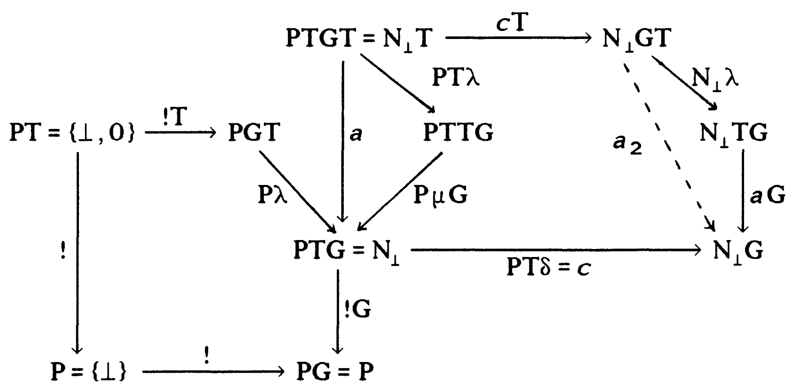
$$(a+b\cdot s)\cdot X\lambda_f = ((a+b)\cdot X\lambda_f)\cdot s,$$

and by letting $a*b\cdot s$ be in $\mathrm{Dom}(\cdot X\lambda_f)$ if $a*b$ is, and

$$(a*b\cdot s)\cdot X\lambda_f = ((a*b)\cdot X\lambda_f + a\cdot X\lambda_f).$$

Specific instances of (3.4) involving the natural numbers may be pictured as follows, noting that the *pointed natural numbers* $N_\perp$ are by definition equal to PTG where $P=\{\perp\}$.

(3.15)



The algebraic structure $a = PT\lambda;P\mu G$ of $N_\perp$ embodies the definition of + and *. In particular, $PT\lambda$ is called the *Peano implementation.*

The coalgebraic structure on the pointed natural numbers $N_\perp$ is the diagonal map $N_\perp \to N_\perp \times N_\perp$ (modulo $a\cdot\perp = \perp = \perp\cdot a$). Such a map is a special case of what in programming languages is called *simple assignment.*

Computer science provides many more examples of bialgebras. For a machine with word size n the endofunctor G will "multiply" by a pointed set with 2^n defined elements, and it is appropriate to call (G,ε,δ) a *space defining comonad.*

Machines are inherently bialgebraic - since machine operations are always defined in terms of predefined space. Input - which occupies new space - is co-algebraic, and output - which releases space - is algebraic.

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