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ON PRIMENESS AND MAXIMALITY OF FILTERS

by Francis BORCEUX and Maria-Cristina PEDICCHIO

RÉSUMÉ. Dans un topos arbitraire, on étudie diverses formulations des notions classiques de filtre premier et maximal ainsi que les implications entre ces définitions. Dans le cas des algèbres booléennes, on associe à chaque notion de filtre maximal une notion de filtre premier qui lui est équivalente dans un topos arbitraire, et inversement. On indique aussi des conditions sur la logique interne à un topos qui forcent certaines équivalences classiquement vraies.

INTRODUCTION.

In this paper we are concerned with the study of prime and maximal filters in a boolean algebra or a distributive lattice in an arbitrary topos. Classically, a filter F is prime when, for every elements a, b

$$a \vee b \in F \Rightarrow a \in F \vee b \in F \qquad \text{(P)}$$

which is equivalent to the fact of being a prime element in the lattice of filters; thus for filters G, H

$$G \cap H \subset F \Rightarrow G \subset F \vee H \subset F.$$

Moreover a prime filter F is very often required to be proper, thus to satisfy

$$\neg(0 \in F) \qquad \text{(p)}$$

On the other hand a filter F is maximal when it is proper and satisfies the condition, for every filter G

$$G \supset F \wedge \neg(0 \in G) \Rightarrow G = F,$$

or equivalently

$$G \supset F \Rightarrow G = F \vee 0 \in G. \qquad \text{(SM)}$$

Notice that the first condition for maximality is equivalent to

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$$G \supset F \wedge (0 \in G \Rightarrow 0 \in F) \Rightarrow G = F \qquad \text{(M)}$$

because of the properness of F.

In a topos, the condition of being a proper filter is very strong and many natural boolean algebras or distributive lattices in a topos do not have a proper filter. The necessity of considering in a topos filters which are not necessarily "globally proper" has already been recognized by several authors (cf. [1,2,9]). Therefore we shall study filters which satisfy the conditions P (prime filter), SP (strongly prime filter), M (maximal filter) or SM (strong maximal filter) without requiring necessarily the properness condition p.

In a first part, we extend some results due to P.T. Johnstone (cf. [7]). For filters in a boolean algebra in an arbitrary topos we have

strongly maximal ⇒ strongly prime ⇒ prime ⇒ maximal

while in the case of distributive lattices we obtain

strongly maximal ⇒ strongly prime ⇒ prime
strongly maximal ⇒ maximal.

The converse implications are generally not valid, but are in fact equivalent to some logical principles in the topos: the implication

maximal ⇒ prime

is equivalent to strong De Morgan's law

$$(\varphi \Rightarrow \Psi) \vee (\Psi \Rightarrow \varphi)$$

in the general case and to De Morgan's law

$$\neg\varphi \vee \neg\neg\varphi$$

in the case of proper filters. The implication

prime ⇒ strongly prime

is equivalent to the booleanness of the topos in the case of distributive lattices, to strong De Morgan's law in the case of arbitrary filters in a boolean algebra and to De Morgan's law in the case of proper filters in a boolean algebra. The implication

maximal ⇒ strongly maximal

is always equivalent to the booleanness of the topos.

In a second part, we extend a result of C.J. Mulvey (cf. [9]) by showing that in the case of a boolean algebra B in an arbitrary topos, each notion of maximal filter is equivalent to an appropriate formulation of the primeness condition, and conversely for prime filters. For example, in the case of a proper filter F, the condition (M) of maximality studied by P.T. Johnstone is equivalent to the primeness condition

$$a \vee b \in F \wedge \neg(a \in F) \Rightarrow b \in F$$

while his notion (P) of primeness is equivalent to the maximality condition:

$$(G \supset F \wedge a \in G) \Rightarrow (a \in F \vee \neg\neg(0 \in G));$$

the strong primeness condition (SP) is equivalent to the maximality condition:

$$G \supset F \Rightarrow G = F \vee \neg\neg(0 \in G)$$

while finally the strong maximality condition (SM) is equivalent to the primeness condition:

$$(G \subset F) \vee (H \subset F) \vee \exists\, b\, (b \in G \cap H \wedge \neg(b \in F)),$$

where a, b are elements of B and G, H filters of B.

1. MAXIMAL AND PRIME FILTERS.

When D is a distributive lattice in an arbitrary topos **E**, we write $F(D) \in \mathbf{E}$ to denote the locale of filters of D in **E** (cf. [1]). When B is a boolean algebra in **E**, we write b^* to denote the complement of $b \in B$.

DEFINITION 1.1. A filter F in a distributive lattice D is

a) *proper* when $\neg(0 \in F)$,

b) *prime* when

$$a \vee b \in F \Rightarrow a \in F \vee b \in F,$$

c) *strongly prime* when

$$G \cap H \subset F \Rightarrow G \subset F \vee H \subset F,$$

d) *maximal* when

$$(G \subset F) \wedge (0 \in G \Rightarrow 0 \in F) \Rightarrow (G = F),$$

e) *strongly maximal* when

$$G \supset F \Rightarrow G = F \vee 0 \in G,$$

where $a, b \in D$ and $G, H \in F(D)$.

It should be noticed that a proper filter F is maximal in the sense of Definition 1.1 precisely when (cf. [7])

$$G \supset F \wedge \neg(0 \in G) \Rightarrow G = F.$$

We recall also the following result.

PROPOSITION 1.2. *For a filter* $F \subset B$ *in a boolean algebra* B *in a topos, the following conditions are equivalent:*

(1) F *is a prime filter;*

(2) F *is an ultrafilter, i.e., for* $b \in F$,

$$b \in F \vee b^* \in F.$$

(1)⇒(2) since $b \vee b^* = 1$. (2)⇒(1) since

$$\begin{aligned} a \vee b \in F &\Rightarrow (a \vee b \in F) \wedge (a \in F \vee a^* \in F) \wedge (b \in F \vee b^* \in F) \\ &\Rightarrow a \in F \vee b \in F \vee (a \vee b \in F \wedge (a \vee b)^* \in F) \\ &\Rightarrow a \in F \vee b \in F \vee 0 \in F \Rightarrow a \in F \vee b \in F. \end{aligned}$$ ■

PROPOSITION 1.3. *For a filter* F *in a distributive lattice* D, *consider the following conditions:*

(SM) F *is strongly maximal,*

(SP) F *is strongly prime,*

(P) F *is prime,*

(M) F *is maximal.*

Then the following implications are valid:

$$(SM) \Rightarrow (SP) \Rightarrow (P) \text{ and } (SM) \Rightarrow (M).$$

Moreover when D *is a boolean algebra*

$$(SM) \Rightarrow (SP) \Rightarrow (P) \Rightarrow (M).$$

(SM)⇒ (SP): If $G \cap H \subset F$, apply (SM) to $F \vee G$ and $F \vee H$, where $\vee$ denotes the supremum in $F(D)$. One gets

$$(F \vee G \subset F) \vee (0 \in F \vee G), \quad (F \vee H \subset F) \vee (0 \in F \vee H),$$

thus also

$$G \subset F \vee H \subset F \vee 0 \in (F \vee G) \cap (F \vee H).$$

But since $F(D)$ is a locale and $G \cap H \subset F$,

$$(F \vee G) \cap (F \vee H) = F \vee (G \cap H) = F,$$

so that

$$0 \in (F \vee G) \cap (F \vee H) \Rightarrow 0 \in F \Rightarrow G \subset F.$$

(SP)$\Rightarrow$(P): If $a \vee b \in F$, apply (SP) to the principal filters generated by a, b.

(SM)$\Rightarrow$(M): obvious.

(P)$\Rightarrow$(M) for boolean algebras. Consider $G \supset F$ such that $0 \in G \Rightarrow 0 \in F$. For an element $a \in G$,

$$a \in F \vee a^* \in F$$

(Proposition 1.2), but

$$a^* \in F \Rightarrow a^* \in G \Rightarrow 0 \in G \Rightarrow 0 \in F \Rightarrow a \in F. \blacksquare$$

2. THE FILTERS OF THE INITIAL ALGEBRA.

The object $2 \approx 1 \amalg 1$ is, in every topos, the initial boolean algebra.

PROPOSITION 2.1. *In every topos,* $F(2) \approx \Omega$.

A filter $F \subset 2$ is determined by the fact that its characteristic map $\varphi : 2 \to \Omega$ satisfies

$$\varphi(1) = 1, \quad \varphi(a \wedge b) = \varphi(a) \wedge \varphi(b).$$

But for elements $a, b \in 2$ we always have

$$a = 0 \vee a = 1, \quad b = 0 \vee b = 1.$$

Therefore the condition $\varphi(1) = 1$ implies that F is completely determined by the element $\varphi(0) \in \Omega$, while the second condition on φ reduces to $\varphi(0) \leq \varphi(1)$, what follows from $\varphi(1) = 1$. So F is characterized by $\varphi(0)$ which is an arbitrary element of Ω. ■

PROPOSITION 2.2. *In a topos:*

(1) $\{1\}$ *is the only proper filter of* 2,

(2) *every filter of* 2 *is prime, thus also maximal,*

(3) *the strongly prime filters of* 2 *correspond bijectively with the prime elements of* Ω,

(4) *the strongly maximal filters of* 2 *correspond bijectively with the widespread elements of* Ω (cf. [8]).

(1) and (2) follow immediately from the relations

$$a = 0 \vee a = 1, \ \neg(0 = 1)$$

for an element $a \in 2$, while (3) is an obvious consequence of 2.1. The element $u \in \Omega$ corresponds with a strongly maximal filter of 2 when, in Ω,

$$v \geq u \Rightarrow v = u \vee v = 1,$$

which is equivalent to

$$v \geq u \Rightarrow v \leq u \vee v = 1,$$

or
$$v \geq u \Rightarrow (v \Rightarrow u) = 1 \vee v = 1$$

or
$$v \geq u \Rightarrow (v \Rightarrow u) \vee v = 1;$$

thus finally, in the locale $\uparrow u$, to the property $\neg v \vee v = 1$, which means precisely the booleanness of $\uparrow u$, thus the fact for u to be widespread (cf. [8]). ∎

PROPOSITION 2.3. (1) *A topos satisfies De Morgan's law iff* $\{1\}$ *is a strongly prime filter of* 2.

(2) *A topos satisfies strong De Morgan's law iff every filter of* 2 *is strongly prime.*

(3) *A topos is boolean iff* $\{1\}$ *is a strongly maximal filter of* 2, *which implies that every filter of* 2 *is strongly maximal.*

De Morgan's law can be expressed equivalently under the form

$$u \wedge v = 0 \Rightarrow \neg u \vee \neg v = 1$$

where $u, v \in \Omega$. But

$$\neg u \vee \neg v = 1 \Leftrightarrow \neg u = 1 \vee \neg v = 1 \Leftrightarrow u = 0 \vee v = 0.$$

So De Morgan's law is equivalent to the primeness of $0 \in \Omega$ and (1) follows from 2.2.3.

Strong De Morgan's law can be expressed equivalently under the form

$$u \wedge v \leq w \Rightarrow (u \Rightarrow w) \vee (v \Rightarrow w) = 1$$

where $u, v, w \in \Omega$. Since $\cdot \Rightarrow w$ is the pseudo-complement in the upper segment $\uparrow w$, strong De Morgan's law means that every element $w \in \Omega$ is prime and (2) follows from 2.2.3.

Ω is a boolean algebra precisely when $0 \in \Omega$ is widespread (cf. [8]), which implies that every upper segment $\uparrow w$ is a boolean algebra, thus every element $w \in \Omega$ is widespread. Thus (3) follows from 2.2.4. ∎

3. THE CONVERSE IMPLICATIONS.

In this paragraph, we study the relations between the properties involved in Proposition 1.3. This extends the result of [7].

PROPOSITION 3.1. *The following conditions are equivalent in a topos:*

(1) *The topos satisfies De Morgan's law.*

(2) *Every proper maximal filter of a distributive lattice is strongly prime.*

(3) *Every proper maximal filter of a distributive lattice is prime.*

(4) *Every proper maximal filter of a boolean algebra is strongly prime.*

(5) *Every proper maximal filter of a boolean algebra is prime.*

(6) *Every proper prime filter of a boolean algebra is strongly prime.*

(1) ⇒ (2): Let F be a proper maximal filter of the distributive lattice D and G, H two filters of D such that $G \cap H \subset F$. The following implications are valid, where $G \vee F$ denotes the supremum in the locale $F(D)$:

$$(0 \in G \vee F) \wedge (0 \in H \vee F) \Rightarrow 0 \in (G \vee F) \cap (H \vee F) = (G \cap H) \vee F = F \Rightarrow \text{false}.$$

Thus, by De Morgan's law

$$\neg(0 \in G \vee F) \vee \neg(0 \in H \vee F).$$

By maximality of F, we deduce

$$(G \vee F \subset F) \vee (H \vee F \subset F),$$

thus also $G \subset F \vee H \subset F$.

(2) ⇒ (3) ⇒ (5) and (2) ⇒ (4) ⇒ (5) are obvious (Propostion 1.3). (5) ⇒ (1) is proved in [7]. (1) ⇒ (6) follows from Proposition 1.3 and (1) ⇒ (2). Finally {1} is a proper prime filter of 2 (Proposition 2.2), thus (6) implies that it is a strongly prime filter and we conclude by 2.3.1. ■

PROPOSITION 3.2. *The following conditions are equivalent in a topos:*

(1) *The topos satisfies strong De Morgan's law.*

(2) *Every maximal filter of a distributive lattice is strongly*

prime.

(3) *Every maximal filter of a distributive lattice is prime.*

(4) *Every maximal filter of a boolean algebra is strongly prime.*

(5) *Every maximal filter of a boolean algebra is prime.*

(6) *Every prime filter of a boolean algebra is strongly prime.*

(1)$\Rightarrow$(2): Let F be a maximal filter of the distributive lattice D and G, H two filters of D such that $G \cap H \subset F$. The following implications are valid, where $G \vee F$ denotes the supremum in the locale $\mathcal{F}(D)$.

$$(0 \in G \vee F) \wedge (0 \in H \vee F) \Rightarrow 0 \in (G \vee F) \cap (H \vee F) = (G \cap H) \vee F = F.$$

Thus by strong De Morgan's law

$$(0 \in G \vee F \Rightarrow 0 \in F) \vee (0 \in H \vee F \Rightarrow 0 \in F).$$

By maximality of F, we deduce

$$(G \vee F \subset F) \vee (H \vee F \subset F)$$

thus also $G \subset F \vee H \subset F$.

(2)$\Rightarrow$(3)$\Rightarrow$(5) and (2)$\Rightarrow$(4)$\Rightarrow$(5) are obvious (Proposition 1.3). We now prove (5)$\Rightarrow$(1). One has $\mathcal{F}(2\times 2) = \mathcal{F}(2)\times \mathcal{F}(2)$ (cf. [1]), thus $\mathcal{F}(2\times 2) = \Omega\times\Omega$ (Proposition 2.1). For elements $u, v \in \Omega$ consider the element

$$(u \Rightarrow v, v \Rightarrow u) \in \Omega\times\Omega.$$

We first prove the maximality of the corresponding filter F of 2×2. Let us choose $(\alpha,\beta) \in \Omega\times\Omega$ which corresponds to a filter G of 2×2. The condition $G \supset F$ means

$$(\alpha,\beta) \geq (u \Rightarrow v, v \Rightarrow u)$$

while the condition $0 \in G \Rightarrow 0 \in F$ means

$$(\alpha,\beta) = (1,1) \Rightarrow (u \Rightarrow v, v \Rightarrow u) = (1,1).$$

Thus equivalently

$$\alpha \wedge \beta = 1 \Rightarrow (u \Rightarrow v) \wedge (v \Rightarrow u).$$

From these relations we deduce

$$\alpha \leq \beta \Rightarrow [(u \Rightarrow v) \wedge (v \Rightarrow u)] = [\beta \Rightarrow (u \Rightarrow v)] \wedge [\beta \Rightarrow (v \Rightarrow u)]$$
$$\leq [(v \Rightarrow u) \Rightarrow (u \Rightarrow v)] \wedge [(v \Rightarrow u) \Rightarrow (v \Rightarrow u)] = [(v \Rightarrow u) \Rightarrow (u \Rightarrow v)$$
$$= [(v \Rightarrow u) \wedge u] \Rightarrow v \leq (u \wedge u) \Rightarrow v = u \Rightarrow v.$$

Thus finally $\alpha = (u \Rightarrow v)$. In the same way $\beta = (v \Rightarrow u)$ and we have proved the required maximality. Applying (5), we deduce that F is a prime filter of 2×2. Since $(1,0) \vee (0,1) \in F$, we deduce

$$(1,0) \in F \vee (0,1) \in F \text{ , or equivalently, } (v \Rightarrow u) = 1 \vee (u \Rightarrow v) = 1$$

thus finally

$$(v \Rightarrow u) \vee (u \Rightarrow v) = 1$$

which is strong De Morgan's law.

Now, (1)⇒(6) follows from Proposition 1.3 and (1)⇒(2). Finally since every filter of 2 is prime (Proposition 2.2), (6) implies that every filter of 2 is strongly prime and we conclude by 2.3.2. ∎

PROPOSITION 3.3. *The following conditions are equivalent in a topos:*

(1) *The topos is boolean.*

(2) *Every prime filter of a distributive lattice is strongly prime.*

(3) *Every proper prime filter of a distributive lattice is strongly maximal.*

(4) *Every maximal filter of a distributive lattice is strongly maximal.*

(5) *Every proper maximal filter of a distributive lattice is strongly maximal.*

(6) *Every maximal filter of a boolean algebra is strongly maximal.*

(7) *Every proper maximal filter of a boolean algebra is strongly maximal.*

(8) *Every prime filter of a boolean algebra is strongly maximal.*

(9) *Every proper prime filter of a boolean algebra is strongly maximal.*

Clearly the classical proofs of conditions (2) to (7) carry over in a boolean topos. Moreover the implications

$$(2) \Rightarrow (3),\ (4) \Rightarrow (5) \Rightarrow (7) \Rightarrow (9),\ (6) \Rightarrow (8) \Rightarrow (9)$$

are obvious (cf. 1.3). So it suffices to prove (3)⇒(1) and (9)⇒(1).

For elements $u, v \in \Omega$,

$$(u \vee v = 1) \Leftrightarrow (u = 1 \vee v = 1)$$

so that (1) is a prime filter in Ω. As moreover $\neg(0 = 1)$ in Ω, $\{1\}$ is a prime filter of Ω. Assuming (3), $\{1\}$ is a strongly prime filter of Ω. For any element $u \in \Omega$ consider the two filters

$$G_u = \{ v \in \Omega \mid v \geq u \}, \quad H_u = \{ v \in \Omega \mid u \vee v = 1 \}.$$

Since $G_u \cap H_u = \{1\}$ we deduce $G_u = \{1\} \vee H_u = \{1\}$; hence, since $u = 1$ implies $0 \in H_u$

$$u = 1 \vee \neg(u = 1), \text{ thus finally } u \vee \neg u = 1$$

which proves the booleanness of Ω.

Now $\{1\}$ is a proper prime filter of 2 (Proposition 2.2); assuming (9), $\{1\}$ is a strongly maximal filter of 2 and we conclude by 2.3.3. ∎

PROPOSITION 3.4. (1) *In a topos, every proper strongly prime filter of a boolean algebra is strongly maximal iff the logical principle*

(De Morgan's law) ⇒ (Booleanity)

is valid.

(2) *In a topos, every strongly prime filter of a boolean algebra is strongly maximal iff every upper segment* $\uparrow u$ *of* Ω *satisfies the principle*

(De Morgan's law) ⇒ (Booleanity) .

Suppose every proper strongly prime filter of a boolean algebra is strongly maximal and De Morgan's law holds. $\{1\}$ is a proper prime filter of 2, thus a proper strongly prime filter (Proposition 3.1), thus a strongly maximal filter (hypothesis). Proposition 2.3.3 implies now the booleanity condition.

Conversely, assume the logical principle

(De Morgan's law) ⇒ (Booleanity) .

Given a proper strongly prime filter F in a boolean algebra B, its characteristic mapping $\varphi : B \to \Omega$ preserves 1 and $\wedge$ since F is a filter, 0 since F is proper and $\vee$ since F is prime (Proposition 1.3). Thus φ factors through 2 since B is a boolean algebra. Now choose elements $u, v \in \Omega$ such that $u \wedge v = 0$ and consider the corresponding filters G_u, G_v of 2 (Proposition 2.1). We deduce $G_u \cap G_v = \{1\}$ where the filter $\{1\}$ corresponds to $\varphi(0) \in \Omega$. Therefore $\varphi^{-1}(G_u) \cap \varphi^{-1}(G_v) = F$ and, since F is strongly prime

$$\varphi^{-1}(G_u) \subset F \vee \varphi^{-1}(G_v) \subset F.$$

This implies

$$u \leq \varphi(0) = 0 \vee v \leq \varphi(0) = 0.$$

That proves the primeness of $0 \in \Omega$, thus De Morgan's law (Proposition 2.3). The hypothesis implies the booleanity and we conclude by Propositions 1.3 and 3.3.

The case of arbitrary filters is analogous and left to the reader. It suffices to replace 2 by its quotient which identifies 0 and $\varphi(0)$ in Ω (notations of the first part of the proof - cf. [3] for more details). ■

It should be noticed that the logical principle in Proposition 3.4.2 is by no way equivalent to

$$\text{(strong De Morgan's law)} \Rightarrow \text{(Booleanity)} .$$

Both conditions in Proposition 3.4 are obviously satisfied in every boolean topos, but also in the topos of sheaves on the Cantor space since, in that case, De Morgan's law is equivalent to the false in every upper segment $\uparrow u$ of Ω (cf. [3]).

4. SOME EQUIVALENCES.

In this last paragraph, we extend a result of C.J. Mulvey (cf. [9]) whose philosophy is the existence of an equivalence between the notions of prime and maximal filter in a boolean algebra in an arbitrary topos ... when a careful choice is made in the formulation of those definitions. To avoid too heavy technicalities, we prove the results in the case of a proper filter and just mention the general results.

PROPOSITION 4.1. *For a proper filter* F *of a boolean algebra* B *in a topos, the following conditions are equivalent:*

(1) F *is maximal.*

(2) *For elements* $a, b \in \mathrm{B}$,

$$(a \vee b \in \mathrm{F}) \wedge \neg(a \in \mathrm{F}) \Rightarrow b \in \mathrm{F}.$$

(1)⇒(2): Assuming the hypothesis of (2) we prove first that $\mathrm{F} \vee \uparrow b$ is proper, where $\uparrow b$ denotes the upper segment of b in B and $\vee$ denotes the supremum in $F(\mathrm{B})$.

$$0 \in \mathrm{F} \vee \uparrow b \Rightarrow \exists c\ (c \in \mathrm{F} \wedge c \wedge b = 0) \Rightarrow \exists c\ (c \in \mathrm{F} \wedge c \leq b^*) \Rightarrow b^* \in \mathrm{F}$$
$$\Rightarrow (a \vee b) \wedge b^* \in \mathrm{F} \Rightarrow a \wedge b^* \in \mathrm{F} \Rightarrow a \in \mathrm{F} \Rightarrow \text{false}.$$

Using the maximality of F, we deduce $\mathrm{F} \vee \uparrow b \subset \mathrm{F}$, thus $b \in \mathrm{F}$.

(2) $\Rightarrow$ (1): Conversely assume (2) and choose a proper filter G which contains F. For any $b \in G$ we have, putting $a = b^*$,

$$b^* \vee b = 1 \in F, \quad b^* \in F \Rightarrow b^* \in G \Rightarrow 0 = b \wedge b^* \in G \Rightarrow \text{false}.$$

Applying (2), we obtain $b \in F$. ∎

In the case of an arbitrary filter F, the maximality of F is equivalent to the primeness condition

$$(a \vee b \in F) \wedge (a \in F \Rightarrow 0 \in F) \Rightarrow (b \in F)$$

for elements a, b of B.

PROPOSITION 4.2. *For a proper filter* F *in a boolean algebra* B *in a topos, the following conditions are equivalent:*

(1) F *is prime.*

(2) *For* $G \in F(B)$ *and* $a \in B$,

$$(G \subset F \wedge a \in G) \Rightarrow (a \in F \vee 0 \in G).$$

Cf. [9]. ∎

Proposition 4.2 carries over without any change in the case of an arbitrary filter F.

PROPOSITION 4.3. *For a proper filter* F *in a boolean algebra* B *in a topos, the following conditions are equivalent:*

(1) F *is strongly prime.*

(2) *For every* $G \in F(B)$,

$$G \subset F \Rightarrow G = F \vee \neg\neg(0 \in G).$$

(1)$\Rightarrow$(2): $F(B)$ is a locale (cf. [1]), thus for a filter $G \supset F$, we can consider the filter:

$$G \Rightarrow F = \{a \in B \mid \forall b \in G \;\; a \vee b \in F\}.$$

From the relation $G \cap (G \Rightarrow F) \subset F$ and the primeness of F, we deduce

$$G \subset F \vee (G \Rightarrow F) \subset F.$$

$G \subset F$ implies $G = F$ since $G \supset F$.

Now consider the condition $(G \Rightarrow F) \subset F$ which means

$$(\forall b \in G \;\; a \vee b \in F) \Rightarrow (a \in F)$$

for every $a \in B$. In particular when $a = 0$ we obtain $G \subset F \Rightarrow 0 \in F$, or in other words $\neg(G \subset F)$ since F is proper. It remains to prove

$$\neg(G \subset F) \Rightarrow \neg\neg(0 \in G)$$

or equivalently

$$\neg(G \subset F) \wedge \neg(0 \in G) \Rightarrow \text{false}.$$

But F is prime (Proposition 1.3) and $G \supset F$; thus (Proposition 4.2)

$$a \in G \Rightarrow (a \in F \vee 0 \in G).$$

Assuming $\neg(0 \in G)$ we deduce $a \in G \Rightarrow a \in F$; thus $G \subset F$; so assuming also $\neg(G \subset F)$ we obtain the false.

(2)$\Rightarrow$(1): Now suppose F satisfies condition (2) and choose filters G, H such that $G \cap H \subset F$. Applying (2) to the filters $G \vee F$ and $H \vee F$, we obtain

$$G \vee F = F \vee \neg\neg(0 \in G \vee F), \quad H \vee F = F \vee \neg\neg(0 \in H \vee F).$$

Combining those two relations we find

$$G \subset F \vee H \subset F \vee (\neg\neg(0 \in G \vee F) \wedge \neg\neg(0 \in H \vee F)).$$

Finally we deduce, since $F(B)$ is a locale, $G \cap H \subset F$ and F is proper,

$$\neg\neg(0 \in G \vee F) \wedge \neg\neg(0 \in H \vee F) \Rightarrow \neg\neg(0 \in (G \vee F) \cap (H \vee F))$$
$$\Rightarrow \neg\neg(0 \in F) \Rightarrow \text{false}. \blacksquare$$

In the case of an arbitrary filter F, the strong primeness of F is equivalent to the maximality condition

$$G \supset F \Rightarrow G = F \vee ((0 \in G \Rightarrow 0 \in F) \Rightarrow 0 \in F)$$

for every filter $G \in F(B)$.

PROPOSITION 4.4. *For a proper filter* F *in a boolean algebra* B *in an topos, the following conditions are equivalent:*

(1) F *is strongly maximal.*

(2) *For every filters* $G, H \in F(B)$,

$$(G \subset F) \vee (H \subset F) \vee \exists b (b \in G \cap H \wedge \neg(b \in F)).$$

Suppose F strongly maximal. We deduce, for arbitrary filters G, H:

$$(G \vee F \subset F) \vee (0 \in G \cap F), \quad (H \vee F \subset F) \vee (0 \in H \vee F).$$

Combining those relations, we obtain

$$(G\subset F)\vee(H\subset F)\vee(0\in(G\cap H)\vee F)\ .$$

We conclude by noticing that, since F is proper

$$\begin{aligned} 0\in(G\cap H)\vee F &\Rightarrow \exists a\in F\ \exists\ b\in G\cap H\ \ a\wedge b=0 \\ &\Rightarrow \exists\, b\in G\cap H\ \vee\ \neg(b\in F)\ . \end{aligned}$$

Conversely suppose F satisfies (2). For each element $a\in B$, choosing $G_a=\uparrow a$ and $H_a=\uparrow a^*$ we deduce, applying (2),

$$a\in F\ \vee\ a^*\in F\ \vee\ \text{false},$$

which implies that F is an ultrafilter. Now for a filter $G\supset F$, let us apply (2) to G and $H=B$; we obtain

$$G\subset F\ \vee\ B\subset F\ \vee\ \exists b(b\in G\wedge\neg(b\in F)).$$

$B\subset F$ implies obviously $G\subset F$. On the other hand since F is a proper ultrafilter contained in G,

$$\begin{aligned} \exists b\ (b\in G\wedge\neg(b\in F)) &\Rightarrow \exists b\ (b\in G\wedge b^*\in F) \\ &\Rightarrow \exists b\ (b\in G\wedge b^*\in G)\Rightarrow\exists b\ (0\in G)\Rightarrow 0\in G. \end{aligned}$$ ∎

In the case of an arbitrary filter F, the strong maximality of F is equivalent to the primeness condition

$$(G\subset F)\vee(H\subset F)\vee\exists b(b\in G\cap H\wedge(b\in F\Rightarrow 0\in F))$$

for arbitrary filters $G,H\in \mathcal{F}(B)$.

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