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# COMPLETIONS OF CATEGORIES AND INITIAL COMPLETIONS

*by Walter THOLEN and Anna TOZZI* [1]

RÉSUMÉ. Des pro-catégories généralisées sont utilisées pour établir une théorie de complétion dans **CAT**/$X$ qui, d'une part, englobe la théorie de complétion pour les catégories abstraites ($X = 1$) et, d'autre part, étend les résultats sur les foncteurs fidèles (complétions initiales, complétions concrètes).

## INTRODUCTION.

When completing a metric space $X$ one has, in principle, two different procedures: one may find the completion $\overline{X}$ as the closure of $X$ in some complete extension of $X$ (the bounded real-valued functions on $X$, say), or one describes the points of $\overline{X}$ more directly as equivalence classes of Cauchy sequences in $X$, thus just collecting the data to which one wished limits to exist and identifying them whenever they give the same limit. By the first method one obtains a quick existence proof, but the second description provides a safer feeling of what the completion really is. How constructive each method is depends on the closure process in the first case and on the identification process in the second case.

The situation with respect to completions of categories is very similar: to find a completion $\overline{K}$ of a category $K$ with respect to a certain type of limits, most people have tried to present $\overline{K}$ as a full subcategory of the dual of the (complete) ca-

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tegory $[K,\mathbf{Set}]$ of **Set**-valued presheaves on $K$ (cf. Lambek [L], Isbell [I1, I2], Kennison [Kn], Gabriel-Ulmer [GU], and for good overviews Popescu-Popescu [P], Kelly [K], Adámek [A]). This way, existence of $\overline{K}$ at least in a higher universe becomes a triviality, and in special cases its description is very simple since one just has to consider those functors in $[K,\mathbf{Set}]$ which preserve certain limits (cf. [A]). But in general, a highly non-constructive closure process is needed, and legitimacy of $K$ (that is the problem whether $\overline{K}$ belongs to the same universe as $K$) remains difficult to decide.

The second method has been followed by Trnková [Tr1, Tr2], adding new data to $K$ by a transfinite process. In the case of cofiltered limits (filtered colimits, dually), there is, however, the classical one-step construction of the pro-category of $K$, due to Grothendieck and Verdier [GV]. This method has been extended by the first author [Th4] to more general diagram schemes, and our first objective in the present paper is to present some improvements of results obtained in [Th1, Th2, Th4]. *So, without any iteration process we show that generalized pro-categories serve as universal completions within the same universe with respect to fairly general (but always small) diagram schemes.* Not only the classical pro-categories, but also Lambek's completion as well as the formal product completion appear as special instances (cf. Sections 1, 2, 5).

In fact, our present approach is much more general as we investigate different sorts of "universal completions" of arbitrary functors $\mathrm{U}\colon A\to X$ in a way that completions of categories appear as the special case that X is the terminal category $\mathbf{1}$. So, our constructions take place in $\mathbf{CAT}/X$, not just $\mathbf{CAT}$. More precisely, we shall construct a universal extension

$$\begin{tikzcd} A \arrow[rr, "\mathrm{E}"] \arrow[dr, "\mathrm{U}"'] & & \mathrm{Pro}(\mathbf{D},\Gamma,\mathrm{U}) \arrow[dl, "\mathrm{T}"] \\ & X & \end{tikzcd}$$

such that T admits *initial liftings* for all T-cones of a given type $\Gamma$ (for instance: limit-cones, mono-cones, arbitrary cones) over a fixed type $\mathbf{D}$ of small diagram schemes (for instance filtered diagrams, discrete diagrams, arbitrary small diagrams). The choice

$$\mathbf{D} = \text{small diagrams and } \Gamma = \text{limit cones}$$

shows that, if $X$ is complete, U admits a universal extension to a continuous functor with complete domain. The case

$$\mathbf{D} = \text{small diagrams and } \Gamma = \text{arbitrary cones}$$

shows that one can associate with U a functor T which is universal with respect to the property that initial lifts to all small data exist (cf. Section 4).

We mention the last two cases explicitly as they demonstrate best that we provide an abstract counter-part of what has been done for concrete functors before: C. Ehresmann [E] and A. Charles Ehresmann [CE] have shown that the theory of concrete (limit-)completions on the one hand (cf. Adámek-Koubek [AK], Herrlich [H2]) and of initial completions on the other hand (cf. Brümer [B], Hoffmann [Ho], Wyler [Wy], Wischnewsky [Wi] for intiality, and Herrlich [H1], Adámek-Herrlich-Strecker [AHS] for completions) admit a common generalized description. All these constructions, however, take place in the category $\mathbf{CAT}_{ff}/X$ of *faithful* functors into $X$ and are, in general, not universal in $\mathbf{CAT}/X$. Nevertheless, once one has established a completion theory in $\mathbf{CAT}/X$ it is fairly easy to see which modifications are needed to get to initial completions (cf. Section 6).

Perhaps the most surprising observation in this paper is that the category $\mathrm{Pro}(\mathbf{D},\Gamma,\mathrm{U})$ lives not only over $X$ but also over the generalized pro-category $\mathrm{Pro}(\mathbf{D},A)$ from which it may inherit certain cocompleteness properties. The reason for this is the existence of a functor

$$\mathrm{S}\colon \mathrm{Pro}(\mathbf{D},\Gamma,\mathrm{U}) \longrightarrow \mathrm{Pro}(\mathbf{D},A)$$

which behaves somehow dually to T and admits certain final liftings (cf. Sections 3,4).

Which properties of the class $\mathbf{D}$ are needed to obtain such a smooth and constructive completion theory is studied in more detail by an embedding of $\mathrm{Pro}(\mathbf{D},K)$ into $\mathbf{CAT}/K^{op}$. It turns out that $\mathrm{Pro}(\mathbf{D},K)$ inherits good properties from $\mathbf{CAT}/K^{op}$ whenever the conglomerate of all categories $J$ for which there is a confinal functor $I \to J$ with some $I \in \mathbf{D}$ is closed under colimits of type $\mathbf{D}$ in $\mathbf{CAT}$ (cf. Section 5).

In this paper we distinguish between (small) sets, classes and conglomerates; sets are elements of classes, classes are elements of conglomerates. **Set** and **Cat** denote the (legitimate) categories of (small) sets and small categories respectively, whereas **SET** and **CAT** denote the (illegitimate) categories of classes and all (illegitimate) categories respectively. Every category (in **CAT**) is assumed to have small hom-sets. Finally, we emphasize that our list of references can contain only a small selection of articles on completions of categories and initial completions.

## 1. REVIEW OF GENERALIZED PRO-CATEGORIES AND INITIALITY.

For a class $\mathbf{D}$ of small categories such that the terminal category $\mathbf{1}$ belongs to $\mathbf{D}$, and any category $K$ one defines the generalized pro-category $\mathrm{Pro}(\mathbf{D},K)$ as in [Th4]: objects are functors $\mathbf{X}$: $I^{\mathrm{op}} \to K$ with $I \in \mathbf{D}$; one writes $\mathbf{X} = (X_i)_{i \in \mathrm{Ob} I}$ (where $X_i$ is the value $\mathbf{X}i$); for a morphism $\nu$: $i \to i'$ in $I$ its value under $\mathbf{X}$ is again denoted by $\nu$: $X_{i'} \to X_i$. A morphism $\boldsymbol{f}$: $\mathbf{X} \to \mathbf{Y} = (Y_j)_{j \in \mathrm{Ob} J}$ in $\mathrm{Pro}(\mathbf{D},K)$ is a family $(\boldsymbol{f}_j)_{j \in \mathrm{Ob} J}$ where each $\boldsymbol{f}_j$ is an equivalence class of the smallest equivalence relation $\sim_j$ on $\sum_{i \in \mathrm{Ob} I} K(X_i, Y_j)$ such that

$$(f,i) \sim_j (f\nu, i') \text{ for all } f\text{: } X_i \to Y_j \text{ and } \nu\text{: } i \to i';$$

the family $\boldsymbol{f}$ must satisfy the coherence condition

$$(\mu f, i) \in \boldsymbol{f}_{j'} \text{ for all } (f,i) \in \boldsymbol{f}_j \text{ and } \mu\text{: } j' \to j.$$

For $\boldsymbol{g}$: $\mathbf{Y} \to \mathbf{Z} = (Z_n)_{n \in \mathrm{Ob} N}$, the composition $\boldsymbol{gf} = \boldsymbol{h} = (\boldsymbol{h}_n)_{n \in \mathrm{Ob} N}$ is defined by

$$\boldsymbol{h}_n = \{(h,i) \mid \exists (g,j) \in \boldsymbol{g}_n,\ (f,i') \in \boldsymbol{f}_j\text{: } (h,i) \sim_n (gf,i')\}.$$

Every $K$-object $X$ may be considered as a $\mathbf{1}$-indexed family, so one has a full embedding $\mathrm{I}_K$: $K \to \mathrm{Pro}(\mathbf{D},K)$.

Without further assumption on $\mathbf{D}$ one has

**1.1. PROPOSITION.** *Every* $\mathrm{Pro}(\mathbf{D},K)$*-object* $\mathbf{X}$ *is a limit of* $K$*-objects. more precisely:* $\mathbf{X}$ *is the limit of the functor* $\mathrm{I}_K\mathbf{X}$: $I^{\mathrm{op}} \to \mathrm{Pro}(\mathbf{D},K)$.

In fact, the $\mathrm{Pro}(\mathbf{D},K)$-morphisms $\xi_i : \mathbf{X} \to X_i$ whose only equivalence class is represented by $(1_{X_i}, i)$ form an $I^{op}$-indexed cone with vertex $\mathbf{X}$ which is easily seen to be a limit cone. ∎

We list further well-known properties of $I_K$ the first of which formally follows from 1.1:

**1.2. PROPOSITION.** *$I_K$ preserves all (existing) colimits of $X$ but in general does not preserve the existing limits of $X$. However, $I_K$ does preserve limits of type $D$ if $D$ is a category such that, in* **SET**, *limits of type $D$ commute with colimits of type $I$ for all $I \in \mathbf{D}$.* ∎

**1.3.** Recall that a functor $F: I \to J$ is *confinal* if for all $j \in \mathrm{Ob}\, J$, the comma category $(j \downarrow F)$ is (not empty and) connected; these are exactly the functors which leave colimits in any category $K$ invariant, that is: $\mathrm{colim}\, H$ with $H: J \to K$ exists iff $\mathrm{colim}\, HF$ exists, and in that case both colimits coincide (up to isomorphism). Therefore. if $\mathbf{D}' \supset \mathbf{D}$ has the property that every $J \in \mathbf{D}'$ admits a confinal functor $F: I \to J$ with $I \in \mathbf{D}$, the full embedding

$$\mathrm{Pro}(\mathbf{D},K) \subset \mathrm{Pro}(\mathbf{D}',K)$$

is actually an equivalence (since, for every $\mathbf{X} \in \mathrm{Ob}(\mathrm{Pro}(\mathbf{D}',K))$, 1.1 gives

$$\mathbf{X}F^{op} \approx \lim(I_K \mathbf{X} F^{op}) \approx \lim(I_K \mathbf{X}) \approx \mathbf{X}$$

in $\mathrm{Pro}(\mathbf{D}',K)$). This remains true even if $J$ is not a small category, so $\mathrm{Pro}(\mathbf{D}',K)$ will live in a higher universe, but will be equivalent to the legitimate category $\mathrm{Pro}(\mathbf{D},K)$.

For the class $\mathbf{D}$ of small categories we denote by $\bar{\mathbf{D}}$ the class of all small categories $J$ which admit a confinal functor $I \to J$ with $I \in \mathbf{D}$.

$\mathbf{D}$-*completeness* (that is, $I^{op}$-completeness for every $I \in \mathbf{D}$) of $\mathrm{Pro}(\mathbf{D}, K)$ can only be shown under an additional hypothesis on $\mathbf{D}$ which we recall from [Th4]: Every diagram

$$H: J^{op} \to \mathrm{Pro}(\mathbf{D},K) \text{ with } J \in \mathbf{D}$$

gives $\mathrm{Pro}(\mathbf{D},K)$-objects $Hj = \mathbf{X}^j = (X_i^j)_{i \in \mathrm{Ob}\, I_j}$ and $\mathrm{Pro}(\mathbf{D},K)$-morphisms

$$f^{\mu} = (f_i^{\mu})_{i \in \mathrm{Ob}I_j} : \mathbf{X}^{j'} \longrightarrow \mathbf{X}^j \text{ for all } \mu\colon j \to j' \text{ in } J.$$

The *related category* $\hat{\mathrm{H}}$ of H has as its objects pairs $(i,j)$ with $j \in \mathrm{Ob}J$, $i \in \mathrm{Ob}I_j$, and as its morphisms pairs $(f,\mu)\colon(i,j) \to (i',j')$ with $\mu\colon j \to j'$ in $J$ and $f\colon X_i^{j'} \to X_i^j$ in $K$ such that $(f, i') \in f_i^{\mu}$; composition is pointwise. $\mathbf{D}$ is called *admissible with respect to* $K$ if, for every H as above, the related category $\hat{\mathrm{H}}$ belongs to $\bar{\mathbf{D}}$ (in [Th4], mistakenly, the more restrictive condition $\hat{\mathrm{H}} \in \mathbf{D}$ was imposed). In this case one has a $\mathrm{Pro}(\bar{\mathbf{D}}, K)$-object

$$\mathrm{L}\colon \hat{\mathrm{H}}^{\mathrm{op}} \longrightarrow K, (f,\mu) \mapsto f,$$

which serves as a limit of H in $\mathrm{Pro}(\bar{\mathbf{D}}, K) \approx \mathrm{Pro}(\mathbf{D},K)$. The proof of the following theorem is now straightforward (cf. [Th4]):

**1.4. THEOREM.** *Let* $\mathbf{D}$ *be admissible with respect to* $K$. *Then* $\mathrm{Pro}(\mathbf{D}, K)$ *is* $\mathbf{D}$*-complete, and every functor* $K \to L$ *into a* $\mathbf{D}$*-complete category* $L$ *admits an extension* $\mathrm{Pro}(\mathbf{D},K) \to L$ *which preserves limits of type* $I^{\mathrm{op}}$, $I \in \mathbf{D}$, *and which is uniquely determined by this property, up to natural equivalence.* G *is in fact, together with the identity* $1\colon \mathrm{GI}_K \to \mathrm{F}$, *a right Kan extension of* F *along* $\mathrm{I}_K$. ∎

**REMARKS.** (1) The universal property, of course, determines $\mathrm{Pro}(\mathbf{D}, K)$ up to equivalence of categories. But note that, in general, $\mathrm{Pro}(\mathbf{D}, K)$ is not equivalent to $K$ if $K$ is already $\mathbf{D}$-complete (since the embedding need not preserve the respective limits); one just has that $K$ is a coreflective subcategory of $\mathrm{Pro}(\mathbf{D},K)$ in this case.

(2) From the presentation of G as a right Kan extension one obtains that the functor category $[K,L]$ is equivalent to the category of functors $\mathrm{Pro}(\mathbf{D},K) \to L$ which are $\mathbf{D}$-*continuous*, i.e., preserve $I^{\mathrm{op}}$-limits for every $I \in \mathbf{D}$.

On the existence of colimits one has the following non-trivial result by Weberpals [We3] which generalizes results by Grothendieck and Verdier [GV] and by Artin and Mazur [AM]:

**1.5. THEOREM.** *Let* $\mathbf{D}$ *be regular in the sense of Gabriel and Ulmer* [GU, We2]. *Then, if* $K$ *is* $\mathrm{L}(\mathbf{D})$*-cocomplete, also* $\mathrm{Pro}(\mathbf{D},K)$

*is; here* L(**D**) *is the conglomerate of categories D such that, in* **SET**, *D-limits commute with I-colimits for all* $I \in$ **D**.

This result is proved in [We3] in the dual situation, that is for the category

$$\mathrm{Ind}(\mathbf{D},K) = (\mathrm{Pro}(\mathbf{D}^{\mathrm{op}},K^{\mathrm{op}}))^{\mathrm{op}} \text{ with } \mathbf{D}^{\mathrm{op}} = \{I \mid I^{\mathrm{op}} \in \mathbf{D}\}. \ \blacksquare$$

Finally, we recall some phrases concerning initiality and topological functors and adapt them to the present context. For a functor $\mathrm{U}: A \to X$, a morphism $f: \mathrm{A} \to \mathrm{B}$ in $A$ is U-*initial* if, for every $g: \mathrm{C} \to \mathrm{B}$ in $A$ and $u: \mathrm{UC} \to \mathrm{UA}$ in $X$ with $\mathrm{U}f \cdot u = g$, there is a unique morphism $h: \mathrm{C} \to \mathrm{A}$ in $A$ with $\mathrm{U}h = u$ and $fh = g$. A morphism $x: \mathrm{X} \to \mathrm{UB}$ in $X$ (considered as an object of the comma-category $(X \downarrow \mathrm{U})$) will be called a U-*morphism*; a U-initial morphism $f: \mathrm{A} \to \mathrm{B}$ with $\mathrm{U}f = x$ is a U-*initial lifting of* $x$. Such a lifting is unique up to isomorphism; it is unique if U is *amnestic*, that is: if $\mathrm{U}h = 1$ with an $A$-isomorphism $h$ implies $h = 1$.

These notions can be used more generally for *cones over* **D** *in* $A$, i.e., $\mathrm{Pro}(\mathbf{D},A)$-morphisms $\boldsymbol{f}: \mathrm{A} \to \mathbf{B}$ with domain in $A$; such a cone is called U-*initial* if it is a $\bar{\mathrm{U}}$-initial morphism in the category $\mathrm{Pro}(\mathbf{D},K)$; here

$$\bar{\mathrm{U}}: \mathrm{Pro}(\mathbf{D},A) \longrightarrow \mathrm{Pro}(\mathbf{D},X)$$

is the canonical extension of U (for $\mathbf{A} \in \mathrm{Pro}(\mathbf{D},A)$, $\bar{\mathrm{U}}\mathbf{A}$ is the composition $\mathrm{U}\mathbf{A}$ of functors). Similarly, a U-*cone over* **D** is a $\bar{\mathrm{U}}$-morphism $\boldsymbol{x}: \mathrm{X} \to \bar{\mathrm{U}}\mathbf{A}$ with $\mathrm{X} \in \mathrm{Ob}X$, and a U-initial $\boldsymbol{f}$ is a U-*initial lifting of* $\boldsymbol{x}$ if $\bar{\mathrm{U}}\boldsymbol{f} = \boldsymbol{x}$. Dual notions: U-*comorphisms*, U-*cocone*, U-*final lifting*.

For a class $\Gamma$ of cones in $X$, the functor $\mathrm{U}: A \to X$ is called $\Gamma$-*topological* if every U-cone in $\Gamma$ admits a U-initial lifting. Since a U-initial lifting of a limit cone is a limit cone, one has:

**1.6. PROPOSITION.** *If, for a* $\Gamma$-*topological functor* $\mathrm{U}: A \to X$, *the category* $X$ *is* **D**-*complete and* $\Gamma$ *contains all corresponding limit cones, then also* $A$ *is* **D**-*complete and* U *preserves the respective limits.* $\blacksquare$

In the dual situation, for a class $\Delta$ of cocones in $X$, U will be called $\Delta$-*cotopological*.

## 2. Pro($\mathbf{D}$,$\Gamma$,U) AS A CATEGORY OVER $\mathcal{X}$.

Throughout Sections 2-5 $\mathrm{U}: \mathcal{A} \to \mathcal{X}$ is a functor, $\mathbf{D}$ a class of small categories with $\mathbf{1} \in \mathbf{D}$, and $\Gamma$ is a class of cones over $\mathbf{D}$ in $\mathcal{X}$ such that, for all $\mathrm{A} \in \mathrm{Ob}\mathcal{A}$, the $\mathbf{1}$-indexed cone $1_{\mathrm{UA}}: \mathrm{UA} \to \mathrm{UA}$ belongs to $\Gamma$.

The category Pro($\mathbf{D}$,$\Gamma$,U) has as its objects all U-cones in $\Gamma$, i.e., triples $(\mathrm{X}, \boldsymbol{x}, \mathbf{A})$ with

$$\mathrm{X} \in \mathrm{Ob}\mathcal{X},\ \mathbf{A} \in \mathrm{Ob}(\mathrm{Pro}(\mathbf{D},\mathcal{A})),\ \text{and}\ \boldsymbol{x}: \mathrm{X} \to \bar{\mathrm{U}}\mathbf{A}\ \text{in}\ \Gamma;$$

morphisms $(u, \boldsymbol{f}): (\mathrm{X}, \boldsymbol{x}, \mathbf{A}) \to (\mathrm{Y}, \boldsymbol{y}, \mathbf{B})$ consist of an $\mathcal{X}$-morphism $u: \mathrm{X} \to \mathrm{Y}$ and a Pro($\mathbf{D}$,$\mathcal{A}$)-morphism $\boldsymbol{f}: \mathbf{A} \to \mathbf{B}$ such that

$$\begin{tikzcd}
\mathrm{X} \arrow[r, "u"] \arrow[d, "\boldsymbol{x}"'] & \mathrm{Y} \arrow[d, "\boldsymbol{y}"] \\
\bar{\mathrm{U}}\mathbf{A} \arrow[r, "\bar{\mathrm{U}}\boldsymbol{f}"] & \bar{\mathrm{U}}\mathbf{B}
\end{tikzcd}$$

commutes; this means that, with

$$\mathbf{A} = (\mathrm{A}_i)_{i \in \mathrm{Ob}I} \text{ and } \mathbf{B} = (\mathrm{B}_j)_{j \in \mathrm{Ob}J}$$

for every $j \in \mathrm{Ob}J$ there exist $i_j \in \mathrm{Ob}I$ and $f: \mathrm{A}_{i_j} \to \mathrm{B}_j$ with $(f, i_j) \in \boldsymbol{f}_j$ such that

$$\begin{tikzcd}
\mathrm{X} \arrow[r, "u"] \arrow[d, "x_{i_j}"'] & \mathrm{Y} \arrow[d, "y_j"] \\
\mathrm{UA}_{i_j} \arrow[r, "\mathrm{U}f"] & \mathrm{UB}_j
\end{tikzcd}$$

commutes.

Pro($\mathbf{D}$,$\Gamma$,U) is a full subcategory of the comma category $(\mathrm{I}_{\mathcal{X}} \downarrow \bar{\mathrm{U}})$ and, as such, admits two projections

$$\mathcal{X} \xleftarrow{\mathrm{T}} \mathrm{Pro}(\mathbf{D},\Gamma,\mathrm{U}) \xrightarrow{\mathrm{S}} \mathrm{Pro}(\mathbf{D},\mathcal{A})\ ;$$

also there is a natural transformation

$$\Sigma: \mathrm{I}_{\mathcal{X}} \to \bar{\mathrm{U}}\mathrm{S},\ \text{defined by}\ \Sigma_{(\mathrm{X},\boldsymbol{x},\mathbf{A})} = \boldsymbol{x}.$$

Finally, there is a full embedding $E: \mathcal{A} \to \mathrm{Pro}(\mathbf{D},\Gamma,U)$ with

$$TE = U \ , \ SE = I_{\mathcal{A}} \text{ and } \Sigma E = 1_{I_{\mathcal{X}}U}$$

(hence $EA = (UA, 1_{UA}, A)$). We emphasize:

**REMARK**. In general, T is not faithful, even when U is faithful.

If the empty category $\emptyset$ belongs to $\mathbf{D}$, then $\mathrm{Pro}(\mathbf{D},\mathcal{A})$ has a terminal object which, again, is denoted by $\emptyset$ ; trivially, it is preserved by $\bar{U}$. For every $X \in \mathrm{Ob}\mathcal{X}$, the only cone $X \to \emptyset$ is called *empty*.

**2.1. PROPOSITION**. (1) *If* $\Gamma$ *contains all empty cones, then* T *has a full and faithful right adjoint and, therefore, preserves all colimits.*

(2) *If* U *has a left adjoint such that* $\Gamma$ *contains all units of the adjunction (as* **1***-indexed cones), then* T *has a full and faithful left adjoint and, therefore, preserves all limits.*

**PROOF.** (1) A T-couniversal arrow for $X \in \mathrm{Ob}\mathcal{X}$ is given by

$$1_X: T(X, X \to \emptyset, \emptyset) \longrightarrow X.$$

(2) A T-universal arrow for $X \in \mathrm{Ob}\mathcal{X}$ is given by

$$1_X: X \longrightarrow T(X, \eta_X: X \to UA, A),$$

with a U-universal arrow $\eta_X: X \to UA$. ∎

Assertion (1) of 2.1 is a special case of a more general result: in 2.3 we shall show that T is $\Gamma$-topological, provided $\Gamma$ satisfies a certain closedness property. For this let $\varphi: X \to \bar{T}H$ $(= TH)$ be a T-cone over $\mathbf{D}$ (so $H \in \mathrm{Pro}(\mathbf{D}, \mathrm{Pro}(\mathbf{D},\Gamma,U))$); if $\mathbf{D}$ is admissible with respect to $\mathcal{A}$ and $\mathcal{X}$, one has a limit cone $\Lambda : \mathbf{L} \to \bar{S}H$ in $\mathrm{Pro}(\mathbf{D},\mathcal{A})$ which is preserved by $\bar{U}$ (by 1.4). Then the unique $\mathrm{Pro}(\mathbf{D},\mathcal{X})$-morphism $\boldsymbol{x}$: $X \to \bar{U}\mathbf{L}$ rendering the diagram

$$\begin{tikzcd} I_{\mathcal{X}}X \arrow[r, "u"] \arrow[d, "\boldsymbol{x}"'] & I_{\mathcal{X}}TH \arrow[d, "\Sigma H"] \\ \bar{U}\mathbf{L} \arrow[r, "\bar{U}\Lambda"'] & \bar{U}SH \end{tikzcd} \qquad (1)$$

commutative, is said to be *induced by* $\varphi$.

**2.2. DEFINITION.** $\Gamma$ is *admissible with respect to* $\mathrm{U}$ if $\mathbf{D}$ is admissible with respect to $A$ and $X$, and if for every T-cone $\varphi$ over $\mathbf{D}$ which, as a cone in $X$, belongs to $\Gamma$ also the induced cone $\boldsymbol{x}$ belongs to $\Gamma$.

Admissibility of $\Gamma$ is discussed further in Section 4. Here we first prove:

**2.3. PROPOSITION.** $\mathrm{T}: \mathrm{Pro}(\mathbf{D},\Gamma,\mathrm{U}) \to X$ *is* $\Gamma$*-topological if* $\Gamma$ *is admissible with respect to* $\mathrm{U}$.

**PROOF.** For the given T-cone $\varphi$ in $\Gamma$ we obtain a $\mathrm{Pro}(\mathbf{D},\Gamma,\mathrm{U})$-object $(\mathrm{X},\boldsymbol{x},\mathbf{L})$. Then $\varphi$ and $\Lambda$ (as above) constitute a cone $\Phi$: $(\mathrm{X},\boldsymbol{x},\mathbf{L}) \to \mathrm{H}$ in $\mathrm{Pro}(\mathbf{D},\Gamma,\mathrm{U})$ which turns out to be T-initial; suppose we are given a cone $\Psi: (\mathrm{Y},\boldsymbol{y},\mathbf{B}) \to \mathrm{H}$ and an $X$-morphism $w: \mathrm{Y} \to \mathrm{X}$ with $\varphi w = \mathrm{T}\Psi$. Then, by the limit property of $\Lambda$, there is a unique $\mathrm{Pro}(\mathbf{D},A)$-morphism $\boldsymbol{g}: \mathbf{B} \to \mathbf{L}$ with $\Lambda \boldsymbol{g} = \mathrm{S}\Psi$. Since the limit is preserved by $\bar{\mathrm{U}}$, one easily checks that

$$(w, \boldsymbol{g}): (\mathrm{Y}, \boldsymbol{y}, \mathbf{B}) \longrightarrow (\mathrm{X}, \boldsymbol{x}, \mathbf{L})$$

is a morphism in $\mathrm{Pro}(\mathbf{D},\Gamma,\mathrm{U})$, and trivially it is the only one over $w$ with $\Phi(w,\boldsymbol{g}) = \Psi$. ∎

**2.4. PROPOSITION.** *If* $\mathbf{D}$ *is admissible with respect to* $A$ *and* $X$, *then the full embedding* $\mathrm{E}: A \to \mathrm{Pro}(\mathbf{D},\Gamma,\mathrm{U})$ *is initially* $\Gamma$*-dense, that is: every* $\mathrm{Pro}(\mathbf{D}, \Gamma,\mathrm{U})$*-object is the vertex of a* T*-initial cone with base in* $A$ *and* T*-image in* $\Gamma$. *Also,* E *transforms all* U*-final cocones into* T*-final cocones.*

**PROOF.** Every $\mathrm{Pro}(\mathbf{D},\Gamma,\mathrm{U})$-object $(\mathrm{X},\boldsymbol{x},\mathbf{A})$ gives a cone

$$\Phi = \Phi_{(\mathrm{X},\boldsymbol{x},\mathbf{A})}: (\mathrm{X}, \boldsymbol{x}, \mathbf{A}) \longrightarrow \bar{\mathrm{E}}\mathbf{A} = \mathrm{E}\mathbf{A}$$

in $\mathrm{Pro}(\mathbf{D},\Gamma,\mathrm{U})$ with $\mathrm{T}\Phi = \boldsymbol{x}$ and $\mathrm{S}\Phi: \mathbf{A} \to \mathrm{I}_A\mathbf{A}$ the limit presentation of $\mathbf{A}$ (see 1.1). T-initiality of $\Phi$ is shown as in 2.3.

For the preservation of finality, let $\varphi: \mathrm{H} \to \mathrm{A}$ be a U-final cocone in $A$ with $\mathrm{H}: D \to A$ ($D \in \mathbf{D}$ is *not* required here). Since we must show that $\mathrm{E}\varphi$ is T-final, let us consider a cocone $\Psi$:

$\mathrm{E}H \to (\mathrm{Y},\boldsymbol{y},\mathbf{B})$ with $\mathbf{B} = (\mathrm{B}_j)_{j \in \mathrm{Ob}J}$ and an $X$-morphism

$$u : \mathrm{UA} \to \mathrm{Y} \text{ with } u \cdot \mathrm{TE}\varphi = \mathrm{T}\Psi .$$

For every $j \in \mathrm{Ob}J$, one can define a cocone

$$\psi_j : \mathrm{H} \longrightarrow \mathrm{B}_j \text{ with } (\psi_j)_d = (\mathrm{S}\Psi_d)_j \text{ for all } d \in \mathrm{Ob}\, D .$$

Since $y_j u \cdot \mathrm{U}\varphi = \mathrm{U}\psi_j$ one has a unique $A$-morphism $f_j : \mathrm{A} \to \mathrm{B}_j$ with

$$\mathrm{U}f_j = y_j u \text{ and } f_j \varphi = \psi_j \text{ for all } j \in \mathrm{Ob}J .$$

Naturality in $j$ gives a $\mathrm{Pro}(\mathbf{D},A)$-morphism $\boldsymbol{f} : \mathrm{A} \to \mathbf{B}$, and $(u,\boldsymbol{f}) : \mathrm{EA} \to (\mathrm{Y},\boldsymbol{y},\mathbf{B})$ is the only $\mathrm{Pro}(\mathbf{D},\Gamma,\mathrm{U})$-morphism over $u$ with $(u,\boldsymbol{f}) \cdot \mathrm{E}\varphi = \Psi$. ■

**REMARK.** $\Phi_{(\mathrm{X},\boldsymbol{x},\mathbf{A})}$ is natural in $(\mathrm{X},\boldsymbol{x},\mathbf{A})$, so for

$$(u,\boldsymbol{f}) : (\mathrm{X},\boldsymbol{x},\mathbf{A}) \to (\mathrm{Y},\boldsymbol{y},\mathbf{B})$$

the diagram

$$\begin{array}{ccc}
(\mathrm{X},\boldsymbol{x},\mathbf{A}) & \xrightarrow{(u,\boldsymbol{f})} & (\mathrm{Y},\boldsymbol{y},\mathbf{B}) \\
{\scriptstyle \Phi_{(\mathrm{X},\boldsymbol{x},\mathbf{A})}} \downarrow & & \downarrow {\scriptstyle \Phi_{(\mathrm{Y},\boldsymbol{y},\mathbf{B})}} \\
\bar{\mathrm{E}}\mathbf{A} & \xrightarrow{\bar{\mathrm{E}}\boldsymbol{f}} & \bar{\mathrm{E}}\mathbf{B}
\end{array}$$

commutes in $\mathrm{Pro}(\mathbf{D},\mathrm{Pro}(\mathbf{D},\Gamma,\mathrm{U}))$, that is: $\Phi : \mathrm{I}_{\mathrm{Pro}(\mathbf{D},\Gamma,\mathrm{U})} \to \bar{\mathrm{E}}\mathrm{S}$ is a natural transformation.

We now prove the universality of the extension E in **CAT**/$X$:

**2.5. THEOREM.** *Let* $\Gamma$ *be admissible with respect to* U, *and let* V: $B \to X$ *be* $\Gamma$*-topological. Then every functor* $\mathrm{F} : A \to B$ *with* $\mathrm{VF} = \mathrm{U}$ *admits an extension*

$$\mathrm{G} : \mathrm{Pro}(\mathbf{D},\Gamma,\mathrm{U}) \to B \text{ with } \mathrm{GE} = \mathrm{F} \text{ and } \mathrm{VG} = \mathrm{T}$$

*and the property that* G *transforms* T*-initial cones with* T*-image in* $\Gamma$ *into* V*-initial cones.* G *is unique up to natural equivalence; it is unique if* V *is amnestic.*

**PROOF.** Every $\mathrm{Pro}(\mathbf{D},\Gamma,\mathrm{U})$-object $(\mathrm{X},\boldsymbol{x},\mathbf{A})$ gives a V-cone $\boldsymbol{x} : \mathrm{X} \to \mathrm{V}(\bar{\mathrm{F}}\mathbf{A})$ which, by assumption, admits a V-initial lifting

$$\Psi_{(\mathrm{X},\boldsymbol{x},\mathbf{A})} : \mathrm{G}(\mathrm{X},\boldsymbol{x},\mathbf{A}) \longrightarrow \bar{\mathrm{F}}\mathbf{A}$$

There is a unique way to define G on morphisms

$$(u,\boldsymbol{f})\colon (X,\boldsymbol{x},\mathbf{A}) \longrightarrow (Y,\boldsymbol{y},\mathbf{B})$$

according to the conditions that $VG(u,\boldsymbol{f}) = u$ and that the following diagram in $\mathrm{Pro}(\mathbf{D},\mathcal{B})$ be commutative:

$$\begin{tikzcd} G(X,\boldsymbol{x},\mathbf{A}) \arrow[r, "G(u,\boldsymbol{f})"] \arrow[d, "\Psi_{(X,\boldsymbol{x},\mathbf{A})}"'] & G(Y,\boldsymbol{y},\mathbf{B}) \arrow[d, "\Psi_{(Y,\boldsymbol{y},\mathbf{B})}"] \\ \bar{F}\mathbf{A} \arrow[r, "\bar{F}\boldsymbol{f}"'] & \bar{F}\mathbf{B} \end{tikzcd} \tag{2}$$

Clearly, G becomes a functor with $GE = F$ and $VG = T$. Once we have shown that it preserves initiality, it is (up to natural equivalence) the only such functor: G must transform the T-initial lifting $\Phi_{(X,\boldsymbol{x},\mathbf{A})}$ of the U-cone $\boldsymbol{x}$ into a V-initial lifting of the V-cone $\boldsymbol{x}$, the vertex of which is unique up to isomorphism, due to V-initiality, and unique if V is amnestic.

It remains to be shown that G preserves initiality. So let

$$\Theta\colon (X,\boldsymbol{x},\mathbf{L}) \to H \text{ with } H\colon \mathcal{J}^{op} \to \mathrm{Pro}(\mathbf{D},\Gamma,U)$$

be T- initial such that $T\Theta \in \Gamma$. In order to show that $G\Theta$ is V-initial. let a cone $\beta\colon B \to GH$ in $\mathcal{B}$ and an $\mathcal{X}$-morphism $u\colon VB \to X$ with $T\Theta \cdot u = V\beta$ be given. By the construction used in 2.3, $S\Theta\colon \mathbf{L} \to SH$ is a limit cone which, by 1.4, is preserved by

$$\bar{F}\colon \mathrm{Pro}(\mathbf{D},\mathcal{A}) \to \mathrm{Pro}(\mathbf{D},\mathcal{B}).$$

Diagram (2) shows that one has a natural transformation $\Psi\colon I_{\mathcal{B}}G \to \bar{F}S$, hence a cone

$$\Psi H \cdot I_{\mathcal{B}}\beta\colon I_{\mathcal{B}}B \longrightarrow \bar{F}(SH).$$

Therefore, the limit property of $\bar{F}S\Theta$ gives a unique $\mathrm{Pro}(\mathbf{D},\mathcal{B})$-morphism

$$\boldsymbol{f}\colon B \to \bar{F}\mathbf{L} \text{ with } \Psi H \cdot I_{\mathcal{B}}\beta = \bar{F}S\Theta \cdot \boldsymbol{f}.$$

Therefore, the limit property of $\bar{F}S\Theta$ yields $\bar{V}\boldsymbol{f} = \boldsymbol{x}u$. Since $\Psi_{(X,\boldsymbol{x},\mathbf{L})}$ is V-initial, there is a unique $\mathcal{B}$-morphism

$$f\colon B \longrightarrow G(X,\boldsymbol{x},\mathbf{L}) \text{ with } \Psi_{(X,\boldsymbol{x},\mathbf{L})} f = \boldsymbol{f} \text{ and } Vf = u.$$

$$\begin{tikzcd}
B \arrow[dr, "f"] \arrow[drr, "\beta_j"] \arrow[ddr, "f"'] & & \\
& G(X,\boldsymbol{x},\mathbf{A}) \arrow[r, "G\Theta_j"'] \arrow[d, "\Psi_{(X,\boldsymbol{x},\mathbf{L})}"] & G(Hj) \arrow[d, "\Psi_{Hj}"] \\
& \bar{F}\mathbf{L} \arrow[r, "\bar{F}S\Theta_j"'] & \bar{F}S(Hj)
\end{tikzcd}$$
■

**REMARK.** In general, $T: \mathrm{Pro}(\mathbf{D},\Gamma,U) \to X$ is not amnestic. But any functor $V: B \to X$ can easily be "made" amnestic without losing $\Gamma$-topologicity: take $B_\mu$ to be a full subcategory of $B$ where $\mathrm{Ob}\, B_\mu$ is a representative system of $B$-objects with respect to the equivalence relation

$$B \sim C \Leftrightarrow Vh = 1 \text{ for some } B\text{-isomorphism } h: B \to C;$$

then, let $B_1 = B/\approx$ where $\approx$ is the smallest compatible equivalence relation on $\mathrm{Mor}\, B_\mu$ such that $k \approx 1$ for every automorphism $k$ with $Vk = 1$, and consider the induced functor $V_1: B_1 \to X$.

## 3. Pro$(\mathbf{D},\Gamma,U)$ AS A CATEGORY OVER Pro$(\mathbf{D},A)$.

Next we shall describe some properties of the functor

$$S: \mathrm{Pro}(\mathbf{D},\Gamma,U) \longrightarrow \mathrm{Pro}(\mathbf{D},A).$$

Under some aspects, it behaves dually to T; sometimes even better than T. For instance, the following statement is easily verified:

**REMARK.** S is faithful iff, for every $\mathrm{Pro}(\mathbf{D},\Gamma,U)$-object $(X,\boldsymbol{x},\mathbf{A})$, $\boldsymbol{x}$ is a mono-cone (so $\boldsymbol{x}u = \boldsymbol{x}v$ implies $u = v$). ■

**3.1. PROPOSITION.** (1) *If $X$ contains an initial object 0 and if $\Gamma$ contains all cones in $X$ (over $\mathbf{D}$) with vertex 0, then S has a full and faithful left adjoint and, therefore, preserves all limits.*

(2) *If* $X$ *is* $\mathbf{D}$*-complete and if* $\Gamma$ *contains the respective limit cones, then* S *has a full and faithful right adjoint and, therefore, preserves all colimits.*

**PROOF.** (1) $$1_{\mathbf{A}}: \mathbf{A} \to S(0, 0 \to \bar{U}\mathbf{A}, \mathbf{A})$$

is an S-universal arrow for $\mathbf{A} \in \mathrm{Pro}(\mathbf{D}, A)$.

(2) With the limit $(L, \mathbf{l})$ of $\mathbf{UA}$ in $X$, $1_{\mathbf{A}}: S(L, \mathbf{l}, \mathbf{A}) \to \mathbf{A}$ is an S-couniversal arrow for $\mathbf{A} \in \mathrm{Pro}(\mathbf{D}, A)$. ∎

There are converses of the above statements:

**REMARKS.** (1) If $\Gamma$ contains all empty cones (cf. 2.1) and if S has a left adjoint, then $X$ contains an initial object.

(2) If $\Gamma$ consists of all cones in $X$ over $\mathbf{D}$ and if S has a right adjoint, then $\bar{U}\mathbf{A}$ has a limit in $X$ for every $\mathbf{A}$ in $\mathrm{Pro}(\mathbf{D}, A)$.

Note that, in both cases, the adjoint need not be assumed to be full and faithful.

For a certain conglomerate $\Delta$ of S-cocones, we now want to show that S is $\Delta$-cotopological. For this, we first consider any S-cocone $\psi: SH \to \mathbf{B}$ with $\mathbf{B} = (B_j)_{j \in \mathrm{Ob} J}$ and $H: D \to \mathrm{Pro}(\mathbf{D}, \Gamma, U)$. We do *not* require $D \in \mathbf{D}$ but assume that there is a colimit $\lambda: TH \to L$ in $X$. By 1.2, this colimit is preserved by $I_X$; so there is a unique cone $\boldsymbol{y}: L \to \bar{U}\mathbf{B}$ which is determined by the commutative diagram

$$\begin{array}{ccc} I_X TH & \xrightarrow{\;\; I_X \lambda \;\;} & I_X L \\ {\scriptstyle \Sigma H} \big\downarrow & & \big\downarrow {\scriptstyle \boldsymbol{y}} \\ \bar{U}SH & \xrightarrow{\;\; \bar{U}\psi \;\;} & \bar{U}\mathbf{B} \end{array} \qquad (3)$$

and said to be *co-induced* by $\psi$.

Let $\Gamma^*$ be the conglomerate of all S-cocones $\psi: SH \to \mathbf{B}$ for which TH has a colimit in $X$ and for which the co-induced cone belongs to $\Gamma$.

**3.2. PROPOSITION.** S *is* $\Gamma^*$*-cotopological.*

**PROOF.** For the given $\psi$ in $\Gamma^*$ we have to find an S-final lifting $\Psi: H \to (L, \boldsymbol{y}, \mathbf{B})$: this is the cocone given by the above diagram.

To show S-finality we consider a cocone $\Theta: H \to (Z, \boldsymbol{z}, \mathbf{C})$ and a $\mathrm{Pro}(\mathbf{D}, A)$-morphism $\boldsymbol{f} : \mathbf{B} \to \mathbf{C}$ with $\boldsymbol{f}\psi = S\Theta$. The colimit property gives a unique $X$-morphism $v: L \to Z$ with $v\lambda = T\Theta$ for which one easily shows that $\boldsymbol{z}v = \bar{U}\boldsymbol{f}\cdot\boldsymbol{y}$. Therefore there is a $\mathrm{Pro}(\mathbf{D}, \Gamma, U)$-morphism

$$(v, \boldsymbol{f}) : (L, \boldsymbol{y}, \mathbf{B}) \longrightarrow (Z, \boldsymbol{z}, \mathbf{C});$$

it is the only one over $\boldsymbol{f}$ with $\psi(v, \boldsymbol{f}) = \Theta$. ∎

**3.3. PROPOSITION.** E *transforms all cones in* $A$ *into* S*-initial cones.* (Note that, as $I_A$ is full and faithful, every cone in $A$ is $I_A$-initial.)

**PROOF.** For every cone $\varphi: A \to H$ with any $H: D \to A$, $E\varphi$ is S-initial: indeed, if $\Psi: (Y, \boldsymbol{y}, B) \to EH$ is a cone and $\boldsymbol{f} : \mathbf{B} \to \mathbf{A}$ a morphism in $\mathrm{Pro}(\mathbf{D}, A)$ such that $SE\varphi\cdot\boldsymbol{f} = S\Psi$, then

$$u := \bar{U}\boldsymbol{f}\cdot\boldsymbol{y} : Y \longrightarrow UA$$

gives the unique $\mathrm{Pro}(\mathbf{D}, \Gamma, U)$-morphism $(u, \boldsymbol{f}) : (Y, \boldsymbol{y}, \mathbf{B}) \to EA$ over $\boldsymbol{f}$ with $E\varphi\cdot(u, \boldsymbol{f}) = \Psi$. ∎

**REMARK.** We do not know whether E, as a functor over the category $\mathrm{Pro}(\mathbf{D}, A)$, is finally dense (cf. 2.4) except in the trivial case that every cone in $\Gamma$ is a limit cone in $X$: in this case, every cocone in $\mathrm{Pro}(\mathbf{D}, \Gamma, U)$ is S-final, in particular the cocone $E\varnothing \to (X, \boldsymbol{x}, \mathbf{A})$ where $\varnothing$ is the empty diagram in $A$ and $(X, \boldsymbol{x}, \mathbf{A})$ is any given $\mathrm{Pro}(\mathbf{D}, \Gamma, U)$-object.

We summarize the completeness properties which we can derive from Propositions 1.6, 2.3 and 3.2 respectively:

**3.4. THEOREM.** (1) *Let* $\Gamma$ *be admissible with respect to* U. *Then, if* $X$ *is* **D**-*complete, also* $\mathrm{Pro}(\mathbf{D}, \Gamma, U)$ *is, and both* T *and* S *preserve the respective limits.*

(2) *Let* $\Gamma$ *consist of all cones in* $X$ *over* **D**. *Then, if* $X$ *and* $\mathrm{Pro}(\mathbf{D}, A)$ *are D-cocomplete (for any category D), also the category* $\mathrm{Pro}(\mathbf{D}, \Gamma, U)$ *is, and both* T *and* S *preserve the respective colimits.*

With 1.5 we conclude:

**3.5. COROLLARY.** *If $X$ and $A$ are* L(**D**)*-cocomplete, also the category* Pro(**D**, Γ, U) *is, provided* **D** *is regular and* Γ *consists of all cones over* **D**. ∎

**REMARK.** As Propositions 2.1 and 3.1 give sufficient conditions under which $X$ or Pro(**D**, $A$) can be, up to equivalence, embedded into Pro(**D**, Γ, U) as (co-)reflective subcategories with (co)reflector T or S, any type of completeness or cocompleteness of Pro(**D**, Γ, U) will be inherited by $X$ and Pro(**D**, $A$) under those conditions.

## 4. SPECIAL CASES.

In this section we discuss possible choices for **D** and Γ.

**4.1. PROPOSITION** (cf. [Th4]). *The following classes* **D** *are admissible with respect to any category:* {**1**}, {*r*-*directed sets*} *or* {*small r-filtered categories*} *for any infinite regular cardinal r*. {*small discrete categories*}. {*all small categories*}*: in this case* Pro(**D**, $K$) *is equivalent to* $K$*, to the usual procategory* Pro-$K$ *as introduced by Grothendieck and Verdier for* $r = {}_{\mu}$, *to the formal product completion of* $K$*, and to the Lambek completion of a category respectively.*

**PROOF.** Except in the case **D** = {$r$-directed sets} one obviously has $\hat{\mathrm{H}} \in \mathbf{D}$ for H: $J \to$ Pro(**D**, $K$), $J \in \mathbf{D}$. But in that case one still has $\hat{\mathrm{H}} \in \bar{\mathbf{D}}$ (see 1.3) since every small $r$-filtered category is the codomain of a confinal functor with $r$-directed domain (cf. [GV] p. 65, and [Th3]). ∎

**4.2. PROPOSITION.** *Let* U: $A \to X$ *be a functor, and let* **D** *be admissible with respect to* $A$ *and* $X$. *For the following classes* Γ *of cones over* **D** *in* $X$, Γ *is admissible with respect to* U*:*

{*limit cones*} , {*mono-cones*}, {*all cones*}.

**PROOF.** Nothing is to be shown in case Γ = {all cones}. For the other two cases, we consider diagram (1) again and assume that $\varphi$ and each $x_j: X_j \to \bar{U}\mathbf{A}_j$ is a mono-cone (limit-cone resp.), with

$$(X_j, \boldsymbol{x}_j, \mathbf{A}_j) = Hj,\ j \in \mathrm{Ob}J,\ H: J^{op} \longrightarrow \mathrm{Pro}(\mathbf{D}, \Gamma, U).$$

If $\boldsymbol{x}\, u = \boldsymbol{x} v$ for $u, v: Y \to X$ in $X$, then $\boldsymbol{x}_j \varphi_j u = \boldsymbol{x}_j \varphi_j v$ for all $j$ since (1) commutes. So $\varphi_j u = \varphi_j v$ since each $\boldsymbol{x}_j$ is a mono-cone, and $u = v$ since $\varphi$ is one. Any cone $\boldsymbol{y} : Y \to \mathbf{L}$ induces a morphism

$$\psi_j: Y \to X_j \text{ with } \boldsymbol{x}_j \psi_j = \bar{U}\Lambda_j \cdot \boldsymbol{y},$$

if $\boldsymbol{x}_j$ is a limit cone. The arising cone $\psi: Y \to T\underline{H}$ gives a morphism $z$ with $\varphi z = \psi$ if $\varphi$ is a limit cone. Since $\bar{U}\Lambda$ is a mono-cone, $\boldsymbol{x}\, z = \boldsymbol{y}$ follows, so $\boldsymbol{x}$ is a limit cone in this case. ∎

Considering the case of limit cones, from 1.6, 2.3, 2.5 and 4.2 one obtains for every functor $U: A \to X$ and every class $\mathbf{D}$ which is admissible with respect to $A$ and $X$:

**4.3. THEOREM.** *If $X$ is $\mathbf{D}$-complete, then there is a full embedding $E$ of $A$ into a $\mathbf{D}$-complete category $\hat{A}$ and a $\mathbf{D}$-continuous (i.e., $I^{op}$-continuous for every $I \in \mathbf{D}$) functor $T: \hat{A} \to X$ with $TE = U$ and the following property: every other functor $F: A \to B$ into a $\mathbf{D}$-complete category $B$ such that there is a $\mathbf{D}$-continuous functor $V$ with $VF = U$, admits a $\mathbf{D}$-continuous extension $G: \hat{A} \to B$ with $GE = F$ and $VG \approx T$ which is unique up to natural equivalence.* ∎

**REMARKS.** (1) In addition, from 2.4 one has that $E$ is $\mathbf{D}$-*condense* in the sense that every $\hat{A}$-object is an $I^{op}$-limit of $A$-objects with $I \in \mathbf{D}$.

(2) We only have $VG \approx T$ (rather than $VG = T$) since, in the given situation, $T$ is $\Gamma$-topological only in a weak sense: initial liftings of limit cones exist only up to isomorphisms, so we are applying here a slightly generalized version of 2.5, in the special case that $\Gamma$ is the class of limit cones over $\mathbf{D}$. One could achieve $VG = T$ if $V$ is assumed to be *transportable* (that is: $VB \approx X$ implies the existence of a $B$-object $C$ with $B \approx C$ and $VC = X$).

(3) Theorem 4.3 entails the case $X = \mathbf{1}$ and therefore the case of the Grothendieck-Verdier completion and of the Lambek completion of a category $A$.

(4) Theorem 4.3 does not entail the Adámek-Koubek Completion Theorem for concrete categories ([AK], see also Herrlich [H2]) which states that, for $\mathbf{D}$ = {all small categories}, Theorem 4.3 remains valid with the additional requirement that U, T, V be faithful and E, F $\mathbf{D}$-continuous.

Also the case $\mathbf{D}$ = {all small categories} and $\Gamma$ = {all cones over $\mathbf{D}$ in $X$} deserves special consideration: a $\Gamma$-topological functor $U: A \to X$ is called *small-topological* in this case. From 2.3 and 2.5, one then obtains:

**4.4. THEOREM.** *For every* $U: A \to X$ *there is a small-topological functor* $T: \check{A} \to X$ *and a full embedding* $E: A \to \check{A}$ *with* $TE = U$ *and the following property: every other functor* $F: A \to B$ *such that there is a small-topological functor* V *with* $VF = U$ *factorizes uniquely (up to natural equivalence) as* $GE = F$ *for a functor* G *with* $VG = T$ *which transforms small* T*-initial cones into* V*-initial cones.* ∎

The category $\check{A}$ admits a $\Gamma^*$-cotopological functor S into Pro($\mathbf{D}$, $A$) (cf. 3.2). If $X$ is $D$-cocomplete (for any category $D$) and if $\Gamma$ = {all cones over $\mathbf{D}$ in $X$}, then $\Gamma^*$ contains all cocones over $D$ in Pro($\mathbf{D}$, $A$), so $D$-cocompleteness is transferred from Pro($\mathbf{D}$, $A$) to $\check{A}$ (cf. 3.4). Corresponding investigations for $\hat{A}$ (cf. 4.3) are more complicated; we shortly consider the cases that $\Gamma$ is the class of all mono-cones or all limit cones, and give sufficient conditions for the existence of certain cocones in $\Gamma^*$.

For a colimit $\lambda: G \to L$ in $X$ (with $G: D \to X$) we say that *bi-pullbacks of* $\lambda$ *are epimorphic*, if for all morphisms $u, v$: $K \to L$ in $X$, the family $(p_d: P_d \to K)_{d \in \mathrm{Ob}D}$ as defined by the following diagram is epimorphic; here the solid lines form three pullbacks, and the dotted lines arise as compositions:

$$\begin{tikzcd}
& & P_d \arrow[dll, dashed, "g_d"'] \arrow[dl] \arrow[dr] \arrow[drr, dashed, "h_d"] \arrow[dd, dashed, "p_d"] & & \\
GD \arrow[r] \arrow[d, "\lambda_d"'] & \cdot \arrow[dr] & & \cdot \arrow[dl] \arrow[r] & GD \arrow[d, "\lambda_d"] \\
L & & K \arrow[ll, "u"'] \arrow[rr, "v"] & & L
\end{tikzcd}$$

Now we consider diagram (3) again, with

$$H\colon D \longrightarrow \mathrm{Pro}(\mathbf{D},\Gamma,\mathrm{U}),\ Hd = (X_d, \boldsymbol{x}_d, \mathbf{A}_d) \text{ for } d \in \mathrm{Ob}D,$$

and $\psi\colon SH \to \mathbf{B}$, so $\psi_d\colon \mathbf{A}_d \to \mathbf{B}$ in $\mathrm{Pro}(\mathbf{D},A)$, and $\boldsymbol{y} : L \to \bar{\mathrm{U}}\mathbf{B}$ with

$$\boldsymbol{y}\lambda_d = \bar{\mathrm{U}}\psi_d \cdot \boldsymbol{x}_d \text{ for all } d \in \mathrm{Ob}D.$$

**4.5. PROPOSITION.** *Let X be D-cocomplete, and let bi-pullbacks of D-colimits in X exist and be epimorphic. Then the following holds (with any* **D** *and* U*):*

(1) *If* $\Gamma$ *consists of all mono-cones over* **D** *in X, then* $\Gamma^*$ *contains all S-cocones* $\psi$ *for which each* $\bar{\mathrm{U}}\psi_d \cdot \boldsymbol{x}_d$ *is a mono-cone.*

(2) *If* $\Gamma$ *consists of all limit cones over* **D** *in X, then* $\Gamma^*$ *contains all S-cocones* $\psi$ *for which each* $\bar{\mathrm{U}}\psi_d \cdot \boldsymbol{x}_d$ *is a mono-cone and at least one of these is a limit cone.*

**PROOF.** (1) Suppose that, for the cone $\boldsymbol{y} : L \to \bar{\mathrm{U}}\mathbf{B}$ which is co-induced by $\psi$, one has $\boldsymbol{y}\, u = \boldsymbol{y}\, v$, so

$$\bar{\mathrm{U}}\psi_d \cdot \boldsymbol{x}_d g_d = \boldsymbol{y}\,\lambda_d g_d = \boldsymbol{y}\, u p_d = \boldsymbol{y}\, v p_d = \bar{\mathrm{U}}\psi_d \cdot \boldsymbol{x}_d\, h_d .$$

Hence $g_d = h_d$ if $\bar{\mathrm{U}}\psi_d \cdot \boldsymbol{x}_d$ is a mono-cone. In this case, $u p_d = v p_d$ follows, hence $u = v$.

(2) Let $\boldsymbol{z} : K \to \bar{\mathrm{U}}\mathbf{B}$ be any cone. If $d \in \mathrm{Ob}D$ is such that

$$\bar{\mathrm{U}}\psi_d \cdot \boldsymbol{x}_d\colon X_d \to \bar{\mathrm{U}}\mathbf{B}$$

is a limit cone, then $\boldsymbol{z}$ factors as $\boldsymbol{y}(\lambda_d w) = \boldsymbol{z}$, and this factorization is unique by (1). ∎

**REMARKS.** (1) In finitary algebraic or topological categories over **Set**, bi-pullbacks of directed colimits are epimorphic since they are so in **Set**.

(2) Suppose, for a certain $d \in \mathrm{Ob}D$, $\psi_d\colon \mathbf{A}_d \to \mathbf{B}$ is induced by a commutative triangle

$$\begin{tikzcd} I_d^{\mathrm{op}} \arrow[dr, "\mathbf{A}_d"'] & & J^{\mathrm{op}} \arrow[ll, "F^{\mathrm{op}}"'] \arrow[dl, "\mathbf{B}"] \\ & A & \end{tikzcd}$$

with a confinal functor $F: J \to I_d$ (so $(\psi_d)_j$ is represented by $(1_{B_j}, Fj)$ for every $j \in \mathrm{Ob}J$ ). Then, if $\boldsymbol{x}_d$ is a limit cone, also $\bar{U}\psi_d \cdot \boldsymbol{x}_d$ is one.

Finally we mention a more classical but rather simple case:

$\mathbf{D} = \{\mathbf{1}\}$ and $\Gamma = \{$all morphisms (= cones over $\mathbf{D}$) in $X\}$;

then a $\Gamma$-topological functor $U: A \to X$ is simply a *fibration* [Gy], and $\mathrm{Pro}(\mathbf{D}, \Gamma, U)$ is the comma-category $(X \downarrow U)$. 2.3 and 2.5 just give well-known properties of the projection $T: (X \downarrow U) \to X$; 3.4 (2) now says:

**4.6. COROLLARY.** *For any category $D$, $(X \downarrow U)$ is $D$-cocomplete if $X$ and $A$ are.* ∎

## 5. WEAKLY ADMISSIBLE CLASSES D.

In this section we shall show that, without losing any of the results in the preceding sections, one may weaken the condition that $\mathbf{D}$ be admissible with respect to the categories in question. For this we first have a second look at the construction of $\mathbf{D}$-limits in $\mathrm{Pro}(\mathbf{D}, K)$ as given in 1.4: certainly, for a functor $H: J^{op} \to \mathrm{Pro}(\mathbf{D}, K)$, the constructed functor $\mathbf{L}: \hat{H}^{op} \to K$ is a limit of H in the category $\mathrm{Pro}(\mathbf{Cat}, K)$, and in order to have a limit of H in $\mathrm{Pro}(\mathbf{D}, K)$ it would suffice that there is a $\mathrm{Pro}(\mathbf{D}, K)$-object $\mathbf{M}$ which is isomorphic to $\mathbf{L}$ in $\mathrm{Pro}(\mathbf{Cat}, K)$. Therefore we call $\mathbf{D} \subset \mathrm{Ob}\,\mathbf{Cat}$ with $\mathbf{1} \in \mathbf{D}$ *weakly admissible with respect to* $K$ if this sufficient condition holds for every H with $J \in \mathbf{D}$. A class $\Gamma$ of cones over $\mathbf{D}$ is called *weakly admissible with respect to* U if, in the Definition 2.2, $\mathbf{D}$ is just weakly admissible with respect to $A$ and $X$.

Clearly, one has:

**5.1. PROPOSITION.** *Admissibility implies weak admissibility, and all previous results hold when "admissible" is replaced by "weakly admissible".* ∎

Since the notion of weak admissibility is certainly not very handy in practice, next we shall look for formal properties of the class **D** which imply weak admissibility with respect to any category $K$. The essential tool for this is the functor

$$W_K \colon \mathrm{Pro}(\mathbf{D},K)^{\mathrm{op}} \longrightarrow \mathbf{CAT}/K^{\mathrm{op}}.$$

$W_K$ sends $\mathbf{X} \in \mathrm{Ob}(\mathrm{Pro}(\mathbf{D},K))$ to the projection $P_{\mathbf{X}} \colon E_{\mathbf{X}} \to K^{\mathrm{op}}$ of the category $E_{\mathbf{X}}$ which is defined as follows: objects are pairs

$$(A, \boldsymbol{a}) \text{ with } A \in \mathrm{Ob}\,K \text{ and } \boldsymbol{a} \colon \mathbf{X} \to A \text{ in } \mathrm{Pro}(\mathbf{D},K),$$

and a morphism $u \colon (A, \boldsymbol{a}) \to (B, \boldsymbol{b})$ in $E_{\mathbf{X}}$ is a $K$-morphism $u \colon B \to A$ with $u\boldsymbol{b} = \boldsymbol{a}$ ; or in other words, $E_{\mathbf{X}}^{\mathrm{op}}$ is the comma-category $(\mathbf{X} \downarrow I_K)$. $W_K$ maps a morphism $\boldsymbol{f} \colon \mathbf{Y} \to \mathbf{X}$ to the commutative triangle

$$\begin{tikzcd} E_{\mathbf{X}} \arrow[rr, "E_{\boldsymbol{f}}"] \arrow[dr, "P_{\mathbf{X}}"'] & & E_{\mathbf{Y}} \arrow[dl, "P_{\mathbf{Y}}"] \\ & K^{\mathrm{op}} & \end{tikzcd}$$

in **CAT** where $E_{\boldsymbol{f}}$ maps $(A, \boldsymbol{a}) \in \mathrm{Ob}\,E_{\mathbf{X}}$ to $(A, \boldsymbol{af}) \in \mathrm{Ob}\,E_{\mathbf{Y}}$.

**REMARK.** $W_K$ decomposes as

$$\begin{array}{ccccc} \mathrm{Pro}(\mathbf{D},K)^{\mathrm{op}} & \longrightarrow & [K, \mathbf{Set}] & \longrightarrow & \mathbf{CAT}/K^{\mathrm{op}} \\ \mathbf{X} & \longmapsto & \mathrm{colim}\, Y_K \mathbf{X}^{\mathrm{op}}, \quad F & \longmapsto & P_F^{\mathrm{op}} \end{array}$$

with $Y_K \colon K^{\mathrm{op}} \to [K, \mathbf{Set}]$ the Yoneda embedding and $P_F \colon el F \to K$ the projection of the category of elements of F (as defined in [K]). But for our purposes, it is easier to consider this composition directly.

**5.2. LEMMA.** *For every $\mathbf{X} \colon I^{\mathrm{op}} \to K$, there is a confinal functor*

$$R_{\mathbf{X}} \colon I \to E_{\mathbf{X}} \textit{ with } \mathbf{X}^{\mathrm{op}} = P_{\mathbf{X}} R_{\mathbf{X}}.$$

**PROOF.** $R_{\mathbf{X}}$ sends $i \in \mathrm{Ob}\,I$ to $(X_i, \xi_i)$ with $\xi_i \colon \mathbf{X} \to X_i$ as in 1.1. Since $\nu \colon i \to i'$ in $I$ induces a morphism $(X_i, \xi_i) \to (X_{i'}, \xi_{i'})$ in $E_{\mathbf{X}}$, $R_{\mathbf{X}}$ is a functor with $\mathbf{X}^{\mathrm{op}} = P_{\mathbf{X}} R_{\mathbf{X}}$. Its confinality is an immediate consequence of the definition of morphisms in $\mathrm{Pro}(\mathbf{D},K)$. ∎

$R_{\mathbf{X}}$ has a remarkable universal property:

**5.3. LEMMA.** *For* $\mathbf{X}, \mathbf{Y} \in \mathrm{Pro}(\mathbf{D}, K)$ *and every functor* $G\colon I \to E_{\mathbf{Y}}$ *with* $P_{\mathbf{Y}} G = \mathbf{X}^{op}$, *there is a unique* $\mathrm{Pro}(\mathbf{D}, K)$*-morphism* $\boldsymbol{f}\colon \mathbf{Y} \to \mathbf{X}$ *with* $E_{\boldsymbol{f}} R_{\mathbf{X}} = G$.

$$\begin{tikzcd} I \arrow[r, "R_{\mathbf{X}}"] \arrow[dr, "G"'] & E_{\mathbf{X}} \arrow[d, "E_{\boldsymbol{f}}"] & \mathbf{X} \\ & E_{\mathbf{Y}} & \mathbf{Y} \arrow[u, "\boldsymbol{f}"'] \end{tikzcd}$$

**PROOF.** There is at most one such $\boldsymbol{f} = (\boldsymbol{f}_i)_{i \in \mathrm{Ob}\, I}$ since

$$(X_i, \boldsymbol{f}_i) = (X_i, \xi_i \boldsymbol{f}) = E_{\boldsymbol{f}}(X_i, \xi_i) = E_{\boldsymbol{f}} R_{\mathbf{X}} i = G i.$$

Vice versa, taking this as a definition for $\boldsymbol{f}_i$, functoriality of G yields that $\boldsymbol{f}$ is a $\mathrm{Pro}(\mathbf{D}, K)$-morphism with the desired properties. ■

For $K$ small, the category $E_{\mathbf{X}}$ is small too, so by 5.2 $E_{\mathbf{X}}$ belongs to $\bar{\mathbf{D}}$ (as defined in 1.3). For arbitrary $K$, $E_{\mathbf{X}}$ belongs to

$$\bar{\bar{\mathbf{D}}} := \{ J \in \mathrm{Ob}\,\mathbf{CAT} \mid \exists\, F\colon I \to J \text{ confinal with } I \in \mathbf{D} \}$$

(so $\bar{\mathbf{D}} = \bar{\bar{\mathbf{D}}} \cap \mathrm{Ob}\,\mathbf{Cat}$). $\bar{\mathbf{D}}$ and $\bar{\bar{\mathbf{D}}}$ define, for every $K$, full subcategories $\bar{\mathbf{D}}/K^{op}$ and $\bar{\bar{\mathbf{D}}}/K^{op}$ of $\mathbf{Cat}/K^{op}$ and $\mathbf{CAT}/K^{op}$ respectively, and $W_K$ maps into them. Lemma 5.3 now says:

**5.4. THEOREM.** $W_K$ *embeds* $\mathrm{Pro}(\mathbf{D}, K)^{op}$ *as a full reflective subcategory into* $\bar{\bar{\mathbf{D}}}/K^{op}$; *even into* $\bar{\mathbf{D}}/K^{op}$ *if K is small.*

**PROOF.** $W_K$ clearly maps objects one-to-one. To see that $W_K$ is full and faithful, one considers $\mathbf{X}, \mathbf{Y} \in \mathrm{Pro}(\mathbf{D}, K)$ and any

$$F\colon E_{\mathbf{X}} \longrightarrow E_{\mathbf{Y}} \text{ with } P_{\mathbf{Y}} F = P_{\mathbf{X}}.$$

With $G = F R_{\mathbf{X}}$, from 5.3 we get a unique

$$\boldsymbol{f}\colon \mathbf{X} \to \mathbf{Y} \text{ with } E_{\boldsymbol{f}} R_{\mathbf{X}} = F R_{\mathbf{X}}.$$

Since $E_{\boldsymbol{f}}$ and F commute with the projections, it suffices to show that they coincide on objects to obtain their equality. But every $\boldsymbol{a}\colon \mathbf{X} \to A$ in $\mathrm{Pro}(\mathbf{D}, K)$ admits a factorization $\boldsymbol{a} = u \xi_i$, for a certain $i \in \mathrm{Ob}I$ and $u: \mathrm{X}_i \to \mathrm{A}$. So one has an $E_{\mathbf{X}}$-morphism $u: (\mathrm{A}, \boldsymbol{a}) \to (\mathrm{X}_i, \xi_i)$ which is mapped to the $E_{\mathbf{Y}}$-morphism

$$u:\ \mathrm{F}(\mathrm{A}, \boldsymbol{a}) \to \mathrm{F}(\mathrm{X}_i, \xi_i) = E_{\boldsymbol{f}}(\mathrm{X}_i, \xi_i) = (\mathrm{X}_i, \xi_i \boldsymbol{f})$$

by F, so

$$\mathrm{F}(\mathrm{A}, \boldsymbol{a}) = (\mathrm{A}, u\xi_i \boldsymbol{f}) = (\mathrm{A}, \boldsymbol{af}) = E_{\boldsymbol{f}}(\mathrm{A}, \boldsymbol{a}).$$

For small $K$, $\mathrm{W}_K$ factors as

$$\mathrm{Pro}(\mathbf{D}, K)^{\mathrm{op}} \longrightarrow \mathrm{Pro}(\bar{\mathbf{D}}, K)^{\mathrm{op}} \longrightarrow \bar{\mathbf{D}}/K^{\mathrm{op}}$$

with the first arrow an equivalence of categories. When applying 5.3 to $\bar{\mathbf{D}}$ (rather than $\mathbf{D}$) we get that $\mathrm{R}_{\mathbf{X}}: \mathbf{X}^{\mathrm{op}} \to \mathrm{W}_K \mathbf{X}$ serves as reflexion for $\mathbf{X}^{\mathrm{op}} \in \mathrm{Ob}(\bar{\mathbf{D}}/K^{\mathrm{op}})$ along $\mathrm{W}_K$ (so the reflector maps $\mathbf{X}^{\mathrm{op}}$ to $\mathbf{X}$). For arbitrary $K$ we must follow the same procedure with $\bar{\bar{\mathbf{D}}}$ (rather than with $\bar{\mathbf{D}}$); this causes no difficulties since, although $\bar{\bar{\mathbf{D}}}$ contains arbitrary categories, the category $E_{\mathbf{X}}$ is still codable by a class, so it is a legitimate category belonging to $\bar{\bar{\mathbf{D}}}$. ∎

Since $\mathrm{W}_K$ has a left adjoint, colimits in $\mathrm{Pro}(\mathbf{D}, K)$ are mapped to limits in $\bar{\bar{\mathbf{D}}}/K^{\mathrm{op}}$. In addition one has:

**5.5. PROPOSITION.** *If* $\mathbf{D}$ *is weakly admissible with respect to* $K$, *then* $\mathrm{W}_K$ *sends* $\mathbf{D}$*-limits in* $\mathrm{Pro}(\mathbf{D}, K)$ *to colimits in* $\mathbf{CAT}/K^{\mathrm{op}}$.

**PROOF.** For $\mathrm{H}: J^{\mathrm{op}} \to \mathrm{Pro}(\mathbf{D}, K)$ with $J \in \mathbf{D}$ we may assume that its colimit is formed as in 1.4, so we have $\mathbf{L}: \hat{\mathrm{H}}^{\mathrm{op}} \to K$ and a limit-cone $\Lambda: \mathbf{L} \to \mathrm{H}$ where each $\Lambda_j: \mathbf{L} \to \mathbf{X}^j$ is induced by the functor

$$\mathrm{L}_j:\ I_j \longrightarrow \hat{\mathrm{H}},\ \nu \mapsto (\nu, 1_j).$$

A cocone $\Phi: \mathrm{W}_K \mathrm{H} \to \mathrm{Q}$ in $\bar{\bar{\mathbf{D}}}/K^{\mathrm{op}}$ with $\mathrm{Q}: D \to K^{\mathrm{op}}$ is given by functors $\mathrm{F}_j$: $E_{\mathbf{X}_j} \to D$ with $\mathrm{QF}_j = \mathrm{P}_{\mathbf{X}_j}$ which are natural in $j \in \mathrm{Ob}J$. One must then find a uniquely determined functor G rendering the following diagram commutative

$$\begin{tikzcd}
 & & K^{\mathrm{op}} \\
E_{\mathbf{X}_j} \arrow[rru, "\mathrm{P}_{\mathbf{X}_j}"] \arrow[r, "E_{\Lambda_j}"] \arrow[rrd, "\mathrm{F}_j"'] & E_{\mathbf{L}} \arrow[ru, "\mathrm{P}_{\mathbf{L}}"] \arrow[rd, dashed, "\mathrm{G}"] & \\
 & & D \arrow[uu, "\mathrm{Q}"']
\end{tikzcd}$$

Let $(\mathrm{A},\boldsymbol{a})$ be an object in $E_{\mathbf{L}}$. As in the proof of 5.4, $\boldsymbol{a}$ can be written as $\boldsymbol{a} = u\xi_i^j\Lambda_j$ for some $(i,j)\in \mathrm{Ob}\,\hat{\mathrm{H}}$ and $u:\mathrm{X}_i^j \to \mathrm{A}$; here $\xi_i^j$: $\mathbf{X}^j \to \mathrm{X}_i^j$ is as in 1.1. Therefore one necessarily has

$$\mathrm{G}(\mathrm{A},\boldsymbol{a}) = \mathrm{G}E_{\Lambda_j}(\mathrm{A},u\xi_i^j) = \Gamma_j(\mathrm{A},u\xi_i^j).$$

For a morphism $h:(\mathrm{A},\boldsymbol{a})\to(\mathrm{B},\boldsymbol{b})$ in $E_{\mathbf{L}}$ and $\boldsymbol{b} = v\xi_{i'}^{j'}\Lambda_{j'}$ one has

$$\boldsymbol{a} = h\boldsymbol{b} = hv\xi_{i'}^{j'}\Lambda_{j'},$$

so one has a morphism

$$h:\ (\mathrm{A},hv\xi_{i'}^{j'}) \to (\mathrm{B},v\xi_{i'}^{j'}),$$

hence necessarily

$$\mathrm{G}h = \mathrm{G}E_{\Lambda_j}h = \mathrm{F}_j h.$$

It is lengthy but straightforward exercise that, vice versa, one obtains a well-defined functor G this way, and that the required properties hold. ∎

**REMARK.** 5.5 can be also shown using the decomposition of $\mathrm{W}_K$ as in the Remark before 5.2: the first arrow preserves $\mathbf{D}$-colimits, and the second preserves all existing pointwise colimits.

We are now ready to prove the main result of this section. Slightly modifying a notion used in [Wel] we call a class $\mathbf{D}\subset\mathrm{Ob}\,\mathbf{Cat}$ with $\mathbf{1}\in\mathbf{D}$ *weakly saturated* if $\bar{\bar{\mathbf{D}}}$, as a full subcategory of **CAT**, is closed under $\mathbf{D}$-colimits. Trivially, $\bar{\bar{\mathbf{D}}}/K^{\mathrm{op}}$ inherits $\mathbf{D}$-cocompleteness from $\bar{\bar{\mathbf{D}}}$, so by 5.4 also $\mathrm{Pro}(\mathbf{D},K)^{\mathrm{op}}$ does that. So, for a weakly saturated class $\mathbf{D}$ and any category $K$, the category $\mathrm{Pro}(\mathbf{D},K)$ is $\mathbf{D}$-complete. This fact, however also follows from 1.4 and the following theorem:

**5.6. THEOREM.** *A weakly saturated class* $\mathbf{D}$ *is weakly admissible with respect to all* $K$.

**PROOF.** For $\mathrm{H}$: $J^{\mathrm{op}} \to \mathrm{Pro}(\mathbf{D},K)$ with $J\in\mathbf{D}$ one forms its limit $\mathbf{L}$ in $\mathrm{Pro}(\mathbf{Cat}, K)$. By 5.5 $\mathrm{W}_K$ transforms $\mathbf{L}$ into a colimit of $\mathrm{W}_K\mathrm{H}$ in $\mathbf{CAT}/K^{\mathrm{op}}$ which actually lives in $\bar{\bar{\mathbf{D}}}/K^{\mathrm{op}}$ since $\mathbf{D}$ is weakly saturated. By 5.4, $\mathrm{W}_K$ has a left adjoint which sends the colimit $\mathrm{W}_K\mathbf{L} = \mathrm{P}_{\mathbf{L}}$ to a limit $\mathbf{M}$ of H in $\mathrm{Pro}(\mathbf{D},K)$ since $\mathrm{W}_K$ is full and faithful, so $\mathbf{L}\approx\mathbf{M}$. Therefore, $\mathbf{D}$ is weakly admissible with respect to $K$. ∎

**REMARKS.** (1) We do not know whether a class $\mathbf{D}$ which is weakly admissible with respect to all $K$ must be weakly saturated. (The classes $\mathbf{D}$ we have considered in Section 4 are all weakly saturated.)

(2) That one has to pass from $\mathbf{D}$ to $\bar{\bar{\mathbf{D}}}$, in the definition of a weakly saturated class, causes considerable inconvenience but is not avoidable. For instance let $\mathbf{D}$ be the smallest subclass of Ob **Cat** which contains $\mathbf{1}$ and

$$(\cdot \rightrightarrows \cdot)$$

and which is closed under $\mathbf{D}$-colimits. One easily verifies that $\mathbf{D}$, beside the two-object categories

$$(\cdot \longrightarrow \cdot) \quad \text{and} \quad (\cdot \rightrightarrows \cdot)$$

contains only monoids (= one-object categories) with at most two generators. Now let $K$ be the category

$$\left(\cdot \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{b} \\ \xrightarrow{c} \end{array} \cdot\right).$$

$\mathrm{Pro}(\mathbf{D},K)$ fails to have equalizers "of the second generation", for instance an equalizer of $ad$ and $cd$ with $d$ the equalizer of $a$ and $b$. So $\mathrm{Pro}(\mathbf{D},K)$ is not $\mathbf{D}$-complete although $\mathbf{D}$ is closed under $\mathbf{D}$-colimits.

Finally we want to mention another consequence of Theorem 5.4: Suppose that $\mathbf{D}$ and $K$ have the property that for every diagram

$$I \longrightarrow K^{op} \longleftarrow J$$

with $I, J \in \bar{\bar{\mathbf{D}}}$ also the pullback $I \times_{K^{op}} J$ belongs to $\bar{\bar{\mathbf{D}}}$, then $\bar{\bar{\mathbf{D}}}/K^{op}$ has binary products and pullbacks. Therefore, the reflective subcategory $\mathrm{Pro}(\mathbf{D},K)^{op}$ has the same limits. Using multiple pullbacks one could even construct arbitrary products this way. If $K^{op}$ belongs to $\bar{\bar{\mathbf{D}}}$ one has a terminal object in $\bar{\bar{\mathbf{D}}}/K^{op}$ hence also in $\mathrm{Pro}(\mathbf{D},K)^{op}$. So with pullbacks one could construct all (finite) limits in $\mathrm{Pro}(\mathbf{D},K)^{op}$.

**5.7. COROLLARY.** *Suppose that* $\mathbf{D}$ *with* $1 \in \mathbf{D}$ *has the property that* $K^{op} \in \bar{\bar{\mathbf{D}}}$ *and that* $\bar{\bar{\mathbf{D}}}$ *is closed under the formation of pullbacks in* **CAT**. *Then* $\mathrm{Pro}(\mathbf{D}, K)$ *is finitely cocomplete, even cocomplete when* $\bar{\bar{\mathbf{D}}}$ *is closed under the formation of multiple pullbacks.* ∎

Of course, for a small category $K$ one may replace $\bar{\bar{\mathbf{D}}}$ by $\bar{\mathbf{D}}$. So, for $\mathbf{D} = \mathrm{Ob}\,\mathbf{Cat} = \bar{\mathbf{D}}$ one obtains the known result:

**5.8. COROLLARY.** *The Lambek completion* $\mathrm{Pro}(\mathbf{Cat}, K)$ *of a small category* $K$ *is complete and cocomplete.* ∎

However, that a class $\mathbf{D}$ satisfies the assumptions of 5.7 seems rare. For instance, for $\bar{\bar{\mathbf{D}}} = \{1\}$, $\mathbf{D}$ consists of the categories which have a terminal object; it is not closed under the formation of pullbacks in **CAT**. For $\mathbf{D} = \{$all small discrete categories$\}$, $\bar{\bar{\mathbf{D}}}$ consists of the categories which have a multiterminal object: again, it is not closed under pullbacks.

## 6. REMARKS ON INITIAL COMPLETIONS.

From now on we leave the context of $\mathbf{D}$ being a class of small categories; first we just consider the case that $\mathbf{D} = \mathbf{D}_\mu$ is the conglomerate of all (not necessarily small) discrete categories and $\Gamma = \Gamma_\mu$ is the conglomerate of all cones over $\mathbf{D}_\mu$ in the category $X$, i.e., all *sources* in $X$. A $\Gamma_\mu$-topological functor $T: B \to X$ is simply called *topological* (or $(B, T)$ is called *initially complete*. cf. [H1, AHS]). A basic observation on topological functors is that they are necessarily faithful (in [BT], a general reason for this is given). Therefore, they admit initial liftings not only to discrete but to all data; in other words: the restriction to discrete data is no restriction once there is no size restriction on the data.

Since $T: \mathrm{Pro}(\mathbf{D}_\mu, \Gamma_\mu, U) \to X$ in general fails to be faithful, even when $U: A \to X$ is faithful, $T$ is not a candidate for an initial completion of $U$, even when we disregard the fact that $\mathrm{Pro}(\mathbf{D}_\mu, \Gamma_\mu, U)$ lives in a higher universe. One therefore considers the non-full subcategory $\mathrm{Pro}^\mu(\mathbf{D}_\mu, \Gamma_\mu, U)$ of $\mathrm{Pro}(\mathbf{D}_\mu, \Gamma_\mu, U)$ which has the same objects, but as its morphisms $(u,\boldsymbol{f})$ one considers only those $\boldsymbol{f} = (\boldsymbol{f}_j)$ for which each $\boldsymbol{f}_j$ is represented by an identity map. One may now simplify the description of $\mathrm{Pro}^{\mu}(\mathbf{D}_{\mu},\Gamma_{\mu},\mathrm{U})$ as follows: objects are pairs $(\mathrm{X},\sigma)$ with X an object in $X$ and $\sigma$ a class of U-morphisms with domain X; a morphism $u$: $(\mathrm{X},\sigma) \to (\mathrm{Y},\tau)$ is an $X$-morphism with the property that $yu: \mathrm{X} \to \mathrm{UB}$ is in $\sigma$ for each $u: \mathrm{Y} \to \mathrm{UB}$ in $\tau$. It is elementary to show that the restriction

$$\mathrm{T}^{\mu}\colon\ \mathrm{Pro}^{\mu}(\mathbf{D}_{\mu},\Gamma_{\mu},\mathrm{U}) \longrightarrow X$$

of T is topological; the $\mathrm{T}^{\mu}$-initial lifting of a $\mathrm{T}^{\mu}$-source (= discrete $\mathrm{T}^{\mu}$-cone) $u_i: \mathrm{X} \to \mathrm{T}^{\mu}(\mathrm{Y}_i,\tau_i)$ $(i \in \mathrm{I})$ provides X with the structure $\sigma$ consisting of all compositions of U-morphisms in $\tau_i$ with $u_i$, $i \in \mathrm{I}$.

The embedding $\mathrm{E}: A \to \mathrm{Pro}(\mathbf{D}_{\mu},\Gamma_{\mu},\mathrm{U})$ does not factor through $\mathrm{Pro}^{\mu}(\mathbf{D}_{\mu},\Gamma_{\mu},\mathrm{U})$; one therefore considers a *new* embedding

$$\mathrm{E}^{\mu}\colon\ A \longrightarrow \mathrm{Pro}^{\mu}(\mathbf{D}_{\mu},\Gamma_{\mu},\mathrm{U})$$

with $\mathrm{E}^{\mu}\mathrm{A} = (\mathrm{UA},\sigma_{\mathrm{A}})$ and $\sigma_{\mathrm{A}}$ the class of all U-morphisms $\mathrm{U}f$: $\mathrm{UA} \to \mathrm{UB}$ $(f: \mathrm{A} \to \mathrm{B}$ in $A)$. Obviously, $\mathrm{T}^{\mu}\mathrm{E}^{\mu} = \mathrm{U}$; however, $\mathrm{E}^{\mu}$ fails to be initially dense (cf. 2.4) and can therefore not be expected to have a universal property like in 2.5. So, quite naturally, one restricts oneself to certain full subcategories of $\mathrm{Pro}^{\mu}(\mathbf{D}_{\mu},\Gamma_{\mu},\mathrm{U})$ and restricts $\mathrm{T}^{\mu}$ and $\mathrm{E}^{\mu}$ appropriately, to obtain

– the *largest initial completion* of U (which is determined by the universal property 2.5, but with all functors to be faithful),

– the *universal initial completion* of U (the embedding into the completion preserves initiality and is universal only with respect to such functors which, as before, have to be also faithful),

– the *least* (= *MacNeille) completion* of U (which may be characterized as the injective hull of U in $\mathbf{CAT}_{\mathrm{ff}}/X$, with respect to full embeddings);

for details we refer to Herrlich [H1] and Adámek-Herrlich-Strecker [AHS]. The latter paper addresses the problem of when these completions belong to the same universe as $A$ and $X$, and gives sufficient and necessary conditions in terms of fibre-smallness.

A. Charles Ehresmann's up-date [CE] on Ehresmann's work [E] shows that, in fact, it is possible to perform these constructions in a generalized context in which one no longer deals solely with discrete data. This way one is able to also include concrete limit-completions as constructed by Adámek-Koubek [AK] and Herrlich [H2] (where one needs small non-discrete data).

The result proved in ([CE], Theorem 4) is as follows: let $\Gamma$ be any conglomerate of cones in $X$ and $\Delta$ a conglomerate of U-initial cones in $A$ such that the faithful functor $U: A \to X$ maps cones in $\Delta$ into $\Gamma$. Then there is a faithful $\Gamma$-topological functor $T: C \to X$ and a concrete full embedding $E: A \to C$ which sends cones in $\Delta$ to T-initial cones; any other concrete functor $F: A \to B$ sending cones in $\Delta$ to V-initial cones with $V: B \to X$ a faithful $\Gamma$-topological functor, factors uniquely (up to isomorphisms) through E by a concrete functor $G: C \to B$ which preserves initiality.

In the proof, which needs transfinite methods, one has to pass from $\Gamma$ to a larger conglomerate $\Gamma^{\mu}$ of sources (whose bases belong to a conglomerate $\mathbf{D}^{\mu}$ of discrete categories) and then consider an appropriate full subcategory $C$ of $\mathrm{Pro}^{\mu}(\mathbf{D}_{\mu}, \Gamma_{\mu}, U)$ through which $E^{\mu}$ factors. The case

– $\Delta = \varnothing$, $\Gamma = \Gamma_{\mu}$ gives the largest initial completion,

– $\Delta$ = {all U-initial sources}, $\Gamma = \Gamma_{\mu}$, gives the universal initial completion,

whereas the MacNeille completion needs some extra considerations (cf. [CE] for details). However, the choice

– $\Delta$ = {all limit-cones in $A$ which are preserved by U} and $\Gamma$ = {all small cones} gives the completion described in [AK] and [H2] which is always legitimate and can be described in a fairly constructive way.

Such a result does not hold for non-faithful functors, i.e., in $\mathbf{CAT}/X$ rather than in $\mathbf{CAT}_{ff}/X$, even when $X = \mathbf{1}$: in general, it is impossible to legitimately (i.e., within the same universe) embed a given category $K$ into one with certain limits such that the embedding preserves the existing limits in $K$ of the given type; there is a counterexample by Trnková [Tr2] concerning finite products, and one by Isbell [I2] concerning equalizers. That is why in the non-faithful context we could not consider an additional parameter $\Delta$ like in Ehresmann's result.

**NOTE ADDED IN PROOF.** Independently from Trnková [Tr2], C. Ehresmann gave transfinite constructions for various types of completions (cf. *Cahiers Top. et Géom. Diff.* IX (1967); reprinted in *"Charles Ehresmann: Œuvres complètes et commentées"* Part IV-1, Amiens 1982 ; see also the "Comments" by A.C. Ehresmann in that volume for further references.

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