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GIUSEPPE PAXIA
IRENE SERGIO

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ON THE CONNECTEDNESS OF SOME TOPOLOGICAL SPACES

by Giuseppe PAXIA and Irene SERGIO

RÉSUMÉ. Cet article étudie une propriété de connexité pour les espaces topologiques. Dans un espace topologique X avec un nombre fini de composantes irréductibles, sans polygones, l'intersection de deux ouverts connexes $\mathcal{U}$ et $\mathcal{U}'$ est aussi connexe. Ensuite on donne des applications aux schémas affines et divers exemples.

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INTRODUCTION.

In this paper a connectedness property of some topological spaces is studied. The idea is suggested by an important work [A] of M. Artin where the notion of an "n-polygon" in a topological space is introduced; Artin characterizes rings whose spectra have no polygons. A class of rings is obtained, called absolutely integrally closed (for short AIC), to be precise the class of rings A such that every polynomial $f(X) \in A[X]$ splits into linear factors.

In her paper [S] I. Sergio, modifying Artin's definition of n-polygon, characterizes the connected rings without polygons as those for which a suitable localization has not locally trivial rank-two étale coverings.

In this paper with a further modification of Artin's definition of n-polygon we will prove the following property: let X be a topological space with finitely many irreducible components without n-polygons, then the intersection of any two connected open sets is still connected. This result is obtained for general topological spaces and by elementary methods. It can be applied, in particular, to affine schemes associated to AIC rings, or those characterized by [S], to affirm that such rings with finitely many prime ideals have the property that $X_f \cap X_g$ is connected in $X = \operatorname{Spec} A$ if X_f and X_g are connected.

The paper has two sections. The first deals with some preliminaries and the second proves the main theorem, from which we deduce some consequences.

I.

As was said in the introduction, in [A] M. Artin has given a general definition of an n-polygon in a topological space X.

DEFINITION (D1). In a topological space X an n-polygon for $n \geq 2$ is a configuration consisting of the following data: an integer n and irreducible closed sets $D_0, \dots, D_{n-1}$, $C_0, \dots, C_{n-1}$ in X such that

$$D_i \subset C_j \Leftrightarrow \quad j = i,\ i+1, \text{ indices mod. } n.$$

The closed sets D_i are said to be *vertices*, while the C_j are said to be *sides*.

Remembering that a ring A is said to be "absolutely integrally closed" (AIC), if every polynomial $f(X) \in A[X]$ factors completely in A, by what is implicitly said in Section 1 of Artin's paper, one can deduce that an AIC ring can be characterized as one whose spectrum has no polygons.

In her paper [S] I. Sergio modifies Artin's definition of n-polygon imposing that the irreducible closed sets C_i are minimal among those which contain D_{i-1} and D_i. We call (D2) the definition of n-polygon given in [S] and we recall that she has proved that the connected rings R whose spectra have no polygons in the sense of (D2) are characterized as those in which a suitable localization has no locally trivial rank-two étale coverings. The separably closed domains and the rings which are a direct product of separably closed domains belong to such a class. In this paper we give the following

DEFINITION (D3). In a topological space X, an n-polygon for $n \geq 2$ is a configuration consisting of the following data: an integer n and irreducibl̀e closed sets $D_0, \ldots, D_{n-1}, C_0, \ldots, C_{n-1}$ such that

$$D_i \subset C_j \Leftrightarrow j = i, i+1, \text{ indices mod. } n;$$

the C_j are irreducible components in X.

Such a definition is motivated by the following characterization of the topological spaces with finitely many irreducible components given in [G], 2.1.10.

PROPOSITION 1. *A topological space X with finitely many irreducible components is connected if and only if for any couple of distinct irreducible components X' and X" there is a sequence $(X_i)_{0 \le i \le n}$ of irreducible components such that*

$$X_0 = X', \; X_n = X'', \text{ and for every } i = 1, \ldots, n, \; X_{i-1} \cap X_i \neq \emptyset.$$

From now on a sequence of the above considered kind will be called a sequence joining X' and X".

REMARK 1. We can say that the definition (D1) of n-polygon given by Artin is more general than the definitions (D2) or (D3), since an n-polygon in the sense of (D2) or (D3) is in the sense (D1), too. On the other hand, applying Zorn's Lemma, one can see at once that from an n-polygon in the sense of (D3) an n-polygon can be constructed in the sense of (D2), but not vice versa. Therefore one can say that if a topological space X has no polygons in the sense of (D1) or (D2) it will not have polygons in the sense of (D3) either.

REMARK 2. It is worth observing that in a topological space X without n-polygons in the sense of (D3), then the not empty intersection of two irreducible components is always irreducible; otherwise there should be 2-polygons.

We recall the following important result. Let $\mathcal{U} \subset X$ be a not empty open set of the topological space X, then there exists a one-to-one correspondence between the irreducible closed sets of X which meet $\mathcal{U}$ and the irreducible closed sets of $\mathcal{U}$ in the induced topology. Such a correspondence is given by $Z \mapsto Z \cap \mathcal{U}$, and induces a bijection on the irreducible components; see [B], Proposition 7.11, § 4. As an immediate consequence we obtain the following:

PROPOSITION 2. *Let* $\mathcal{U} \subset X$ *be a not empty open set in the topological space* X *and* X *has no polygons in the sense of* (D3), *then* $\mathcal{U}$ *has no polygons in the induced topology.*

II.

Let X be a topological space such that $X = \mathcal{U} \cup \mathcal{U}'$ where $\mathcal{U}$ and $\mathcal{U}'$ are not empty open sets such that $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$. The topology on X induces a topology on $\mathcal{U}$, $\mathcal{U}'$ and $\mathcal{U} \cap \mathcal{U}'$. In the sequel for every irreducible closed set $P \subset X$ we will put

$$\bar{P} = P \cap \mathcal{U},\ \bar{P}' = P \cap \mathcal{U}' \text{ and } P^* = P \cap \mathcal{U} \cap \mathcal{U}'.$$

THEOREM 1. *Let* X *be a topological space with finitely many irreducible components and without n-polygons in the sense of* (D3), *for all* $n \geq 2$. *Then the intersection of any two connected open sets is still connected.*

PROOF. We consider two not empty open sets $\mathcal{U}$ and $\mathcal{U}'$ both connected and such that $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$. Without loss of generality, we can assume $X = \mathcal{U} \cup \mathcal{U}'$, by Proposition 2. Let us suppose that $\mathcal{U} \cap \mathcal{U}'$ be not connected and prove it is not possible. Then it follows that $\mathcal{U} \cap \mathcal{U}'$ is the disjoint union of finitely many connected components. Let F and G be two such components, then, by Proposition 1, there exist two irreducible components $P^* \subset F$ and $Q^* \subset G$ such that there will not be any sequence joining P^* and Q^* in $\mathcal{U} \cap \mathcal{U}'$. Let P and Q be the irreducible components of X which cut P^* and Q^* in $\mathcal{U} \cap \mathcal{U}'$. By hypothesis $\mathcal{U}$ and $\mathcal{U}'$ are connected and in addition

$$\bar{P} = P \cap \mathcal{U},\quad \bar{Q} = Q \cap \mathcal{U}.\quad \bar{P}' = P \cap \mathcal{U}',\quad \bar{Q}' = Q \cap \mathcal{U}'$$

are not empty and are irreducible components respectively in $\mathcal{U}$ and $\mathcal{U}'$. It means that there exist irreducible components

$$(X_i)_{1 \leq i \leq l} \text{ and } (Y_j)_{1 \leq j \leq m}$$

in X such that

$$X_1 = Y_1 = P,\quad X_l = Y_m = Q$$

so as to satisfy the following conditions:

a) $\bar{X}_i \cap \bar{X}_j \neq \emptyset \Leftrightarrow j = i,\ i+1,\ \bar{Y}'_i \cap \bar{Y}'_j \neq \emptyset \Leftrightarrow j = i, i+1.$

b) Indices i_1 and i_2 must exist so that

$$X_{i_1} \cap X_{i_1+1} \subset \mathcal{U} \setminus \mathcal{U}' \text{ and } Y_{i_2} \cap Y_{i_2+1} \subset \mathcal{U}' \setminus \mathcal{U}.$$

Condition a derives from the connectedness of $\mathcal{U}$ and $\mathcal{U}'$ while b holds since P^* and Q^* cannot be joined in $\mathcal{U} \cap \mathcal{U}'$. Now by induction on the number $k = l + m$ we prove that the previous data are inconsistent with the hypotheses. We can distinguish different cases.

1) If $k = 4$, then $l = m = 2$ and $X_1 = Y_1$, $X_2 = Y_2$; by b it follows

$$X_1 \cap X_2 \subset (\mathcal{U} \setminus \mathcal{U}') \cap (\mathcal{U}' \setminus \mathcal{U}) = \emptyset$$

which contradicts a.

2) If $k = 5$, then $l = 2$, $m = 3$ and $X_1 = Y_1$, $X_2 = Y_3$. By b one has obtained $X_1 \cap X_2 \subset \mathcal{U} \setminus \mathcal{U}'$ and one of the following:

$$Y_1 \cap Y_2 \subset \mathcal{U}' \setminus \mathcal{U} \quad \text{or} \quad Y_2 \cap Y_3 \subset \mathcal{U}' \setminus \mathcal{U}.$$

In both cases it follows immediately that $Y_1 \cap Y_2 \cap Y_3 = \emptyset$. Then Y_1, Y_2, Y_3 are sides of a 3-polygon since $Y_1 \cap Y_2$, $Y_2 \cap Y_3$, $Y_1 \cap Y_3$ are irreducible not empty closed sets; this follows from Remark 2.

3) Suppose $k > 5$. We will apply the induction principle on k where it is necessary. We will now discuss all the cases which can occur.

i) $X_2 \cap Y_2 = \emptyset$. In this case there exist r and s such that

$$X_r \cap Y_s \neq \emptyset \text{ while } X_i \cap Y_j = \emptyset \text{ for } 1 < i < r \text{ and } 1 < j < s.$$

We notice an immediate contradiction since the irreducible components $X_1 \ldots\ldots X_r, Y_s \ldots\ldots Y_2$ form an $(r+s-1)$-polygon, in contradiction with the hypothesis.

ii) $X_2 \cap Y_2 \neq \emptyset$. In this case the condition $i_1 = i_2 = 1$ cannot occur; otherwise X_1, X_2, Y_2 would be a 3-polygon, as a consequence of the fact that

$$X_1 \cap X_2 \cap Y_1 \cap Y_2 = \emptyset.$$

Now let's consider the two possibilities: I) $i_1, i_2 > 1$ and II) $i_1 = 1$, $i_2 > 1$.

I) $i_1, i_2 > 1$. If $X_1 \cap Y_2 \cap \mathcal{U} \neq \emptyset$ we can apply the inductive hypothesis to the sequences $Y_2, X_2, \ldots, X_l$ and $Y_2, \ldots, Y_m$ which have the same extremes, global length $< k$, $i_1, i_2 > 1$. So they satisfy the required conditions for induction applicability. We notice that in this case the irreducible component Y_2 meets $\mathcal{U}\cap\mathcal{U}'$ and so $Y_2^*\subset F$; we have a contradiction. Hence this case cannot occur. Similarly we can verify the same if

$$X_2\cap Y_2\cap \mathcal{U}'\neq\emptyset.$$

Now let us consider the case II: $i_1=1$, $i_2>1$. By the hypotheses one obtains $X_1\cap X_2\subset\mathcal{U}\setminus\mathcal{U}'$. We shall focus our attention on $X_1=Y_1, X_2, Y_2$. Since in X there are no polygons, one of the following conditions must be true:

$$a_1)\ X_1\cap Y_2\subset X_2;\ a_2)\ X_2\cap Y_2\subset X_1;\ a_3)\ X_1\cap X_2\subset Y_2.$$

a_1 implies

$$Y_1\cap Y_2\subset X_1\cap X_2\subset\mathcal{U}\setminus\mathcal{U}'$$

from which $Y_1\cap Y_2\cap\mathcal{U}'=\emptyset$; but this is in contradiction with the hypothesis $\bar{Y}'_1\cap\bar{Y}'_2\neq\emptyset$.
a_2 implies $X_2\cap Y_2\subset X_1\cap X_2$ from which $X_2\cap Y_2\cap\mathcal{U}'=\emptyset$ follows. Then $X_2\cap Y_2\subset\mathcal{U}\setminus\mathcal{U}'$. On the other hand $i_2>1$, then, considering the sequences $Y_2,\dots,Y_m$, $Y_2, X_2,\dots,X_l$ one can deduce that they satisfy the required conditions for induction applicability. So there must be a contradiction. Finally, let's consider case a_3. First of all we obtain $Y_1\cap Y_2\cap\mathcal{U}\neq\emptyset$, and since $Y_1\cap Y_2\cap\mathcal{U}'\neq\emptyset$, necessarily

$$Y_1\cap Y_2\cap\mathcal{U}\cap\mathcal{U}'\neq\emptyset,$$

as $Y_1\cap Y_2$ is an irreducible closed set. On the other hand it is clear that $X_2\cap Y_2\cap\mathcal{U}\neq\emptyset$. Now if $X_2\cap Y_2\subset\mathcal{U}\setminus\mathcal{U}'$ we can apply the inductive hypothesis to the sequences $Y_2, X_2,\dots,X_l$, $Y_2,\dots,Y_m$ and we will have a contradiction.

If $X_2\cap Y_2\cap\mathcal{U}\cap\mathcal{U}'\neq\emptyset$, then there must exist an index i' with $2\leq i'\leq l-1$ such that $X_{i'}\cap X_{i'+1}\subset\mathcal{U}\setminus\mathcal{U}'$, otherwise the sequence $Y_1, Y_2, X_2, \dots, X_l$ would join P^* and Q^* in $\mathcal{U}\cap\mathcal{U}'$. By applying the induction to the sequences $Y_2, X_2,\dots,X_l$ and $Y_2,\dots,Y_m$ we will have a contradiction.

The above result has a natural and useful application if applied to affine schemes. Actually we know a wide class of rings whose spectra have no polygons. In view of this, it is convenient to remember the main result of Sergio's paper [S].

THEOREM 2. *Let* R *be a ring, then* Spec R *has no polygons in the sense* (D2) *and only if* $R'=R_s/a$ *has no locally trivial rank* *two étale converings, where* $a \subset \mathrm{Rad}\,R_s$ *and* $S = R \setminus \cup p_i$ *with* p_i *prime ideals in* R.

The separably closed rings and those which are a product of finitely many separably closed rings belong to the above class along with the AIC rings of Artin. For details see [S], §2.

Remembering what has been said in Remark 1, Section 1, namely: to be without n-polygons in the sense of (D1) or (D2) implies being without n-polygons in the sense of (D3), then we get the following.

COROLLARY 1. *Let* R *be a ring which satisfies the conditions of Theorem 2 with finitely many prime ideals. If* $X = \mathrm{Spec}\,R$ *and* f, $g \in R$ *are such that* X_f *and* X_g *are not empty connected open sets, then* $X_{fg} = X_f \cap X_g$ *is connected too.*

REMARK 3. In fact Corollary 1 can be deduced as a consequence of Théorème 2 and Proposition 2.23 of [G-S]. The interest of our Theorem 1 consists in the fact that our result can be applied to general topological spaces, and not only to affine schemes. In addition we achieved the result by very elementary methods.

Now to give meaning to Theorem 1 it is important to show by some examples that all the hypotheses are necessary and cannot be dropped.

EXAMPLE 1. The following is an example of a topological space with polygons in the sense of (D3) where the property expressed by Theorem 1 doesn't hold. We consider the ring $R = k[X,Y]$ with k an algebraically closed field. Let $a \subset R$ be the ideal product of $p_1 = (Y)$ and $p_2 = (Y - X^2 + 1)$ and $X = \mathrm{Spec}\,R/a$. In X we have a polygon of vertices $P_1 = (1, 0)$ and $P_2 = (-1, 0)$. Let f and g be the images of $X-1$ and $X+1$ in R/a ; it is easy to show that X_f and X_g are not empty connected open sets in X while $X_{fg} = X_f \cap X_g$ is not connected.

EXAMPLE 2. We construct a topological space S, with no polygons for which Theorem 1 is false (of course S must have infinitely many irreducible components). Our S will be an affine algebraic k-scheme, where k is an algebraically closed field. Let us start with a subscheme S_3 of the affine plane $\mathbf{A}^2_k$ consisting of three lines L_1, L_2, L meeting at three different points

$$P_1 = L \cap L_1, \; P_2 = L \cap L_2, \; Q = L_1 \cap L_2 .$$

Let φ: $X \to \mathbf{A}^2_k$ be the blow-up of $\mathbf{A}^2_k$ at Q (see [H] for general facts on the blow-up used below); then X is a smooth surface, and

$$\varphi^{-1}(S_3) = \tilde{L}_1 \cup \tilde{L}_2 \cup \tilde{L} \cup E$$

where "~" denotes "proper transform" and $E \approx \mathbf{P}^1$ is the exceptional divisor. It is clear that E meets $\tilde{L}_1$ and $\tilde{L}_2$ respectively in two distinct points Q_1, Q_2.

Now, let S_4 be the affine scheme obtained from $\varphi^{-1}(S_3)$ by deleting one point $R \in E$, $R \neq Q_1$, $R \neq Q_2$ and let φ_3: $S_4 \to S_3$ be the canonical morphism (clearly surjective). Intuitively S_4 can be considered as the "4-sided polygon" obtained from S_3 by replacing the "vertex" Q with the "side" E-R, which is isomorphic to an affine line. We repeat the procedure by blowing up the smooth surface X-{R} at either Q_1 or Q_2 to obtain S_5 ; and continue by blowing up, at each step, one of two "vertices" lying on the exceptional divisor deleting each time one point on it. We obtain a sequence of affine k-schemes and surjective morphisms

$$\dots \longrightarrow S_n \longrightarrow S_{n-1} \longrightarrow \dots \longrightarrow S_3$$

which corresponds to a sequence of affine k-algebras

$$\dots \supset A_n \supset A_{n-1} \supset \dots \supset A_3$$

Let

$$A = \varinjlim A_n, \; S = \operatorname{Spec} A = \varprojlim S_n.$$

Then S is connected and has no polygons (according to Definition D3). Note also that S has infinitely many irreducible components. Now, let m_1, m_2, m be the maximal ideals of A_3 corresponding to the points P_1, P_2, Q and let $f, g \in A_3 \subset A$ be such that

$$f \in m_1, \; f \notin m_2 \cup m, \quad g \in m_2, \; g \notin m_1 \cup m.$$

By our construction the open sets $\operatorname{Spec}(A_n)_f$ and $\operatorname{Spec}(A_n)_g$ of S_n are connected and not empty for every $n \geq 3$; and since direct limits commute with fractions, it follows that

$$S_f = \mathrm{Spec}(A_f) \text{ and } S_g = \mathrm{Spec}(A_g)$$

are not empty connected open subsets of S. On the other hand $\mathrm{Spec}(A_3)_{fg}$ is not connected, and hence there is a non trivial idempotent $e \in (A_3)_{fg} \subset A_{fg}$. Hence $S_f \cap S_g = \mathrm{Spec}(A_{fg})$ is not connected.

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DIPARTIMENTO DI MATEMATICA
DELL' UNIVERSITA DI CATANIA
VIALE A. DORIA 6
CATANIA. ITALIE