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A CONSTRUCTIVE "CLOSED SUBGROUP THEOREM" FOR LOCALIC GROUPS AND GROUPOIDS

by Peter T. JOHNSTONE

RÉSUMÉ. Cet article introduit de nouveaux concepts de fermeture et densité pour les sous-locales de locales sur une base locale B. Si B est le locale terminal, ces notions se réduisent classiquement aux notions usuelles, mais elles sont différentes de manière constructive. En les utilisant, nous montrons une version constructive du théorème selon lequel tout sous-groupe localique d'un groupe localique est fermé, et une extension de ce résultat aux groupoïdes localiques.

INTRODUCTION.

The "closed subgroup Theorem" for localic groups, first proved in [2] (and subsequently, by a shorter proof, in [6]), asserts that any localic subgroup of a localic group G is closed as a sublocale of G. It is well known that this result, as ordinarily stated, is not true constructively (we shall give a counterexample shortly); in view of the importance of a constructive approach in locale theory, as emphasized in [5], this has for some time been a matter for regret (to the present author, at least). Recently, it has emerged that the fault lies in the notion of closedness: in this paper we shall introduce a new notion of "weak closedness" for sublocales, which reduces to the usual notion in the presence of the Law of Excluded Middle (but is otherwise strictly weaker), and we shall prove

THEOREM 1. *Let* H *be a localic subgroup of a localic group* G, *and suppose* H *is an open locale. Then* H *is weakly closed in* G.

Since classically every locale is open, this theorem genuinely reduces to the main theorem of [2] and [6] in the presence of classical logic. However, it yields new results even in the classical context, about "fibrewise localic groups" (that is, group objects in **Loc**/B, where B is a fixed base locale), and hence should illuminate the relation between fibrewise localic and fibrewise topological groups (cf. [3]) in much the same way as the original closed subgroup theorem illuminates the relationship between localic and topological groups.

This seems an appropriate point at which to give our promised example to show why the closed subgroup theorem, as usually formulated, cannot be constructively true. Let B be a base locale (which may as well be spatial, so long as it is not discrete), and G a nontrivial discrete group. Then (the sheaf of sections of) the projection $G\times B \to B$ is a group object in the topos **Sh**(B) of sheaves on B, and we may think of it as a discrete localic group. Let U be an open sublocale of B which is not closed, and let H be the open sublocale $(G\times U)\cup(\{e\}\times B)$ of $G\times B$. Clearly H is a subgroup of $G\times B$ in **Sh**(B), but if we regard it as a (discrete) localic subgroup it is not closed, because closedness is "absolute" — i.e., it is preserved and reflected by the equivalence between internal locales in **Sh**(B) and external locales over B.

On the other hand, there is clearly a sense in which H *is* closed in $G\times B$, at least if B is spatial: it is closed in each fibre of the projection $G\times B \to B$, since these fibres are discrete. Our first aim in this paper is thus to develop a notion of "fibrewise closedness" (and a corresponding notion of fibrewise denseness) for locales over an arbitrary base locale B; our definition of weak closedness will then be as fibrewise closedness over the terminal locale Ω. (There is a certain spurious generality about developing fibrewise closedness over an arbitrary base B, rather than simply over Ω, since the equivalence from locales over B to internal locales in **Sh**(B) maps B itself to Ω; but since we shall require a number of results involving change of base, it will be convenient to have the base locale explicitly present throughout our arguments — in the first two sections, at least.)

Reverting to Theorem 1, it turns out that there is still more that can be said, concerning localic groupoids: in view of the recent interest in localic groupoids [10,11,12] arising from the Joyal-Tierney representation Theorem [8] for Grothendieck toposes, this result seems very likely to be of use in the future.

THEOREM 2. *Let*

$$\mathbf{G} = (G_1 \rightrightarrows G_0)$$

be a localic groupoid, and

$$\mathbf{H} = (H_1 \rightrightarrows H_0)$$

a subgroupoid such that the domain map (or equivalently the codomain map) $H_1 \to H_0$ *is open (i.e.,* **H** is a regular localic groupoid, in the terminology of [9]). *Then the inclusion* $H_1 \to G_1$ *is fibrewise closed over* $G_0 \times G_0$.

Theorem 1 is of course a special case of Theorem 2, in which G_0 and H_0 are both taken to be Ω. However, we shall give (in Section 3 below) a separate proof of Theorem 1, both in order to motivate the argument employed in proving Theorem 2 and to emphasize the extent to which the proof of Theorem 1 follows exactly the same lines as the proof of the classical result of [6]. The proof of Theorem 2 occupies Section 4; the final section of the paper discusses other potential applications of weak and fibrewise closedness, some of which will be investigated further in later papers.

The results described in this paper were obtained during A. Kock's visit to Cambridge in April-May 1988, and it is a pleasure to acknowledge the stimulation provided by conversations with him, which contributed significantly to the paper's final form.

1. FIBREWISE DENSE AND FIBREWISE CLOSED INCLUSIONS.

Throughout this section we shall be working (constructively) in the category **Loc**/B of locales over a fixed base B; results which involve a change of base will be relegated to Section 2. We begin with the definition of fibrewise denseness.

DEFINITION 1.1. Let

$$\begin{tikzcd} A' \arrow[rr, hook, "f"] \arrow[dr, "h"'] & & A \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

be an inclusion in **Loc**/B. We say f is B-*fibrewise dense* (or fibrewise dense over B, or — if it is necessary to name the structure map $A \to B$ explicitly — fibrewise dense over g) if the following (clearly equivalent) conditions are satisfied:

(i) For every $b \in B$, $f_* h^*(b) = g^*(b)$.

(ii) For every $a \in A$ and $b \in B$,

$$f^*(a) \leq h^*(b) \text{ implies } a \leq g^*(b).$$

(iii) The nucleus $j = f_* f^*$ on A fixes all elements of the form $g^*(b)$.

(iv) As a subset of A, A' contains the image of g^*.

In the case when B is the terminal locale Ω, we say that the inclusion f is *strongly dense*. Note that this implies that f is dense in the usual sense (i.e., that f_* preserves 0); if we assume classical logic, so that $\Omega = \{0,1\}$, it is equivalent to ordinary denseness, since the equality $f_* h^*(1) = g^*(1)$ holds automatically.

From condition (iv) of the definition, we immediately obtain:

LEMMA 1.2. *Any locale over* B *has a smallest* B-*fibrewise dense sublocale.*

PROOF. The smallest B-fibrewise dense sublocale of $(g\colon A \to B)$ is simply the intersection of all sublocales of A which contain the image of g^* (cf. [4], II 2.5). ∎

The nucleus corresponding to the sublocale in Lemma 1.2 can be writtern down explicitly; it is the map

$$a \mapsto \bigwedge\{((a \Rightarrow g^*(b)) \Rightarrow g^*(b)) \mid b \in B\}.$$

However, this explicit description does not appear to be of any practical use.

LEMMA 1.3. *Let j and k be nuclei on a locale* A *over* B *with $j \geq k$, so that we have a diagram*

$$\begin{tikzcd}
A_j \arrow[rr, hook, "f"] \arrow[dr, "d"] \arrow[ddr, "h"'] & & A \arrow[ddl, "g"] \\
& A_k \arrow[ur, "e"] \arrow[d, "l"] & \\
& B &
\end{tikzcd}$$

Then d is B*-fibrewise dense iff the nuclei j and k agree on the image of g .*

PROOF. Suppose $jg^* = kg^*$. Then for any $b \in$ B we have

$$e_* \, d_* h^*(b) \;=\; f_* \, h^*(b) = jg^*(b) = kg^*(b) = e_* \, l^*(b),$$

whence

$$d_* h^*(b) \;=\; l^*(b)$$

since e_* is injective. The converse is similar. ∎

COROLLARY 1.4. *Given a diagram*

$$\begin{tikzcd} A' \arrow[rr, hook, "f"] \arrow[dr, "h"'] & & A \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

there is a unique largest sublocale of A *(the* B*-fibrewise closure of* A_j*) which contains* A_j *as a* B*-fibrewise dense sublocale.*

PROOF. By Lemma 1.3, finding this largest sublocale is equivalent to finding the smallest nucleus on A which agrees with j at all elements of the form $g^*(b)$. But meets in the lattice of nuclei are computed pointwise ([4], II 2.5), so we simply take the meet of all nuclei which agree with j on the image of g^*. ∎

It is clear from the construction that the B-fibrewise closure operation is idempotent.

DEFINITION 1.5. We say a sublocale of a locale over B is B—*fibrewise closed* if it coincides with its B-fibrewise closure. In the particular case $B = \Omega$, we say the sublocale is *weakly closed* (and refer to the *weak closure* of an arbitrary sublocale).

I have not been able to find an intrinsic characterization of the nuclei on A which correspond to B-fibrewise closed sublocales. By analogy with the classical case, one might expect the B-fibrewise closure of a nucleus j on A to be the map

$$a \mapsto a \vee j(g^* g_*(a)),$$

and it is not hard to show that if this map is a nucleus then it is the right one; but there seems to be no reason why it should be a nucleus in general. However, this lack of an explicit formula turns out to be no barrier to further progress, as the following results show.

LEMMA 1.6. *Given a commutative diagram*

$$\begin{tikzcd}
C' \arrow[r, hook, "f'"] \arrow[d, "h'"'] & C \arrow[d, "h"] \\
A' \arrow[r, hook, "f"] \arrow[dr, "g'"'] & A \arrow[d, "g"] \\
 & B
\end{tikzcd}$$

with f B-fibrewise closed and f' B-fibrewise dense, there exists a unique l: $C \to A'$ with $fl = h$ and $lf' = h'$.

PROOF. First we consider the case when both h and h' are inclusions. Let j,k be the nuclei $f_* f^*$ and $h_* h^*$ respectively; then we need to show that $j \leq k$. or equivalently that $j = j \wedge k$. But since j is B-fibrewise closed, it suffices by the construction of 1.4 to show that j and $j \wedge k$ agree on elements of the form $g^*(b)$. Now the fact that f' is B-fibrewise dense means that, as a subset of C, C' contains all elements of the form $h^* g^*(b)$; and since $C' \subset A' \cap C$ as a subset of A, this means that $kg^*(b) \in A'$ for all b, or equivalently $jkg^*(b) = kg^*(b)$. But this clearly implies $jg^*(b) \leq kg^*(b)$, so the result is proved in this case.

In the general case, let

$$C \xrightarrow{q} D \xhookrightarrow{i} A \text{ and } C' \xrightarrow{q'} D' \xhookrightarrow{i'} A'$$

be the image factorizations of h and h', and let f'': $D' \to D$ be the map induced by f and f' (note that f'' is an inclusion, since $if'' = fi'$ is). It suffices to show that f'' is B-fibrewise dense, since then we can apply the previous argument to f and f''. But we have

$$\begin{aligned}
i^* g^* &= q_* q^* i^* g^* && \text{since } q \text{ is an epimorphism} \\
&= q_* h^* g^* = q_* f'_* h'^* g'^* && \text{since } f' \text{ is fibrewise dense} \\
&= f''_* q'_* q'^* i'^* g'^* && \\
&= f''_* i'^* g'^* && \text{since } q' \text{ is an epimorphism. } \blacksquare
\end{aligned}$$

COROLLARY 1.7. B-*fibrewise closure is functorial on the full subcategory of* $(\mathbf{Loc}/\mathrm{B})^2$ *whose objects are inclusions over* B. ∎

COROLLARY 1.8. B-*fibrewise closed inclusions are stable under composition and pullback in* $\mathbf{Loc}/\mathrm{B}$.

PROOF. Both assertions follow easily if we form the fibrewise closure of the composite or pullback and then apply Lemma 1.6 to the appropriate commutative square. ∎

It is immediate from Definition 1.1 (i) that B-fibrewise dense inclusions are stable under composition; but we should not expect them to be stable under arbitrary pullback, since this is not true for the classical notion of denseness. However:

LEMMA 1.9. B-*fibrewise dense inclusions are stable under pullback along open maps.*

PROOF. Consider a diagram

$$\begin{tikzcd}
\mathrm{C}' \arrow[rr, hook, "f'"] \arrow[d, "h'"'] & & \mathrm{C} \arrow[d, "h"] \\
\mathrm{A}' \arrow[rr, hook, "f"] \arrow[dr, "g'"'] & & \mathrm{A} \arrow[dl, "g"] \\
& \mathrm{B} &
\end{tikzcd}$$

where f is B-fibrewise dense, h open and the square is a pullback. Then h' is open, and the Beck condition $h^*f_* = f'_*h'^*$ holds for the pullback square, so we have

$$f'_*h'^*g'^* = h^*f_*g'^* = h^*g^*. \quad ∎$$

COROLLARY 1.10. B-*fibrewise closedness is reflected under pullback along open surjections, i.e., given a diagram*

$$\begin{tikzcd}
\mathrm{C}' \arrow[rr, hook, "f'"] \arrow[d, "h'"'] & & \mathrm{C} \arrow[d, "h"] \\
\mathrm{A}' \arrow[rr, hook, "f"] \arrow[dr, "g'"'] & & \mathrm{A} \arrow[dl, "g"] \\
& \mathrm{B} &
\end{tikzcd}$$

in which f' is B*-fibrewise closed, h is an open surjection and the square is a pullback, we may conclude that f is* B*-fibrewise closed.*

PROOF. By Corollary 1.8 and Lemma 1.9, the B-fibrewise closure of f is preserved under pullback along h; so its dense part pulls back to an isomorphism. But h is an epimorphism, so pullback along it reflects isomorphisms. ■

LEMMA 1.11. *Let*

$$\begin{tikzcd} A' \arrow[rr, hook, "f"] \arrow[dr, "h"'] & & A \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

be a B*-fibrewise dense inclusion. Then*

(i) g is epimorphic iff h is.

(ii) g is open iff h is.

PROOF. (i) It is clear that g must be epimorphic if h is; conversely, if g is epimorphic (i.e., g^* is injective), then h^* is injective since $f^*h^* = g^*$.

(ii) Suppose h is open. Then h^* has a left adjoint $h_!$, so $g^* = f_* h^*$ has a left adjoint $g_! = h_! f^*$. Moreover, for any $a \in$ A and $b \in$ B, we have

$$\begin{aligned} g_!(g^*(b) \wedge a) &= h_! f^*(g^*(b) \wedge a) \\ &= h_!(h^*(b) \wedge f^*(a)) \quad \text{since } f^* \text{ preserves finite meets} \\ &= b \wedge h_! f^*(a) \quad \text{by Frobenius reciprocity for } h \\ &= b \wedge g_!(a), \end{aligned}$$

so Frobenius reciprocity holds for g. Conversely, if g is open, let us define $h_! = g_! f_*$, where $g_!$ is the left adjoint of g^*. Then for $a' \in$ A' and $b \in$ B we have

$$\begin{aligned} h_!(a') \leq b \Leftrightarrow g_! f_*(a') \leq b &\Leftrightarrow f_*(a') \leq g^*(b) = f_* h^*(b) \\ &\Leftrightarrow a \leq h^*(b) \qquad \text{since } f_* \text{ is injective,} \end{aligned}$$

so $h_!$ is left adjoint to h^*. And once again Frobenius reciprocity for h follows from the corresponding condition on g, since

$$\begin{aligned} h_!(h^*(b) \wedge a') &= g_! f_*(h^*(b) \wedge a') = g_!(f_* h^*(b) \wedge f_*(a')) \\ &= g_!(g^*(b) \wedge f_*(a')) = b \wedge g_! f_*(a') = b \wedge h_!(a'). \ ■ \end{aligned}$$

PROPOSITION 1.12. *Let*

$$A' \hookrightarrow A'' \hookrightarrow A \quad \textit{and} \quad C' \hookrightarrow C'' \hookrightarrow C$$

be the B*-fibrewise closures of two inclusions in* **Loc**/B, *and suppose the locales* A' *and* C' *are open over* B. *Then* $A''\times_B C''$ *is the* B*-fibrewise closure of* $A'\times_B C'$ *in* $A\times_B C$.

PROOF. The inclusion $A''\times_B C'' \hookrightarrow A\times_B C$ may be obtained by composing pullbacks of the inclusions

$$A'' \hookrightarrow A \quad \text{and} \quad C'' \hookrightarrow C$$

and so is B-fibrewise closed by Corollary 1.8. Similarly, $A'\times_B C' \hookrightarrow A''\times_B C''$ is the composite of the pullbacks of

$$A' \hookrightarrow A'' \quad \text{and} \quad C' \hookrightarrow C''$$

along the maps

$$A''\times_B C' \longrightarrow A'' \quad \text{and} \quad A''\times_B C'' \longrightarrow C''$$

respectively, and these are open maps (the latter by an application of Lemma 1.11 (ii)); so $A'\times_B C' \hookrightarrow A''\times_B C''$ is B-fibrewise dense by Lemma 1.9. ■

COROLLARY 1.13. *Let* **T** *be a finitary algebraic theory; let* (g: $A \to B$) *be a* **T***-algebra in* **Loc**/B, *and let* $A' \hookrightarrow A$ *be a sub-***T***-algebra such that* $A' \to B$ *is open. Then the* B*-fibrewise closure of* A' *in* A *is also a sub-***T***-algebra.*

PROOF. To define the operations of **T** on the fibrewise closure, combine Proposition 1.12 with the functoriality of fibrewise closure (Corollary 1.7). ■

It seems quite probable that the openness hypotheses in 1.12 and 1.13 could be considerably weakened, but I have not been able to dispense with them entirely.

2. CHANGE OF BASE.

We begin this section with a further batch of characterizations of fibrewise denseness, which perhaps help to justify the name "fibrewise dense".

LEMMA 2.1. *Given*

$$\begin{tikzcd} A' \arrow[rr, hook, "f"] \arrow[dr, "h"'] & & A \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

the following are equivalent:

(i) *f is* B*-fibrewise dense.*

(ii) *The pullback of f along any closed inclusion* $B' \hookrightarrow B$ *is dense.*

(iii) *The pullback of f along any locally closed inclusion* $B' \hookrightarrow B$ *is dense.*

(iv) *The pullback of f along any locally closed inclusion* $B' \hookrightarrow B$ *is* B'*-fibrewise dense.*

PROOF. (i) $\Leftrightarrow$ (ii): Let

$$B' = B_{c(b)} = \uparrow(b) \subset B.$$

Then the pullbacks of A and A' along $B' \to B$ are respectively

$$\uparrow(g^*(b) \subset A \text{ and } \uparrow(h^*(b)) \subset A',$$

and the direct image of the pullback of f is just the restriction of f_* to these subsets. So the assertion that this pullback of f is dense says precisely that $f_* h^*(b) = g^*(b)$;

(iii) $\Rightarrow$ (ii) is trivial; and (ii) $\Rightarrow$ (iii) since a pullback of a dense inclusion along an open inclusion is dense.

(iv) $\Rightarrow$ (iii) is again trivial; given that (iii) implies (i), (iii) $\Rightarrow$ (iv) follows from the fact that a composite of locally closed inclusions is locally closed. ∎

PROPOSITION 2.2. *Fibrewise denseness is a local property: i.e., given*

$$\begin{tikzcd} A' \arrow[rr, hook, "f"] \arrow[dr, "h"'] & & A \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

and an open covering $\{b_i | i \in I\}$ *of* B, *f is* B*-fibrewise dense iff its pullback along each* $B_{u(b_i)} \to B$ *is* $B_{u(b_i)}$*-fibrewise dense.*

PROOF. One direction is contained in the implication (i) $\Rightarrow$ (iv) of Lemma 2.1. For the converse, we use condition (ii) of Definition 1.1: suppose given $a \in A$ and $b \in B$ with $f^*(a) \leq h^*(b)$. If we identify the open sublocale $B_{u(b_i)}$ not with its direct image but with the principal ideal $\downarrow(b_i) \subset B$ (and similarly for its pullbacks $\downarrow(g^*(b_i)) \subset A$ and $\downarrow(h^*(b_i)) \subset A'$), we deduce from the fibrewise density of the pullback of f that

$$a \wedge g^*(b_i) \leq g^*(b \wedge b_i) \leq g^*(b),$$

since we have

$$f^*(a \wedge g^*(b_i)) = f^*(a) \wedge h^*(b_i) \leq h^*(b) \wedge h^*(b_i) = h^*(b \wedge b_i);$$

But

$$a = \bigvee \{a \wedge g^*(b_i) \mid i \in \mathrm{I}\},$$

since the b_i form a covering of B, so we deduce $a \leq g^*(b)$. ■

LEMMA 2.3. *Suppose given a diagram*

$$\begin{tikzcd}
\mathrm{A'} \arrow[rr, hook, "f"] \arrow[dr, "h"'] & & \mathrm{A} \arrow[dl, "g"] \\
& \mathrm{B} \arrow[d, "l"] & \\
& \mathrm{B'} &
\end{tikzcd}$$

(i) *If f is* B-*fibrewise dense, then it is* B'-*fibrewise dense.*

(ii) *If f is* B'-*fibrewise closed, then it is* B-*fibrewise closed.*

PROOF. (i) is immediate from Definition 1.1, and (ii) follows directly from (i). ■

In particular, we note that a weakly closed inclusion is B-fibrewise closed for any B over which it is defined, and a B-fibrewise dense inclusion (for some B) is strongly dense.

LEMMA 2.4. *In the diagram of Lemma 2.3 , suppose l is either an inclusion or a local homeomorphism. Then the converse implications to those of 2.3 are valid.*

PROOF. Once again, the implication for "fibrewise closed" follows from that for "fibrewise dense", so we need only discuss the latter. If l is an inclusion then l^* is surjective, so

$$f^* h^* l^* = g^* l^* \text{ implies } f_* h^* = g^*.$$

If l is a local homeomorphism, then B has a covering by open sublocales which map isomorphically to open sublocales of B', and the result follows from Proposition 2.2. ■

COROLLARY 2.5. *If g*: A→B *is the composite of an inclusion and* *a local homeomorphism, then every sublocale of* A *is* B*-fibrewise closed.*

PROOF. By Lemma 2.4, it suffices to show that every sublocale of A is A-fibrewise closed, or (almost) equivalently that the only A-fibrewise dense sublocale of A is A itself. But this is immediate from Definition 1.1. ∎

PROPOSITION 2.6. (i) *If* $f: A' \hookrightarrow A$ *is* B*-fibrewise closed, then the pullback of* f *along any* $B' \to B$ *is* B'*-fibrewise closed.*

(ii) *Fibrewise closedness is a local property* (cf. 2.2).

PROOF. (i) The pullback of f along $B' \to B$ is B-fibrewise closed by Corollary 1.8 ; so it is B'-fibrewise closed by Lemma 2.3 (ii).

(ii) Suppose $\{b_i \mid i \in I\}$ is an open cover of B such that the pullback of f along each $B_{u(b_i)} \hookrightarrow B$ is $B_{u(b_i)}$-fibrewise closed. Then these pullbacks are B-fibrewise closed by Lemma 2.4 ; and since the family of maps $B_{u(b_i)} \hookrightarrow B$ is jointly epimorphic, an easy modification of Corollary 1.10 shows that f is B-fibrewise closed. ∎

Regarding the stability of fibrewise denseness under arbitrary change of base, we have

PROPOSITION 2.7. *Let*

$$\begin{tikzcd} A' \arrow[rr, hook, "f"] \arrow[dr, "h"'] & & A \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

be a B*-fibrewise dense inclusion, and suppose* g *(or equivalently* h*, by* 1.11 (ii)*) is open. Then the pullback of* f *along an arbitrary locale map* $l: B' \to B$ *is* B'*-fibrewise dense.*

PROOF. Form the diagram

$$\begin{tikzcd} C' \arrow[rr, "l''"] \arrow[dr, "f'"] \arrow[dd, "h"'] & & A' \arrow[dr, "f"] \arrow[dd, "h" near start] & \\ & C \arrow[rr, "l'" near start] \arrow[dl, "g'"] & & A \arrow[dl, "g"] \\ B' \arrow[rr, "l"] & & B & \end{tikzcd}$$

in which the parallelograms are pullbacks. Then g' and h' are open, so in order to show that $g'^* = f'_* h'^*$, it suffices to prove the equality $g'_! = h'_! f'^*$ of their left adjoints. Moreover, since the left adjoints preserve joins, it suffices to verify that $g'_!(c) = h'_! f'^*(c)$ not for an arbitrary $c \in C$ but merely for an "open rectangle" $g'_*(b') \wedge l'^*(a)$, since every element of C is a join of such elements. And for such an element we have

$$
\begin{aligned}
g'_!(g'^*(b') \wedge l'^*(a)) &= b' \wedge g'_! l'^*(a) && \text{by Frobenius reciprocity}\\
&= b' \wedge l^* g_!(a) && \text{by the Beck condition}\\
&= b' \wedge l^* h_! f^*(a) && \text{since } f \text{ is fibrewise dense}\\
&= b' \wedge h'_! l''^* f^*(a) && \text{by the Beck condition}\\
&= h'_!(h'^*(b') \wedge f'^* l'^*(a)) && \text{by Frobenius reciprocity}\\
&= h'_! f'^*(g'^*(b') \wedge l'^*(\mathrm{a})),
\end{aligned}
$$

as required. ■

We conclude this section with a portmanteau result which assembles all the justification we currently possess for the use of the term "fibrewise", in the case of locales over a spatial base locale.

PROPOSITION 2.8. *Let* B *be a spatial locale (equivalently, a sober space); let*

$$
\begin{tikzcd}
A' \arrow[rr, hook, "f"] \arrow[dr, "h"'] & & A \arrow[dl, "g"] \\
& B &
\end{tikzcd}
$$

be an inclusion in **Loc**/B, *and for each point* $p: \Omega \to B$ *write* $f_p: A'_p \hookrightarrow A_p$ *for the pullback of* f *along* p. *Assume classical logic, i.e., that* $\Omega = \{0,1\}$.

(i) *If* f *is* B*-fibrewise closed, then each* f_p *is closed.*

(ii) *Assume either that* B *is a* T_D*-space (i.e., that all its points are locally closed) or that* h *is open. If* f *is* B*-fibrewise dense, then each* f_p *is dense.*

(iii) *Assume one of the hypotheses of* (ii) *and additionally that* A *(respectively, the* B*-fibrewise closure of* A' *in* A*) is a spatial locale. Then the converse of* (ii) *(respectively,* (i)*) holds.*

PROOF. (i) follows from Proposition 2.6 (i), and (ii) from either Lemma 2.1 or Proposition 2.7.

(iii) Let A" be the B-fibrewise closure of A' in A. Then, by the first two parts, each A''_p is the closure of A'_p in A_p. So if each f_p is dense, then each $A''_p \to A_p$ is an isomorphism; to deduce that $A'' \to A$ is an isomorphism, we need to know that the family of maps $A_p \to A$, $p \in \mathrm{pt}(B)$, is jointly epimorphic. But this follows from the hypothesis that A is spatial, since every point of A (lies over some point of B, and hence) factors through some $A_p \to A$. The proof of the converse of (i) is similar. ∎

It should be emphasized that the pullbacks considered in Proposition 2.8 must be computed in **Loc**, and not in the category of spaces: even if A and A' are spatial, the pullbacks A_p and A'_p need not be spatial in general (though they will be if B is a T_D-space, since then they are locally closed sublocales of A and A').

3. PROOF OF THEOREM 1.

Let G be a localic group, and H a localic subgroup of G which is open as a locale. By Corollary 1.13 the weak closure of H is a subgroup of G, and by Lemma 1.11 (ii) it is open as a locale. So, to prove Theorem 1, it suffices to prove that an open localic group cannot have a nontrivial strongly dense subgroup. As in [6], we do this by proving

PROPOSITION 3.1. *Let* G *be an open localic group, and let* S *and* T *be any two strongly dense sublocales of* G. *Then the product* S.T *(i.e., the image of the composite*

$$S \times T \hookrightarrow G \times G \xrightarrow{m} G)$$

is the whole of G.

PROOF. As in [6], we form the diagram

$$\begin{array}{ccccc} P & \hookrightarrow & G\times G & \xrightarrow{\pi_2} & G \\ \downarrow & & \downarrow{\scriptstyle (\pi_1, m(i\times 1), \pi_2)} & & \downarrow{\scriptstyle \Delta} \\ S\times T\times G & \hookrightarrow & G\times G\times G & \xrightarrow[m\times 1]{} & G\times G \end{array}$$

where both squares are pullbacks (the left-hand one being the definition of P). The image of the bottom composite may not be exactly $S.T{\times}G$ (since image factorization is not stable under pullback in **Loc**), but it is surely contained in it; so if we can show that the top composite $P \to G$ is epimorphic, then we may conclude that the diagonal Δ factors through $S.T{\times}G$, which forces $S.T = G$. But $\pi_2 : G{\times}G \to G$ is epimorphic (being split by Δ), so it suffices by Lemma 1.11 (i) to show that the inclusion: $P \hookrightarrow G{\times}G$ is G-fibrewise dense.

Now $S{\times}T{\times}G$ is the intersection of the sublocales $S{\times}G{\times}G$ and $G{\times}T{\times}G$ of $G{\times}G{\times}G$; pulling these back along the middle vertical map, we deduce that P is the intersection of $S{\times}G \hookrightarrow G{\times}G$ and

$$T{\times}G \hookrightarrow G{\times}G \xrightarrow{(\mathrm{im}(1{\times}i),\pi_2)} G{\times}G.$$

But the inclusions

$$S{\times}G \hookrightarrow G{\times}G \text{ and } T{\times}G \hookrightarrow G{\times}G$$

are G-fibrewise dense by Proposition 2.7, and the "twist map" $G{\times}G \to G{\times}G$ is an isomorphism in **Loc**/G; so $P \hookrightarrow G{\times}G$ is G-fibrewise dense in Lemma 1.2. ∎

On comparing the above proof with that given in [6] (particularly bearing in mind the result of Lemma 2.1), it will be apparent how close the author came to discovering the concept of fibrewise denseness (without, at the time, realizing it) while writing [6].

We conclude this section by mentioning a couple of simple corollaries of Theorem 1. The first is the result on fibrewise localic groups mentioned in the Introduction, which is obtained by interpreting Theorem 1 in the topos **Sh**(B):

COROLLARY 3.2. *Let* $(G \to B)$ *be a group object in* **Loc**/B, *and let* H *be a sublocale which is a subgroup of* G *in* **Loc**/B, *such that the composite* $H \hookrightarrow G \longrightarrow B$ *is an open map. Then the inclusion* $H \hookrightarrow G$ *is* B-*fibrewise closed.* ∎

Of course, Corollary 3.2 may also be regarded as a special case of Theorem 2, since a fibrewise localic group is the same thing as a localic groupoid "in which every morphism is an endomorphism", i.e., one whose domain and codomain maps are equal. (Theorem 2 would produce the conclusion that $H \hookrightarrow G$ is fibrewise closed over $B{\times}B$, but since the structure map $G \to B{\times}B$

factors through the diagonal $B \to B \times B$, the result as stated follows from Lemma 2.3 (ii).)

The second corollary is, like Theorem 1 itself, the constructive version of a result which is well-known classically:

COROLLARY 3.3. *Any point of a localic group is weakly closed.*

PROOF. The identity point $e: \Omega \to G$ is a localic subgroup, and Ω is (trivially) an open locale; so e is weakly closed by Theorem 1. But an arbitrary point $g: \Omega \to G$ may be written as the composite

$$\Omega \xrightarrow{\quad e \quad} G \xrightarrow{\quad l_g \quad} G$$

where l_g (left multiplication by g) is an isomorphism; so it too is weakly closed. ∎

Classically, the closedness of the points of a localic group is deduced from its regularity, which in turn follows from its uniformizability (cf. [2]). Constructively, localic groups are uniformizable, but uniformizability does not imply regularity (as usually formulated). We shall have more to say about this in Section 5 below.

4. PROOF OF THEOREM 2.

The proof of the result for localic groupoids is similar to that for localic groups, except that we have to make more use of the base change results in Section 2. Let

$$\mathbf{G} = (G_1 \rightrightarrows G_0)$$

be a localic groupoid, and let

$$\mathbf{H} = (H_1 \rightrightarrows H_0)$$

be a subgroupoid (by which we mean that we have a functor $\mathbf{H} \to \mathbf{G}$ whose components $H_0 \to G_0$ and $H_1 \to G_1$ are inclusions, not just monomorphisms) such that the domain and codomain maps $d_0, d_1: H_1 \to H_0$ are open. The reader might expect us to begin by considering the $(G_0 \times G_0)$-fibrewise closure of H_1 in G_1; but it seems impossible to prove that this object defines a subgroupoid of $\mathbf{G}$, unless we make the (unpleasantly restrictive) assumption that

$$(d_0,d_1)\colon H_1 \longrightarrow H_0 \times H_0$$

is an open map. We shall therefore take a slightly different approach.

Let $H_1^l \hookrightarrow G_1$ be the fibrewise closure of $H_1 \hookrightarrow G_1$ over $d_0\colon G_1 \to G_0$, and $H_1^r \hookrightarrow G_1$ the fibrewise closure of the same inclusion over d_1. Let $\overline{H}_1 \hookrightarrow G_1$ be the intersection of these two sublocales; then the inclusion $H_1 \hookrightarrow \overline{H}_1$ is fibrewise dense over both d_0 and d_1 (though not necessarily over

$$(d_0,d_1)\colon G_1 \longrightarrow G_0 \times G_0),$$

and $H_1 \hookrightarrow G_1$ is $(G_0 \times G_0)$-fibrewise closed, since both

$$H_1^l \hookrightarrow G_1 \text{ and } H_1^r \hookrightarrow G_1$$

are $(G_0 \times G_0)$-fibrewise closed by Lemma 2.3 (ii). Now $H_0 \hookrightarrow G_0$, being an inclusion, is G_0-fibrewise closed by Corollary 2.5; so by applying Lemma 1.6 to the square

$$\begin{tikzcd}
H_1 \arrow[r, hook] \arrow[dd, "d_0"'] & H_1^l \arrow[d] \\
 & G_1 \arrow[d, "d_0"] \\
H_0 \arrow[r, hook] & G_0
\end{tikzcd}$$

we deduce that $d_0\colon G_1 \to G_0$ restricts to a map $H_1^l \to H_0$ and hence to a map $\overline{H}_1 \to H_0$. Similarly, $d_1\colon G_1 \to G_0$ restricts to a map $\overline{H}_1 \to H_0$. Moreover, the inclusion $H_1 \hookrightarrow \overline{H}_1$ is fibrewise dense over either of these maps $\overline{H}_1 \to H_0$, by Lemma 2.4, and so they are both open maps by Lemma 1.11 (ii).

LEMMA 4.1. $\overline{\mathbf{H}} = (\overline{H}_1 \rightrightarrows H_0)$ *is a subgroupoid of* $\mathbf{G}$.

PROOF. The composite

$$H_0 \hookrightarrow G_0 \xrightarrow{\ e\ } G_1$$

factors through H_1 and hence through $\overline{H}_1$. Since the inverse map $i\colon G_1 \to G_1$ maps H_1 into itself, and interchanges d_0 and d_1, it maps H_1^l into H_1^r and vice versa; so it maps $\overline{H}_1$ into itself.

Thus it remains to verify that the multiplication

$$m\colon G_1 \times_{G_0} G_1 \longrightarrow G_1$$

maps $\overline{H}_1 \times_{G_0} \overline{H}_1$ into $\overline{H}_1$ (note that, since $H_0 \to G_0$ is an inclusion, we do not have to distinguish between fibre products over G_0 and over H_0).

Now the inclusion

$$H_1 \times_{G_0} H_1 \hookrightarrow \overline{H}_1 \times_{G_0} H_1$$

is fibrewise dense over the projection

$$d_0\pi_1\colon \overline{H}_1 \times_{G_0} H_1 \longrightarrow \overline{H}_1 \longrightarrow H_0 \hookrightarrow G_0,$$

by Lemma 1.9. And the inclusion

$$\overline{H}_1 \times_{G_0} H_1 \hookrightarrow \overline{H}_1 \times_{G_0} \overline{H}_1$$

is fibrewise dense over $\pi_1\colon \overline{H}_1 \times_{G_0} \overline{H}_1 \to \overline{H}_1$, by Proposition 2.7, and hence over

$$d_0\pi_1 : \overline{H}_1 \times_{G_0} \overline{H}_1 \longrightarrow \overline{H}_1 \longrightarrow H_0 \hookrightarrow G_0,$$

by Lemma 2.3 (i). Putting these together, we see that

$$H_1 \times_{G_0} H_1 \hookrightarrow \overline{H}_1 \times_{G_0} \overline{H}_1$$

is fibrewise dense over $d_0\pi_1$, and so by Lemma 1.6 we deduce that the composite

$$\overline{H}_1 \times_{G_0} \overline{H}_1 \hookrightarrow G_1 \times_{G_0} G_1 \xrightarrow{m} G_1$$

factors through $H_1^l \hookrightarrow G_1$. By symmetry, it also factors through $H_1^r \hookrightarrow G_1$, and hence through $\overline{H}_1 \hookrightarrow G_1$. ∎

Thus we may reduce the general case of Theorem 2 to the particular case where $H_0 \to G_0$ is an isomorphism and the inclusion $H_1 \hookrightarrow G_1$ is fibrewise dense over both d_0 and d_1 (so that, by Lemma 1.11 (ii), these maps $G_1 \to G_0$ are both open); our aim in this case is to prove that H_1 is the whole of G_1. To do this, we use the following generalization of Proposition 3.1:

PROPOSITION 4.2. *Let*

$$\mathbf{G} = (G_1 \rightrightarrows G_0)$$

be a localic groupoid such that d_0 and $d_1\colon G_1 \to G_0$ are open, *and let* S *and* T *be sublocales of* G_1 *such that* $S \hookrightarrow G_1$ *is fibrewise dense over* $d_0 : G_1 \to G_0$ *and* $T \hookrightarrow G_1$ *is fibrewise dense over* $d_1 : G_1 \to G_0$. *Then the composite*

$$S \times_{G_0} T \hookrightarrow G_1 \times_{G_0} G_1 \longrightarrow G_1$$

is epimorphic.

PROOF. By Lemma 1.11 (i), it suffices to show that the inclusion

$$S \times_{G_0} T \hookrightarrow G_1 \times_{G_0} G_1$$

is fibrewise dense over m, since m is clearly (split) epimorphic. But $S \times_{G_0} T$ is the intersection of $S \times_{G_0} G_1$ and $G_1 \times_{G_0} T$; and we have a diagram

$$\begin{tikzcd} S \times_{G_0} G_1 \arrow[r, hook] \arrow[d, "\pi_1"] & G_1 \times_{G_0} G_1 \arrow[r, "m"] \arrow[d, "\pi_1"] & G_1 \arrow[d, "d_0"] \\ S \arrow[r, hook] & G_1 \arrow[r, "d_0"'] & G_0 \end{tikzcd}$$

in which both squares are pullbacks (the right-hand one because **G** is a groupoid — [9] attributes this observation to D. Bourn), so by Proposition 2.7, $S \times_{G_0} G_1 \hookrightarrow G_1 \times_{G_0} G_1$ is fibrewise dense over m. Similarly, $G_1 \times_{G_0} T \hookrightarrow G_1 \times_{G_0} G_1$ is fibrewise dense over m. So the result follows, as before, from Lemma 1.2. ■

We note that the proof of Theorem 2 just completed actually yields a slightly stronger conclusion about the inclusion $H_1 \hookrightarrow G_1$ than that stated in the Theorem: namely, if we have any factorization

$$H_1 \hookrightarrow K \hookrightarrow G_1$$

such that $H_1 \hookrightarrow K$ is fibrewise dense over both d_0 and $d_1 : G_1 \to G_0$, then $H_1 \hookrightarrow K$ is an isomorphism. It is not clear whether this extra strength is useful in practice. Note also that in general, if we are given a diagram of the form

$$\begin{tikzcd} A' \arrow[rr, hook, "f"] \arrow[dr, "h"'] & & A \arrow[dl, "g"] \\ & B_1 \times B_2 & \end{tikzcd}$$

the assertion that f is fibrewise dense over both B_1 and B_2 is equivalent to saying that the maps $f_* h^*$ and g^* agree on all "open rectangles" in $B_1 \times B_2$ — but, because f_* does not preserve joins, this does not imply that the two maps are equal.

On comparing the proofs of Propositions 3.1 and 4.2, it will be apparent that the former contains an unnecessary detour: we could have proved directly that the inclusion $S \times T \hookrightarrow G \times G$ is fibrewise dense over m: $G \times G \to G$, instead of first applying the functor $(-) \times G$ and pulling back along the diagonal. We retained the roundabout proof in 3.1 in order to emphasize the extent to which it is the same as the proof in [6]; but when we came to the groupoid case, it seemed better to omit the detour.

5. CONCLUDING REMARKS.

It is clear that the "classical" notions of closedness and denseness are not supplanted by the notions of weak closedness and strong denseness introduced in this paper, since there are many constructive contexts in which the original notions are clearly the right ones to use (a case in point being Lemma 2.1 in this paper, where we definitely need to consider closed inclusions $B' \hookrightarrow B$ rather than weakly closed ones). However, there are many other areas within (constructive) locale theory where it may be profitable to consider the effect of replacing the old notions by the new ones.

One such area concerns the separation axioms. Although, once again, the usual definition of regularity for locales ([4], III 1.1) is clearly the "right" one in many contexts (for example, in the presence of compactness), we have already mentioned the unfortunate fact that, constructively, not every uniformizable locale is regular in this sense. (Indeed, not every discrete locale is regular: a discrete locale Ω^X is regular iff the object X is decidable.) We may now define a locale A to be *weakly regular* if every open sublocale $A_{u(a)}$ is expressible as a join of open sublocales whose weak closures are contained in $A_{u(a)}$; it is then clear from Corollary 2.5 that, at least, every discrete locale is weakly regular.

We may also define a locale A to be *weakly Hausdorff* if the diagonal $A \to A \times A$ is weakly closed (if the terminology of [1] and [4] were followed, this should be "weakly strongly Hausdorff", but this concatenation of adverbs is clearly unacceptable). Note that if a localic group G is an open locale, then it follows at once from Theorem 1 that G is weakly Hausdorff, strengthening Corollary 3.3.

Further aspects of these weak separation axioms, and their relations with uniformizability, will be investigated in a subsequent paper [7].

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