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On the geometry of computations, II

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CAHIERS DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE CATÉGORIQUES *VOL. XXIX-4 (1988)*

ON THE GEOMETRY OF COMPUTATIONS, II

by René GUITART

RÉSUMÉ. Cet article complète [7] par les résultats suivants:

1. Les propriétés d'orthogonalité de morphismes sont incorporables dans l'esquisse des structures.
2. Il existe une esquisse dont les modèles dans *TOP* sont les corps topologiques.
3. En termes de "diagrammes localement libres" on peut décrire des ultraproduits utiles pour des théories non finitaires.
4. On peut esquisse la finitude des modèles, et définir correctement les esquisses finies.
5. L'arithmétique élémentaire (Bézout) et la logique élémentaire (Boole) sont des aspects spéciaux du "diagramme localement libre".

Ces résultats établissent *le contact entre la théorie abstraite des esquisses et programmes de* [7] *et la théorie élémentaire des modèles.*

INTRODUCTION.

This paper is a companion to [7].

In §1 we explain how we can sketch the properties of morphisms, and, especially for the analysis of topological partial algebras, we introduce the "continuous cutting".

In §2 we develop a (new) theory of ultraproducts (available for infinitary algebras) in terms of mixed limits and of locally free diagrams.

In §3 we explain how we can sketch the finiteness of models and also what are the finite sketches. Iterative categories are introduced here in order to explain the calculus of "domain's errors".

In §4, Arithmetic and Boolean Algebra are seen as aspects of the "locally free diagram" construction.

CONTENTS.

1. PROJECTIVE AND INDUCTIVE SPECIFICATION OF STRUCTURES AND OF MORPHISMS.

1.1. Two sketches for PILE. In sorting problems it is useful to know how to specify by a sketch the datum of the set PILE(C) of finite stacks or piles (without repetitions) of elements of C. The set C^* of arbitrary finite tuples is specified as $C^* = \coprod_{n\in N} C^n$. For PILE(C) we proceed as follows: let p_i^n: $C^n \to C$ be the i^{th}-projection from C^n to C; then

$$\mathrm{PILE}(C) = \left[\coprod_{n\geq 1} \cap_{1\leq i<j\leq n} \complement_{C^n}(\ker(p_i{}^n, p_j{}^n))\right] \amalg \{\emptyset\}.$$

With the help of this specification we get a sketch $S^{\flat}{}_{PILE}$ with projective cones and only sums as inductive cones, equipped with two objects X and Y such that for every realization R we have

$$R(Y) = \mathrm{PILE}(R(X)).$$

In this description PILE(C) is analyzed as a subset of C^*, and so the natural transformations between realizations of $S^{\flat}{}_{PILE}$ are described by applications f: $C \to D$ such that f^*: $C^* \to D^*$ sends PILE(C) in PILE(D); such an f is just an injection. In fact we can construct another sketch $S^{*}{}_{PILE}$ specifying PILE(C) and such that the natural transformations between realizations of $S^{*}{}_{PILE}$ are described by arbitrary applications f: $C \to D$. For that we have to analyse PILE(C) as a quotient of C^* by the identification with the empty word $\emptyset$ of every word with at least one repetition; i.e., PILE(C) is the cokernel of

$$\left[\cup_{n\in N} \cup_{1\leq i<j\leq n} \ker(p_i{}^n, p_j{}^n)\right] \underset{\subset}{\overset{\ulcorner \emptyset \urcorner}{\rightrightarrows}} C^*$$

This more natural presentation exhibits the theory in which PILE(C) is the free algebra generated by C: it is the theory of monoids in which

$$(\forall x,y,z,t)[xyzyt \equiv 1].$$

REMARK. A slightly different way of constructing PILE is as

$$\{\emptyset\} \amalg \mathrm{PILE}^+(C)$$

where $\mathrm{PILE}^+(C)$ is the U_T-free object generated by C, with U_T: $\underline{T} \to \mathrm{PAR}$ defined as follows: PAR is the category of sets and partial maps; $\underline{T}$ has for objects the (M, α: M×M $\to$ M) with M a set, α a partial map such that

$$\alpha\cdot(\alpha\times M) = \alpha\cdot(M\times\alpha) \quad \text{and} \quad \alpha\cdot\Delta_M = O_{M,M}$$

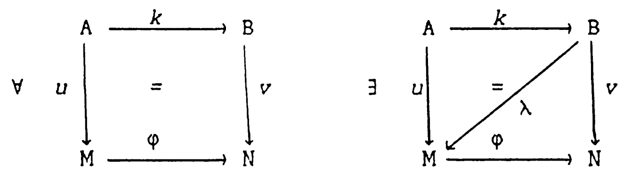
(for Δ_M: $M \to M\times M$ the "diagonal map" and $O_{A,B}$: $A \to B$ the "empty map"); a morphism in $\underline{T}$ is an h: $M \to N$ a partial map such that

$$\beta\cdot(h\times h) \leq h\cdot\alpha.$$

PROPOSITION 1. *Although* $S^{\flat}_{PILE}$ *and* $S^{\#}_{PILE}$ *have the same models in* SET, $SET^{S^{\flat}_{PILE}}$ *is only* $(SET^{S^{\#}_{PILE}})_{Mono}$ *the subcategory of* $SET^{S^{\#}_{PILE}}$ *whose morphisms are monomorphisms in* $SET^{S^{\#}_{PILE}}$.

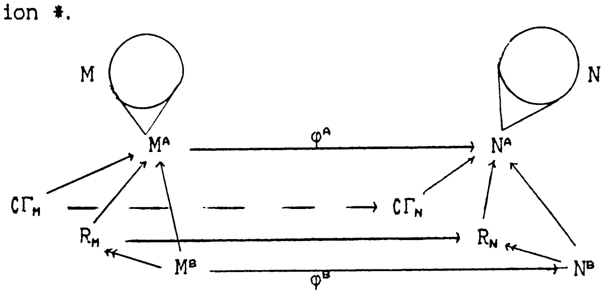
1.2. Specifications of morphisms, categories of monomorphisms.

PROPOSITION 2. *Let* $\underline{S}$ be a category, k: A → B *a morphism of* $SET^{\underline{S}}$ *and* $(SET^{\underline{S}})_k$ the subcategory of $SET^{\underline{S}}$ whose morphisms are the φ: M → N *such that*

* 

Then there is a mixed sketch S⟨k⟩ *such that* $(SET^{\underline{S}})_k \approx SET^{S\langle k\rangle}$.

In fact the proof is concentrated in the picture below, where the dashed arrow, corresponding to the global naturality, is equivalent to the condition *.



We have

$$M^A = Hom_{SET^{\underline{S}}}(A,M),$$

and this object is specified as the projective limit of the system

$$(M(s)^{A(s)} \longrightarrow M(s')^{A(s)} \longleftarrow M(s')^{A(s')})_{f:s\to s'}$$

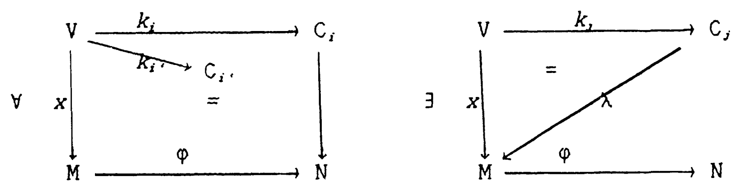
where each $M(s)^{A(s)}$, $M(s')^{A(s)}$ is a product indexed by the set $A(s)$ (independent of M) of the objects $M(s)$, $M(s')$. We have also

$$M^A \approx C\Gamma_M + R_M, \quad N^A = C\Gamma_N + R_N,$$

where R_M (resp. R_N) is the image of $M^B \to M^A$ (resp. of $N^B \to N^A$).

If k is replaced by $\underline{k} = (k_i: V \to C_i)_{i \in I}$ a set of morphisms of $SET^{\underline{S}}$ with the same domain V, we consider $(SET^{\underline{S}})_k$ the subcategory of $SET^{\underline{S}}$ whose morphisms are the φ: $M \to N$ such that

*



i.e., $$\forall x \; [\exists i, v \; (\varphi \cdot x = v . k_i) \Rightarrow \exists j, \lambda \; (x = \lambda \cdot k_j)].$$

PROPOSITION 3. *If* $S = (\underline{S}, P, Y)$ *is a sketch and if* $\underline{k}$ *is as hereabove with, for every* $i \in I$, $k_i \in SET^S$ *we define in the same way the category* $(SET^S)_{\underline{k}}$ *;* and the same proof shows that *there is a sketch* $S\langle \underline{k} \rangle$ *such that* $(SET^S)_{\underline{k}} \simeq \underline{SET}^{S\langle \underline{k} \rangle}$.

For example if S is such that $SET^S \simeq RING$ and if

$$(k_n: Z[X] \to Z[X]/X^n)_{n \in N},$$

then $RING_{\underline{k}}$ is the subcategory of RING whose morphisms are the φ: $M \to N$ such that

$$\forall x \in M \; [\varphi(x) \text{ nilpotent} \Rightarrow x \text{ nilpotent}].$$

So this category $RING_{\underline{k}}$ is naturally sketchable.

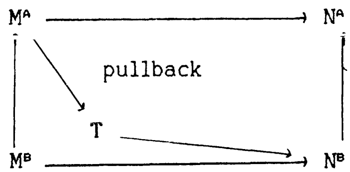
If Yon: $\underline{S}^{op} \to SET^{\underline{S}}$ is the Yoneda embedding, let for every $s \in \underline{S}_0$, k_s: $Yon(s)+Yon(s) \to Yon(s)$ be the codiagonal. Then the monomorphisms of $SET^{\underline{S}}$ are the elements of $\cap_s (SET^{\underline{S}})_{k_s}$, and so they constitute a naturally sketchable category. And this works again for the category of monomorphisms of SET^S with S an arbitrary sketch:

PROPOSITION 4. *For very sketch* S *there is a sketch* S_{mono} *such that*

$$SET^{S_{mono}} \simeq (SET^S)_{Mono}.$$

REMARK. If k is an epimorphism, then in * the morphisms $M^B \to M^A$, $N^A \to N^A$ are monomorphisms, and it is no longer necessary to specify the epimorphism $M^B \twoheadrightarrow R_M$, $N^B \twoheadrightarrow R_N$: so $S\langle k \rangle$ is constructed by adding to S only some projective limits and some sums.

REMARK. We have not to take $(SET^S)_k$ for $k(SET^S)$ the full subcategory of $(SET^S)^2$ whose objects are the $\varphi \in (SET^S)_k$. $k(SET^S)$ is sketchable following the picture



Especially if k is an epi, the specification of the epi becomes the specification of an isomorphism, and the sketch is projective:

PROPOSITION 5. *If* X = SET^S *for* S *a sketch, then* X_{Mono} *is sketchable but generally is not algebraic in* X *but* Mono($\underline{X}$) *is algebraic in* X^2.

1.3. Partial maps, partial laws, continuous cutting.

REMARK. There are different kinds of partial laws:

- The first situation is sketched in the form

$$C \underset{v}{\overset{u}{\leftleftarrows}} B \xleftarrow{ker(u,v)} A \xrightarrow{l} D \qquad (1)$$

Here l is a total law on a subobject A of B defined by an equation $u = v$. In this case we use only of a kernel specification, and the sketch stays a projective sketch.

- The second situation is sketched in the form

$$B \xleftarrow{i} A \xrightarrow{l} D \qquad (2)$$

Here l is a total law on a subobject A of B. In this case we use only of a pullback (in order to specify that i is a monomorphism) and the sketch stays a projective sketch.

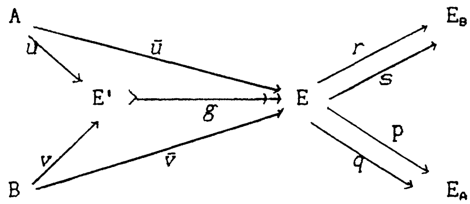
But this second situation is also sketched in the form

$$B \xrightarrow{\bar{l}} D+1 \qquad (3)$$

where is specified a final element 1 and a sum D+1: so here the sketch is a mixed one (i.e., with projective and inductive limits).

Although the specifications (1), (2) and (3) are equivalent, if we realize in SET or in a topos, *we have to be careful if we have in mind to realize elsewhere*, e.g., in various categories of topological spaces.

Finally, for the description of partial topological structures as models of a sketch in TOP (category of topological spaces) we shall use of the *continuous cutting* specification (c.c.). It consists of the sketch S_{cc} pictured by



with the specifications: E' ≃ A+B, g epi and mono, (p,q) cokernel pair of $\bar{u} = g.u$, $\bar{u} = ker(p,q)$, (r,s) cokernel pair of $\bar{v} = g.v$, $\bar{v} = ker(r,s)$. A model in SET of S_{cc} is a set with a subset A and its complement B.

PROPOSITION 6. *A model in* TOP *of* S_{cc} *is a space* E *with two subspaces (i.e., subsets equipped with the induced topology)* A *and* B, *such that the set* B *is the complement of the set* A *in* E.

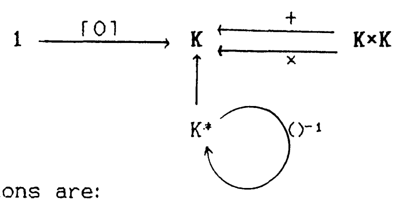
Shortly we will say that

$$A \xrightarrow{\bar{u}} E \xleftarrow{\bar{v}} B$$

"is" a continuous cutting, and we write that $cc_E A = B$.

1.4. Bad and good sketches for the fields.

A field is a model of the sketch generated by (among other things):



where the specifications are:

projective: 1 is final, K×K is a product of K and K,

inductive: K is a sum of 1 and K*.

Let us call this sketch S_{field}. So in a model we have

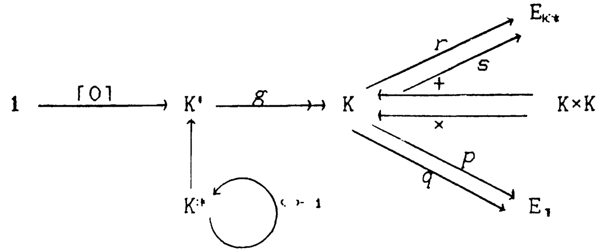
$$K = \{0\} \amalg K^* \quad \text{and so} \quad K^* = \{x \in K \mid x \neq 0\}.$$

Then the inversion operator $()^{-1}$ is defined just for the $x \in K$ which are not 0. This sketch is good only for the study of fields in SET;

the theorem in §4.2 in the special case could be used.

But we have to notice that a model of this sketch in the category TOP of topological spaces *is not* a topological field (e.g., in TOP we have not $\mathbf{R} \simeq \{0\} \amalg \mathbf{R}^{+}$, because 0 is glued to $\mathbf{R}^{*}$, 0 is not isolated).

Fortunately we can modify this sketch and consider the following sketch:



where the specifications are as in S_{field} hereover except that $K = 1+K^{*}$ is to be replaced by: " $cc_K 1 = K^{*}$ " (continuous cutting specification defined in §1.3). Let us call this sketch S^{g}_{field}.

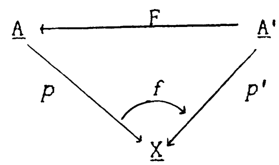
PROPOSITION 7. *A model of* S^{g}_{field} *in* SET *is a field, and a model of* S^{g}_{field} *in* TOP *is a topological field.*

REMARK. Because in S^{g}_{field} there are other specifications than sums, with S^{g}_{field} we can't see that we are in the special case in Theorem §4.2. On the other side S^{g}_{field} allows us to study $\mathrm{TOP}^{S^{g}_{field}}$ as the category of topological fields. And as in fact the theorem in §4.2 is extendable to situations where SET^{S} is replaced by TOP^{S} (cf. Guitart-Lair [12b]), the existence of S^{g}_{field} is useful for example to study the link between topological rings and topological fields.

2. ULTRAPRODUCTS AND THE WEAK REPRESENTATIONS OF GERMS OF LOCAL CONCEPTS.

2.1. Diagrams and machines.

Let $\underline{X}$ be a locally small category. Then $D_{\leftarrow}\underline{X}$ is the locally small category of small diagrams in $\underline{X}$. So an object of $D_{\leftarrow}\underline{X}$ is a functor p: $\underline{A} \to \underline{X}$ where $\underline{A}$ is a small category, and a morphism in $D_{\leftarrow}\underline{X}$ from p to p' is a (F, f) with F: $\underline{A}' \to \underline{A}$ a functor and f: $p.F \to p'$ a natural transformation.



$$p \dashrightarrow p'$$

We define

$$D_{\rightarrow}\underline{X} := [D_{\leftarrow}(\underline{X}^{op})]^{op} .$$

Then if $\underline{X}$ is complete (resp. cocomplete) we have the functor

$$\lim: D_{\leftarrow}\underline{X} \to \underline{X} \quad (\text{resp. } \operatorname{colim}: D_{\rightarrow}\underline{X} \to \underline{X}).$$

The fundamental property of $D_{\rightarrow}\underline{X}$ is that it is the *strong lax-cocompletion* of $\underline{X}$ (Guitart-Van den Bril [9]). This fact can also be related to the facts that $CAT^{X^{op}}$ is a 2-cocompletion of $\underline{X}$ and the existence of a "good" 2-functor (cf. Guitart [10], p. 474):

$$\mathcal{J}: D_{\rightarrow}\underline{X} \to CAT^{X^{op}}: p \mapsto [(-)\downarrow p: \underline{X}^{op} \to CAT].$$

A *machine* from a category $\underline{U}$ to a category $\underline{X}$ is a functor M: $\underline{U} \to D_{\leftarrow}\underline{X}$, or equivalently, a functor $M^{op}: \underline{U}^{op} \to D_{\rightarrow}(\underline{X}^{op})$. Machines can be composed in series or in parallele, and we get the bicategory MAC of machines (Guitart {8]). In fact in [8] $D_{\rightarrow}$ is denoted by D, and a machine is a functor, from $\underline{U}$ to $D\underline{X}$. But now, in order to agree with the terminology on fibrations and cofibrations, I prefer to call these old machines by the name comachines.

The main idea with diagrams and machines is that

" MAC/CAT = REL/SET ",

i.e., the machines play in CAT the role played by relations in SET, and in particular $D_{\leftarrow}\underline{X}$ is an analogue of $\mathcal{P}$E.

2.2. Spaces and germs of local concepts.

We think of a functor F: $\underline{X}^{op} \to$ SET as a *concept* on $\underline{X}$. For example if $\underline{X}^{op}$ is the category *Eucl* of euclidean spaces, the concept of a sphere is described by the data for every space E of the set Sph(E) of the spheres in E, and Sph is a functor from *Eucl* to SET. Let $\mathbb{R}^+$ be the set of positive real numbers. For $x,y \in \mathbb{R}^+$ with $x < y$, we define

$Sph_{x,y}(E)$ as being, for every euclidean space E, the set of spheres Σ in E such that

$$x < \text{radius}(\Sigma) < y .$$

Clearly $Sph = \text{colim}_{(x,y)} Sph_{x,y}$, and we say that Sph is the ultimate concept associated to the local (system of) concept(s) $(Sph_{x,y})_{x,y\in R+}$, or the germ "at the infinite" of $(Sph_{x,y})_{x,y\in R+}$.

From now on, we consider that a machine "is" a space, because we think of a machine M: $\underline{U} \to D_{\leftarrow}\underline{X}$ as a parametrization (by $\underline{U}$) of a local system (i.e., a family of variable diagrams) in $\underline{X}$.

A classical topological space T = (E,$\underline{O}$) with E a set and $\underline{O}$ the ordered set of opens (for T) in E, is actually a space in this generalized form, because T is the machine

$$\underline{O}^{op} \rightarrowtail \mathcal{P}(E)^{op} \rightarrowtail D_{\leftarrow}E.$$

In this case the composition with $\int^{p}{}_{E^{op}}$ provides the functor

$$\underline{O} \xrightarrow{\text{eva}} \{\emptyset,1\}^{E} \xrightarrow{\text{cano}} CAT^{E^{op}}$$

with

$$\text{eva}(U)(x) = 1 \quad \text{iff} \quad x \in U.$$

In [11] I introduce the category SPA of spaces, a *space* being a triple ($\underline{X}$,$\underline{U}$,R) where $\underline{X}$ is a category (of "points"), $\underline{U}$ a category (of "opens") and R: $\underline{U} \to CAT^{\underline{X}^{op}}$ a functor "realization" (for $U \in \underline{U}_0$ and $x \in \underline{X}^{op}$, $R(U)(x) \in CAT_0$ is the value of the sentence "U is around x"). The structure of a space is precisely the dialectic between opens and points described by R. So this point of view is different from the point of view of Grothendieck (where the points are forgotten). Fuzzy spaces and stochastic spaces live naturally as objects of SPA.

The following description of local concepts and germs works in fact in SPA. But in this paper I choose to work only with the (special case of spaces associated to) machines.

DEFINITION. Let $\underline{U}$, $\underline{X}$ and $\underline{Y}$ be categories.

1° A *machine* from $\underline{U}$ to $\underline{X}$ is a functor M: $\underline{U} \to D_{\leftarrow}\underline{X}$.

2° A *$\underline{X}$-local concept on $\underline{Y}$ is a functor C: $D_{\leftarrow}\underline{X} \to SET^{\underline{Y}^{op}}$.*

3° For a machine M and a local concept C we define the *germ of* C *at* M by

$$g_M C = \text{colim}_{U\in\underline{U}} [\underline{U} \xrightarrow{M} D_{\leftarrow}\underline{X} \xrightarrow{C} SET^{\underline{Y}^{op}}]$$

so $g_M C$: $\underline{Y}^{op} \to SET$ is a concept on $\underline{Y}$.

2.3. Projective cones as a local concept.

The Yoneda embedding

$$\mathrm{Yon}_{\underline{X}}: \underline{X} \to \mathrm{SET}^{\underline{X}^{op}}: x \mapsto \mathrm{Yon}_{\underline{X}}(x) = \mathrm{Hom}_{\underline{X}}(-,X)$$

does not preserve inductive limits which exist in $\underline{X}$, but it preserves projective limits. If d: $\underline{A} \to \underline{X}$ is a diagram in $\underline{X}$ and if $p_A = (P \to d(A))_{A \in \underline{A}}$ is a projective limit of d, then $\mathrm{Hom}_{\underline{X}}(-,P)$ is the projective limit of the $\mathrm{Hom}_{\underline{X}}(-,d(A))$. But in fact, for $X' \in \underline{X}_0$,

$$\mathrm{Hom}_{\underline{X}}(X',P) = \mathrm{Cones}_{\underline{X}}(X',d)$$

where the right expression is the set of projective cones in $\underline{X}$ with top X' and base d. In this way we get $\mathrm{Cones}_{\underline{X}}$: $D_{\leftarrow}\underline{X} \to \mathrm{SET}^{\underline{X}^{op}}$ as an $\underline{X}$-local concept on $\underline{X}$, and a diagram d in $\underline{X}$ has a projective limit in $\underline{X}$ iff the functor $\mathrm{cones}_{\underline{X}}(-,d)$: $\underline{X}^{op} \to \mathrm{SET}$ is representable. So, for an arbitrary $\underline{X}$ and an arbitrary d: $\underline{A} \to \underline{X}$ we can substitute $\mathrm{Cones}_{\underline{X}}(-,d)$ for lim d (because

$$\mathrm{cones}_{\underline{X}}(-,d) = \lim(\mathrm{Yon}_{\underline{X}} \circ d)\ ).$$

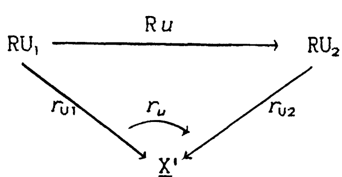
In fact if for $X' \in \underline{X}_0$ we denote by $\ulcorner X' \urcorner$: $1 \to \underline{X}$ the diagram constant on X', we have

$$\mathrm{Cones}_{\underline{X}}(X',d) = \mathrm{Hom}_{D_{\leftarrow}\underline{X}}(\ulcorner X' \urcorner, d),$$

so $\mathrm{Cones}_{\underline{X}}$ is the trace in $\underline{X}$ of the notion of homomorphism in $D_{\leftarrow}\underline{X}$, and because of that it can be considered in $\underline{X}$ as a notion of local homomorphism.

2.4. Mixed limits, ultraproducts, weak products.

From Guitart-Lair ([12 a], p. 62 and p. 101), we recall the description of a *mixed limit*. For that we suppose that $\underline{U}$ is a category and R: $\underline{U} \to \mathrm{CAT}$ is a functor and we suppose that for all u: $U_2 \to U_1 \in \underline{U}^{op}$ we have a datum (Ru, r_u) (as on the picture)



which is couniversal in the sense that every similar datum toward a $\underline{X}''$ factors uniquely as $h.r$ with $h: \underline{X}' \to \underline{X}''$ (so r is a presentation of $\underline{X}' \in CAT_0$ as a 2-colim of R in CAT).

Now we suppose that $F: \underline{X}' \to \underline{X}$ is a functor. Then the *mixed limit* of (F,r) is (if it exists) the object

$$\ell =: \mathrm{colimlim}(F,r) =: \mathrm{colim}_{U \in \underline{U}^{op}}(\lim_{x' \in RU}(F.r_U(X'))).$$

We recover the classical notion of ultraproduct in the case where $\underline{U}$ is the category associated to the ordered set $(\mathcal{U},\leq)$ with $\mathcal{U}$ an ultrafilter on a set D. Cf. e.g. [3]. If $(X_i)_{i \in D}$ is a family of objects in $\underline{X}$ indexed by D, then the ultraproduct of this family is

$$\prod X_i/\mathcal{U} = \mathrm{colim}_{U \in \mathcal{U}}(\prod_{i \in U} X_i).$$

The notion of a weak product of I. Sain has in fact the following category-theoretic characterization: Let $(X_j)_{j \in J}$ be a family of objects in $\underline{X}$ and F the Fréchet filter on J ($A \in F$ iff $A \subset J$ and $C_J A$ is finite). For each $U \in F$, we denote by

$$K_U \overset{p_U}{\underset{q_U}{\rightrightarrows}} \prod_J X_j$$

the kernel pair of the restriction $\prod_J X_j \to \prod_U X_j$, and by $\dot{K}_U$ the pullback of p_U and $0 \to \prod_J X_j$ (with 0 the initial object of $\underline{X}$). The canonical map from $\dot{K}_U$ to $\prod_J X_j$ deduced from q_U is denoted by $\dot{q}_U: \dot{K}_U \to \prod_J X_j$. With $\dot{K} = \mathrm{colim}_{U \in F}\dot{K}_U$ and $q: \dot{K} \to \prod_J X_j$ the map deduced from $(\dot{q}_U)_{U \in F}$, we get the weak product

$$WP_J X_j = \mathrm{image}(q).$$

In this form *the weak product appears as a mixed limit.*

REMARK. It is also possible to describe $WP_J X_j$ as an ordinary limit:

$$WP_J X_j = \lim_{r \in R} A_r$$

where the A_r are all possible objects in a position

$$(\dot{K}_U \xrightarrow{r} A_r \rightarrowtail \prod_J X_j) = \dot{q}_U$$

for a U. But R is constructed from the data J *and* all the objects X_j.

2.5. Weak representations.

DEFINITION. If $G \in SET^{X^{op}}$ has a reflexion ε: $G \to Hom_X(-,W)$ in the representable functors, we say that W is a *weak representation* of G.

PROPOSITION 8. $colim_I X_i$ *is a weak representation of* $colim_I Hom_X(-,X_i) = \Lambda$ *and in fact* Λ *is representable iff* $colim_I X_i$ *is an absolute colimit; so generally* G *could have a weak representation but no representation.*

2.6. Ultraproduct as a weak representation of "germ of cone".

J. Rosický raised the problem of the indirectness (i.e. based on products and colimits) of the categorical expression of the ultraproduct (cf. Okhuma or [3]). H. Hien, I. Nemeti and I. Sain have introduced, in contrast with the "product-colim"-ultraproduct, another object called the *universal ultraproduct*, in which products and colimits are not explicitly used. They do that "by hands". In fact (Guitart [8]) the datum of R, r and F in §2.4 is equivalent to the datum of a machine M_U: $\underline{U}^{op} \to D_{\leftarrow}\underline{X}$ (determined by $M_U(\underline{U}) = F \cdot r_U$) and we can consider the germ of "cones" (cf. §2) at M_U (cf. §1):

$$g_U =: g_{M_U}\text{cones} = \text{colim}_{\underline{U}^{op}}\{\underline{U}^{op} \xrightarrow{M_U} D_{\cdot}\underline{X} \xrightarrow{\text{cones}_X} SET^{X^{op}}\}.$$

With a more compact notation we have, in $SET^{X^{op}}$,

$$g_U = \text{colim}_{U \in U}\text{cones}_X(-,U) = \text{colim}_U C_U.$$

PROPOSITION 9. *For each* $U \in U$, *let* N_U: $J(U) \to \underline{X}$ *be an arbitrary functor such that*

$$C_U = \text{colim}_{j \in J(U)} Hom_X(-,N_U(j)).$$

Then g_U *is weakly representable iff* $W = \text{colim}_{U,j} N_U(j)$ *exists in* $\underline{X}$, and g_U *is representable iff this colimit is absolute, i.e.,*

$$\varepsilon: \text{colim } Hom_X(-,N_U(j)) \longrightarrow Hom_X(-,W)$$

is an isomorphism.

PROPOSIITION 9. *If the "product-colim"-ultraproduct exists, it is a representation of* g_U, *and of course a weak representation of* g_U. *A weak representation of* g_U *is nothing else than a "universal ultraproduct".*

2.7. Local formula, local model, covered colimit, Łos Lemma.

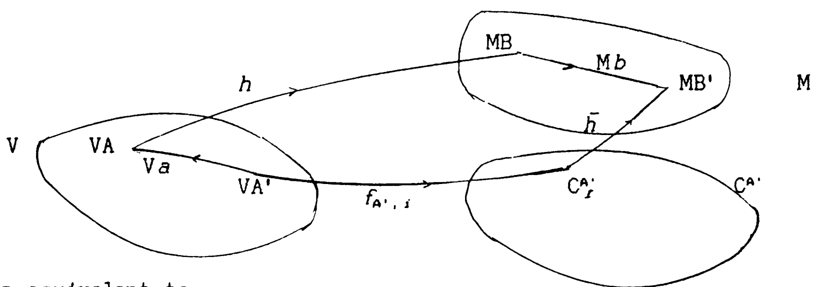
Let $\underline{Z}$ be a category. A *local formula* in $\underline{Z}$ is a datum

$$(V: \underline{A} \to \underline{Z}\ ;\ (f_{A,i}: V(A) \to C^A{}_i)_{i\in EA})$$

in short $(f_{A,i})_{A\in\underline{A},\, i\in EA}$, or f, where V is a functor from a category $\underline{A}$ to $\underline{Z}$, and, for every $A \in \underline{A}_0$, $f_{A,i}$ is a discrete projective cone in $\underline{Z}$ from $V(A)$ to a family $(C_{A,i})_{i\in A}$.

A diagram $M: \underline{B} \to \underline{Z}$ is called a *local model* of f, and this is written $M \models f$, iff

$$[\forall\ A \in \underline{A}_0,\ \forall\ B \in \underline{B}_0,\ \forall\ h: VA \to MB,\ \exists\ a: A' \to A,\ \exists\ b \in B \to B',$$
$$\exists\ i \in E_{A'},\ \exists\ \bar{h}: C^{A'}{}_i \to MB']\ (\bar{h}.f_{A',i} = M(b).h.V(a)).$$



This is equivalent to

$$\mathrm{colim}_{i,A}\lim_B \mathrm{Hom}_Z[C^A{}_i, M(B)] \twoheadrightarrow \mathrm{colim}A\,\mathrm{colim}_B \mathrm{Hom}_Z[V(A), M(B)].$$

This condition is sketchable and so, modulo the adjunction of new symbols, it is no stronger than the case of formula in the sense of Andreka-Nemeti [1], which is the case $\underline{A} = \mathbf{1}$, $\underline{B} = \mathbf{1}$. But it is stronger if we decide to stay inside $\underline{Z}$. In any case it is a more flexible language.

EXAMPLE 1. Let U be an ultrafilter on a set D and $(X_i)_{i\in D}$ a family of objects in $\underline{X}$. In $SET^{\underline{X}^{op}} = \underline{Z}$ we denote by X_U the functor $\mathrm{cone}(-, (X_i)_{i\in U})$ (represented by $\prod_{i\in U} X_i$ if this product exists in $\underline{X}$) and we consider the diagram

$$X_-: U \to \underline{Z}: U \mapsto X_U.$$

Let $(f_j: V \to C^j)_{j\in E}$ be a formula in $\underline{X}$, with E finite. Then $X_- \models f$ if for every $i \in D$, $X_i \models f$ (with

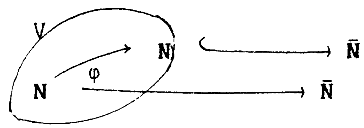
$$U_j = \{i \in U \mid \mathrm{proj}^U{}_i.h \text{ factorize through } f_j\}$$

we have $\cup_{j<n} U_j = U$, and there exists j such that $U_j \in U$. Then $\mathrm{proj}^U{}_{U_j}.h$

factorizes through f_j).

EXAMPLE 2. In the category $\underline{Z}$ of metric spaces and continuous maps, we consider the formula

$f \equiv$



where V consists of all the strictly non decreasing maps from **N** to **N**. Then M |= f (case B = 1) iff M is compact.

Let $u = (u_i\colon X_i \to S)_{i \in \underline{I}}$ be a cocone in $\underline{X}$ and $\underline{V}_0 \subset \underline{X}_0$. Then S is said to be *$\underline{V}_0$-covered* by u iff for every $v \in \underline{V}_0$ and every h: $V \to S \in \underline{X}$ there is a $\bar{h}$ in $\mathrm{colim}_i \mathrm{Hom}_{\underline{X}}[X,X_i]$ with $\varepsilon_V(\bar{h}) = h$ (i.e., ε_V is an epi), where

$$\varepsilon\colon \mathrm{colim}_i \mathrm{Hom}_{\underline{X}}[-,X_i] \to \mathrm{Hom}_{\underline{X}}[-,S]$$

is deduced from u).

In particular if $\underline{V}_0 = \underline{X}_0$ this means that ε is an epimorphism.

If $S = \mathrm{colim}_i X_i$, we shall say that colim X_i is a $\underline{V}_0$-covered colim iff S is $\underline{V}_0$-covered by u.

EXAMPLE 3. In the category of Sets every colim is $\{1\}$-covered and every filtered colim is finitely covered.

Looking to the diagram

$$\begin{array}{ccc}
\mathrm{colim}_u \mathrm{Hom}_{\underline{X}}[V,X_u] & \xrightarrow{\varepsilon_V} & \mathrm{Hom}_{\underline{X}}[V,\mathrm{colim}_u X_u] \\
\uparrow & = & \uparrow \\
\mathrm{colim}_j \mathrm{colim}_u \mathrm{Hom}_{\underline{X}}[C^j,X_u] & \longrightarrow & \mathrm{colim}_j \mathrm{Hom}_{\underline{X}}[C^j,\mathrm{colim}_u X_u]
\end{array}$$

it is easy to verify the following avatar of Łos Lemma:

PROPOSITION 11. *Let $(f_j\colon V \to C^j)_{j \in E}$ be a formula in $\underline{X}$ and let $(X_u)_{u \in \underline{U}}$ be a diagram in $\underline{X}$; then we have:*

1. If $(X_u)_{u \in \underline{U}} \models f$ and if $\mathrm{colim}_u X_u$ is $\{v\}$-covered, then

$$\mathrm{colim}_u X_u \models f$$

1. If $(X_u)_{u \in \underline{U}} \models f$, if $\mathrm{colim}_u X_u$ is $\{C^j \mid j \in E\}$-covered, if ε_V is a mono, and if $(X_u)_{u \in \underline{U}}$ is filtered, then $(X_u)_{u \in \underline{U}} \models f$.

In the case of ultraproducts (cf. Examples 1 and 3 hereover) in the category of models of a finitary first order theory, this Lemma reduces to the usual Łos Lemma. *So a convenient definition of ultraproduct should be* (as in §6) *as a weak representation of a germ of cone*; but after that, in order to obtain interesting properties like the Łos Lemma it is necessary to add new condiitions on ε: for some V, ε_V has to be epi, or mono (see Lemma). But in general the condition "ε iso" (i.e., the representability) is too strong and under this condition the ultraproduct does not exist.

Now, with the idea of locally free diagrams (cf. §4), it seems natural to propose to *define the ultraproduct as the locally free diagram generated by* g_U *in the subcategory of* $SET^{X^{op}}$ *which consists of the representables.* So the ultraproduct of $(X_j)_{j\in D}$ with respect to the ultrafilter U on D is defined as a small diagram $(L_a)_{a\in A}$ of objects $L_a \in \underline{X}_0$ equipped with a natural family

$$(\varepsilon_a\colon \operatorname{colim}_U C_U \to \operatorname{Hom}_{\underline{X}}(-,L_a))_{a\in A}$$

inducing an isomorphism for every $Z \in \underline{X}_0$:

$$Hom(\operatorname{colim}_{U\in U}C_U, \operatorname{Hom}_{\underline{X}}(-,Z)) \simeq \operatorname{colim}_{a\in A}\operatorname{Hom}_{\underline{X}}(L_a,Z).$$

3. FINITE SKETCHES, SKETCH OF FINITENESS.

3.1. Creation of FSET by free propagation of errors.

Let **N** be the set of natural numbers, and for $n \in \mathbf{N}$ let $\underline{n}$ *be the set* $\underline{n} = \{0,1,\dots,n-1\}$. *We denote by FSET the category with objects the* $\underline{n}$, $n \in \mathbf{N}$, and where a morphism from $\underline{n}$ *to* $\underline{m}$ is just a map from $\underline{n}$ *to* $\underline{m}$. In this category, $\underline{1}$ is a terminal object, and the objects $\underline{n}$ *and* $\underline{m}$ have a sum $\underline{n}+\underline{m}$ such that $\underline{n}+\underline{m} = \underline{n+m}$.

An *iterative category* is a category $\underline{C}$ such that:

1. In $\underline{C}$ there is a terminal object 1.

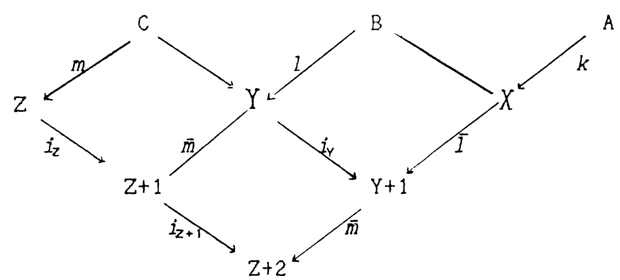
2. For every object X of $\underline{C}$, there is a sum of X and 1 in $\underline{C}$, denoted by X+1, with the two canonical morphisms

$$i_X\colon X \to X+1 \quad \text{and} \quad e_X\colon 1 \to X+1.$$

PROPOSITION 12. *Every category* $\underline{X}$ *generates freely an iterative category* It$(\underline{X})$ *which is the free calculus of the propagation of errors of domains in the description of programs working in the "abstract type"* $\underline{X}$. *In particular* It$(\emptyset)$ = FSET.

Let us explain that. The initiality of FSET is clear, a morphism of iterative categories being a functor which preserves 1 and (−)+1; and on the other side It(∅), obtained by successive construction of 1 and of the sums 1+1, 1+1+1, ... *and* the morphisms between them coming from the universal property of sums, furnishes a construction of FSET by "free propagation of errors".

In an abstract type $\underline{X}$ let us consider a fragment like



and the "program"

(P) 1. $x = k(a)$, 2. $y = l(x)$, 3. $z = m(y)$.

This program has two "domain errors": $l(x)$ is computable only if $x \in$ X is in fact in B, and $m(y)$ is computable only if $y \in$ Y is in fact in C. So a correct version of (P) would be:

(corP) 1. $x = k(a)$,
2. If $x \in$ B then $y = l(x)$, otherwise $y = \text{error}_Y$,
3. If $y \in$ C then $z = m(y)$, otherwise if $y \in$ Y\C then $z = \text{error}_Z$, otherwise if $y = \text{error}_Y$ then $z = \text{error}_{Z+1}$.

With $\bar{l}/B = l$ and $\bar{l}/X\backslash B = e_Y$, and with $\bar{m}$ and $\bar{\bar{m}}$ defined in the same way, (corP) becomes:

(CORP) 1. $x = k(a)$, 2. $y = \bar{l}(x)$, 3. $z = \bar{\bar{m}}(y)$.

So (P) which is not correct in $\underline{X}$ becomes correct in It($\underline{X}$), by creating at each step (at Y, at Z) an "error term" (e_Y, e_Z) and by propagating the preceding error terms. The false path (k,l,m) becomes a correct path $(k,\bar{l},\bar{\bar{m}})$. If (P) contains n instructions with possible errors of domain, the correction of (P) in (CORP) would necessitate the objects X+1,...X+n for X $\in \underline{X}_0$.

3.2. Finiteness.

It is easy to sketch the structure consisting of a set equipped with a bijection to **N**. Now let us sketch the structure "finite set". Let NSET be the full subcategory of SET which contains FSET (cf. §1) and **N**. We add to $NSET^{op}$:

- two objects called FINITE and ENUMERABLE,
- two inductive cones δ and φ with base $FSET^{op}$ and top ENUMERABLE and FINITE,
- a morphism k: FINITE $\to$ ENUMERABLE such that we have for every $n \in \mathbf{N}$, $\delta_n = k.\varphi_n$,
- a (formal) inverse k^{-1} for k.

Let $\underline{S}_{fin}$ the category generated by these data. We get a sketch S_{fin} by specifying in $\underline{S}_{fin}$:

- the projective cones deduced from the inductive cones of NSET which define

$$\underline{n} = \amalg_n\{1\} \quad \text{and} \quad \mathbf{N} = \amalg_{\mathbf{N}}\{1\},$$

- the inductive cones φ and δ.

Then if R: $S_{fin} \to$ SET is a realization of S_{fin} we have

$$R(\text{FINITE}) \approx \{X \subset R(1),\ \bar{X} < \omega\},$$
$$R(\text{ENUMERABLE}) \approx \{X \subset R(1),\ \bar{X} \leqslant \omega\}.$$

And $R(k)$ invertible means that a subset of R(1) is enumerable iff it is finite, in other words R(1) is finite. So a realization R of S_{fin} consists exactly of a finite set E = R(1).

As a consequence for a sketch S and an object A $\in \underline{S}$, by glueing S and S_{fin} (at the point A), we can add the specification "R(A) is finite" (for a realization R of S).

PROPOSITION 13. *For every sketch* S *there is a sketch* fin(S) *such that*

$$SET^{fin(S)} \simeq FSET^{S}.$$

3.3. Specification of initial algebras.

If F: $S_1 \to S_2$ is a morphism of projective sketches, then

$$SET^F: SET^{S2} \longrightarrow SET^{S1}$$

has a left adjoint Ext_F: if R: $S_1 \to$ SET, let $\bar{R}$: $S_2 \to$ SET be defined by

$$\bar{R}(X_2) = \text{colim}_{F\downarrow X2}[F\downarrow X_2 \longrightarrow S_1 \xrightarrow{R} SET]$$

with $F\downarrow X_2$ the category whose objects are the $(X_1, f\colon F(X_1) \to X_2)$. Then $Ext_F(R) = a(\bar{R})$ with a left adjoint to the inclusion $SET^{s2} \hookrightarrow SET^{S2}$. *a is the "associated sheaf functor"*; it is actually a special case of the locally free diagram functor (cf. [7] and here, §4) and it is computed by inductive limits again. So $Ext_F(R)$ *is specifiable in terms of inductive limits.*

Especially if $F\colon S_1 \to S_2$ is given, it is possible to sketch the property

$$M \approx Ext_F(M \cdot F) \qquad *$$

for $M\colon S_2 \to SET$.

A statement like * is an "initial algebra specification". It says that M is the SET^F-free object generated by the restriction $M \cdot F$ of M to $\underline{S}_1$. Example: **N** is the free monoid generated by 1.

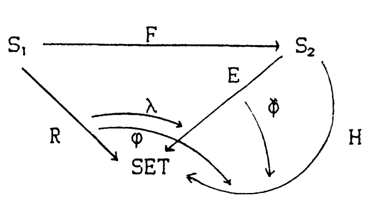
Conversely every inductive limit could be specified in terms of Ext_F: for a given $\underline{S}$ and a $B\colon \underline{I} \to \underline{S}$, $colim(R \cdot B)$ for $R\colon \underline{S} \to SET$ is $Ext_F(R)(B)$ for $F\colon \underline{S} \to \underline{S}\langle B' \rangle$ ("mapping cone"), with $\underline{S}\langle B' \rangle$ the category obtained by formally adding to $\underline{S}$ an object B and an inductive cone $(b_i\colon B_i \to (B))_{i \in I}$.

PROPOSITION 14. *The specification of* lim *and* colim *is equivalent to the specification of* lim *and free models of an essentially algebraic theory (or "initial algebra specifications"); i.e., both allow to describe the same categories as categories of models.*

REMARK. Now, in the optics of locally free diagram it seems natural to propose to get specifications in terms of lim and locally free models of mixed sketches, or "*equations and local specifications*".

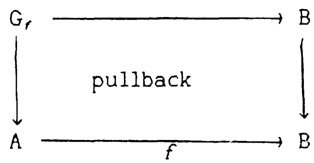
3.4. SPECIFICATION OF OBJECTS UP TO ISO, OF MORPHISMS ON THE NOSE.

For some given R, F, H, φ, $Ext_F(R)$ is determined up to isomorphism only by its universal property. And then for a given choice of $Ext_F(R) = E$ with $\lambda\colon R \to E \cdot F$ there is a unique $\bar{\varphi}\colon E \to H$ such that $(\bar{\varphi}F)\lambda = \varphi$.



It is important to observe these *two very different parts in the use of universal specifications: to construct a new object* E *up to iso, to construct exactly new morphisms* $\bar{\varphi}$.

For example in order to confirm this point, let us recall that if $f: A \to B$ is given, then the object G_f "graph of f is only defined up to isomorphism by the specification:



But in the other direction, G_f, though given only up to iso, defines exactly f. This remark will be meaningful for th description of recursion: a priori il will be different to speak of recursivity of a map or to speak of recursivity of the graph of a map.

3.5. Correct definition of a finite sketch.

The first idea would be: the sketch $S = (\underline{S},P,Y)$ is finite iff $\underline{S}$ is a finite category, P and Y finite families of finite cones and cocones (i.e., cones and cocones with finite bases). A better idea is to work as follows:

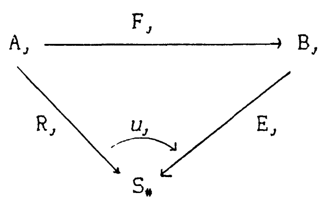
DEFINITIONS. 1. A *finite projective sketch* is a sketch

$$S = (\mathrm{Path}(G,k),P,\emptyset)$$

where G is a multiplicative graph and P a finite family of finite projective cones (i.e., cones with finite bases) in the category $\mathrm{Path}(G,k)$ free generated by (G,k).

We denote by FPSKE the category of morphisms of sketches between finite projective sketches.

2. A *sketch* is a datum $S = (S_*, (E_j,R_j,F_j,u_j)_{j\in J})$ where $S_* \in PSKE_0$ is a projective cone, and $(E_j,R_j;F_j,u_j)_{j\in J}$ is a family indexed by a set J of data of the form



where

$$A_j,\ B_j \in \mathrm{PSKE}_0,\quad E_j,R_j,F_j \in \mathrm{PSKE}_1,\quad u_j\colon R_j \to E_j\cdot F_j.$$

3. A model of S is a morphism M in PSKE

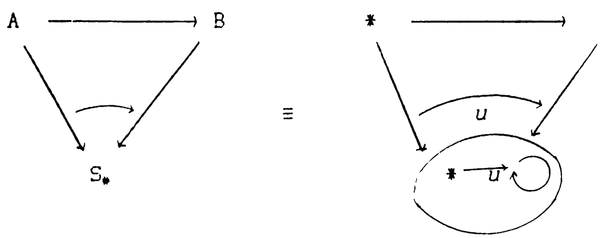
$$M\colon S_* \longrightarrow (\mathrm{SET}, \mathit{Lim}, \emptyset)$$

such that for every j we have $M\cdot E_j \simeq \mathrm{Ext}_{F_j}(M\cdot R_j)$ (of course via Mu_j).

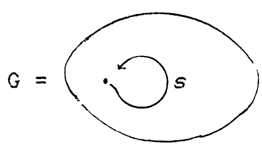
4. A *finite sketch* is a sketch S as above such that J is a finite set, for every $j \in$ J, A_j and B_j are finite projective sketches and S_* is a finite projective sketch.

In fact in programming we are only concerned with finite models (i.e., models in FSET) of finite sketches.

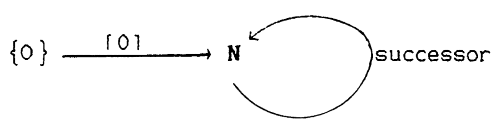
Let us consider the following example of a finite sketch:



A and B are finite projective sketches (A = 1 = {*}) equipped with the specification that * is a final object and the same for S_*: in S_*, * is a final object. B is Path(G) with



the multiplicative graph with just the arrow s; and $\bar{u}_* = u$. A model of this sketch in a category $\underline{X}$ is of course a Natural Number Object (à la Péano-Lawvere) in $\underline{X}$, like **N** in the case $\underline{X}$ = SET:



We denote this sketch by S_N. It is important not to confuse S_N with the "bad" sketch for **N** which consists to specify that $\mathbf{N} = \amalg_{n \in \mathbf{N}} 1$. This is a sketch of **N**, but a tautological on (i.e., without syntactitcal information on **N**) because it is not a finite sketch.

PROPOSITION 15. *Herbrand's schemes are the same thing as programs*

$$\alpha \xrightarrow{\quad I \quad} \rho \xleftarrow{\quad J \quad} \beta$$

in the sense of [7] p. 132, *with* α, ρ *and* β *finite sketches in the above sense.*

4. FREE MODELS AND BOOLEAN ASPECTS OF SKETCHES.

4.1. The underlying category of a model.

If $\underline{X}$ is sketched by S, i.e., if

$$\underline{X} = \mathrm{Real}(S,SET) \simeq Set^{s},$$

there is no natural forgetting functor from $\underline{X}$ to SET, but there is a natural forgetting functor H_S: $\underline{X} \to$ CAT. If R: S $\to$ SET is an object of $\underline{X}$, its *underlying category* $H_S(R)$ is the cofibred (discrete) category associated to R (= Grothendieck's construction), also called the category of hypermorphisms of R (Ehresmann's definition): an object of $H_S(R)$ is a (S,s), with $S \in \underline{S}_0$ and $s \in R(S)$; a morphism in $H_S(R)$ from (S,s) to (S',s') is an

$$f\colon S \to S' \text{ in } \underline{S} \text{ such that } R(f)(s) = s'.$$

So the morphisms of $H_S(R)$ are denoted by (f,s) ; and the composition law in $H_S(R)$ is

$$(f',s').(f,s) = (f'.f,s) \quad \text{iff} \quad s' = R(f)(s).$$

Very often, if it is not ambiguous, $R(f)(s)$ is denoted by fs.

Now if m: R $\to$ R' is a morphism in $\underline{X}$, i.e., a natural transformation from R to R', $H_S(m)$: $H_S(R) \to H_S(R')$ is a functor defined by

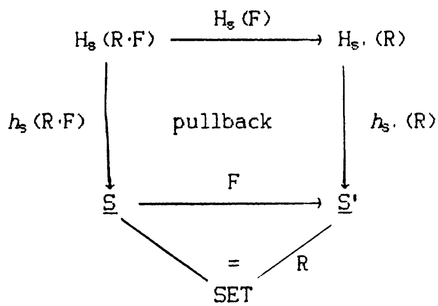
$$H_S(m)(S,s) = m_S(s), \quad H_S(m)(f) = f.$$

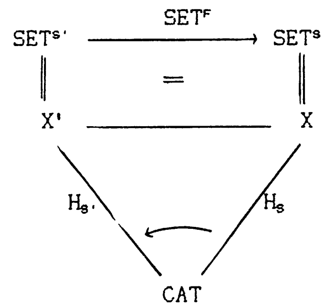
REMARK. Each sketch S of $\underline{X}$ could define a different forgetting functor H_S: $\underline{X} \to$ CAT; so the study of various sketches of $\underline{X}$ is related to the study of forgetting functors from $\underline{X}$ to CAT. More generally if F:

$S \to S'$ is a morphism of sketches, we have, with

$$h_S(R): H_S(R) \to \underline{S} : \quad (f,s) \mapsto f \quad \text{and} \quad H_S(F)(f,s) = (F(f),s)$$

the two diagrams:





Let Diag(CAT) be defined exactly as $D_{\leftarrow}(\underline{X})$ in §2.1, except that the sources of diagrams are not necessary small, and let SKE be the category of sketches.

PROPOSITION 16. H: SKE $\to$ Diag(CAT) *is a functor.*

REMARK. We consider H as a fundamental semantical functor, and in particular for every sketch S and realization R of S, *we think of* $H_S(R)$ *as the support of the structure* R. An immediate application of this point of view (useful thanks to the Zig-zag Theorem of Isbell which tests if a functor is an epimorphism) is:

PROPOSITION 17. *m:* $R \to R'$ *is an epimorphism in* $\underline{X} = SET^S$ *if* $H_S(m)$: $H_S(R) \to H_S(R')$ *is an epimorphism in* CAT.

Now let R: $\underline{S} \to$ SET be an arbitrary functor. It is a model of S (with $S = (\underline{S},P,Y)$) iff:

1. For every projective cone $p = (p_i : V \to B_i)_{i \in I}$ in P we have:

$$\forall\ \bar{B}: \underline{I} \to H_S(R)\ \left[(h_S(R)\cdot\bar{B} = B) \Rightarrow \exists!\ v(\bar{B}) \in R(V)\quad \forall\ i \in \underline{I}\ (\bar{B}_i = (B_i, p_i(v(\bar{B}))))\right].$$

2. For every inductive cone $y = (y_i : B_i \to V)_{i \in L}$ in Y we have

a) $\forall\ v \in R(V)\quad \exists\ i[v] \in \underline{L}_0 \quad \exists\ (B_i, b[v]) \in H_S(R)\ (y_{i[v]}(b[v]) = v)$.

b) If (B_i, b_i) and $(B_{i'}, b_{i'})$ satisfy to a, then in $H_S(R)$ there is a zig-zag $Z((B_i,b_i),(B_{i'},b_{i'}))$ from (B_i,b_i) to $(B_{i'},b_{i'})$.

Hence R is a model of S iff in $H_S(R)$ the operator $\bar{B} \mapsto v(\bar{B})$ and the multivalued operators

$$v \mapsto (i[v], b[v]) \quad \text{and} \quad ((B_1, b_1), (B_{1'}, b_{1'})) \mapsto Z((B_1, b_1), (B_{1'}, b_{1'}))$$

are well defined.

4.2. The locally free diagram.

If R: $\underline{S} \to$ SET is not a model of S, we can try to generate freely a model of S with R: this can be performed by a transfinite formal saturation of $H_S(R)$ with respect to the operators $v(\ )$, $i[\ ]$, $b[\ ]$, $Z(\ ,\ )$; a precise construction is given in Guitart-Lair [12 a]. At each step of the transfinite saturation we have to make a choice (for the part played by some new formal elements), and the ultimate model produced is of course subordinate to this transfinite (but bounded) sequence of choices. Just in order to make this paper self contained (for the readers who have not in hand the paper Guitart [7]), let us reformulate the theorem of existence of locally free diagrams as:

PROPOSITION 18. *Let* S *be a small sketch, and* R: $\underline{S} \to$ SET *a functor. There is a small category* $\underline{A}$ *and a diagram* D: $\underline{A} \to \mathrm{SET}^{S}$ *and a projective cone* $d = (d_A: R \to D_A)_{A \in \underline{A}}$ *in* $\mathrm{SET}^{\underline{S}}$ *with base* D *and top* R, *such that naturally for every* $G \in \mathrm{SET}^{S}$

$$\mathrm{Hom}_{\mathrm{SET}^{\underline{S}}}(R,G) \xrightarrow{\ \simeq\ } \mathrm{colim}_{A \in \underline{A}^{op}} \mathrm{Hom}_{\mathrm{SET}^{S}}(D_A,G).$$

Specially if the only distinguished inductive cones in S *are discrete (i.e., are sums), then* $\underline{A}$ *will be a discrete category (i.e., a set) and*

$$\mathrm{Hom}_{\mathrm{SET}^{\underline{S}}}(R,G) \xrightarrow{\ \simeq\ } \coprod_{A \in \underline{A}} \mathrm{Hom}_{\mathrm{SET}^{S}}(D_A,G).$$

D *is called the locally free diagram on* R.

We may think of $\underline{A}$ as a spectrum of R generated by the successive possible forks in the choices in the attempt to construct the free model. The description of $\underline{A}$ is the "arithmetic of R with respect to S ": $\underline{A} =: \mathrm{Arith}_S(R)$.

4.3. Free fields on Z.

Starting with the ring Z of integers we could try to freely generate a field. For that we choose a map $c: Z\backslash\{0\} \to \{0,1\}$ and by inductive limits in the category of unitary commutative rings we construct

Z_c and u_c: $\mathbf{Z} \to \mathbf{Z}_c$ such that:

1. For every $n \in Z$, $u_c(n) = 0$ if $c(n) = 0$.
2. For every $n \in Z$, $u_c(n)$ is invertible if $c(n) = 1$.
3. u_c is a unitary homomorphism of rings, and $\{u_c(n), n \in Z\}$ generates Z_c.

Then there are three cases:

Case 1. If for every $n \neq 0$, $c(n) = 1$, then $Z_c = \mathbf{Q}$.

Case 2. If for some $n \neq 0$ and some $k \neq 0$ we have $c(n) = 0$ and $c(kn) = 1$, then $Z_c = \mathbf{1}$ (the "field" with one element, where $0 \equiv 1$).

Case 3. If for some $n \neq 0$, $c(n) \neq 0$ and if for every $n \neq 0$ and $k \neq 0$

$$(c(n) = 0 \Rightarrow c(kn) = 0),$$

then we define

$$n_0 = \inf \{n > 0,\ c(n) = 0\} .$$

In this case, if $n > n_0$, $n = kn_0 + r$, $r < n_0$, and

$$c(n) = 0 \quad \text{iff} \quad n_0 | n \quad (n_0 \text{ divise } n).$$

If $m < n_0$, we have an m' in Z_c such that $mm' = 1$, i.e., if m' is not formal, $mm' = 1 \bmod n_0$ or $mm' + kn_0 = 1$ and (Bezout) m and n_0 are relative primes. If this is the case for every $m < n_0$, then n_0 itself is prime. In fact if n_0 is not prime, one of the m' is formal: $n_0 = m.p$, therefore $m.p = 0$ in Z_c and $m'.m = 1$ gives

$$m'.m.p = p = 0, \text{ so } \mathbf{Z}_c = \mathbf{1}.$$

PROPOSITION 19.

$$\mathrm{Arith}_{\mathrm{Field}}(Z) = \{0\} \cup \{1\} \cup \{p \geqslant 2,\ p \text{ prime}\}$$

and with $D_0 = \mathbf{Q}$, $D_1 = \mathbf{1}$, $D_p = Z/pZ$ *we have*

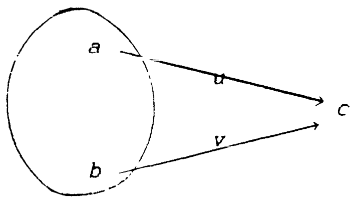
$$\mathrm{Hom}_{\mathrm{Ring}}(Z,K) = \coprod_{a \in \mathrm{ArithField}Z} \mathrm{Hom}_{\mathrm{Field}}(D_a, K).$$

4.4. Free sums and free epis.

Let V be the sketch consisting of the category

$$a \xrightarrow{\ u\ } c \xleftarrow{\ v\ } b \qquad \equiv \underline{V}$$

where we specify that c is isomorphic to the sum $a + b$, via the cocone



The inclusion functor $SET^{V} \hookrightarrow SET^{\underline{V}}$ has no adjoint. Let

$$A \xrightarrow{u} C \xleftarrow{v} B$$

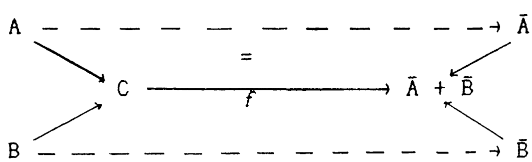
be an object in $SET^{\underline{V}}$.

Case 1: $u(A) \cap u(B) \neq \emptyset$. Then there is no morphism from (u,v) toward a sum diagram, and therefore the locally free diagram on this object (u,v) is the empty diagram.

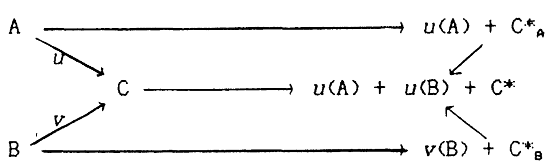
Case 2: $u(A) \cap u(B) = \emptyset$. Then write

$$C^{*} = C \setminus (u(A) \cup u(B)),\ C^{*}_{A} = \{c \in C^{*} \mid f(c) \in \bar{A}\},$$
$$C^{*}_{B} = \{c \in C^{*} \mid f(c) \in \bar{B}\},$$

for f, $\bar{A}$ et $\bar{B}$ as in the diagram



Then f factorizes through

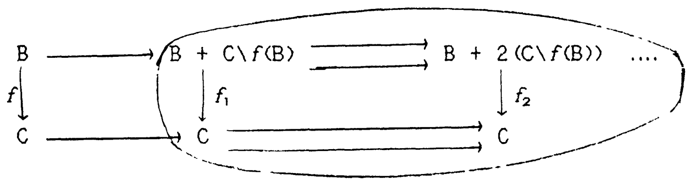


Hence we get

PROPOSITION 20. *The locally free diagram on* (u,v) *above is indexed by* *the set of bipartitions of* C^*.

PROPOSITION 21. *If* I *is a set and if we try to freely construct an* I-*sum diagram generated by an* I-*cocone* $(u_i\colon A_i \to C)$, *we get a locally free diagram indexed by the set of* I-*partitions of* $C^* = C\setminus\coprod u_i(A_i)$, *in the case where for each* i, j, $u(A_i)\cap u(A_j) = \emptyset$, *and else we get the empty diagram.*

PROPOSITION 22. *If in* $\mathrm{SET}^{(\to)}$ *starting with the object* $f\colon B \to C$ *we try to freely generate an object* $g\colon D \to E$ *which is an epimorphism as a map in* SET, *we get the locally free diagram:*



In fact the above diagram is indexed by FSET.

REMARK. These two constructions FS (free sums) and FE (free epis) could be iterated in a sweeping process, i.e., following the sequence:

FS - FE - FS - FE - ....,

eventually with anologous FC (free cokernels), FP (free products), FK (free kernels), and this will give a new proof for the existence of locally free diagram at least for sketches with a finite number of universal specifications.

4.5. Boolean glueings of sketches.

In §4.3 above we have seen that the description of locally free diagrams is a kind of arithmetic of systems of "choices", and so in §4.4 that it is a kind of boolean algebra (cf. $u(A) + C^*_A$, $B + C\setminus f(B)$, etc). In Guitart-Lair [12c] we construct a functorial boolean calculus at the level of sketches, i.e., we exhibit functors

$$\neg\colon \text{L-SKE} \longrightarrow \text{L-SKE}, \quad \wedge_{i\in I}\colon (\text{L-SKE})^I \longrightarrow \text{L-SKE},$$

where L-SKE is the category of L-sketches, an L-sketch being a sketch $S = (\underline{S},P,Y)$ equipped with a morphism $R\colon L \to S^*$ with $S^* = (\underline{S},P^*,Y^*)$ where $P^* \subset P$, $Y^* \subset Y$, such that

$$\mathrm{GrReal}(R)\colon \mathrm{GrReal}(S^*) \longrightarrow \mathrm{GrReal}(L)$$

is an equivalence (GrReal(L) being the groupoid of isomorphisms between models of L in SET)

¬GrReal(S) is the groupoid complementary of GrReal(S) in GrReal(S). Therefore the operators ¬ and ∧ on L-sketches are such that:

$$\neg\mathrm{GrReal}(S) \simeq \mathrm{GrReal}(\neg S), \quad \cap_{i\in I}\mathrm{GrReal}(S_i) \simeq \mathrm{GrReal}(\wedge S_i).$$

4.6. Sheaves and models: Stone duality in terms of sketches.

Let S be a *site*, i.e., a category $\underline{S}$ in which for every object A there is given a set CovA of families $(f_i\colon A_i \to A)_{i\in I}$ called covering families of A.

A sheaf on S is a functor $F\colon \underline{S}^{op} \to \mathrm{SET}$ such that for every $(f_i\colon A_i \to A)_{i\in I} \in \mathrm{Cov}A$ we have

$$FA \simeq \lim FA_k, \quad \text{with } k = i \text{ or } k = (i,j), \quad A_{(i,j)} = A_i\times_A A_j.$$

A *model* of S is a functor $M\colon S \to \mathrm{SET}$ such that for every family $(f_i\colon A_i \to A)_{i\in I} \in \mathrm{Cov}A$ the family $(Mf_i\colon MA_i \to MA)_{i\in I}$ is epimorphic, and M commutes with finite projective limits (cf. Makkai-Reyes [13]).

Let $\underline{K}$ be the category whose objects are the

$$i, \; i\times_A j, \; i_1\times_A\ldots\times_A i_q, \; \text{etc.},$$

and whose morphisms are the canonical maps $i\times_A j \to i$, etc. And

$$\mathbf{A}\colon \underline{K} \to \underline{S}\colon i_1\times_A\ldots\times_A i_q \mapsto A_{i1}\times_A\ldots\times_A A_{iq}\ .$$

Then M is a model of the site S iff M commutes with finite lim and

$$\mathrm{colim}\ \mathrm{Hom}_{\mathrm{SET}\underline{S}}(A_k,M) \simeq \mathrm{Hom}_{\mathrm{SET}\underline{S}}(\mathbf{A},M), \quad \text{i.e.,} \quad \mathrm{colim}(M\cdot\mathbf{A}) \simeq M(A).$$

On the other hand F is a sheaf iff $\lim(F\cdot\mathbf{A}^{op}) \simeq F(A)$.

So: *a model of a site* S *is a sheaf on* S *which commutes with finite* colim *and with values in* SET^{op}.

But $2^{(-)}\colon \mathrm{SET}^{op} \to \mathrm{SET}$ (cf. §1.1) has for adjoint $(2^{(-)})^{op}$ and the associated monad on SET, denoted just by $\prod = 2^{2^{(-)}}\colon \mathrm{SET} \to \mathrm{SET}$, has for algebras the complete atomic boolean algebras. This category of algebras is isomorphic to SET^{op}. So SET^{op} is isomorphic to the subcategory of SET whose objects are the 2^X and whose morphisms are the 2^f; then: a model of S is a presheaf $m\colon \underline{S}^{op} \to \mathrm{SET}$ such that $2^{(-)}\cdot m =: M^{op}$ is a sheaf commuting with finite colim. As $2^{(-)}$ preserves and reflects colim, a model of S is also an m such that $\prod\cdot m$ is a sheaf and m commutes with finite colim. So:

PROPOSITION 23. *The Stone duality in* SET *"induces" for each site an injection*

$$(\text{Models } S)_0 \rightarrowtail (\text{sheaves } S)_0\text{:} \quad M \mapsto (2^{(-)})\cdot M^{op} = \prod\cdot m.$$

We see that the models of a site are some sheaves of complete boolean algebras. Their theory is contained in that of sheaves of Heyting algebras and more generally of sheaves of topos. Let us finally remark that this works for the models of a sketch, because in Guitart-Lair [12a] it is shown that models of a sketch are models of a "big site" in which the covering families are not necessarily $(f_i\colon A_i \to A)_{i\in I}$, i.e., families

$$(\text{Hom}(f_i,-)\colon \text{Hom}_{\underline{S}}(A,-) \longrightarrow \text{Hom}_{\underline{S}}(A_i,-))_{i\in I}$$

which are representable, *but* families $(V \to C_i)_{i\in I}$ with $V \in \text{SET}^{\underline{S}}$ *and for every* $i \in I$, $C_i \in \text{SET}^{\underline{S}}$. A model of such a big site is an $R \in \text{SET}^{\underline{S}}$ such that

$$\text{Hom}_{\text{SET}^{\underline{S}}}(V,R) \longrightarrow\!\!\!\!\rightarrow \coprod_{i\in I}\text{Hom}_{\text{SET}^{\underline{S}}}(C_i,R)$$

(notion of axiomatization and model of an axiomatization due to Andreka-Nemeti).

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