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CATEGORICAL DIFFERENTIAL CALCULUS FOR INFINITE DIMENSIONAL SPACES

by L. D. NEL

RÉSUMÉ: Nous introduisons une théorie catégorique du calcul différentiel pour des espaces de dimension infinie. Nous partons de quatre propriétés postulées pour le corps des scalaires (réel ou complexe) considéré en tant qu'objet d'une catégorie cartésienne fermeé avec structures initiales. Les concept centraux du calcul différentiel sont alors définis et les résultats classiques démontrés de manière purement algébrique. Nous démontrons aussi les lois exponentielles pour les espaces d'applications lisses. M.R. Classification: 18D20, 58C20.

Introduction

The well known category $\mathsf{C_s}$ (sequential convergence spaces, defined by three Fréchet-Urysohn axioms) is a typical example of a category which upholds the theory of this paper. But several categories are suitable and so it is appropriate to adopt as general frame of reference a variable cartesian closed topological category C (whose morphisms are 'continuous' maps) in which the scalar field $\mathbb{K}$ is postulated to have four crucial properties (given in section 1). The stated program calls for all concepts to be expressed categorically i.e. effectively in terms of continuous maps. For 'continuous differentiability' of a C-map $f : U \to F$, where E and F are 'injectable' linear C-spaces and $U \subset E$ a 'primary domain', we introduce the categorical concept 'difference factorizer' for f: a C-map $\Phi : U \times U \to [E, F]$ such that $\Phi(x,y)\cdot(y-x) = f(y) - f(x)$. This definition is motivated by the expression $\Phi(x,y)\cdot(y-x) = \int_0^1 Df(x+\theta(y-x))\cdot(y-x)d\theta = f(y)-f(x)$, well known in analysis. Difference factorizers enable us to study continuous differentiability without explicit reference to convergence of difference quotients: the 'continuity' of C-maps already encode all the relevant information. Derivatives become the values of difference factorizers at points on the diagonal. The standard results about differentiation such as the chain rule and properties of partial derivatives follow in elegant manner.

Integration of C-curves $c : \mathbb{R} \to E$ requires on the one hand spaces E which are separated and complete enough and on the other hand our approach requires a category with virtually all the attributes of the familiar category LC of all linear C-spaces. These two independent demands are simultaneously satisfied by the category oLC of 'optimal linear C-spaces', a concept we developed in [16] and [18]. We construct a natural transformation av ('average value') in oLC, characterized by two algebraic identities for its components, which embodies the standard features of Riemann integration of curves (theorems 3c1, 3c2). The construction is largely based on the reflexiveness of the space of scalar valued curves – a new technique for vector valued integration.

In section 4 we structure the spaces of C^r-maps to become oLC-spaces and we derive the exponential law $\mathsf{C}^\infty(U \times V, G) \simeq \mathsf{C}^\infty(U, \mathsf{C}^\infty(V, G))$ for smooth maps via seven useful auxiliary results. An algebraic characterization of difference factorizers (new to the theory of calculus) provides the link between differentiation and integration, leading to a Fundamental Theorem of calculus expressed as a natural transformation.

Up to this point our study of C^∞-maps makes no use of higher order derivatives. We introduce them in section 5 for their own sake. We derive the Taylor formula and the symmetry of higher order derivatives via the further new concept of higher order difference factorizer.

Specializations of C to various choices ($\mathsf{C_s}$, $\mathsf{C_c}$, $\mathsf{C_d}$, $\mathsf{C_h}$ and Cgt) are discussed in section 6, also how the present paper relates to earlier theories of calculus in special cartesian closed setting: [2], [19] and [6]. $\mathsf{C_s}$ yields a realization of calculus not studied before. $\mathsf{C_c}$ is the filter convergence analogue of $\mathsf{C_s}$ defined by the Choquet axioms [5], perhaps the most important special case because its theory includes all complete locally convex spaces (cf [2]). These two are the principal motivating examples. $\mathsf{C_d}$ yields a calculus closely related to the recent theory of A. Frölicher and A. Kriegl [6]. In general, the smooth maps between locally convex spaces and the functional analysis intrinsic to these special categories differ conspicuously from one to another. In view of these differences (tabulated for $\mathsf{C_c}$ and $\mathsf{C_d}$ in 6c) it is clear that one single category can never be blessed with all the virtues one would like to see. This being the case, the study of calculus in a variable category C, as done here, becomes the more relevant.

Readers are assumed to be familiar with basic categorical concepts: natural transformations, adjunctions and (co-)limits [8]. Verification of the scalar field axioms for $\mathsf{C_s}$ requires very elementary real analysis and just a touch of freshman-level one dimensional calculus. Up to these modest prerequisites this paper can also serve, with considerable logical economy, as a first introduction to infinite dimensional calculus.

1. The category C and its vector spaces

Calculus requires continuous maps and linear continuous maps. In this section we set up the categories of these basic maps.

1a. The category C. In all that follows C will denote a category satisfying the following axioms.

1a1. AXIOM. *C is a category of well structured sets and functions.*

We refer to [15] for a more detailed description, if needed, of the first two axioms. We will follow the usual practice of denoting a C-space (object) and its underlying set by the same symbol, so also a C-map (morphism) and its underlying function. The above axiom includes the fact that C has constant maps.

1a2. AXIOM. *C has initial structures.*

This implies, as well known, that C has final structures too.

1a3. AXIOM. *C is cartesian closed.*

Thus C upholds the exponential law $\mathsf{C}(W\times X, Y) \simeq \mathsf{C}(W, \mathsf{C}(X, Y))$ for its canonical spaces $\mathsf{C}(X, Y)$ of C-maps $X \to Y$.

The remaining four axioms involve the scalar field $\mathbb{K}$ (real or complex) which is supposed structured as C-space once and for all.

1a4. AXIOM. *The arithmetical maps, addition, subtraction and multiplication, are C-maps $\mathbb{K} \times \mathbb{K} \to \mathbb{K}$.*

Λ will always denote a variable *convex subset of* $\mathbb{K}$ *with at least two points* structured as C-subspace of $\mathbb{K}$.

1a5. AXIOM. *If $f, g \in \mathsf{C}(\Lambda, \mathbb{K})$, $\alpha \in \Lambda$ and $f(\xi) = g(\xi)$ for all $\xi \neq \alpha$, then $f = g$.*

In view of axiom 1a4 we can form in the usual way the category LC of linear C-spaces (vector spaces in C) and linear maps. This category has cotensor products: canonical spaces $\mathsf{C}(X, F)$ of C-maps $X \to F$, where $X \in \mathsf{C}$ and $F \in \mathsf{LC}$. For our purposes, *affine map* $\Lambda_1 \to \Lambda_2$ will mean a map $\xi \mapsto \mu\xi + \gamma$ with $\mu \neq 0$.

1a6. AXIOM. *A unique linear C-map $\mathsf{av}_{\Lambda\mathbb{K}} : \mathsf{C}(\Lambda, \mathbb{K}) \to \mathsf{C}(\Lambda \times \Lambda, \mathbb{K})$ exists such that the following two identities hold:*

$$(\beta - \alpha)\mathsf{av}(f)(\alpha, \beta) + (\gamma - \beta)\mathsf{av}(f)(\beta, \gamma) + (\alpha - \gamma)\mathsf{av}(f)(\gamma, \alpha) = 0,$$
$$\mathsf{av}(f)(\xi, \xi) = f(\xi).$$

Moreover, $\mathsf{av}_{\Lambda\mathbb{K}}$ is natural in Λ with respect to affine maps.

The LC-subspace $[E, F] \subset \mathsf{C}(E, F)$ consists of all linear C-maps $E \to F$. In particular, $[E, \mathbb{K}]$ is the canonical dual space and E is called *reflexive* if the map $@_E : E \to [[E, \mathbb{K}], \mathbb{K}]$, $@(x)(u) = u(x)$, is an isomorphism.

1a7. AXIOM. *The space $\mathsf{C}(\Lambda, \mathbb{K})$ is reflexive.*

The purpose of 1a6 is to provide, via the formula $\int_\alpha^\beta f(\theta)d\theta = (\beta - \alpha)\mathsf{av}(f)(\alpha, \beta)$, a categorical expression for the integral of scalar valued curves. Axiom 1a7 will enable us to extend this to vector valued curves so as to retain the essential properties of the integral.

The above axioms should not be considered enough to derive the whole theory of calculus in all its elaborations: they are just what is needed for the results dealt with in this paper: results common to real and complex scalars.

The following well known facts are stated for convenient reference and notation.

1a8. THEOREM. *The following are natural isomorphisms in* C:
(1) $\dagger_{WXY} : \mathsf{C}(W \times X, Y) \to \mathsf{C}(W, \mathsf{C}(X, Y))$; $\dagger(f)(w)(x) \stackrel{\text{def}}{=} f(w, x)$.
(2) $\ddagger_{WXY} : \mathsf{C}(W, \mathsf{C}(X, Y)) \to \mathsf{C}(W \times X, Y)$; $\ddagger(g)(w, x) \stackrel{\text{def}}{=} g(w)(x)$.
(3) $\S_{WXY} : \mathsf{C}(W, \mathsf{C}(X, Y)) \to \mathsf{C}(X, \mathsf{C}(W, Y))$; $\S(f)(x)(w) \stackrel{\text{def}}{=} f(w)(x)$.
(4) $\mathsf{xch}_{XY} : X \times Y \to Y \times X$; $\mathsf{xch}(x, y) \stackrel{\text{def}}{=} (y, x)$ ('exchange').
The following are natural transformations in C
(5) $\mathsf{eval}_{XY} : \mathsf{C}(X, Y) \times X \to Y$; $\mathsf{eval}(f, x) \stackrel{\text{def}}{=} f(x)$.
(6) $@^Y_X : X \to \mathsf{C}(\mathsf{C}(X, Y), Y)$; $@(x)(f) \stackrel{\text{def}}{=} f(x)$ ('at').

Naturality is always to be understood in all subscripted variables (that is why Y is not a subscript on $@$: this map is not natural in Y.) The map $\dagger(f)$ will occasionally also be written $f^\dagger$ where convenient and similarly we put $g^\ddagger = \ddagger(g)$, $f^\S = \S(f)$.

1b. The category iLC. A space $E \in \mathsf{LC}$ may fail to have non-zero linear C-maps $E \to \mathbb{K}$. So we form the subcategory iLC of *injectable linear* C*-spaces* determined by all E for which the map $@_E : E \to [[E, \mathbb{K}], \mathbb{K}]$ is an injection. It is equivalent to demand that E should admit a monomorphism of the form $E \to \mathsf{C}(X, \mathbb{K})$. Then iLC has all the categorical completeness and closedness properties known for LC (see [16]). In particular, if $F \in \mathsf{iLC}$, then so are all $\mathsf{C}(X, F)$ and all $[E, F]$.

> Subcategories must be understood to be full and isomorphism closed in this paper, unless otherwise stated.

2. Difference factorizers, C^r-maps and derivatives

In this section we begin with differential calculus, deriving results which do not yet require completeness of the spaces.

2a. Primary domains. Two kinds of domains for maps appear in theories of differential calculus. Primary domains are those on which differentiability of functions are defined from first principles. Secondary domains are constructed out of primary domains e.g. by gluing them together, as in the definition of manifolds, or by taking subspaces. Smooth maps on secondary domains are then constructed out of smooth maps on primary domains. In this paper we study maps only on primary domains, defined as follows.

Let E be an iLC-space. By a *primary domain* in E, will be meant a convex C-subspace U such that for every $x \in U$ the set $U - x$ spans E. Thus every $h \in E$ can be expressed as a linear combination of the form $h = \sum_{i=1}^{n} \lambda_i(y_i - x)$ with $\lambda_i \in \mathbb{K}$ and $y_i \in U$. It follows at once that *primary domains in* $\mathbb{K}$ *are precisely the convex subspaces with at least two distinct points* i.e. precisely the sets Λ appearing in axioms 1a4 through 1a7. In particular, the closed line segment $[0,1]$ is a primary domain in $\mathbb{K}$.

2a1. PROPOSITION. *Let E and F be* iLC*-spaces.*
(1) *If U and V are primary domains in E and F respectively, then $U \times V$ is a primary domain in $E \times F$.*
(2) *If U is a primary domain in E, then there exists a primary domain I in $\mathbb{K}$ such that for every pair $x, y \in U$ the line segment $\{(1-\lambda)x + \lambda y | \lambda \in I\}$ lies within U and contains both x and y.*
(3) *E is a primary domain in itself.*

Proof . Simple algebraic verification.□

The defining property of primary domain is needed for creation of our C^1-maps, property (1) for the definition of C^r-maps and partial derivatives, property (2) for certain later constructions (see 4b2) while property (3) is used all over. Had we chosen to define primary domains in E to be nothing but E itself, then proposition 2a1 would still be valid - one would then use $\mathbb{K}$ itself in the role of I in (2) - and a parallel but weaker theory would follow.

Useful primary domains in E are furnished by those convex subspaces U which are *c-open* i.e. open in the final topology on (the underlying set of) E induced by the family of all C-curves $c : \mathbb{K} \to E$, where $\mathbb{K}$ carries its usual topology for this purpose. In the motivating special cases, $\mathsf{C} = \mathsf{C_c}$ or $\mathsf{C_s}$ with $\mathbb{K} = \mathbb{R}$, c-open subspaces of Fréchet spaces are just the usual open sets, but in more general spaces the c-open sets need not be open in the underlying locally convex topology.

In an earlier draft we attempted to use 'convex c-open subspace' as definition of primary domain, but awkward exceptional cases then had to be put in the statements of the axioms as well as certain propositions.

When $\mathbb{R}$ is the scalar field, useful primary domains in $\mathbb{R}^n$ are also furnished by n-dimensional rectangles which can be open or closed or compact. But note that no subspace of $\mathbb{R}^k$ $(k < n)$ can be a primary domain in $\mathbb{R}^n$.

> Henceforth we assume that E, F and G denote iLC-spaces, while $U \subset E$, $V \subset F$, $W \subset G$ and $\Lambda \subset \mathbb{K}$ denote primary domains in the respective spaces.

In section 3 the spaces E, F and G will become further restricted to be complete in a suitable sense. The value of a linear map $u : E \to F$ will often be written $u \cdot h$ rather than $u(h)$.

2b. Difference factorizers. Suppose $f : U \to F$ and $\Phi : U \times U \to [E, F]$ are

C-maps. We call Φ a *difference factorizer* for f if the following identity holds:

$$f(y) - f(x) = \Phi(x, y) \cdot (y - x).$$

We define the concept of C^r-map recursively as follows. C^0-map means C-map, we call f a C^r-map if there exists a C^{r-1}-map which is a difference factorizer for f $(r = 1, 2, \ldots)$; we call f a C^∞-map if it is a C^r-map for all r. It follows by induction that

$$\mathsf{C}^0 \supset \mathsf{C}^1 \supset \ldots \mathsf{C}^r \supset \mathsf{C}^{r+1} \supset \ldots \supset \mathsf{C}^\infty.$$

2b1. REMARK. The concept of difference factorizer is motivated by the expression $f(y) - f(x) = \int_0^1 Df(x + \theta(y - x))d\theta \cdot (y - x)$, long known in analysis. If $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable in the classical sense then the integral can easily be seen to be the nothing but the difference quotient $(f(y) - f(x))/(y - x)$ when $x \neq y$ and nothing but $Df(x)$ when $y = x$. However, the analytic expression, unlike difference quotients, still makes sense for infinite dimensional domains. Thus difference factorizers are effective substitutes for difference quotients. The following algebraic characterization will be important later.

2b2. PROPOSITION.
For a C^r-map $\Phi : U \times U \to [E, F]$ the following statements are equivalent:
(α) Φ is a difference factorizer for some C^{r+1}-map $f : U \to F$.
(β) Φ upholds the following triangle identity*:*
$\Phi(x, y) \cdot (y - x) + \Phi(y, z) \cdot (z - y) + \Phi(z, x) \cdot (x - z) = 0.$

Proof . Let $\mathsf{tri}(\Phi)(x, y, z)$ denote the expression on the left. If (α) holds, then $\mathsf{tri}(\Phi)(x, y, z) = f(y) - f(x) + f(z) - f(y) + f(x) - f(z) = 0$. If (β) holds, choose $a \in U$ and put $f(x) = \Phi(a, x) \cdot (x - a)$. By applying the identities $\mathsf{tri}(\Phi)(x, y, a) = 0$ and $\mathsf{tri}(\Phi)(a, y, a) = 0$ one verifies readily that Φ is a difference factorizer for f.

2c. Derivatives. In general a C-map may have infinitely many difference factorizers. This is intuitively clear from the fact that $\Phi(x, y)$ is specified only on the one dimensional vector subspace spanned by $y - x$, allowing different extensions to a complementary subspace. Construction of an explicit counterexample in the case of 2-dimensional domains is no more than an elementary exercise in linear algebra. Nevertheless, two important uniqueness properties are present, as follows.

2c1. PROPOSITION.
(a) *A C^1-map with a one dimensional domain has precisely one difference factorizer.*
(b) *If Φ and Ψ are difference factorizers of the same map $f : U \to F$, then $\Phi(x, x) = \Psi(x, x)$ for all $x \in U$.*

Proof . (a) Suppose $\Phi, \Psi : \Lambda \times \Lambda \to [\mathbb{K}, F]$ are difference factorizers for f. Then for $\xi \neq \eta$ and all $\lambda \in \Lambda$ we have $\Phi(\xi, \eta) \cdot \lambda = \Psi(\xi, \eta) \cdot \lambda = [f(\eta) - f(\xi)] \cdot \lambda/(\eta - \xi)$.

For each fixed ξ and λ the functions $\eta \mapsto \Phi(\xi,\eta)\cdot\lambda$ and $\eta \mapsto \Psi(\xi,\eta)\cdot\lambda$ are C-maps which agree on the set $\xi \neq \eta$. Hence they agree everywhere (this is where axiom 1a5 enters the picture). It follows that $\Phi = \Psi$. (b) Take any $x \in U$. It is enough to show that $l(\Phi(x,x)\cdot h) = l(\Psi(x,x)\cdot h)$ holds for all $h \in E$ and all linear C-maps $l : F \to \mathbb{K}$, since the latter form a monomorphic family. So consider such h and l. Since U is a primary domain, we have $h = \sum_{i=1}^{n} \lambda_i(y_i - x)$ as a linear combination with $y_i \in U$. Let I denote the closed line segment from 0 to 1 in $\mathbb{K}$. Define for each i the C-maps $c_i : I \to U$, $c_i(\theta) = x + \theta(y_i - x)$ and $\phi_i : I \times I \to [\mathbb{K}, \mathbb{K}]$, where

$$\phi_i(\xi,\eta)\cdot\lambda = l(\Phi(c_i(\xi), c_i(\eta))\cdot\lambda(c_i(1) - c_i(0))).$$

Since $(\eta - \xi)(c_i(1) - c_i(0)) = c_i(\eta) - c_i(\xi)$, we have

$$\phi_i(\xi,\eta)(\eta - \xi) = (l \circ f \circ c_i)(\eta) - (l \circ f \circ c_i)(\xi).$$

This shows ϕ_i to be a difference factorizer for $l \circ f \circ c_i$. It follows that $\phi_i(0,0).1 = l(\Phi(x,x)\cdot(y_i - x))$. Now define ψ_i to be the map obtained when Φ is replaced by Ψ in the definition of ϕ_i. It then follows as before that ψ_i is likewise a difference factorizer for $l \circ f \circ c_i$. By uniqueness of difference factorizers for 1-dimensional domains (part a) we conclude that $\phi_i = \psi_i$. By summing $\phi_i(0,0).\lambda_i = \psi_i(0,0).\lambda_i$ over i we obtain the equation $l(\Phi(x,x)\cdot h) = l(\Psi(x,x)\cdot h)$ as required. □

In view of the preceding proposition, the following definition makes sense. For every C^1-map $f : U \to F$ we define the C-map

$$\mathsf{D}f : U \to [E, F]$$

by putting $(\mathsf{D}f)(x) = \Phi(x,x)$, where Φ is an arbitrary difference factorizer for f.

2c2. THEOREM.
(a) *Every constant map $f : U \to F$ is a C^∞-map and $\mathsf{D}f(x) = 0$.*
(b) *Every linear C-map $f : U \to F$ is a C^∞-map and $\mathsf{D}f(x) = f$.*
(c) *Every n-linear C-map $f : U^n \to F$ is a C^∞-map and*

$$\mathsf{D}f(x)\cdot(h_1,\ldots,h_n) = \sum_{i=1}^{n} f(x_1,\ldots,x_{i-1},h_i,x_{i+1},\ldots,x_n).$$

(d) *If Λ is such that the function $f : \Lambda \to \mathbb{K}$, $f(\xi) = 1/\xi$ is a C-map, then f is a C^∞-map and $\mathsf{D}f(\xi)\cdot\eta = -\eta/\xi^2$.*
(e) *(Chain Rule) If $f : U \to V$ and $g : V \to G$ are C^r-maps ($r \geq 1$) then $g \circ f$ is a C^r-map and $\mathsf{D}(g \circ f)(x) = \mathsf{D}g(f(x)) \circ \mathsf{D}f(x)$.*
(f) *If $f_i : U \to F_i$ ($i \in I$) are C^r-maps ($r \geq 1$) then the map $g = (f_i) : U \to \prod_{i\in I} F_i$, induced by the product, is a C^r-map and $\mathsf{D}g(x) = (\mathsf{D}f_i(x))_{i\in I}$.*
(g) *The set $\mathsf{C}^r(U,F)$ of all C^r-maps $U \to F$ forms a vector space (an algebra when F is an algebra) under the pointwise defined vector operations and $\mathsf{D} : \mathsf{C}^r(U,F) \to \mathsf{C}^{r-1}(U,[E,F])$ is a linear function.*

Proof . (a) and (b) are trivial. In (c) we obtain a difference factorizer for f by putting

$$\Phi(x,y)\cdot(h_1,\ldots,h_n)=\sum_{i=1}^{n} f(x_1,\ldots,x_{i-1},h_i,y_{i+1},\ldots,y_n).$$

In (d) a difference factorizer is provided by $\Phi(\xi,\eta)\cdot\lambda=-\lambda/\xi\eta$. To obtain (e) we take difference factorizers Φ_f for f and Φ_g for g build a difference factorizer for $g\circ f$ by putting

$$\Phi_{g\circ f}(x,y)=\Phi_g(f(x),f(y))\circ\Phi_f(x,y).$$

In each of the above cases the formula for the derivative follows at once by applying the definition to the provided difference factorizer. The proofs of the remaining statements are left as a pleasant exercise for the reader. The last three are obtained by induction, using the already established results. In the case of (e), one uses also the fact that composition of linear maps is a bilinear operation.

2d. Partial derivatives. Given the C-maps $f: U\times V\to G$ and $\Phi_1:(U\times U)\times V\to[E,G]$, we call Φ_1 a *difference factorizer in the first variable* for f if the identity

$$\Phi_1(w,x,y)\cdot(x-w)=f(x,y)-f(w,y)$$

holds. A *difference factorizer in the second variable* for f is defined similarly as a C-map $\Phi_2: U\times(V\times V)\to[F,G]$. Supposing such Φ_i to exist for f, we define

$$\begin{aligned}\partial_1 f(x,y) &= \Phi_1(x,x,y),\\ \partial_2 f(x,y) &= \Phi_2(x,y,y).\end{aligned}$$

Putting $f_y=f(-,y)$ we note that $\partial_1 f(x,y)\cdot h=\mathsf{D}f_y(x)\cdot h$. Similarly $\partial_2 f(x,y)\cdot k=\mathsf{D}f_x(y)\cdot k$. The following proposition will be generalized to the case of C^r-maps in section 4, when we will have structures available for the spaces of C^r-maps. The details of the present proof will serve also as basis for that more general proof. It is convenient in this context to introduce the map $\dagger 2\stackrel{\text{def}}{=}\dagger\circ\mathsf{xch}$, so that $f^{\dagger 2}(x)(w)=f(w,x)$ and for the sake of symmetry we put $\dagger 1\stackrel{\text{def}}{=}\dagger$.

2d1. PROPOSITION. *For a* C*-map $f: U\times V\to G$ the following statements are equivalent:*

(α) *f is a* C^1*-map.*

(β) *$f^{\dagger 1}: U\to\mathsf{C}(V,G)$ and $f^{\dagger 2}: V\to\mathsf{C}(U,G)$ are* C^1*-maps.*

Moreover, if f is a C^1*-map, then*

$$\mathsf{D}f(x,y)\cdot(h,k)=\partial_1 f(x,y)\cdot h+\partial_2 f(x,y)\cdot k.$$

Proof . Suppose (α) and take a difference factorizer $\Phi:(U\times V)\times(U\times V)\to[U\times V,G]$ for f. To get a difference factorizer for $f^{\dagger 1}$ we begin by constructing the map $\Psi_1:(U\times U)\times V\to[E,G]$ as the following composition (in which $\mathsf{pro}_1: U\times V\to U$ denotes the canonical projection and pro_2 similarly).

$$\begin{array}{lr}
(U \times U) \times V & ((w,x),y) \\
\quad \downarrow \; (\mathsf{id},\mathsf{id}) \times (\mathsf{id},\mathsf{id}) & \downarrow \\
(U \times U) \times (V \times V) & ((w,x),(y,y)) \\
\quad \downarrow \; \mathsf{iso} & \downarrow \\
(U \times V) \times (U \times V) & ((w,y),(x,y)) \\
\quad \downarrow \; \Phi & \downarrow \\
[E \times F, G] & \Phi((w,y),(x,y))(-,-) \\
\quad \downarrow \; [(\mathsf{id},0),G] & \downarrow \\
[E,G] & \Phi((w,y),(x,y))(-,0)
\end{array}$$

Now put $\Phi_1 = \S \circ \dagger 1(\Psi_1)$, a composition of the form

$$U \times U \to \mathsf{C}(F,[E,G]) \to [E,\mathsf{C}(V,G)].$$

The map Φ_2 is constructed in the obvious similar way. Thus we obtain maps such that

$$\begin{aligned}
(\Psi_1(w,x) \cdot h)(y) &= \Phi((w,y),(x,y)) \cdot (h,0) \\
(\Psi_2(y,z) \cdot k)(x) &= \Phi((x,y),(x,z)\cdot)(0,k) \\
(\Psi_1(x,w) \cdot (w-x))(y) &= f(w,y) - f(x,y) \\
(\Psi_2(y,z) \cdot (z-y))(x) &= f(x,z) - f(x,y).
\end{aligned}$$

Hence Ψ_1 and Ψ_2 are difference factorizers for $f^{\dagger 1}$ and $f^{\dagger 2}$ respectively and $\partial_1 f(x,y) \cdot h = \mathsf{D}f(x,y) \cdot (h,0)$, $\partial_2 f(x,y) \cdot k = \mathsf{D}f(x,y) \cdot (0,k)$. Suppose ($\beta$) and let Ψ_1, Ψ_2 be given difference factorizers of $f^{\dagger 1}$ and $f^{\dagger 2}$ as above. Compose the map Φ as follows.

$$\begin{array}{l}
(U \times V) \times (U \times V) \\
\qquad \downarrow \; \mathsf{iso} \\
(U \times U) \times (V \times V) \\
\qquad \downarrow \; \Phi_1 \times \Phi_2 \\
[E, \mathsf{C}(V,G)] \times [F, \mathsf{C}(U,G)] \\
\qquad \downarrow \; [\mathsf{pro}_1, \mathsf{C}(U,G)] \times [\mathsf{pro}_2, \mathsf{C}(V,G)] \\
[E \times F, \mathsf{C}(V,G)] \times [E \times F, \mathsf{C}(U,G)] \\
\qquad \downarrow \; [E \times F, \mathsf{C}(\mathsf{pro}_2, G)] \times [E \times F, \mathsf{C}(\mathsf{pro}_1, G)] \\
[E \times F, \mathsf{C}(U \times V, G)] \times [E \times F, \mathsf{C}(U \times V, G)]
\end{array}$$

$$\downarrow +$$
$$[E \times F, \mathsf{C}(U \times V, G)]$$

This construction yields a map Φ such that

$$\Phi((w,y),(x,z)) \cdot (h,k) = (\Psi_1(w,x) \cdot h)(y) + (\Psi_2(y,z) \cdot k)(x).$$

Direct verification shows that Φ is a difference factorizer for f.

2e. Differentials. For C^1-maps $f : U \to F$, with derivative $\mathsf{D}f : U \to [E,F]$, we define the *differential* $\mathsf{d}f : U \times E :\to F$ to be the map $\ddagger(\mathsf{D}f)$. Thus $\mathsf{d}f$ is linear in the second variable. The exponential law of C allows us to recover $\mathsf{D}f$ from this differential. We thus have two equivalent concepts and every statement about one of them translates automatically into a statement about the other. We could adapt the definition of difference factorizer in the obvious way to lead to a differential rather than a derivative. This could be useful if one wishes, for pedagogical reasons, to develop differential calculus temporarily in a restricted context which does not uphold an exponential law of mapping spaces.

3. The category oLC and integration of curves.

The map av in axiom 1a6 provides an integral for scalar valued C-curves. The main business of this section is to create corresponding integral-providing maps for vector valued curves in the form of a natural transformation. First we must create a subcategory of iLC whose spaces are sufficiently complete to allow this. There is no hope that iLC-spaces will do: it is well known that a continuous curve $f : [\alpha, \beta] \to E$ into a normed space may fail to be Riemann integrable when E is not complete.

3a. Optimal LC-spaces. Recall that a monomorphism m in any category is called extremal if it allows a factorization $m = k \circ e$ through an epimorphism e only if e is an isomorphism. We define the subcategory oLC of *optimal* LC*-spaces*, to be determined by all E such that $@_E : E \to [[E, \mathbb{K}], \mathbb{K}]$ is an extremal monomorphism in iLC. It is equivalent to demand that there exists some extremal monomorphism $m : E \to \mathsf{C}(X, \mathbb{K})$. We know from [16] and [18] that oLC is a reflective subcategory with epimorphic adjunctions and $\mathsf{C}(X,F)$ and $[E,F]$ lie in oLC whenever F does. See [16] for fourteen good categorical properties of oLC.

Let us emphasize that in the definition of oLC it is imperative that extremal monomorphisms in iLC be considered rather than in LC. In LC the extremal monomorphisms coincide with embeddings and their use in the definition would result in the category eLC of *embeddable* LC*-spaces*. The categories in the chain

$$\mathsf{oLC} \subset \mathsf{eLC} \subset \mathsf{iLC} \subset \mathsf{LC}$$

have very similar properties as categories, but the quality of their spaces differs significantly.

To illustrate further the nature of oLC, we mention that in the special case $\mathsf{C} = \mathsf{C_c}$, $\mathbb{K} = \mathbb{R}$, all extremal monomorphisms in $\mathsf{iLC_c}$ are closed embeddings. Since all spaces $\mathsf{C}(X, \mathbb{R})$ are complete, all $\mathsf{oLC_c}$-spaces are complete. But not all complete $\mathsf{iLC_c}$-spaces are $\mathsf{oLC_c}$-spaces. In fact, the subcategory of all regular complete $\mathsf{iLC_c}$-spaces is not closed under formation of spaces $[E, F]$, as a recent (still unplublished) example of H.-P. Butzmann shows. Thus $\mathsf{oLC_c}$ automatically selects only the good complete spaces. It is just a fluke that completeness, being a 'topological' condition, defines an algebraically well behaved subcategory in the case of normed spaces.

3b. The spaces $X \odot \mathbb{K}$ and $X \hat{\odot} \mathbb{K}$. The category iLC upholds the external exponential laws

3c0.
$$[X \odot E, F] \simeq [E, \mathsf{C}(X, F)] \overset{\S}{\simeq} \mathsf{C}(X, [E, F]),$$

where $X \in \mathsf{C}$, $E, F \in \mathsf{iLC}$ (see [16]). The explicit description of $X \odot E$ in LC given in [14], applies also in iLC because the space $X \odot E$ constructed in LC lies in iLC. For the moment we need only recall three facts about this interesting space. (i) The underlying vector space of $X \odot E$ consists of all functions $X \to E$ (not C-maps) vanishing on complements of finite subsets. (ii) $X \odot \mathbb{K}$ is the free iLC-space on X; in other words, the functor $(-) \odot \mathbb{K}$ is left adjoint to the underlying space functor $\mathsf{iLC} \to \mathsf{C}$. (iii) $\mathsf{C}(X, \mathbb{K})$ is isomorphic to the canonical dual space of $X \odot \mathbb{K}$. In fact, by just putting $E = F = \mathbb{K}$ in the above isomorphism we conclude $\mathsf{C}(X, \mathbb{K}) \simeq [X \odot \mathbb{K}, \mathbb{K}]$.

Let us build the commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "@_X"] \arrow[dr, "\mathsf{fa}_X"'] & & {[\mathsf{C}(X, \mathbb{K}), \mathbb{K}]} \\
& X \odot \mathbb{K} \arrow[r, "\mathsf{oc}_X"'] & X \hat{\odot} \mathbb{K} \arrow[u, "\mathsf{rz}_X"']
\end{tikzcd}$$

as follows. The map fa_X ('free adjunction') is the unit for adjunction of the left adjoint $(-) \odot \mathbb{K}$; it maps each point to its characteristic function. By the universal property, $@_X$ induces a linear map $\mathsf{lin}@_X$ (not displayed in the diagram) such that $\mathsf{lin}@ \circ \mathsf{fa} = @$. Now define $X \hat{\odot} \mathbb{K}$ to be the intermediate space which arises in the (epi, extremal mono)-factorization of $\mathsf{lin}@_X$; we will call the factors oc_X and rz_X. As such, $X \hat{\odot} \mathbb{K}$ is the oLC reflection ('completion') of $X \odot \mathbb{K}$, thus the free oLC-space on X (see [16]). In this connection we interpret $X \hat{\odot} \mathbb{K}$ as a space of abstract Radon measures, $X \odot \mathbb{K}$ as a space of point measures, oc as the oLC-completion map which interprets point measures as Radon measures and we look upon rz as an abstract Riesz representation map. In the special case where $\mathsf{oLC} = \mathsf{oLC_c}$ and

X is a locally compact space, this is precisely what these spaces and maps turn out to be. Notice that all maps in the diagram are C-maps and that all but fa and @ are linear C-maps.

3b1. PROPOSITION. *For a C-space X the following statements are equivalent.*
(a) $\mathsf{lin}@_X$ *is an epimorphism in* iLC.
(b) rz_X *is an isomorphism in* oLC.
(c) $\mathsf{C}(X, \mathbb{K})$ *is a reflexive space.*
(d) *If* $u, v : [\mathsf{C}(X, \mathbb{K}), \mathbb{K}] \to \mathbb{K}$ *are linear* C*-maps such that* $u \circ @_X = v \circ @_X$*, then* $u = v$.

Proof . The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) follow readily from the facts presented in the preceding discussion. To show that (c) $\Rightarrow$ (d), let us suppose $@_{\mathsf{C}(X,\mathbb{K})}$: $\mathsf{C}(X, \mathbb{K}) \to [[\mathsf{C}(X, \mathbb{K}), \mathbb{K}], \mathbb{K}]$ is an isomorphism. If u and v are maps with $u \circ @_X = v \circ @_X$, then we can find $f, g \in \mathsf{C}(X, \mathbb{K})$ such that $@_{\mathsf{C}(X,\mathbb{K})}(f) = u$ and $@_{\mathsf{C}(X,\mathbb{K})}(g) = v$, hence for all $x \in X$ we have $(u \circ @_X)(x) = @_{\mathsf{C}(X,\mathbb{K})}(f)(@_X(x)) = @_X(x)(f) = f(x)$ and similarly $(v \circ @_X)(x) = g(x)$; we conclude $f = g$ and therefore $u = v$. Suppose this time that (d) holds and consider $u, v : [\mathsf{C}(X, \mathbb{K}), \mathbb{K}] \to G$ such that $u \circ @_X = v \circ @_X$. Then every $w : G \to \mathbb{K}$ gives $w \circ u \circ @_X = w \circ v \circ @_X$. By ($\epsilon$), $w \circ u = w \circ v$. Since the maps w form a monomorphic family, we conclude $u = v$ and $@_X$ is an epimorphism.

3b2. REMARK. It is worth pointing out that proposition 3b1 holds in greater generality than the present context. Its proof made no use of axioms 1a5,6,7. In this connection, see 3.5 in [13] where the implication (d) $\Rightarrow$ (c) was proved in a much more general context. The proof of the converse implication, given above, also applies in that context.

Henceforth E, F and G will denote oLC-spaces.

3c. The natural transformation av. We have now prepared the way for the promised construction of integral-providing maps for vector valued curves out of the corresponding one for scalar valued curves.

3c1. THEOREM.
There exists a unique natural transformation

$$\mathsf{av}_{\Lambda E} : \mathsf{C}(\Lambda, E) \to \mathsf{C}(\Lambda \times \Lambda, E),$$

natural in Λ with respect to affine maps and natural in E with respect to oLC*-maps, such that the following average value identities are upheld:*

$$(\beta - \alpha)\mathsf{av}(f)(\alpha, \beta) + (\gamma - \beta)\mathsf{av}(f)(\beta, \gamma) + (\alpha - \gamma)\mathsf{av}(f)(\gamma, \alpha) = 0,$$
$$\mathsf{av}(f)(\xi, \xi) = f(\xi).$$

Proof . We begin by assembling the building blocks needed for the construction of av_E: (1) the linear map $\mathsf{av}_{\Lambda\mathbb{K}} : \mathsf{C}(\Lambda, \mathbb{K}) \to \mathsf{C}(\Lambda \times \Lambda, \mathbb{K})$ provided by axiom 1a6, (2) the linear map $\mathsf{lin}@^E_\Lambda : \Lambda\hat{\odot}\mathbb{K} \to [\mathsf{C}(\Lambda, E), E]$ induced by $@^E_\Lambda : \Lambda \to [\mathsf{C}(\Lambda, E), E]$ via the universal property of $\Lambda\hat{\odot}\mathbb{K}$ (see 3b) and (3) the LC-isomorphism $\mathsf{rz} : \Lambda\hat{\odot}\mathbb{K} \to [\mathsf{C}(\Lambda, \mathbb{K}), \mathbb{K}]$ provided by 3b1. Apply $\S : [\mathsf{C}(\Lambda, \mathbb{K}), \mathsf{C}(\Lambda \times \Lambda, \mathbb{K})] \to \mathsf{C}(\Lambda \times \Lambda, [\mathsf{C}(\Lambda, \mathbb{K}), \mathbb{K}])$ (cf. 3b0) to form $\S(\mathsf{av})$ and compose the map tav^E as follows.

$$\mathsf{tav}^E : \quad \Lambda \times \Lambda \xrightarrow{\S(\mathsf{av})} [\mathsf{C}(\Lambda, \mathbb{K}), \mathbb{K}] \xrightarrow{\mathsf{rz}^{-1}} \Lambda\hat{\odot}\mathbb{K} \xrightarrow{\mathsf{lin}@} [\mathsf{C}(\Lambda, E), E].$$

Then $\mathsf{tav}^E \in \mathsf{C}(\Lambda \times \Lambda, [\mathsf{C}(\Lambda, E), E]))$. Put

$$\mathsf{av}_E \stackrel{\text{def}}{=} \S(\mathsf{tav}^E) \in [\mathsf{C}(\Lambda, E), \mathsf{C}(\Lambda \times \Lambda, E)].$$

Naturality of av in E means that for every oLC-map $w : E \to F$ the following diagram should commute.

$$\begin{tikzcd}
\mathsf{C}(\Lambda, E) \arrow[r, "\mathsf{av}_E"] \arrow[d, "{\mathsf{C}(\Lambda, w)}"'] & \mathsf{C}(\Lambda \times \Lambda, E) \arrow[d, "{\mathsf{C}(\Lambda \times \Lambda, w)}"] \\
\mathsf{C}(\Lambda, F) \arrow[r, "\mathsf{av}_F"'] & \mathsf{C}(\Lambda \times \Lambda, F)
\end{tikzcd}$$

In other words, we should have $w \circ \mathsf{av}_E(f) = \mathsf{av}_F(w \circ f)$ for all $f \in \mathsf{C}(\Lambda, E)$. To prove this, we begin by noting that the following diagram commutes:

$$\begin{tikzcd}
\Lambda\hat{\odot}\mathbb{K} \arrow[r, "\mathsf{lin}@^E_\Lambda"] \arrow[d, "\mathsf{lin}@^F_\Lambda"'] & {[\mathsf{C}(\Lambda, E), E]} \arrow[d, "{[\mathsf{C}(\Lambda, E), w]}"] \\
{[\mathsf{C}(\Lambda, F), F]} \arrow[r, "{[\mathsf{C}(\Lambda, w), F]}"'] & {[\mathsf{C}(\Lambda, E), F]}
\end{tikzcd}$$

This follows from the universal property of $\Lambda\hat{\odot}\mathbb{K}$ via the following equation $[\mathsf{C}(\Lambda, E), w] \circ @^E_\Lambda = [\mathsf{C}(\Lambda, w), F]$ i.e. $w \circ @(\xi) = @(\xi) \circ \mathsf{C}(\Lambda, w)$, established by direct verification. It now follows at once that

$$[\mathsf{C}(\Lambda, E), w] \circ \mathsf{tav}^E = [\mathsf{C}(\Lambda, w), F] \circ \mathsf{tav}^F.$$

Evaluation of the left side gives us $w(\mathsf{tav}^E(\alpha, \beta))(f) = w(\mathsf{av}(f)(\alpha, \beta))$ while evaluation at the same points on the right side gives us $(\mathsf{tav}(\alpha, \beta) \circ \mathsf{C}(\Lambda, w))(f) =$

$\mathsf{tav}^F(\alpha,\beta)(w \circ f) = \mathsf{av}_F(w \circ f)(\alpha,\beta)$. The naturality in E is thus established. To verify that av_E upholds the average value identities, put $F = \mathbb{K}$ in the naturality diagram above. Note that the linear functionals $w : E \to \mathbb{K}$ form a monomorphic family and that the functors $\mathsf{C}(\Lambda, -)$ and $\mathsf{C}(\Lambda \times \Lambda, -)$ preserve monomorphic families. By using these facts it can readily be seen that since $\mathsf{av}_{\Lambda\mathbb{K}}$ upholds the average value identities, av_E must do so as well. The stated uniqueness and the naturality in Λ follows from the corresponding facts in the scalar valued case (axiom 1a6). via the mentioned monomorphic families.

3c2. LEMMA. *The following diagram commutes*

$$\begin{tikzcd}
\mathsf{C}(\Lambda, \mathsf{C}(X,F)) \arrow[r, "\mathsf{av}_{\Lambda\mathsf{C}(X,F)}"] \arrow[d, "\S_{\Lambda XF}"'] & \mathsf{C}(\Lambda \times \Lambda, \mathsf{C}(X,F)) \arrow[d, "\S_{\Lambda\times\Lambda, XF}"] \\
\mathsf{C}(X, \mathsf{C}(\Lambda, F)) \arrow[r, "\mathsf{C}(X, \mathsf{av}_{\Lambda F})"'] & \mathsf{C}(X, \mathsf{C}(\Lambda \times \Lambda, F))
\end{tikzcd}$$

Proof . We establish by straight forward verification that $\S_{X,\Lambda\times\Lambda,F} \circ \mathsf{C}(X, \mathsf{av}_{\Lambda F}) \circ \S_{\Lambda XF}$ is a natural map which satisfies the identities of 3e1 for $E = \mathsf{C}(X,F)$. By uniqueness of such a map, the composition just given must equal $\mathsf{av}_{\Lambda\mathsf{C}(X,F)}$. We now show how the natural transformation av provides in categorical manner for integration of vector valued curves.

3c3. THEOREM. *Define* $\int_\alpha^\beta f(\theta)d(\theta) = (\beta-\alpha)\cdot\mathsf{av}(f)(\alpha,\beta)$ *for* $f \in \mathsf{C}(\Lambda, E)$. *Then the following hold:*

(a) $u \int_\alpha^\beta f(\theta)d\theta = \int_\alpha^\beta u(f(\theta))d\theta$ *for linear* C*-maps* u.

(b) $f \mapsto \int_\alpha^\beta f$ *is a* C*-map* $\mathsf{C}(\Lambda, E) \to E$.

(c) $\int_\alpha^\beta = \int_\alpha^\gamma + \int_\gamma^\beta$.

(d) *For* $g : \Lambda \times X \to F$ *we have*

$$\int_\alpha^\beta g(\theta,x)d\theta \stackrel{\text{def}}{=} \int_\alpha^\beta g^{\dagger_2}(x)(\theta)d\theta = \int_\alpha^\beta g^{\dagger_1}(\theta)d\theta(x).$$

Proof . Property (a) just restates the naturality of $\mathsf{av}_{\Lambda E}$ in the variable E. The map in (b) is $(\beta-\alpha)\cdot\mathsf{eval}(-,(\alpha,\beta)) \circ \mathsf{av}$. (In classical theory, with E a Banach space, one usually arrives at the continuity of this map via the inequality $|\int_\alpha^\beta f| \leq |\beta-\alpha| \sup_{\alpha\leq\tau\leq\beta} |f(\tau)|$). Property (c) just restates the first average value identity. Property (d) follows from lemma 3c2: it is essentially the equation derived in its proof, where we take $E = \mathsf{C}(X,F)$ and $g = f^\dagger$.

We return to integration of curves after the exponential laws for smooth maps have been established. It will put us in a position to derive quickly a natural Fundamental Theorem of calculus.

3d. Generalizations uLC of oLC. One could replace the class of extremal monomorphisms in the definition of oLC by any class u of monomorphisms such that the following three conditions are satisfied: (1) u is preserved by the functors $\mathsf{C}(X,-)$; (2) there is a matching class of epimorphisms, u-dense maps (say) such that every LC-map v has an essentially unique factorization $v = m \circ e$ with $m \in \mathsf{u}$ and e a u-dense map; (3) the map $\mathsf{lin}@_\Lambda : \Lambda \odot \mathbb{K} \to [\mathsf{C}(\Lambda,\mathbb{K}),\mathbb{K}]$ is always u-dense. By properties (1) and (2), u is then an 'upgrading class' (see [16]). *The subcategory* uLC *determined by all E for which $@_E : E \to [[E,\mathbb{K}],\mathbb{K}]$ is a* u*-map can be substituted for* oLC *in the present theory.* The closed embeddings in $\mathsf{iLC_c}$ (in the sense of filter convergence) form a typical example of an upgrading class f, leading to the category $\mathsf{fLC_c}$ (the "functionally complete" spaces of [16]).

4. Spaces of smooth maps

In this section we provide a C-structure for the vector spaces $\mathsf{C}^r(U,F)$ introduced in section 2. We will show that these spaces are always oLC-spaces. We will establish a natural version of the familiar Fundamental Theorem of calculus for curves into oLC-spaces and the exponential law for the spaces of C^∞-maps.

The notations of section 3 remain in force; in particular, E, F and G are always oLC-spaces. We remind that oLC could be replaced throughout by a more general category uLC as explained in 3d.

4a. The LC-spaces $\mathsf{C}^r(U,F)$. We structure $\mathsf{C}^r(U,F)$ $(r \in \mathbb{N})$ recursively as LC-spaces as follows: $\mathsf{C}^0(U,F) \stackrel{\text{def}}{=} \mathsf{C}(U,F)$ and $\mathsf{C}^r(U,F)$ is defined to carry the initial C-structure induced by the linear functions $D : \mathsf{C}^r(U,F) \to \mathsf{C}^{r-1}(U,[E,F])$ and $\mathsf{inc} : \mathsf{C}^r(U,F) \to \mathsf{C}^{r-1}(U,F)$ $(r \geq 1)$. The space $\mathsf{C}^\infty(U,F)$ is defined to carry the initial C-structure induced by the family of all linear inclusions

$$\mathsf{inc}^{\infty r} : \mathsf{C}^\infty(U,F) \to \mathsf{C}^r(U,F) \ (r \in \mathbb{N}).$$

These C-structures are evidently compatible with the vector operations and yield LC-spaces. More generally, we define $\mathsf{C}^r(U,V)$ to be the obvious C-subspace of $\mathsf{C}^r(U,F)$. But it can readily be seen that $\mathsf{C}^r(U,V)$ need not be a primary domain when V is not a vector space. For this reason the exponential law to be established will be restricted to mapping spaces having codomain in oLC.

4a1. PROPOSITION. *For all $r > s \in \mathbb{N}$ the triangles*

$$\begin{tikzcd} & \mathsf{C}^\infty(U,F) \arrow[dl, "\mathsf{inc}^{\infty r}"'] \arrow[dr, "\mathsf{inc}^{\infty s}"] & \\ \mathsf{C}^r(U,F) \arrow[rr, "\mathsf{inc}^{rs}"'] & & \mathsf{C}^s(U,F) \end{tikzcd}$$

commute and constitute a projective limit diagram in LC.

Differentiation is a natural transformation in the following restricted way.

4a2. PROPOSITION. *Let* C^r *denote the category of* oLC*-spaces with* C^r*-maps between them. Then*

$$\mathsf{D}_{UF} : \mathsf{C}^1(U,F) \to \mathsf{C}^0(U,[E,F])$$

is a natural transformation from $\mathsf{C}^1(-,-)$ *to* $\mathsf{C}^0 \circ (\mathsf{Id}, [-,-]) : \mathsf{C}^{1\mathrm{op}} \times \mathsf{oLC} \to \mathsf{oLC}$.

4b. The exponential law for smooth maps. A problem to overcome is the failure of $\dagger$ to lift (by restriction) as a C-map $\mathsf{C}^r(U \times V, G) \to \mathsf{C}^r(U, \mathsf{C}^r(V,G))$. This failure is not surprising because a C^1-map f will clearly not in general yield $\partial_1\partial_2 f$ as C-map. We will show however that $\ddagger$ does lift and that $\dagger$ at least lifts in a weakened form to the C^r situation. A peculiar technical maneuver is used for this: neither these two liftings nor any one of five other auxiliary results about C^r-maps could be proved in isolation. But together they allow proof by induction.

The derivation of the exponential law will include a study of the *standard difference factorizer* function

$$\mathsf{dfac}^r_F : \mathsf{C}^r(U,F) \to \mathsf{C}^{r-1}(U \times U, [E,F])$$

defined by the assignment $\mathsf{dfac}(g)(x,y) \cdot h = \int_0^1 \mathsf{D}g(x + \theta(y-x)) \cdot h d\theta$.

In what follows, $q, r, s \in \mathbb{N}$ and $p \in \mathbb{N} \cup \{\infty\}$. C^p will denote the category formed by oLC-spaces and C^p-maps between them. In the theorem to follow, naturality must be understood to be with respect to C^r-maps as far as the variables U and V are concerned and with respect to linear C-maps as far as the variable G is concerned. The latter naturality will be extended to C^∞-maps in the proof of theorem 4b3.

4b1. THEOREM. *The statements* (a^r) *through* (g^r) *hold for each* $r \in \mathbb{N}$ *and all* U, V *and* G.
(a^r) *The functor* $\mathsf{C}(U,-) : \mathsf{oLC} \to \mathsf{oLC}$ *lifts to a functor* $\mathsf{C}^r(U,-) : \mathsf{oLC} \to \mathsf{oLC}$ *and the latter preserves* C*-initial families of linear* C*-maps; hence it preserves cartesian products in* oLC.

(b^r) *There exists a natural* oLC*-isomorphism* $\S^r_{EVG} : [E, \mathsf{C}^r(V,G)] \to \mathsf{C}^r(V,[E,G])$ *which lifts* $\S_{EVG} : [E, \mathsf{C}(V,G)] \to \mathsf{C}(V,[E,G])$.

(c^r) *A map* $f \in \mathsf{C}(U \times V, G)$ *is a* C^r*-map if and only if for all* $s = 0,1,\ldots,r$ *we have* $f^{\dagger 1} \in \mathsf{C}^s(U, \mathsf{C}^{r-s}(V,G))$ *and* $f^{\dagger 2} \in \mathsf{C}^s(V, \mathsf{C}^{r-s}(U,G))$.

(d^r) *For every* $s = 0,1,\ldots,r$ *there is a natural transformation* $\dagger 1^{r,s}_{UVG} : \mathsf{C}^r(U \times V, G) \to \mathsf{C}^s(U, \mathsf{C}^{r-s}(V,G))$ *which lifts* $\dagger 1$ *and a similar statement holds for* $\dagger 2^{r,s}$.

(e^r) *There exists a natural map* $\ddagger^r : \mathsf{C}^r(U, \mathsf{C}^r(V,G)) \to \mathsf{C}^r(U \times V, G)$ *which lifts the corresponding map* $\ddagger$ *in* C.

(f^r) *The function* $\mathsf{comp}^r_{UVG} : \mathsf{C}^r(U,V) \times \mathsf{C}^r(V,G) \to \mathsf{C}^r(U,G)$,
$\mathsf{comp}^r(f,g) = g \circ f$, *is a* C*-map.*

(g^r) $\mathsf{C}^r(U,F)$ *is a* oLC*-space and when* $r > 0$, *the function* $\mathsf{dfac}^r_F : \mathsf{C}^r(U,F) \to \mathsf{C}^{r-1}(U \times U, [E,F])$ *is a natural transformation in* oLC *such that* $\mathsf{dfac}^r(g)$ *is a difference factorizer for* g.

Proof . In all cases except dfac the naturality follows at once from that of the transformations in C being lifted. We use induction. For $r = 0$ there is nothing to prove since all of the above facts are known for the category C; indeed, these properties motivated the concept of functional analytic category (see [16]). So fix $r \in \mathbb{N}$ and suppose (a^r) through (g^r) hold for all spaces.

Proof (a^{r+1}). It is clear from theorem 2c2 that for every LC-map $w : F \to G$, the linear function $\mathsf{C}(U,w)$, $f \mapsto w \circ f$, carries $\mathsf{C}^{r+1}(U,F)$ into $\mathsf{C}^{r+1}(U,G)$. Thus the lifted function $\mathsf{C}^{r+1}(U,w) : \mathsf{C}^{r+1}(U,F) \to \mathsf{C}^{r+1}(U,G)$ exists as a linear function. Let us show this function is a C-map. The argument is rather simple, but we present it in detail in order to identify typical steps to be encountered repeatedly. We have to show that the compositions

$$\mathsf{C}^{r+1}(U,F) \xrightarrow{\mathsf{C}^{r+1}(U,w)} \mathsf{C}^{r+1}(U,G) \xrightarrow{\mathsf{inc}} \mathsf{C}^r(U,G),$$
$$\mathsf{C}^{r+1}(U,F) \xrightarrow{\mathsf{C}^{r+1}(U,w)} \mathsf{C}^{r+1}(U,G) \xrightarrow{\mathsf{D}} \mathsf{C}^r(U,[E,G])$$

are C-maps. We do this by setting up commutative diagrams which factorize these functions as compositions of known C-maps, known from the inductive hypothesis. The first diagram (called inc-*check diagram*):

$$\begin{tikzcd}
\mathsf{C}^{r+1}(U,F) \arrow[r, "\mathsf{C}^{r+1}(U,w)"] \arrow[d, "\mathsf{inc}"'] & \mathsf{C}^{r+1}(U,G) \arrow[d, "\mathsf{inc}"] \\
\mathsf{C}^r(U,F) \arrow[r, "\mathsf{C}^r(U,w)"'] & \mathsf{C}^r(U,G)
\end{tikzcd}$$

obviously commutes. The second diagram (called D-*check diagram*):

$$\begin{tikzcd}
\mathsf{C}^{r+1}(U,F) \arrow[r, "\mathsf{C}^{r+1}(U,w)"] \arrow[d, "\mathsf{D}"'] & \mathsf{C}^{r+1}(U,G) \arrow[d, "\mathsf{D}"] \\
\mathsf{C}^r(U,[E,F]) \arrow[r, "\mathsf{C}^r(U,[E,w])"'] & \mathsf{C}^r(U,[E,G])
\end{tikzcd}$$

commutes because $\mathsf{D}(w \circ f)(x) = w \circ \mathsf{D}f(x)$ when w is linear. Thus $\mathsf{C}^{r+1}(U, -)$ is a functor. It is shown by similar arguments that this functor preserves initial C-structures induced by families of LC-maps. In the construction of the D-check diagram one makes use of the known fact that functors of the form $[E, -]$ preserve such initial structures.

Proof (b^{r+1}). Take $w \in [F, \mathsf{C}^{r+1}(U, G)]$. First we must verify that $\S(w) \in \mathsf{C}^{r+1}(U, [F, G])$. Since $\mathsf{D} \circ w \in [F, \mathsf{C}^r(U, [E, G])]$, we have $\S^r(\mathsf{D} \circ w) \in \mathsf{C}^r(U, [F, [E, G]])$. Using the isomorphism $\S_{FEG} : [F, [E, G]] \to [E, [F, G]]$, we form $\mathsf{C}^r(U, \S_{FUG})$ and after evaluating this map at the point $\S^r(\mathsf{D} \circ w)$ in its domain, we arrive at

$$v \stackrel{\mathrm{def}}{=} \mathsf{C}^r(U, \S_{FUG})(\S^r(\mathsf{D} \circ w)) \in \mathsf{C}^r(U, [E, [F, G]]).$$

Put $\Phi(x, y) \cdot h = \int_0^1 v(x + \theta(y - x)) \cdot h d\theta$. Then $\Phi(x, y) \cdot (y - x) \cdot k = w(k)(y) - w(k)(x) = (\S(w)(y) - \S(w)(x)) \cdot k$ by direct calculation. We conclude that Φ is a C^r difference factorizer for $\S(w)$ (indeed that D ($\S$(w))=v). Thus $\S_F^{r+1} EG$ is well defined as a function and it is clearly linear. It is a C-map by construction of inc-check and D-check diagrams via the result assumed for r, as in proof (a^{r+1}). The first diagram is straight forward. Construction of the D-check diagram proceeds via the isomorphism $\mathsf{C}^r(U, \S_{VUG})$. In the opposite direction the first problem is again to show that $\S$ lifts to $\S^{r+1}$ as a function. Take $f \in \mathsf{C}^{r+1}(U, [F, G])$ and let Φ be a C^r difference factorizer for f. It is readily seen that for every $k \in F$ the map $\Phi_k = [E, \mathsf{eval}(-, k)] \circ \Phi$ is then a C^r difference factorizer for $\S(f) \cdot k$. Thus $\S_{UVG}^{r+1}$ exists as a function and it is clearly linear. It is a C-map, by construction of inc-check and D-check diagrams much as before.

Proof (c^{r+1}). Suppose (α^{r+1}). Then we have $\Phi : (U \times V) \times (U \times V) \to [E \times F, G]$, a C^r difference factorizer. As in the proof of 2d1 we construct the maps $\Psi_1 \in \mathsf{C}^r((U \times U) \times V, [E, G])$ and $\Psi_2 \in \mathsf{C}^r((V \times V) \times U, [F, G])$. By applying ($\mathrm{d}^r$), we obtain the map $\Phi_{1s} = \dagger 1^{r,s}(\Psi_1) \in \mathsf{C}^s(U \times U, \mathsf{C}^{r-s}(V, [E, G]))$ to serve as the required difference factorizer for $f^{\dagger 1}$. Similarly for $f^{\dagger 2}$. Conversely, suppose (β^{r+1}). Then $f^{\dagger 1} \in \mathsf{C}^{r+1}(U, \mathsf{C}(V, G))$ and we have a C^r-map $\Phi_1 : U \times U \to [E, \mathsf{C}(V, G)]$ and similarly a map Φ_2 to use for construction of a C^r-map Φ as in the proof of 2d1.

Proof (d^{r+1}). That $\dagger 1^{r+1,s}$ and $\dagger 2^{r+1,s}$ exist as functions can readily be concluded from (c^{r+1}), just proved. To show that $\dagger 1^{r+1,s+1}$ is a C-map, the construction of an inc-check diagram is quite straight forward via (d^r); the construction of a D-check diagram proceeds via (d^r), (b^{r-s}) and the map $[(\mathsf{id}, 0), G] : [E \times F, G] \to [E, G]$. Then $\dagger 1^{r+1,0}$ is dealt with separately and it is much simpler. Similarly for $\dagger 2$.

Proof (e^{r+1}). Existence of $\ddagger^{r+1}$ as a function follows from (c^{r+1}) already proved. Construction of the D-check diagram from $\ddagger^{r+1}$ proceeds via (e^r), (b^r) with $[F, G]$ in the role of G, and repeated application of (a^r) to embeddings of the form $\mathsf{inc} : [F, G] \to \mathsf{C}^r(V, G)$. Construction of the inc-check diagram is straight forward.

Proof (f^{r+1}): We express $\mathsf{D} \circ \mathsf{comp}_{UVW}$ as the following composition of C-maps:

$$\begin{array}{l}\mathsf{C}^{r+1}(U,V)\times\mathsf{C}^{r+1}(V,W)\\ \qquad\downarrow\ (\mathsf{D},\mathsf{inc})\times\mathsf{D}\\ \mathsf{C}^{r}(U,[E,F])\times\mathsf{C}^{r}(U,V)\times\mathsf{C}^{r}(V,[F,G])\\ \qquad\downarrow\ \mathsf{id}\times\mathsf{comp}^{r}\\ \mathsf{C}^{r}(U,[E,F])\times\mathsf{C}^{r}(U,[F,G])\\ \qquad\downarrow\ \mathsf{C}^{r}(U,\mathsf{comp})\\ \quad\mathsf{C}^{r}(U,[E,G]).\end{array}$$

The inc-check diagram is straight forward to construct.

Proof (g^{r+1}): To show $\mathsf{C}^{r+1}(U,F)$ is in oLC, we will construct a section S_a : $\mathsf{C}^{r+1}(U,F)\to G$ in LC with $G\in\mathsf{oLC}$. This is enough because every section is a regular monomorphism, hence in $\mathbf{u}$. As a prelude to this construction we show that the linear function $\mathsf{dfac}_F^{r+1}:\mathsf{C}^{r+1}(U,F)\to\mathsf{C}^{r}(U\times U,[E,F])$ is a C-map by expressing it as a composition of C-maps. For this purpose, let I denote the line segment $[0,1]\subset\mathbb{K}$, $c:I\to\mathsf{C}^{r}(U\times U,U)$ the C-map $c(\theta)(x,y)=x+\theta(y-x)$, $[u]$ the constant function with value u and define the C-map $\mathsf{const}:Y\to\mathsf{C}(X,Y)$ by $\mathsf{const}(y)=[y]$. The factorization of dfac that we are looking for can now be formed as follows:

$$\begin{array}{lr}\mathsf{C}^{r+1}(U,F) & g\\ \qquad\downarrow\ \mathsf{D} & \downarrow\\ \mathsf{C}^{r}(U,[E,F]) & \mathsf{D}g\\ \qquad\downarrow\ ([c],\mathsf{const}) & \downarrow\\ \mathsf{C}(I,\mathsf{C}^{r}(U\times U,U))\times\mathsf{C}(I,\mathsf{C}^{r}(U,[E,F])) & (c,[\mathsf{D}g])\\ \qquad\downarrow\ \mathsf{iso} & \downarrow\\ \mathsf{C}(I,\mathsf{C}^{r}(U\times U,U)\times\mathsf{C}^{r}(U,[E,F])) & \theta\mapsto(c(\theta),\mathsf{D}g)\\ \qquad\downarrow\ \mathsf{C}(I,\mathsf{comp}^{r}) & \downarrow\\ \mathsf{C}(I,\mathsf{C}^{r}(U\times U,[E,F])) & \theta\mapsto\mathsf{D}g\circ c(\theta)\\ \qquad\downarrow\ \int_0^1 & \downarrow\\ \mathsf{C}^{r}(U\times U,[E,F]) & \int_0^1\mathsf{D}g\circ c(\theta)d\theta\end{array}$$

By considering $\int_0^1\mathsf{D}g\circ c(\theta)d\theta$ as member of $\mathsf{C}(U\times U,[E,F])$ so that 3c3(d) can be applied, we see that $\int_0^1 \mathsf{D}g\circ c(\theta)d\theta(x,y) = \int_0^1 \mathsf{D}g(x+\theta(y-x))d\theta = \mathsf{dfac}(g)(x,y)$. Let $\mathsf{C}^r\mathsf{DF}(U\times U,[E,F])$ denote the null space of the LC-map $\mathsf{tri}:\mathsf{C}^r(U\times U,[E,F])\to \mathsf{C}(U\times U\times U,F)$ that was defined in the proof of 2b2. Then $\mathsf{C}^r\mathsf{DF}(U\times U,[E,F])$ is a oLC-space and it consists of precisely all difference factorizers of maps in $\mathsf{C}^{r+1}(U,F)$. Fix some $a\in U$ and define the function

$$S_a:\mathsf{C}^{r+1}(U,F)\to \mathsf{C}^r\mathsf{DF}(U\times U,[E,F])\times F$$

by putting $S_a(g) = (\mathsf{dfac}(g), g(a))$. S_a is clearly a linear C-map. One verifies directly that the map dfac^{r+1} carries g to a C^r difference factorizer for g i.e. a member of $\mathsf{C}^r\mathsf{DF}(U\times U,[E,F])$, moreover that it is natural. To show S_a is a section, we produce a left inverse R_a for it by putting

$$R_a(\Phi,z)(x) = \Phi(a,x)\cdot(x-a)+z.$$

We can now conclude that $\mathsf{C}^{r+1}(U,F)\in \mathsf{oLC}$ $(r\in\mathbb{N})$.

4b2. REMARK. The appearance of $I=[0,1]$ in the proof of 4b1(g^{r+1}), marks a situation where we could not avoid the use of a non-open primary domain. For each fixed pair $(x,y)\in U\times U$ there is clearly an open convex neighborhood Ω_{xy} of I such that $x+\theta(y-x)\in U$ for all $\theta\in\Omega_{xy}$, but no single such neighborhood works for all pairs (x,y). Note that this difficulty does not arise when $U=E$, the entire vector space and for calculus of maps with such domains the use of $\Lambda=\mathbb{K}$ for axioms 1a6 and 1a7 will be enough.

4b3. THEOREM.
(a^∞) *The functor* $\mathsf{C}(U,-):\mathsf{oLC}\to\mathsf{oLC}$ *lifts to a functor* $\mathsf{C}^\infty(U,-):\mathsf{oLC}\to\mathsf{oLC}$ *and the latter preserves* C*-initial families of linear* C*-maps; hence it preserves cartesian products in* oLC.

(b^∞) *There exists a natural* oLC*-isomorphism* $\S^\infty_{EVG}:[E,\mathsf{C}^\infty(V,G)]\to\mathsf{C}^\infty(V,[E,G])$ *which lifts* $\S_{EVG}:[E,\mathsf{C}(V,G)]\to\mathsf{C}(V,[E,G])$.

(c^∞) *A map* $f\in\mathsf{C}(U\times V,G)$ *is a* C^∞*-map if and only if* $f^{\dagger 1}\in\mathsf{C}^\infty(U,\mathsf{C}^\infty(V,G))$ *and* $f^{\dagger 2}\in\mathsf{C}^\infty(V,\mathsf{C}^\infty(U,G))$.

(d^∞) *The function* $\dagger 1^\infty_{UVG}:\mathsf{C}^\infty(U\times V,G)\to\mathsf{C}^\infty(U,\mathsf{C}^\infty(V,G))$ *is a natural map in* oLC *which lifts* $\dagger 1$ *and a similar statement holds for* $\dagger 2$.

(e^∞) *There exists a natural transformation* $\ddagger^\infty:\mathsf{C}^\infty(U,\mathsf{C}^\infty(V,G))\to\mathsf{C}^\infty(U\times V,G)$ *which lifts* $\ddagger$.

(f^∞) *The function* $\mathsf{comp}^\infty_{UVG}:\mathsf{C}^\infty(U,V)\times\mathsf{C}^\infty(V,G)\to\mathsf{C}^\infty(U,G)$, $\mathsf{comp}^\infty(f,g)=g\circ f$, *is a* C^∞*-map.*

(g^∞) $\mathsf{C}^\infty(U,F)$ *is an* oLC*-space,* $\mathsf{dfac}^\infty_F:\mathsf{C}^\infty(U,F)\to\mathsf{C}^\infty(U\times U,[E,F])$ *is a natural transformation in* oLC *and* $\mathsf{dfac}(g)$ *is a difference factorizer for* g.

Proof . The statements (a^∞), (b^∞), (c^∞) and (g^∞) follow at once by 4a1, the completeness of the category oLC and the corresponding statements in 4b1. The remaining statements require further reasoning, as follows. If $f \in C^\infty(U \times V, G) \subset C^r(U \times V, G)$, then by 4b1($d^r$) we have $\dagger(f) \in C^s(U, C^{r-s}(V, G))$. Since this holds for all $s \leq r \in \mathbb{N}$, we conclude that $\dagger^\infty$ exists as linear function. By applying 4b1(a^r), we see that both of the families

$$\mathsf{inc}^r : C^\infty(U, C^\infty(V, G)) \to C^r(U, C^\infty(V, G)) \ (r \in \mathbb{N}),$$
$$C^r(U, \mathsf{inc}^s) : C^r(U, C^\infty(V, G)) \to C^r(U, C^s(V, G)) \ (s \in \mathbb{N})$$

are C-initial. One readily concludes from this that $\dagger 1^\infty$ is an LC-map, similarly for $\dagger 2^\infty$. In a similar manner, using 4b1, one shows that $\ddagger^\infty$ is an LC-map. In view of 4b1 we have naturality in U and V with respect to C^∞-maps, but in G the naturality is so far known only for LC-maps. If $g : G \to H$ is a C^∞-map, then the function $C^\infty(U, g) : C^\infty(U, G) \to C^\infty(U, H)$ clearly exists and the diagram expressing naturality in G with respect to C^∞-maps will clearly commute as a diagram of functions. What remains to be proved, is that the functions $C^\infty(U, g) : C^\infty(U, G) \to C^\infty(U, H)$ will always be C^∞-maps. To do this, let us cast the space $C^\infty(V, G)$ in the role of E in the isomorphism $\ddagger^\infty_{UV}$ established above (without regard to naturality for the moment). We obtain the isomorphism

$$\ddagger : C^\infty(C^\infty(V, G), C^\infty(V, G)) \to C^\infty(C^\infty(V, G) \times V, G).$$

By evaluating this map at the point id in its domain, we obtain the C^∞-map

$$\mathsf{eval}_{VG} = \ddagger(\mathsf{id}) : C^\infty(V, G) \times V \to G.$$

The composition map

$$\mathsf{comp}^\infty_{UGH} : C^\infty(G, H) \times C^\infty(U, G) \to C^\infty(U, H)$$

can now be constructed as a C^∞-map by putting $\mathsf{comp}_{UGH} = \dagger 1(\mathsf{eval}_{GH} \circ (\mathsf{id} \times \mathsf{eval}_{UG}))$. Finally we are in a position to see that $C^\infty(E, g)$ is a C^∞-map, because we can express $C^\infty(U, g) = \mathsf{comp}_{UGH} \circ ([g] \times \mathsf{id})$, where $[g]$ is the constant function with value g.

4c. Integration of curves revisited. We are now in a position to elaborate on several earlier results. Let us call attention to the fact that the map

$$\mathsf{eval}_{\mathbb{K}F}(-, 1) : [\mathbb{K}, F] \to F, \quad g \mapsto g(1),$$

is a natural isomorphism in oLC; let up denote its inverse. We define $\mathsf{grad}_F = C(\mathbb{K}, \mathsf{eval}_{\mathbb{K}F}(-, 1)) \circ D_{\mathbb{K}F}$ and $f'(\xi) \stackrel{\text{def}}{=} \mathsf{grad}(f)(\xi) = Df(\xi) \cdot 1$.

4c1. THEOREM. (Fundamental Theorem of Calculus) *The maps*

$$\mathsf{grad}_F : C^{r+1}(\Lambda, F) \to C^r(\Lambda, F), \ \mathsf{grad}_F : C^\infty(\Lambda, F) \to C^\infty(\Lambda, F)$$

are natural retractions in oLC. *Moreover, for a given* $f \in \mathsf{C}^r(U, F)$ *the* g *that satisfies* $g' = f$ *is unique up to an additive constant and upholds the formula* $\int_\alpha^\beta f(\theta)d\theta = g(\beta) - g(\alpha)$.

Proof. Since $\mathsf{av}(f) : \Lambda \times \Lambda \to F$ upholds the first average value identity, the characterization 2b2 shows that $\mathsf{up} \circ \mathsf{av}(f) : \Lambda \times \Lambda \to [\mathbb{K}, F]$ is the difference factorizer of some $g \in \mathsf{C}^r(\Lambda, F)$. It follows at once that

$$\int_\alpha^\beta g' = (\beta - \alpha)\mathsf{av}(f)(\alpha, \beta) = (\mathsf{up} \circ \mathsf{av}(f))(\alpha, \beta) \cdot (\beta - \alpha) = g(\beta) - g(\alpha).$$

In particular, if $g' = 0$, then g must be constant and the stated uniqueness up to a constant follows readily. Finally, that $g \mapsto g'$ is a linear C-map, is immediate from the definitions.

By applying the chain rule we obtain at once the following familiar formula.

4c2. COROLLARY. *We have* $\int_{\phi(\alpha)}^{\phi(\beta)} f(\tau)d\tau = \int_\alpha^\beta f(\phi(\theta))\phi'(\theta)d\theta$, *for* C^1*-maps* $\phi : \Lambda \to \Lambda$ *and* C*-maps* $f : \Lambda \to F$.

4c3. THEOREM. *The natural transformation* $\mathsf{av}_{\Lambda E} : \mathsf{C}(\Lambda, E) \to \mathsf{C}(\Lambda \times \Lambda, E)$ *lifts to natural transformations*

$\mathsf{av}^r_{\Lambda E} : \mathsf{C}^r(\Lambda, E) \to \mathsf{C}^r(\Lambda \times \Lambda, E)$ *and* $\mathsf{av}^\infty_{\Lambda E} : \mathsf{C}^\infty(\Lambda, E) \to \mathsf{C}^\infty(\Lambda \times \Lambda, E)$.

Proof. The first identity of 3e1 in conjunction with characterization 2b2 shows that every value $\mathsf{av}(f)$ is a difference factorizer. By uniqueness of difference factorizers where one dimensional domains occur (2c1) and by 4b1(g^{r+1}) we have $\mathsf{av}^r \circ \mathsf{grad} = \mathsf{iso} \circ \mathsf{dfac}^{r+1}$. Since grad is a retraction in oLC (4c1), av^r is an LC-map.

5. Higher order derivatives

Our concept of C^r-map did not require higher order derivatives for its definition nor for its further development in the preceding sections. But higher order derivatives are important for their own sake. We show in this section that their basic properties can be established effectively with the present approach.

5a. Sum closed products and higher order differences. For $r \in \mathbb{N}$ we put $\langle r \rangle = \{1, 2, \ldots, r\}$ while $\mathsf{P}X$ will denote the set of all subsets of X and $|X|$ the cardinality of X. We define the *sum closed product* to be the following C-subspace of $U \times E^r$:

$$U \bowtie E^r = \{(x, h_J) : \; x + \sum_{i \in J} \theta_i h_i \in U\}$$

where J varies as subset of $\langle r \rangle$, h_J denotes $(h_i)_{i \in J} \in E^J$ and $0 \leq \theta_i \leq 1$. The spaces $U \bowtie E^r$ serve well as domains for the usual higher order difference maps of $f : U \to F$, namely $\Delta^r f : U \bowtie E^r \to F$, defined recursively by the formulas:

$\Delta^1 f(x, h) = f(x + h) - f(x),$

$$\Delta^{m+1}f(x, h_{\langle m+1\rangle}) = \Delta^m f(x + h_{m+1}, h_{\langle m\rangle}) - \Delta^m f(x, h_{\langle m\rangle}).$$

5a1. PROPOSITION. *$\Delta^r f(x, h_{\langle r\rangle}) = \sum_{J\in \mathsf{P}\langle r\rangle}(-1)^{|\langle r\rangle\setminus J|}f(x + \sum_{i\in J} h_i)$. Hence $\Delta^r f(x, h_{\langle r\rangle})$ is symmetric in the variables $h_1, \ldots, h_r$.*

Proof . By induction on r. To match up the sums in the inductive step one uses the fact that for $J \subset \langle r\rangle$ and $i \in \langle r\rangle$

either $i \in J$ and $|\langle r\rangle \setminus J| = |\langle r\rangle \setminus (\{i\} \cup J)|$

or $i \in \langle r\rangle \setminus J$ and $|\langle r\rangle \setminus J| = |\langle r\rangle \setminus (\{i\} \cup J)| + 1$

but not both.

5b. Symmetry of higher order derivatives. For a C^r-map $f : U \to F$ we define higher order derivatives recursively as follows: $\mathsf{D}^1 f = \mathsf{D}f$, $\mathsf{D}^{m+1}f = \mathsf{D}(\mathsf{D}^m f)$. To simplify notation we define recursively $[E^1; F] = [E, F]$, $[E^{m+1}; F] = [E, [E^m; F]]$. Then $\mathsf{D}^r f \in \mathsf{C}(U, [E^r; F])$. It follows quickly from the definitions that $\mathsf{D}^s(\mathsf{D}^{r-s}f) = \mathsf{D}^r f$ $(s = 0, 1, \ldots, r)$. There is an obvious natural isomorphism

$$r\mathsf{lin} : [E^r; F] \to [E^r,_r F], \; r\mathsf{lin}(u)(h_{\langle r\rangle}) = u(h_r) \cdot h_{r-1} \cdot \ldots \cdot h_1,$$

where $[E^r,_r F]$) denotes the oLC-subspace of $\mathsf{C}(E^r, F)$ formed by all r-linear maps.

A C-map $\Phi : U \bowtie E^r \to [E^r; F]$ such that $\Phi(x, h_{\langle r\rangle}) \cdot h_{\langle r\rangle} = \Delta^r f(x, h_{\langle r\rangle})$ will be called an *r-th order difference factorizer* for $f : U \to F$. Let us point out that the map $(x, y) \mapsto (x, y - x)$ is an isomorphim $U \times U \to U \bowtie E^1$ and that the concept of difference factorizer defined in section 2 agrees up to this isomorphism with the present concept of first order difference factorizer. We will not study higher order difference factorizers in any depth in this paper. They will make only a brief but important appearance in the next proof.

5b1. THEOREM. *For every C^r-map $f : U \to F$, every $h_{\langle r\rangle} \in E^r$ and every permutation $\pi : \langle r\rangle \to \langle r\rangle$ we have*

$\mathsf{D}^r f(x) \cdot h_r \cdot \ldots \cdot h_1 = \mathsf{D}^r f(x) \cdot h_{\pi(1)} \cdot \ldots \cdot h_{\pi(r)}$.

Proof . Put

$$I_r(f)(x, h_{\langle r\rangle}) \cdot k_{\langle r\rangle} = \int_0^1 \ldots \int_0^1 \mathsf{D}^r f(x + \sum_{i=1}^r \theta_i h_i) d\theta_r \ldots d\theta_1 \cdot k_{\langle r\rangle}.$$

Then $I_r(f)$ is clearly a C-map $U \bowtie E^r \to [E^r; F]$. Let us verify inductively that it is an r-th order difference factorizer for f. For $r = 1$ this is immediate from the definition. Assume validity for $r = m$. We have

$$\begin{aligned} &I_{m+1}(f)(x, h_{\langle m+1\rangle}) \cdot h_{m+1} = \\ &= \int_0^1 I_m(\mathsf{D}f)(x + \theta_{m+1}h_{m+1}, h_{\langle m\rangle}) \cdot h_{m+1} d\theta_{m+1} \\ &= \int_0^1 \Delta^m(\mathsf{D}f)(x + \theta_{m+1}h_{m+1}, h_1, \ldots, h_m) \cdot h_{m+1} d\theta_{m+1} \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \sum_{J \in \mathsf{P}\langle m\rangle} (-1)^{|\langle m\rangle \setminus J|} \mathsf{D} f(x + \theta_{m+1} h_{m+1} + \sum_{i \in J} h_i) \cdot h_{m+1} d\theta_{m+1} \\
&= \sum_{J \in \mathsf{P}\langle m\rangle} (-1)^{|\langle m\rangle \setminus J|} f(x + h_{m+1} + \sum_{i \in J} h_i) \\
&\qquad - \sum_{J \in \mathsf{P}\langle m\rangle} (-1)^{|\langle m\rangle \setminus J|} f(x + \sum_{i \in J} h_i) \\
&= \Delta^m f(x + h_{m+1}, h_{\langle m\rangle}) - \Delta^m f(x, h_{\langle m\rangle}) \\
&= \Delta^{m+1} f(x, h_{\langle m+1\rangle}).
\end{aligned}
$$

Thus we obtain the required $m+1$-th order difference factorizer. For any permutation $\pi : \langle r\rangle \to \langle r\rangle$, we put

$$
q_\pi(\lambda) = I_r(f)(x, \lambda h_{\pi(1)}, \ldots, \lambda h_{\pi(r)}) \cdot h_{\pi(r)} \cdot \ldots \cdot h_{\pi(1)}.
$$

It follows at once that

$$
q_\pi(\lambda) = \begin{cases} \Delta^r f(x, \lambda h_{\pi(1)}, \ldots, \lambda h_{\pi(r)})/\lambda^r & \text{if } \lambda \neq 0 \\ \mathsf{D}^r f(x) \cdot h_r \cdot \ldots \cdot h_1 & \text{if } \lambda = 0. \end{cases}
$$

Since $\Delta^r f(x, h_1, \ldots, h_r)$ is symmetric in the variables h_i, it follows that, for any two permutations π and σ, we have $q_\pi(\lambda) = q_\sigma(\lambda)$ whenever $\lambda \neq 0$. By axiom 1a5, $q_\pi(0) = q_\sigma(0)$.

5b2. COROLLARY. $\partial_1\partial_2 f(x_1, x_2) \cdot h_1 \cdot h_2 = \partial_2\partial_1 f(x_1, x_2) \cdot h_2 \cdot h_1$ *for all* C^2*-maps* $f : U \times V \to G$.

Proof . By 5b1, $\mathsf{D}^2 f(x_1, x_2) \cdot (h_1, 0) \cdot (0, h_2) = \mathsf{D}^2 f(x_1, x_2) \cdot (0, h_2) \cdot (h_1, 0)$. So the result follows by 2d1.

5c. The Taylor formula. We will abbreviate $\mathsf{D}^r f(x) \cdot h_r \cdot \ldots \cdot h_1$ to $\mathsf{D}^r f(x) \cdot h^r$ when all $h_i = h$.

5c1. THEOREM.
If $f : U \to F$ *is a* C^{r+1}*-map, then for all* $(x, h) \in U \bowtie E^1$,

$$
f(x + h) = f(x) + \mathsf{D}^1 f(x) \cdot h^1 + \ldots + (1/r!)\mathsf{D}^r f(x) \cdot h^r + \mathrm{Rem}_r(x, h),
$$

where $\mathrm{Rem}_r(x, h) = (1/r!) \int_0^1 (1 - \theta)^r \mathsf{D}^{r+1} f(x + \theta h) \cdot h^{r+1} d\theta$.

Proof . One proceeds by induction on r. For $r = 0$, the result follows at once from 4b3. Assume validity for $r = m$ and put

$$
g(\theta) = (1/(m+1)!)(1 - \theta)^{m+1} \mathsf{D}^{m+1} f(x + \theta h) \cdot h^{m+1}.
$$

By computing $g'(\theta)$ via the product rule (2c2(c) with n=2) and by applying the Fundamental Theorem (4c2) to g', one obtains the recursive relation
$\mathrm{Rem}_r(x, h) = (1/(r+1)!)\mathsf{D}^{r+1} f(x) \cdot h^{r+1} + \mathrm{Rem}_{r+1}(x, h)$.

6. Special theories of calculus.

It is long known that the classical setting of locally convex spaces has conspicuous inadequacies as framework in which to cast calculus: there is no natural way to structure spaces of smooth maps so that evaluation $(f, x) \mapsto f(x)$ is always a smooth map. In fact, it is not even possible to make evaluation a continuous operation on spaces of linear continuous maps [9]. The studies of [2], [19] and [6] showed that these inadequacies can be overcome in suitable cartesian closed topological categories. Thus there emerged parallel theories of calculus. In this section we briefly discuss some special choices of C and $\mathbb{K}$ to which the preceding theory applies and we indicate very briefly how they relate to the previous studies just mentioned. In all cases, $\mathbb{K}$ will be given the obvious 'usual' structure for the category in question, so we will make no further mention of this structure.

The peculiar features of special theories are interesting: results which fail for arbitrary C but hold in the special model of C because the category in question is cooperative beyond the call of duty, so to speak. Such results, which require real analysis for their proofs, may determine which special category is best suited for a particular kind of problem.

6a. Calculus in $\mathsf{C_c}$. The category $\mathsf{C_c}$ is defined by axioms which are simply the filter convergence analogues of the three Fréchet-Urysohn axioms for sequential convergence, thus equivalent to the axioms postulated by Choquet [5] (cf. 6b).

Our first task is to verify that $\mathsf{C_c}$ upholds the axioms 1a. It is well known that 1a1, 1a2 and 1a3 are satisfied (see [3] or [15] for background about this). With $\mathbb{K} = \mathbb{R}$ in its usual topological structure, axioms 1a4 and 1a5 are obviously satisfied. As regards the remaining two, let us recall that every topological space is a $\mathsf{C_c}$- space via its convergent filters and topological products of topological spaces agree with their products in $\mathsf{C_c}$. The spaces $\mathsf{C_c}(X, Y)$ carry the continuous convergence structure. If X is a locally compact Hausdorff topological space, then $\mathsf{C_c}(X, \mathbb{R})$ carries the topology of uniform convergence on compact subsets of X. We need this background for the following verification of 1a5.

6a1. PROPOSITION. *There exists a unique $\mathsf{LC_c}$-map $\mathsf{av}_{\Lambda\mathrm{R}} : \mathsf{C_c}(\Lambda, \mathbb{R}) \to \mathsf{C_c}(\Lambda \times \Lambda, \mathbb{R})$ such that the following two identities hold:*

$$(\beta - \alpha)\mathsf{av}(f)(\alpha, \beta) + (\gamma - \beta)\mathsf{av}(f)(\beta, \gamma) + (\alpha - \gamma)\mathsf{av}(f)(\gamma, \alpha) = 0,$$
$$\mathsf{av}(f)(\xi, \xi) = f(\xi).$$

Moreover, $\mathsf{av}_{\Lambda\mathrm{R}}$ is natural in the variable Λ with respect to affine maps.

Proof . Put
$$\mathsf{av}(f)(\lambda, \mu) = \begin{cases} \int_\lambda^\mu f(\theta)d(\theta)/(\mu - \lambda) & \text{if } \lambda \neq \mu \\ f(\lambda) & \text{if } \lambda = \mu. \end{cases}$$

It is a well known elementary fact of Riemann integration that for each f the image function $\mathsf{av}(f) : \Lambda \times \Lambda \to \mathbb{R}$ is continuous. Thus av is well defined as a function and it is clearly linear. Since finite powers of Λ are locally compact, the canonical mapping spaces $\mathsf{C_c}(\Lambda^n, \mathbb{R})$ carry the topology of uniform convergence on compact sets. The continuity of av therefore follows readily from the estimate $\sup_{I\times I} |\mathsf{av}(f)(\lambda,\mu)| \leq \sup_I |f(\lambda)|$, where I is a compact subinterval of Λ. Via the well known fact that $\int_\alpha^\beta + \int_\beta^\gamma + \int_\gamma^\alpha = 0$ we arrive at the first identity. The second follows by continuity. Suppose $\mathsf{av}_{1\mathbb{R}}$ and $\mathsf{av}_{2\mathbb{R}}$ are both average value maps. Fix $\alpha \in \Lambda$ and put

$$g_i(\xi) = (\xi - \alpha)\mathsf{av}_i(f)(\alpha, \xi), \;\; (i = 1, 2).$$

From the first identity one obtains quickly that $(g_i(\eta) - g_i(\xi))/(\eta - \xi) = \mathsf{av}_i(f)(\xi, \eta)$. Let $\eta \to \xi$ to obtain $g_i'(\xi) = \mathsf{av}_i(f)(\xi, \xi) = f(\xi)$. By elementary one dimensional differential calculus we deduce that $g_1 - g_2$ is a constant function. Since $g_1(\alpha) = g_2(\alpha) = 0$, we conclude $g_1 = g_2$. It follows that $\mathsf{av}_1(f)(\alpha, \xi) = \mathsf{av}_(f)(\alpha, \xi)$ for all $\xi \neq \alpha$, hence that $\mathsf{av}_1 = \mathsf{av}_2$. The stated naturality just means that we have $\mathsf{av}_{\Lambda\mathbb{R}}(f \circ \phi) = \mathsf{av}_{\Theta\mathbb{R}}(f) \circ (\phi \times \phi)$ for affine maps $\phi : \Lambda \to \Theta$. This follows from the elementary properties of integration by straight forward verification.

That the remaining axiom 1a7 is satisfied, is implied by the striking result [4] that $\mathsf{C_c}(X, \mathbb{R})$ is reflexive for all $\mathsf{C_c}$-spaces X. This can also be proved via the Krein-Milman theorem (see [3]). Alternatively, 1a7 can be verified as for $\mathsf{C_s}$ in 6b1.

For functions between locally convex spaces, numerous definitions of 'continuously differentiable map' emerged in the literature which, by the late sixties, led to "an impression of chaos" [1]. Nine of these concepts were selected for special study in [10]. The question now arises: which one agrees with the categorical concept of $\mathsf{C_c}^1$-map? For all concepts in [10] the formula $f(y) - f(x) = \int_0^1 Df(x + \theta(y - x)) \cdot (y - x)$ holds, so the difference factorizer exists as a function. Whether this function is continuous, depends on the structure used for the space of linear continuous maps. Since [2] used continuous convergence (the canonical $\mathsf{C_c}$-structure), it comes as no surprise that the concept in [2], while expressed differently, agrees with ours (see [1] and [10] for more details, including a reference to an even earlier equivalent concept due to A.D. Michal in 1938). $\mathsf{C_c}^1$-maps were also used (apparently rediscovered) in the important paper [7] in the context of Fréchet spaces: in this context, or any which has at least finite cartesian products, one can replace the difference factorizer $\Phi : U \times U \to [E, F]$ by its natural image $\Phi^\ddagger : (U \times U) \times E \to F$ for the definition of continuously differentiable.

6b. Calculus in $\mathsf{C_s}$. A $\mathsf{C_s}$-space is a set X structured with convergent sequences subject to the following three (Fréchet-Urysohn) axioms. (1) every constant sequence converges to its constant value; (2) if u_n converges to x, then so does every subsequence $u_{s(n)}$; (3) if u_n is such that every subsequence $u_{s(n)}$ has a subsequence $u_{t(s(n))}$ which converges to x, then u_n converges to x. $\mathsf{C_s}$-maps are functions which preserve convergent sequences.

It is well known that $\mathsf{C_s}$ satisfies axioms 1a1, 1a2 and 1a3 and it is immediate that 1a4 and 1a5 are satisfied, where of course $\mathbb{K} = \mathbb{R}$ in its usual structure is used. It is also known and readily verified that the convergent sequences in $\mathsf{C_s}(X, \mathbb{R})$ are those that converge uniformly on sequentially compact subsets of X. Thus the verification of 1a6 is almost word for word as in 6a1 above. However, the verification of 1a7 requires argument, as follows.

6b1. PROPOSITION. $\mathsf{C_s}(\Lambda, \mathbb{R})$ *is reflexive.*

Proof . In view of 3b1 it is enough to show that $\mathsf{lin@}_\Lambda$ is an epimorphism in iLC. We will do this by showing that for every continuous linear map $w : \mathsf{C_c}(\Lambda, \mathbb{R}) \to \mathbb{R}$ there is a sequence v_n in $\mathsf{lin@}(\Lambda \odot \mathbb{R})$ which converges to w in $[\mathsf{C_s}(\Lambda, \mathbb{R}), \mathbb{R}]$, where limits are unique. By the classical Riesz theorem, w is represented by a Riemann-Stieltjes integral $w(f) = \int_\alpha^\beta f d\phi$, where ϕ is a function of bounded variation. Since ϕ is a difference of non-decreasing functions, we may just as well assume ϕ to be non-decreasing. For such ϕ one can readily construct a sequence ϕ_n of step functions such that $\phi_n(\lambda)$ converges to $\phi(\lambda)$ at every point $\lambda \in [\alpha, \beta]$ where ϕ is continuous. By putting $v_n = \int_\alpha^\beta f d\phi_n$ we obtain a sequence in the image set of $\mathsf{lin@}_\Lambda$, because the integral with respect to the step function ϕ_n reduces to a finite Riemann-Stieltjes sum $\sum_i f(\xi_i)\cdot[\phi_n(\tau_i) - \phi_n(\tau_{i-1})] = \sum_i @(\xi_i)(f)\cdot[\phi_n(\tau_i) - \phi_n(\tau_{i-1})]$. By using the fact that the convergence of $\phi_n(\lambda)$ to $\phi(\lambda)$ takes place for λ that form a dense subset of $[\alpha, \beta]$, it can be shown by a typical '$\epsilon/3$ argument' that for any sequence f_n that converges uniformly on $[\alpha, \beta]$ to f, we have $v_{s(n)}(f_n)$ convergent to $w(f)$. This means v_n converges to w in the $\mathsf{C_s}$-space $[\mathsf{C_s}(\Lambda, \mathbb{R}), \mathbb{R}]$, as required.

The $\mathsf{C_s}$ based calculus is a new theory. Very little is known so far about analysis peculiar to $\mathsf{C_s}$. We hope to report some results in this connection in a future paper.

6c. Calculus in $\mathsf{C_d}$ and the Frölicher-Kriegl theory. The category $\mathsf{C_d}$ of diffeological spaces [17] has spaces which are sets structured with abstract smooth curves. There is a category of the form $\mathsf{uLC_d}$ ('derivative complete spaces'), larger than $\mathsf{oLC_d}$, which is isomorphic [17] to the category of convenient vector spaces of [6]. That $\mathsf{C_d}$ upholds axioms 1a1, 1a2 and 1a3 is clear from [17] while 1a4 and 1a5 are obvious ($\mathsf{C_d}$-maps $\Lambda \to \mathbb{R}$ are just the usual smooth functions). That 1a6 holds, can readily be deduced from the calculus already established for $\mathsf{C_c}$ and $\mathsf{C_c}^\infty$-maps. The last axiom 1a7 on the other hand, requires lengthy proof. For the case where Λ is an open interval, this can be found in [6] and it can also be deduced from the theory of distributions: $\Lambda \hat{\odot} \mathbb{R}$ turns out to be the space of distributions of compact support. On this basis we know that the calculus of [6], at least for maps on entire vector spaces, can be interpreted as a realization of the calculus of this paper. However, the mentioned proof in [6] rests on a considerable amount of theory, including calculus. Therefore there arises the interesting problem of finding, for arbitrary primary domains Λ, a direct verification of 1a7. It will make the calculus of convenient vector spaces [6] accessible via a different route which may offer a degree of logical economy. We plan to study this in a separate paper.

In terms of what is known, it is interesting to compare the $\mathsf{C_c}$-based theory with the $\mathsf{C_d}$-based theory as regards the following four statements expressing peculiar features. Statement (1) means every bilinear function $u : E \times F \to G$ which is a C-map in each variable separately is a C-map. It can also be expressed by saying that $[E, G]$ carries the initial structure induced by the family of all point evaluations into G. Statement (2) means $[E, \mathbb{R}]$ agrees with the classical Banach space dual when E is Banach. Statement (3) means C^∞-maps $E \to F$ between locally convex spaces in $\mathsf{oLC_c}$ (resp. $\mathsf{oLC_d}$) are continuous in the underlying locally convex topologies.

Statement	$\mathsf{oLC_c}$	$\mathsf{oLC_d}$
(1) Bilinear separately C-map implies C-map	false	true
(2) E Banach $\Rightarrow$ $[E, \mathbb{R}]$ Banach	false	true
(3) C^∞-maps are continuous	true	false
(4) All spaces $\mathsf{C}(X, \mathbb{R})$ are reflexive	true	false

For $\mathsf{oLC_c}$ the statements (1), (2) and (4) are well known results, while (3) is clear from definition. For convenient vector spaces, hence for $\mathsf{oLC_d}$-spaces, the statement (4) will be found in [12] and the remaining three in [6]. The reflexiveness of all $\mathsf{C_c}(X, \mathbb{R})$ forces the *false* response to (2) and the *true* response to (3) forces the *false* response to (1). Conversely, the *true* response of the $\mathsf{C_d}$ based theory to (1) and (2) precludes such responses to (3) and (4) (cf. [12]). Thus the two theories complement each other.

6d. Calculus in Cgt. It is well known that the category Cgt of compactly generated topological spaces satisfies 1a1, 1a2 and 1a3 while 1a4 and 1a5 are obvious since $\mathbb{R}$ in its usual topology is compactly generated. Verification of both 1a6 and 1a7 is similar to the case $\mathsf{C_s}$ done above. One can also obtain 1a7 as corollary of prop. 4.2 in [13]. The category Cgt also underlies the theory of [19]. The linear spaces used for calculus in [19] (linear Cgt-spaces whose locally convex modifications are sequentially complete) is not of the form oLC or uLC. Thus the calculus of [19] is close to but differerent from ours.

6e. Calculus in $\mathsf{C_h}$. This is the category of holological spaces introduced in [11] (sets structured with abstract holomorphic curves). It provides the only known special case where the axioms 1a are satisfied when $\mathbb{K}$ is chosen to be the complex field. Here $\mathsf{C}(\Lambda, \mathbb{K})$ becomes a space of holomorphic functions. Axioms 1a1, 1a2 and 1a3 are verified in [11] while 1a4 and 1a5 are again obvious from the definition. The first identity of axiom 1a6 is essentially a version of Cauchy's theorem. The verification of 1a7 (at least for open primary domains Λ) rests on Runge's theorem. We hope to report about this in a separate paper.

6f. Remark about the Fréchet derivative. The Fréchet differential calculus for functions between Banach spaces is the version of infinite dimensional calculus first encountered by most students. Since Banach spaces are among the oLC-spaces in every special case so far encountered, moreover as a subcategory closed under finite products, the following question arises. Is there a choice of category C whose concept of C^1-map (section 2) will agree, for functions between Banach spaces, with the classical concept of 'Fréchet continuously differentiable function'? In this connection one should bear in mind the known fact that continuity of the Fréchet derivative $Df : E \to [E, F]$ is not equivalent to continuity of the differential $df : E \times E \to F$ as one would like to see. Therefore, a general theory can recover the Fréchet calculus as special case only if that theory is likewise afflicted with this pathology. Fortunately, the C-theories of this paper are not.

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