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SOME COMBINATORIAL CALCULUS ON LIE DERIVATIVE

by M. Carmen MINGUEZ*)

RÉSUMÉ. Dans le cadre de la Géométrie Différentielle Synthétique, on introduit les notions de dérivée de Lie et de produit intérieur d'une forme relativement à un champ X sur M. On définit aussi l'action d'une p-forme sur un p-uple de champs de vecteurs, et on étudie leurs propriétés et relations. Ces notions améliorent les notions classiques, à savoir que les formules obtenues synthétiquement sont valables pour n'importe quel objet linéaire infinitésimal, par exemple des objets de dimension infinie (comme des espaces vectoriels "convenables") ou des objets avec singularités.

ABSTRACT. In the context of Synthetic Differential Geometry we introduce the notions of Lie derivative and interior product of a form with respect to a field X on M. We also define the action of a p-form on a p-tuple of vector fields and we obtain their properties and relations. These notions improve the classical ones, namely the synthetically derived formulas apply to any infinitesimally linear objects, e.g., infinite dimensional objects (like convenient vector spaces) or objects with singularities.

As in the case of the exterior differential we have not been able to define the Lie derivative if we take the quasi-classical notion of form, and therefore we consider a wider notion (see [5]).

We refer the reader to [2] as general reference and to [4] for the proofs based on combinatorial calculus derived from the following Axiom 1 (stronger than Axiom 1 and weaker than Axiom 1", both in [2]). In what follows we assume that the object M is infinitesimally linear and has the property W, although this property is not needed in some situations.

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Let R stand for the basic ring, "the line", and $D(n) \subset R^n$ the "set" of n-tuples

$$\underline{d} = (d_1,\ldots,d_n) \quad \text{with} \quad d_i d_j = 0, \quad 1 \leq i, j \leq n.$$

AXIOM 1. For any $n \geq 1$, the map $\alpha: R^{n+1} \to R^{D(n)}$ defined by

$$\alpha(a_0,a_1,\ldots,a_n)(\underline{d}) = a_0 + a_1 d_1 + \ldots + a_n d_n$$

is a bijection.

DEFINITION 2. A p-form on M is a map $\omega: M^{D^p} \to R$ which is

(i) homogeneous: $\omega(\lambda\cdot_i\tau) = \lambda\omega(\tau)$, where $\lambda \in R$, $i = 1,\ldots,p$ and

$$\lambda\cdot_i\tau(d_1,\ldots,d_p) = \tau(d_1,\ldots, \lambda d_i,\ldots,d_p),$$

(ii) alternating: $\omega(\tau o D^\sigma) = sig(\sigma)\omega(\tau)$, where $\sigma \in S_p$ and

$$D^\sigma(d_1,\ldots,d_p) = (d_{\sigma(1)},\ldots,d_{\sigma(p)}) .$$

We may consider several additive structures on M^{D^p}, namely if ξ and η are p-tangents such that

$$\xi(d_1,\ldots,d_{i-1},0,d_{i+1},\ldots,d_p) = \eta(d_1,\ldots,d_{i-1},0,d_{i+1},\ldots,d_p)$$

we can define the p-tangent $\xi\oplus_i\eta$ by

$$\xi\oplus_i\eta(d_1,\ldots,d_p) = 1(d_i,d_i)(d_1,\ldots,d_{i-1},0,d_{i+1},\ldots,d_p)$$

where $1: D(2) \to M^{D^p}$ is unique since M^{D^p} is infinitesimally linear. Condition (i) implies that ω is additivie [3] with respect to these additive structures, that is

$$\omega(\xi\oplus_i\eta) = \omega(\xi) + \omega(\eta) .$$

Forms can be differentiated with respect to a vector field. If X: $M\times D \to M$ is a field, for each $h \in D$ we denote by X_h: $M \to M$ the map defined by

$$X_h(m) = X(m,h) .$$

If ω is a p-form, we consider the map

$$X_h^*(\omega) = \omega o X_h^{D^p} : M^{D^p} \to M^{D^p} \to R .$$

When τ is fixed, by applying Axiom 1 to the function sending $h \in D$ into

$$X_h^*(\omega)(\tau) = \omega(X_h o \tau) ,$$

we obtain a unique element on R, denoted $L_X\omega(\tau)$. It is easy to show that $L_X\omega(\tau)$ is a p-form.

DEFINITION 3. Given a field on M, the *Lie derivative with respect to* X is defined to be the R-linear operator

$$L_X: \Lambda^p(M) \to \Lambda^p(M), \qquad p \geq 0,$$

which assigns to each p-form ω the unique p-form $L_X\omega$ satisfying

$$\forall h \in D, \forall \tau \in M^D: \quad \omega(X_h o \tau) = \omega(\tau) + hL_X\omega(\tau) .$$

The interior multiplication with respect to a field X can be defined explicitly. Given

$$X: M \times D \to M \quad \text{and} \quad \tau: D^{p-1} \to M$$

we denote by $X \cdot \tau: D^p \to M$ the p-tangent defined by

$$X \cdot \tau(\underline{d}) = X(\tau(\underline{d}^{\hat{1}}), d_1)$$

where $\underline{d}^{\hat{1}}$ is obtained from $\underline{d}$ by omitting the first coordinate d_1.

DEFINITION 4. Let X be a vector field on M, we call *interior multiplication with respect to* X the R-linear operator

$$i_X: \Lambda^p(M) \to \Lambda^{p-1}(M)$$

defined by

$$i_X\omega(\tau) = \omega(X \cdot \tau) \text{ if } p \geq 1, \quad \text{and} \quad i_X f = 0 \text{ if } p = 0 .$$

Both operators are connected by means of the exterior differential

$$d: \Lambda^p(M) \to \Lambda^{p+1}(M)$$

PROPOSITION 5. *For each vector field* X *on* M, *the following equality is satisfied:*

$$L_X = i_X d + d i_X$$ □

We show that i_X is an antiderivation in a similar way as we proved Leibniz's formula in [5], and therefore L_X is a derivation.

PROPOSITION 6. *Given a vector field* X, *a p-form* ω *and a q-form* θ, *we have*

(i) $$i_X(\omega\wedge\theta) = i_X\omega\wedge\theta + (-1)^p\omega\wedge i_X\theta .$$

(ii) $$L_X(\omega\wedge\theta) = L_X\omega\wedge\theta + \omega\wedge L_X\theta .$$ □

When M is infinitesimally linear and has the property W, the set $\chi(M)$ of vector fields on M is a R-Lie algebra and, with respect to this structure, the operators L_- and i_- have the following properties.

PROPOSITION 7. *For every* $f\colon M \to R$ *and vector fields* X,Y *on* M, *the following equalities are satisfied:*

(i)	$i_{fX} = f i_X$,	(i)'	$L_{fX} = fL_X + df\wedge i_X$,
(ii)	$i_{X+Y} = i_X + i_Y$,	(ii)'	$L_{X+Y} = L_X + L_Y$,
(iii)	$i_{[X,Y]} = [L_X, i_Y]$,	(iii)'	$L_{[X,Y]} = [L_X, L_Y]$.

PROOF. Note that the properties in the second column are obtained from the first one by applying Proposition 5. As an example of the techniques used we will prove (iii).

Given a p-form ω and a $(p-1)$-tangent τ, for every $h \in D$ we have

$$h i_{[X,Y]}\omega(\tau) = \omega(\underline{d} \to [X,Y]_{hd_1}(\tau(\underline{d}^{\hat{1}})))$$

and

$$h[L_X, i_Y]\omega(\tau) = \omega(Y\cdot(X_h\circ\tau)) - \omega(X_h\circ(Y\cdot\tau)) = \omega(\underline{d} \to [X,Y]_{hd_1}\circ X_h(\tau(\underline{d}^{\hat{1}}))) .$$

The last equality follows because ω is additive [3] and

$$(Y.(X_h\circ\tau) \ominus_1 X_h\circ(Y\cdot\tau)))(\underline{d}) = [X,Y]_{hd_1}\circ X_h(\tau(\underline{d}^{\hat{1}})) .$$

Now we complete the proof with the help of the following lemma:

LEMMA 8. Let $X^1,\ldots,X^p$, Y be fields and $m \in M$. For each $h \in D$ and $i = 1,\ldots,p$ fixed, we consider the p-tangents ζ_h, ξ_h given by

$$\zeta_h(\underline{d}) = X^p_{d_p}\circ\ldots\circ X^i_{hd_i}\circ\ldots\circ X^1_{d_1}(m) ,$$
$$\xi_h(\underline{d}) = X^p_{d_p}\circ\ldots\circ Y_h\circ\ldots\circ X^i_{hd_i}\circ\ldots\circ X^1_{d_1}(m) .$$

Then $\omega(\zeta_h) = \omega(\xi_h)$ holds for every p-form ω on M.

PROOF. The map φ: D(2) $\to$ R defined by

$$\varphi(\delta_1,\delta_2) = \omega(\underline{d} \to X^p{}_{dp}\circ\ldots\circ Y_{\delta_1}\circ\ldots\circ X^i{}_{(\delta_1+\delta_2)d_i}\circ\ldots\circ X^1{}_{d_1}(m)),$$

satisfies

$$\varphi(\delta,0) = \omega(\xi_\delta), \quad \varphi(0,\delta) = \omega(\zeta_\delta) \quad \text{and} \quad \varphi(0,0) = 0.$$

Then

$$\varphi(h,-h) = \omega(\zeta_h) - \omega(\xi_h)$$

and $\varphi(h,-h) = 0$ since ω is applied on a p-tangent independent of d_i. □

Some of the preceding results have been found independently by R. Lavendhomme [1]. Different proofs for Proposition 6 and the parts (i), (ii) of Proposition 7 are also in [1], pp. 132-135.

When we consider forms operating on vector fields, we obtain an explicit description of $L_X\omega$, in the following way:

If $X^1,\ldots,X^p$ are fields on M, let $X^p{}_{dp}\ldots X^1{}_{d_1}(m)$ denote the p-tangent at $m \in$ M defined by

$$X^p{}_{dp}\ldots X^1{}_{d_1}(m)(\underline{h}) = X^p{}_{hp}\circ\ldots\circ X^1{}_{h_1}(m), \quad \underline{h} \in D^p$$

and let $X^1\ldots X^p$ be the p-field

$$M \to M^{D^p}, \quad m \mapsto X^p{}_{dp}\ldots X^1{}_{d_1}(m).$$

Each p-form ω: $M^{D^p} \to$ R acts on $\chi(M)^p$ by composition, that is

$$\omega: \chi(M)\times\overset{p)}{\cdots\cdots}\times\chi(M) \to R^M, \quad \omega(X^1,\ldots,X^p)(m) = \omega(X^p{}_{dp}\ldots X^1{}_{d_1}(m)).$$

In relation to the structure of R^M-module $\chi(M)$, this action has the following properties:

PROPOSITION 9.

(a) $\qquad \omega(X^1,\ldots,fX^i,\ldots,X^p) = f\omega(X^1,\ldots,X^p)$.

(b) $\qquad \omega(X^1,\ldots,X^i+Y^i,\ldots X^p) = \omega(X^1,\ldots,X^i,\ldots X^p) + \omega(X^1,\ldots,Y^i,\ldots,X^p)$.

(c) $\qquad \omega(X^{\sigma(1)},\ldots,X^{\sigma(p)}) = \mathrm{sgn}(\sigma)\omega(X^1,\ldots,X^p)$ for each $\sigma \in S_p$. □

Finally, we give a description of the operators i_X, d and L_X with respect to the action on fields.

PROPOSITION 10. *Let* ω *be a p-form and* $X, X^1, \ldots, X^{p-1}$ *vector fields on* M. *Then we have:*

(a) $$i_X\omega(X^1,\ldots,X^{p-1}) = \omega(X,X^1,\ldots,X^{p-1}) .$$

(b) $$d\omega(X^1,\ldots,X^{p+1}) = \sum_{1\le i\le p+1}(-1)^{i+1}X^i\omega(X^1,\ldots,\hat{X}^i,\ldots,X^{p+1}) + \sum_{1\le j<i\le p+1}(-1)^{i+j}\omega([X^j,X^i],X,X^1,\ldots,\hat{X}^j,\hat{X}^i,\ldots,X^{p+1}) .$$

(c) $$L_X\omega(X^1,\ldots,X^p) = X\omega(X^1,\ldots,X^p) + \sum_{1\le i\le p}(-1)^i\omega([X,X^i],X^1,\ldots,\hat{X}^i,\ldots,X^p) .$$

PROOF. (a) is straightforward; (c) follows from (a), (b) and Proposition 5. Koszul's formula (b) can be proved by using the following lemma:

LEMMA 11. Let $X^1,\ldots,X^{p+1}$ be vector fields on M and for each $m \in$ M, $h \in$ D and $i \in \{1,\ldots,p+1\}$ let $\xi_i(h)$ be the tangent defined by

$$\xi_i(h)(\underline{d}) = X^{p+1}{}_{dp}\circ\ldots\circ X^{i+1}{}_{di}\circ X^i{}_h\circ X^{i-1}{}_{di-1}\circ\ldots\circ X^1{}_{d1}(m) .$$

Then, if ω is a p-form, we have:

$$\omega(\xi_i(h)) = \omega(X^1,\ldots,\hat{X}^i,\ldots,X^{p+1})(X^i{}_h(m)) + h\sum_{j<i}(-1)^j\omega([X^i,X^j],X,X^1,\ldots,\hat{X}^j,\hat{X}^i,\ldots,X^{p+1})(m) .$$

PROOF. It is proved by induction on i. It is clear that the formula is true when $i = 1$. We suppose that it is true for $i-1$; taking into account that

$$[X^{i-1},X^i]_{hdi-1} = X^i{}_{-h}\circ X^{i-1}{}_{-di-1}\circ X^i{}_h\circ X^{i-1}{}_{di-1}$$

we have

$$\xi_i(h) = X^{p+1}{}_{dp}\ldots X^{i+1}{}_{di}X^{i-1}{}_{di-1}X^i{}_h[X^{i-1},X^i]_{hdi-1}X^{i-2}{}_{di-2}\ldots X^1{}_{d1}(m) .$$

We consider the p-tangents at $m \in$ M, $\alpha(h)$, $\beta(h)$ given by

$$\alpha(h) = X^{p+1}{}_{dp}\ldots X^{i+1}{}_{di}X^{i-1}{}_{di-1}X^i{}_hX^{i-2}{}_{di-2}\ldots X^1{}_{d1}(m)$$
$$\beta(h) = X^{p+1}{}_{dp}\ldots X^{i+1}{}_{di}X^i{}_h[X^{i-1},X^i]_{hdi-1}X^{i-2}{}_{di-2}\ldots X^1{}_{d1}(m) .$$

Since

$$\xi_i(h) = \alpha(h)\oplus_{i-1}\beta(h)$$

we have

$$\omega(\xi_i(h)) = \omega(\alpha(h)) + \omega(\beta(h)) .$$

Now, we calculate $\omega(\beta(h))$ by applying Lemma 8 to the fields

$$X^1,\dots,X^{i-2},[X^{i-1},X^i],X^{i+1},\dots,X^{p+1} \quad \text{and} \quad Y = X^i :$$

$$\omega(\beta(h)) = h\omega(X^1,\dots,X^{i-2},[X^{i-1},X^i],X^{i+1},\dots,X^{p+1})(m) = \\ = (-1)^{i-1}h\omega([X^i,X^{i-1}],X^1,\dots,X^{i-2},X^{i+1},\dots,X^{p+1}(m) .$$

Finally, $\omega(\alpha(h))$ is calculated by applying induction and observing that $\alpha(h)$ is like $\xi_{i-1}(h)$, but with the fields

$$X^1,\dots,X^{i-2},X^i,X^{i+1},\dots,X^{p+1} . \qquad \square$$

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