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ORTHOGONAL AND PREREFLECTIVE SUBCATEGORIES

by Jiří ROSICKÝ) and Walter THOLEN**)*

Dedicated to the memory of Evelyn Nelson

RÉSUMÉ. La notion de préréflexivité, originellement introduite par R. Börger, est utilisée pour étudier les intersections de sous-categories réflexives. Parmi les résultats généraux et les contr'exemples présentés dans cet article, on a: sous de faibles hypothèses sur la catégorie, les intersections de petites familles de sous-categories réflexives sont préréflexives ou, de manière équivalente, bien-pointées (au sens de Kelly), mais il y a des sous-catégories orthogonales (au sens de Freyd et Kelly) qui ne sont pas préréflexives.

This paper deals with limit-closed but not necessarily reflective subcategories. Examples of such categories in the category *Top* of topological spaces and the category of compact Hausdorff spaces were given, partly under set-theoretic restrictions, by Herrlich [H] (cf. also [K-R]) and by Trnková [Tr 1,2], Koubek [Ko] and Isbell [I] respectively. Our interest in the subject stems from recent results in the study of intersections of reflective subcategories (cf. [A-R], [A-R-T]), and from the desire for better understanding of the concept of a prereflective subcategory (cf. [B], [Th 3]). We present some general results which add to those given in the survey articles [Ke 2], [Th 4] as well as some new counter-examples.

1. ORTHOGONAL SUBCATEGORIES.

1.1. Recall [F-K] that a morphism h: M → N in a category $\mathcal{C}$ is *orthogonal* to an object B, written as $h \perp B$, if the map

$$\mathcal{C}(h,B): \mathcal{C}(N,B) \to \mathcal{C}(M,B)$$

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is bijective. For a class H of morphisms in C, one puts

$$H^{\perp} = \{B \mid h \perp B \text{ for all } h \in H\}.$$

A subcategory B of C (which is always assumed to be full and replete) is called *orthogonal* if its class of objects is $H^{\perp}$ for some class H; if H can be chosen to be small or even as a singleton set, then B is called *small-orthogonal* or *simply-orthogonal* respectively. The latter two notions are of interest to us only since every orthogonal subcategory is the intersection of a possibly large collection of simply-orthogonal subcategories; indeed for every class H one has

$$H^{\perp} = \cap_{h \in H} \{h\}^{\perp} .$$

We shall get back to this point in Section 4.

1.2. Orthogonality represents a fundamental notion for us since

(1) one has the implications

reflective ⇒ orthogonal ⇒ limit-closed ,

(2) arbitrary intersections of orthogonal subcategories are orthogonal,

according to the equation:

$$\cap (H_i)^{\perp} = (\cup H_i)^{\perp} .$$

The proof of the implications (1) is straightforward. That these implications are proper is less easy. However, since one now has good examples of non-reflective intersections of reflective subcategories, it is clear by (2) that there are non-reflective but orthogonal subcategories, for instance in *Top* (cf. [A-R]). Our further interest here will be in studying the gap between reflectivity and orthogonality (since intersections of reflective subcategories are orthogonal). For the gap between orthogonality and limit-closedness we just mention the following counter-examples:

1.3. (H. Herrlich, private communication 1986). Let Ω be the class of ordinal numbers, provided with the dual order, and add a least element. Then Ω is closed under small limits (i.e., small infima) in the resulting category C, but not orthogonal. If one adds to Ω three new elements a,b,c with

$$a < b,\ a < c \quad \text{and} \quad a,b,c < i \quad \text{for all } i \in \Omega,$$

then Ω is even closed under arbitrary limits in the resulting category $\mathcal{D}$, yet not orthogonal. Note that $\mathcal{C}$ is complete and cocomplete, but neither wellpowered nor cowellpowered; $\mathcal{D}$ is not even finitely cocomplete.

1.4. The following example of a non-orthogonal but (small-)limit-closed subcategory $\mathcal{B}$ in a category $\mathcal{C}$ with better properties than those of the category $\mathcal{C}$ in 1.3 was, in fact, given in [R, Ex. 1.4].

Let L be the language consisting of unary relation symbols R_i where i runs through the class Ω of ordinal numbers. Consider the category $\mathcal{C}$ whose objects are all L-structures which satisfy sentences

$$(\forall x,y)(R_i(x)\wedge R_i(y) \to x = y) \text{ for all } i \in \Omega,$$
$$(\forall x)(R_i(x)\wedge R_j(y) \to R_k(x)) \quad \text{for all} \quad i,j,k \in \Omega,\ i < j,k,$$
$$(\forall x)(R_i(x) \to (\exists y)R_j(y)) \quad \text{for all} \quad i,j \in \Omega\ ;$$

this means that L-structures are either sets (all R_i's are empty) or sets equipped with constants c_i (the only ones in R_i) such that

$$(c_i = c_j \to c_i = c_k) \quad \text{for all} \quad i,j,k \in \Omega, \quad i < j,k,$$

holds; morphisms in $\mathcal{C}$ preserve the R_i's.

$\mathcal{C}$ is complete, cocomplete, wellpowered and cowellpowered (even solid and strongly fibre-small over *Set*). Let $\mathcal{B}$ be the subcategory consisting of the objects of the second kind, i.e. sets with constants. It is easy to see that $\mathcal{B}$ is closed under small limits in $\mathcal{C}$. We shall show that it is not orthogonal in $\mathcal{C}$: if $h\colon M \to N$ is such that $h \perp B$ for any B in $\mathcal{B}$, then either M,N $\in$ Ob $\mathcal{B}$ or h is the identity; hence $\mathcal{C}$ is the orthogonal hull of $\mathcal{B}$ in $\mathcal{C}$.

2. WELL-POINTED AND PREREFLECTIVE SUBCATEGORIES

2.1. Recall that a pointed endofunctor (T,η) of a category $\mathcal{C}$ consists of a functor T and a natural transformation $\eta\colon Id_{\mathcal{C}} \to T$; it is *well-pointed* [K] if $T\eta = \eta T$, and it is a *prereflection* ([B], [Th 3]) if, for all $f\colon X \to Y$ and $h\colon TX \to TY$ in $\mathcal{C}$,

$$h\cdot\eta X = \eta Y\cdot f \quad \text{implies} \quad h = Tf\ .$$

A (full and replete) subcategory $\mathcal{B}$ of $\mathcal{C}$ is called *well-pointed* (*prereflective* resp.) if $\mathcal{B}$ is the subcategory $Fix(T,\eta)$ of all $\mathcal{C}$-objects B with ηB an isomorphism, for some well-pointed endofunctor (prereflection resp.) (T,η) of $\mathcal{C}$. Obviously one has the implications

$$\text{reflective} \Rightarrow \text{prereflective} \Rightarrow \text{well-pointed} \Rightarrow \text{orthogonal:}$$

for the last implication one shows that

$$\mathrm{Ob}\ \mathcal{B} = \{\eta X \mid X \in \mathrm{Ob}\ \mathcal{C}\}^{\perp}$$

if $\mathcal{B} = \mathrm{Fix}(T,\eta)$ with a well-pointed (T,η), observing that ηX is an isomorphism as soon as it is split-monomorphism.

2.2. Well-pointed subcategories are stable under binary intersections (cf. [B-K]). Indeed, for well-pointed endofunctors (T,η) and (S,ε) of $\mathcal{C}$, also

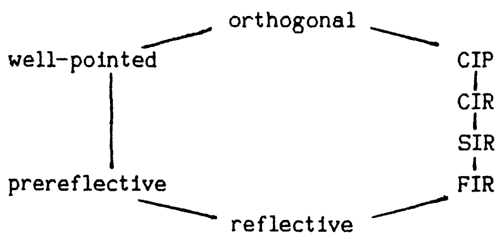
$$(ST,\ \varepsilon T\cdot\eta) = (ST,\ S\eta\cdot\varepsilon)$$

is well-pointed, and

$$\mathrm{Fix}(ST,\ \varepsilon T\cdot\eta) = \mathrm{Fix}(T,\eta)\cap\mathrm{Fix}(S,\varepsilon)\ .$$

2.3. A subcategory $\mathcal{B}$ of $\mathcal{C}$ is called FIR (SIR, CIR resp.) if it is the intersection of a finite (small, class-indexed resp.) collection of reflective subcategories of $\mathcal{C}$. It is called CIP if it is the intersection of a class-indexed collection of prereflective subcategories.

From 1.2, 2.1 and 2.2, one obtains the following system of implications for full subcategories in any category $\mathcal{C}$ (to be read from bottom to top):



Our next aim will be in simplifying this scheme when $\mathcal{C}$ satisfies mild completeness and smallness assumptions.

3. TRANSFINITE CONSTRUCTIONS.

3.1. Kelly [Ke 1, Prop. 9.1]) shows that, for any collection of well-pointed endofunctors (T_i, η_i) of a category $\mathcal{C}$, the fibred coproduct η: $\mathrm{Id}_{\mathcal{C}} \to T$ of the family $(\eta_i\colon \mathrm{Id}_{\mathcal{C}} \to T_i)_i$ gives a well-pointed endofunctor with

$$\mathrm{Fix}(T,\eta) = \cap_i \mathrm{Fix}(T_i,\eta_i) .$$

We have another result of this kind:

3.2. PROPOSTION. *In any category having colimits of chains, any intersection of a set of well-pointed subcategories is well-pointed.*

PROOF. Let $\mathcal{C}$ have colimits of chains, n be a cardinal number and (T_i,η_i), $i < n$, a set of well-pointed endofunctors of $\mathcal{C}$. We define a chain of endofunctors S_i, $i \leq n$, with bounding morphisms $\sigma_{ij}\colon S_i \to S_j$, $0 \leq i \leq j \leq n$, as follows:

$$S_0 = \mathrm{Id}_{\mathcal{C}} ,$$
$$S_{j+1} = T_j S_j, \quad \sigma_{i,j+1} = \eta_i S_j \cdot \sigma_{ij} ,$$
$$S_k = \mathrm{colim}_{i<k}\, S_i , \text{ with canonical injections } \sigma_{ik}$$
(for a limit ordinal k).

Setting $\varepsilon_j = \sigma_{0j}$, it is enough to show that (S_j,ε_j), $j \leq n$, are well-pointed endofunctors and

$$\mathrm{Fix}(S_j,\varepsilon_j) = \cap_{i<j} \mathrm{Fix}(T_i,\eta_i) .$$

The isolated step follows from 2.2. For the limit step, one has

$$\begin{aligned}
S_k\varepsilon_k \cdot \sigma_{ik} &= S_k\sigma_{ik} \cdot S_k\varepsilon_i \cdot \sigma_{ik} && (\text{since } \varepsilon_k = \sigma_{ik}\varepsilon_i)\\
&= S_k\sigma_{ik} \cdot \sigma_{ik}S_i \cdot S_i\varepsilon_i \\
&= S_k\sigma_{ik} \cdot \sigma_{ik}S_i \cdot \varepsilon_i S_i && ((S_i,\varepsilon_i) \text{ is well-pointed})\\
&= S_k\sigma_{ik} \cdot \varepsilon_k S_i \\
&= \varepsilon_k S_k . \sigma_{ik}
\end{aligned}$$

for all $i < k$, so (S_k,ε_k) is well-pointed. Let now X be in $\mathrm{Fix}(S_k,\varepsilon_k)$, so $\varepsilon_k X$ is an isomorphism, that is: $\sigma_{ik}X \cdot \varepsilon_i X$ is an isomorphism for all $i < k$. Since (S_i,ε_i) is well-pointed, the split-monomorphism $\varepsilon_i X$ must be an isomorphism, hence X is in $\mathrm{Fix}(S_i,\varepsilon_i)$ for all $i < k$. Consequently, X belongs to

$$\mathrm{Fix}(S_{i+1},\varepsilon_{i+1}) \subset \mathrm{Fix}(T_i,\eta_i)$$

for all $i < k$. Vice versa, if we assume that $\eta_i X$ is an isomorphism for all $i < k$, then also $\varepsilon_k X$ is an isomorphism (since the colimit is constructed pointwise). □

3.3. Suppose now that we are given a large collection of well-pointed endofunctors (T_i,η_i), with $i \in I = \Omega$, say. Individually for each object X, we may still form the chain from 3.2, and under the assumption that

$(*)$ $\forall X \in \mathrm{Ob}\,\mathcal{C}\ \exists i_X \in I\ \forall j \geq i_X : \eta_j S_j X : S_j X \to T_j S_j X$ is an isomorphism,

we may define

$$S_\infty = S_{i_X} X, \quad \varepsilon_\infty X = \varepsilon_{i_X} X .$$

Since, by $(*)$, $\sigma_{ij} X$ is an isomorphism for all $j \geq i \geq i_X$, S_∞ becomes a functor and ε_∞ a natural transformation when we set

$$S_\infty f = (\sigma_{i_Y j} Y)^{-1} \cdot S_j f \cdot \sigma_{i_X j} X$$

for $f: X \to Y$ in $\mathcal{C}$ and $j \geq \max\{i_X, i_Y\}$. It is easy to check that $(S_\infty,\varepsilon_\infty)$ is well-pointed and

$$\mathrm{Fix}(S_\infty,\varepsilon_\infty) = \cap_{i\in\Omega} \mathrm{Fix}(T_i,\eta_i) .$$

The construction contains, as a special case, the iteration process as considered for a well-pointed endofunctor in [Ke 1] and for a prereflection in [Th 4]: taking (T_i,η_i) always the given (T,η) one obtains the chain of powers of (T,η). The "convergence" condition $(*)$ can be simplified to

$$\forall X \in \mathrm{Ob}\,\mathcal{C}\ \exists i_X \in I : \eta S_{i_X} X \text{ is an isomorphism.}$$

A category is called *weakly cowellpowered* if it is cowellpowered with respect to strong epimorphisms.

3.4. PROPOSITION. *In each weakly cowellpowered category with connected colimits, prereflective subcategories coincide with the well-pointed ones.*

PROOF. For a given well-pointed endofunctor (T,η) on $\mathcal{C}$ we shall construct a natural transformation $\varphi = \varphi(T): T \to T'$ as follows: for each object A in $\mathcal{C}$ let $\varphi A: TA \to T'A$ be the joint coequalizer of all pairs

$$Tf, h: TX \to TA \quad \text{with} \quad f: X \to A,\ h\cdot\eta X = \eta A\cdot f .$$

By Kelly's [Ke 1, Prop. 9.1], one has that, with $\eta' = \varphi\eta$, (T',η') is well-pointed and

$$\mathrm{Fix}(T',\eta') = \mathrm{Fix}(T,\eta) \cap \{A \mid \varphi A \text{ is an isomorphism}\}.$$

In our situation we even have

$$\mathrm{Fix}(T',\eta') = \mathrm{Fix}(T,\eta)$$

since, if ηA is an isomorphism, for any pair f,h with $h\cdot\eta X = \eta A\cdot f$, one has

$$T\eta A\cdot h = \eta TA.h = Th\cdot\eta TX = Th\cdot T\eta X = T\eta A\cdot Tf\ ,$$

so $h = Tf$. Hence the joint coequalizer φA is an isomorphism, too.

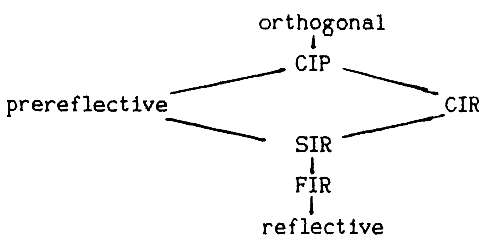
We can now iterate the '-operation, defining a chain of endofunctors R_i, $i \in \Omega$, with bounding morphisms ρ_{ij} by

$$R_0 = T,$$
$$R_{j+1} = (R_j)', \quad \rho_{ij+1} = \varphi(R_j)\cdot\rho_{ij}\ ,$$
$$R_k = \mathrm{colim}_{i<k}\, R_i\ ,\ \text{with canonical injections } \rho_{ik}$$
(for a limit ordinal k).

Since $\mathcal{C}$ is weakly cowellpowered, the above chain becomes stationary. So very similarly to 3.3, we can define an endofunctor R_∞ and a natural transformation ρ_∞ with $\rho_\infty A = \rho_{0 i_A} A\cdot\eta A$ for some i_A (depending on $A \in \mathrm{Ob}\,\mathcal{C}$). It is evident that (R_∞,ρ_∞) is a prereflection and

$$\mathrm{Fix}(R_\infty,\rho_\infty) = \mathrm{Fix}(T,\eta)\ .$$

3.5. From 3.4 and 2.3 on has that, in a weakly cowellpowered category with connected colimits, the scheme 2.3 simplifies to



4. ORTHOGONALITY VERSUS PREREFLECTIVITY.

We have seen in 1.1 that every (small-)orthogonal subcategory is the intersection of a (small) collection of simply-orthogonal subcategories. Kelly [Ke 1, §10] gives an elegant proof that in a cocomplete category simply-orthogonal subcategories are well-pointed (see 4.2) below, using the following result due to Wolff [W]:

4.1. For an adjunction $F \dashv G\colon \mathcal{C} \to \mathcal{D}$ and a well-pointed endofunctor (S,ε) of $\mathcal{D}$, let (T,η) be constructed pointwise by the pushout

$$\begin{array}{ccc} FG & \xrightarrow{\;F\varepsilon G\;} & FSG \\ \downarrow & & \downarrow \\ \mathrm{Id} & \xrightarrow{\;\eta\;} & T \end{array}$$

with the left vertical arrow the counit of the adjunction. Then (T,η) is a well-pointed endofunctor of $\mathcal{C}$ with

$$\mathrm{Fix}(T,\eta) = G^{-1}\mathrm{Fix}(S,\varepsilon) .$$

4.2. For a given morphism $h\colon M \to N$ in $\mathcal{C}$, let $G\colon \mathcal{C} \to \mathit{Set}^2$ be the functor sending A to the map $\mathcal{C}(h,A)\colon \mathcal{C}(N,A) \to \mathcal{C}(M,A)$. If $\mathcal{C}$ is cocomplete, its left adjoint F sends a set mapping $u\colon X \to Y$ to the $\mathcal{C}$-object Fu, constructed by the pushout

$$\begin{array}{ccc} X\cdot M & \xrightarrow{\;u\cdot M\;} & Y\cdot M \\ {\scriptstyle X\cdot h}\downarrow & & \downarrow \\ X\cdot N & \longrightarrow & Fu \end{array}$$

in $\mathcal{C}$; here $X\cdot M$ is the coproduct of X copies of M in $\mathcal{C}$. There is a (pre)reflection (S,ε) on Set^2 with $Su = 1_Y$ and ε_u given by the square

$$\begin{array}{ccc} X & \xrightarrow{\;u\;} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle 1} \\ Y & \xrightarrow{\;1\;} & Y \end{array}$$

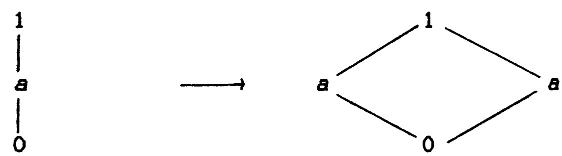
Obviously, u is in Fix(S,ε) iff u is bijective. Therefore

$$A \in \{h\}^{\perp} \Leftrightarrow \mathcal{C}(h,A) \text{ bijective} \Leftrightarrow A \in G^{-1}\mathrm{Fix}(S,\varepsilon)\ .$$

So, from 3.2 and 4.2 one obtains (it also follows by 3.1):

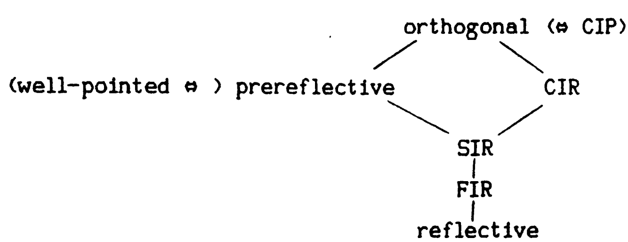
4.3. PROPOSITION. *In a cocomplete category $\mathcal{C}$, every small-orthogonal subcategory is well-pointed.*

4.4. EXAMPLE. The category *CBool* of complete Boolean algebras is a simply-orthogonal subcategory in the category *Frm* of frames (complete lattices with $x \wedge \bigvee y_i = \bigvee x \wedge y_i$, morphisms preserve $\wedge$ and $\vee$). Indeed, in a frame B, complements exist iff B is orthogonal to the embedding



By 4.3 *CBool* is prereflective in *Frm* (but not reflective since, otherwise, free complete Boolean algebras would exist). A suitable prereflection is given by the frame of nuclei on a frame (cf. [J]) or, equivalently, by adjoining complements in a straightforward manner (cf. [J-T]). The prereflectivity of *CBool* was mentioned in [Th 3]; we do not know whether *CBool* is an intersection of reflective subcategories of *Frm*.

4.5. From 3.5 and 4.3 one has that, in a cocomplete and weakly cowell-powered category, the scheme 2.3 simplifies to



We suppose that each of these implications is proper and that there are no other implications in general. Part of this conjecture will be proved by the example we present in Section 5.

Before presenting the example, however, we want to point out that any such example has to be found in "unranked" categories since, under rank-conditions, one needs set-theoretic hypotheses to find a non-reflective limit-closed subcategory (cf. [A-R-T]).

5. AN ORTHOGONAL BUT NON-PREREFLECTIVE SUBCATEGORY.

5.1. Let $\mathcal{L}$ be the language consisting of unary relation symbols P_i and binary relation symbols R_i where i runs through the class Ω of ordinal numbers. Consider the category $\mathcal{C}$ whose objects are all $\mathcal{L}$-structures which satisfy the sentences

(1) $(\forall x)(P_i(x) \wedge P_j(x) \to P_k(x))$ for all $i,j,k \in \Omega$, $i \neq j$,

(2) $(\forall x,y,z)(R_i(x,y) \to (\exists ! t) R_i(z,t))$ for all $i \in \Omega$,

(3) $(\forall x,y,z,t)(R_i(x,y) \wedge R_j(x,y) \wedge R_i(z,t) \to R_j(z,t))$ for all $i,j \in \Omega$.

(4) $(\forall x,y)(R_i(x,y) \wedge R_j(x,y) \to R_k(x,y))$ for all $i < j$, $i \leq k$.

Axiom (2) says that whenever R_i is defined somewhere, then it must determine a unary operation; (3) means that, whenever R_i and R_j coincide somewhere, then they must coincide everywhere. By (4) and (3), if R_i and R_j coincide for $i < j$ then $R_i = R_k$ for any $k \geq i$. The morphisms of $\mathcal{C}$ are the usual homomorphisms, i.e. mappings which preserve the given relations. It is easy to check that $\mathcal{C}$ is a complete, cocomplete, weakly wellpowered and weakly cowellpowered (it is moreover solid over *Set*).

Let $\mathcal{B}$ be the subcategory of $\mathcal{C}$ consisting of all $\mathcal{L}$-structures which, for all $i \in \Omega$, satisfy the sentence

$$(4_i) \qquad (\forall x)(P_i(x) \to (\exists ! y) R_i(x,y)).$$

$\mathcal{B}$ is orthogonal in $\mathcal{C}$; it is even the intersection of reflective subcategories $\mathcal{B}_i$ in $\mathcal{C}$ where, for each $i \in \Omega$, $\mathcal{B}_i$ is the subcategory of objects satisfying (4_i) (any small subcategory is reflective in $\mathcal{C}$ by [Ke_1, Theorems 10.1 and 10.2] applied to E = epis, M = extremal monos).

5.2. We shall prove that $\mathcal{B}$ is not prereflective in $\mathcal{C}$, hence

(CIR $\not\Rightarrow$ prereflective) and (orthogonal $\not\Rightarrow$ prereflective) in the scheme 4.5.

Assume that there is a prereflection (T,η) of $\mathcal{C}$ with $\mathrm{Fix}(T,\eta) = \mathcal{B}$. Consider the following $\mathcal{L}$-structures A_i, A_∞ and B_i for all $i \in \Omega$, determined by their underlying sets $|*|$ and relations as follows (we indicate only those which are present in these structures):

$$|A_i| = \{0\}, \quad P_i(0) \ ,$$
$$|A_\infty| = \{0\}, \quad P_i(0) \quad \text{for all } i \in \Omega \ ,$$
$$|B_i| = \mathbf{N} \text{ (the integers} \geqslant 0), \ P_i(0),$$
$$R_i(n,n+1) \text{ for all } i \in \Omega \text{ and } n \in \mathbf{N}.$$

Clearly, all A_i, A_∞ and B_i belong to $\mathcal{C}$, but only the B_i's belong to $\mathcal{B}$. We shall write η_i, η_∞ instead of ηA_i, ηA_∞ resp.

Let

$$S = \{i \in \Omega \mid R_i(\eta_\infty(0),a) \text{ for some } a \in TA_\infty\}.$$

If S is a proper class then

$$R_i(\eta_\infty(0),a) \quad \text{and} \quad R_j(\eta_\infty(0),a)$$

for different ordinals i,j and $a \in TA_\infty$. But this is impossible since there certainly is a morphism $h\colon A_\infty \to B$ with $B \in \mathrm{Ob}\,\mathcal{B}$ such that

$$R_i(h(0),b) \quad \text{and} \quad R_j(h(0),c)$$

with different $b,c \in |B|$, and we cannot factorize h through η_∞. Hence S must be a small set, and we have an ordinal i with $i \notin S$. In what follows we shall analyze η_i for such an $i \in \Omega$.

Since the morphism $f_i\colon A_i \to B_i$ with $f_i(0) = 0$ factorizes through η_i, one cannot have $a \in |TA_i|$ or $b,c \in |TA_i|$ such that $P_j(a)$ or $R_j(b,c)$ resp. for $j \neq i$. However, $R_i(b,c)$ cannot hold either since $i \notin S$ and since there is a morphism $A_i \to A_\infty$ sending 0 to 0 (notice that, by (2), $R_i(u,v)$ does not hold for any $u,v \in |A_\infty|$). Hence the constant mapping g with value 0 gives a homomorphism $TA_i \to A_i$. Since g must be a retraction of η_i, η_i must be actually an isomorphism, so $A_i \in \mathrm{Ob}\,\mathcal{B}$; this is a contradiction.

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